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A Direct Proof for the Existence of Best Approximation of Random Signals

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Abstract
We give a direct proof in this paper for the existence of best approximation of random signals in unit disc, using the asymptotic behavior of the elliptic integral of the first kind.

Keywords Takenaka–Malmquist system · Random signal · Best approximation · Stochastic Hardy space

Mathematics Subject Classification 30B99 · 30E10 · 41A20 · 60G10

1 Introduction

Let D be the unit disc in the complex plane, $H^2(D)$ be the Hardy space consisting of holomorphic functions f on D that satisfy

$$\sup_{0\le r<1}\int_0^{2\pi}|f(re^{it})|^2dt<\infty.$$

This research is dedicated in memory of Frank Sommen.

Communicated by Irene Sabadini.

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The inner product of $f, g \in H^2(D)$ is defined by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\overline{g(e^{it})}dt,$$

and the induced norm is denoted by $\|\cdot\|$.

For any $f \in H^2(D)$ we have

$$f(z) = \sum_{k=0}^{\infty} c_k z^k = \sum_{k=0}^{\infty}\langle f, z^k\rangle z^k, \text{ and } \|f\|^2 = \sum_{k=0}^{\infty}|c_k|^2.$$

If $f \in L^2(\partial D)$ (square integrable on ∂D) is real valued, it can be related to $H^2(D)$ by the following relation

$$f(e^{it}) = 2\mathrm{Re}f^+(e^{it}) - c_0$$

for a.e. $t \in [0, 2\pi)$, where $f^+(e^{it})$ is the boundary limit of $f^+(z) = \sum_{k=0}^{\infty} c_k z^k \in H^2(D)$, and c_k $(k = 0, 1, 2, \ldots)$ is the Fourier coefficient of f.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of points in D, the Takenaka–Malmquist (TM) system $\{B_n\}_{n=1}^{\infty}$ is defined by

$$B_n(z) = B_{a_1,\ldots,a_n}(z) := \frac{\sqrt{1-|a_n|^2}}{1-\overline{a_n}z}\prod_{k=1}^{n-1}\frac{z-a_k}{1-\overline{a_k}z}, \quad z \in D.$$

TM system is orthonormal in $H^2(D)$, and is complete in $H^p(D)$ $(1 \le p < \infty)$ if and only if $\sum_{n=1}^{\infty}(1-|a_n|) = \infty$. If $a_n = 0$ for all n, $\{B_n\}_{n=1}^{\infty}$ reduces to the Fourier system $\{z^k\}_{k=0}^{\infty}$.

A function f is called a Blaschke form of order n, if f can be written as

$$f = \sum_{k=1}^{n} c_k B_k,$$

where $c_1, \ldots, c_n$ are constants and $c_n \neq 0$.

In [9, 14] the adaptive approximation theory in $H^2(D)$ associated to $\{B_n\}_{n=1}^{\infty}$ was built up. By that theory, every $f \in H^2(D)$ admits the expansion

$$f = \sum_{n=1}^{\infty}\langle f, B_n\rangle B_n$$

in both H^2 and pointwise sense, where $\{a_n\}_{n=1}^{\infty}$ was adaptively chosen step by step according to the Maximum Selection Principle:

$$a_n = \arg\max_{a\in D}|\langle f, B_n\rangle|^2 = \arg\max_{a\in D}|\langle f, B_{a_1,\ldots,a_{n-1},a}\rangle|^2$$

for previously selected $a_1, \ldots, a_{n-1}$, and $\langle f, B_n \rangle$ can be calculated by the formula:

$$\langle f, B_n \rangle = \langle f_n, B_n \rangle = \langle \tilde{f}_n, e_{a_n} \rangle = \sqrt{1 - |a_n|^2}\, \tilde{f}_n(a_n),$$

where

$$e_{a_n}(z) = \frac{\sqrt{1 - |a_n|^2}}{1 - \overline{a_n} z}$$

is the normalized Szegö kernel of D,

$$f_n := f - \sum_{k=1}^{n-1} \langle f, B_k \rangle B_k$$

is the n-th standard remainder, and

$$\tilde{f}_n(z) := f_n(z) \prod_{k=1}^{n-1} \frac{1 - \overline{a_k} z}{z - a_k}$$

is the n-th reduced remainder. The maximum selection principle makes each component attain its maximal energy to best match the original signal, it is the heart of the adaptive approximation.

The adaptive approximation theory has been generalized to many other situations, including Drury–Arveson space and matrix-valued functions ([1, 2]), slice quaternionic regular functions ([7]), Hardy space on polydisk ([11]), complex stochastic Hardy space ([12]), weighted Bergman space ([18]), and Clifford monogenic Hardy space ([21]) etc, and has been applied to control theory and signal processing. Note that for a given n, step-by-step selection of $a_1, \ldots, a_n$ according to maximum selection principle can not guarantee that the total energy $\sum_{k=1}^{n} |\langle f, B_k \rangle|^2$ attains it maximum over all possible selections of $a_1, \ldots, a_n$ as an n-group. In [15] the authors generalized the maximum selection principle to the case of n parameters by proving the following theorem:

Theorem 1.1 *Let $f \in H^2(D)$, n be any given positive integer. If f is not a Blaschke form of order less than n, then there exists $a_1, \ldots, a_n \in D$ such that*

$$\left\| \sum_{k=1}^{n} \langle f, B_{a_1,\ldots,a_k} \rangle B_{a_1,\ldots,a_k} \right\|^2 = \sum_{k=1}^{n} |\langle f, B_{a_1,\ldots,a_k} \rangle|^2$$

attains its maximum.

Theorem 1.1 implies the existence of the solution (which has been proved in [3, 8]) to the n-best rational approximation problem: for $f \in H^2(D)$, find polynomials p and q of which the degrees do not exceed n, such that $\|f - p/q\|$ attains its minimum. The problem has a long history ([4, 19]), it is closely related to Z-transform that has applications in control theory. Theorem 1.1 was generalized to random signals in [17]

by the method of Poisson integral used in [10], and to weighted Bergman space by subtle estimates on the TM systems generalized by the kernels ([18]). We note here that the proof of Theorem 1.1 given in [22] made use of pseudohyperbolic distance.

In this paper, we shall demonstrate a direct proof for an analogue of Theorem 1.1 for random signals, by the asymptotic behavior of the elliptic integral of the first kind. There are two reasons for doing this. The first one is due to the study of the generalization of best approximation to higher dimensions and other kinds of function spaces. The main difficulty lies in the estimate of the Gram–Schmidt (GS) orthogonalization of the reproducing kernel, as it does not have a clear or regular expression like TM systems. The second one is due to the need of searching for practical and effective algorithms for best rational approximation, that is an NP hard problem. Interested readers can refer to [5, 10, 13].

2 Random Signals and the Best Approximation

Let (Ω, μ) be a probability space, a function $F(t, \omega)$ $((t, \omega) \in \partial D \times \Omega)$ is called a random signal in the unit disc, if for a.s. sample point $\omega \in \Omega$, the function $F(\cdot, \omega)$ is a deterministic signal of finite energy, and for a.e. $t \in \partial D$, $F(t, \cdot)$ is a random variable. For convenience we also write $F(t, \omega)$ as $f(e^{it}, \omega)$, or $f_\omega(e^{it})$ $((t, \omega) \in [0, 2\pi) \times \Omega)$. Since f is a.s. of finite energy, we have the Fourier expansion

$$f(e^{it}, \omega) = \sum_{k=-\infty}^{\infty} c_k(\omega) e^{ikt}, \quad \text{with } c_k = \frac{1}{2\pi} \int_0^{2\pi} f(e^{is}, \omega) e^{-iks} ds.$$

By Plancherel theorem we get

$$\|f_\omega\|^2 = \sum_{k=-\infty}^{\infty} |c_k(\omega)|^2.$$

The energy expectation norm (see [12]) of f is defined by

$$\|f\|_{\mathcal{N}} := (E_\omega(\|f_\omega\|^2))^{\frac{1}{2}} = \left(\sum_{k=-\infty}^{\infty} E_\omega(|c_k(\omega)|^2) \right)^{\frac{1}{2}},$$

where E_ω denotes the mathematical expectation. Random signals of finite energy expectation norm form a Hilbert space, denoted by $L^2_\omega(\partial D)$, i.e.,

$$L^2_\omega(\partial D) = \{f : f \text{ is a random signal and } \|f\|_{\mathcal{N}} < \infty\}.$$

To make use of complex analysis, we also consider the stochastic Hardy space $H^2_\omega(D)$, defined by

$$H^2_\omega(D) = \left\{ f : f(z,\omega) = f_\omega(z) = \sum_{k=0}^{\infty} c_k(\omega) z^k, \text{ and } \|f\|^2_{\mathcal{N}} = \sum_{k=0}^{\infty} E_\omega(|c_k(\omega)|^2) < \infty \right\}.$$

Functions in $H^2_\omega(D)$ are also referred as random holomorphic (or analytic) signals. Similar to the deterministic case, for a real valued random signal $f \in L^2_\omega(\partial D)$, we have

$$f_\omega(e^{it}) = 2\mathrm{Re} f^+_\omega(e^{it}) - c_0(\omega),$$

where $f^+_\omega(e^{it})$ is the boundary limit of $f^+_\omega(z) = \sum_{k=0}^{\infty} c_k(\omega) z^k \in H^2_\omega(D)$, and $c_k(\omega)$ $(k = 0, 1, 2, \ldots)$ is the Fourier coefficient of f_ω.

A random analytic signal $f \in H^2_\omega(D)$ is called a random Blaschke form of order n, if f can be written as

$$f = \sum_{k=1}^{n} c_k(\omega) B_k,$$

where $c_1(\omega), \ldots, c_n(\omega)$ are random variables and $c_n(\omega) \neq 0$.

The purpose of this paper is to give a direct proof of the existence of best approximation by Blaschke forms in $H^2_\omega(D)$ (cf. [17]):

Theorem 2.1 *Let $f \in H^2_\omega(D)$, n be any given positive integer. If f is not a random Blaschke form of order less than n, then there exists $a_1, \ldots, a_n \in D$ such that*

$$\left\| \sum_{k=1}^{n} \langle f_\omega, B_{a_1,\ldots,a_k} \rangle B_{a_1,\ldots,a_k} \right\|^2_{\mathcal{N}} = \sum_{k=1}^{n} E_\omega \left(|\langle f_\omega, B_{a_1,\ldots,a_k} \rangle|^2 \right)$$

attains its maximum.

Proof First, we note that $B_{a_1,\ldots,a_k}$ $(1 \leq k \leq n)$ can be obtained through GS orthogonalization process applying to the Szegö kernels or higher order Szegö kernels of D (see [12, 20]), which implies that the total energy $\sum_{k=1}^{n} E_\omega(|\langle f_\omega, B_k \rangle|^2)$ is independent of the arrangement order of $a_1, \ldots, a_n$. So, to prove Theorem 2.1, it suffices to show that the energy of the last term satisfies

$$\lim_{|a_n| \to 1} E_\omega(|\langle f_\omega, B_n \rangle|^2) = \lim_{|a_n| \to 1} E_\omega(|\langle f_\omega, B_{a_1,\ldots,a_n} \rangle|^2) = 0 \tag{1}$$

uniformly for $a_1, \ldots, a_{n-1} \in D$. For convenience, we may assume that f_ω is a.s. bounded. Otherwise, we can choose some k large enough and let

$$g_\omega := \sum_{l=0}^{k} \langle f_\omega, z^l \rangle z^l$$

which is a.s. bounded, such that $\|f_\omega - g_\omega\|_{\mathcal{N}}^2 = E_\omega(\|f_\omega - g_\omega\|^2)$ is small enough. Since

$$\begin{aligned}|\langle f_\omega, B_n\rangle|^2 &\le (|\langle f_\omega - g_\omega, B_n\rangle| + |\langle g_\omega, B_n\rangle|)^2\\ &\le (\|f_\omega - g_\omega\| + |\langle g_\omega, B_n\rangle|)^2\\ &\le 2\left(\|f_\omega - g_\omega\|^2 + |\langle g_\omega, B_n\rangle|^2\right),\end{aligned}$$

we get

$$E_\omega(|\langle f_\omega, B_n\rangle|^2) \le 2E_\omega\left(\|f_\omega - g_\omega\|^2 + |\langle g_\omega, B_n\rangle|^2\right),$$

then (1) will follow if

$$\lim_{|a_n|\to 1} E_\omega(|\langle g_\omega, B_n\rangle|^2) = 0.$$

Now, let f_ω be a.s. bounded, i.e., there is a random variable C_ω such that $|f_\omega(z)| \le C_\omega$ for all $z \in D$. Then

$$\begin{aligned}|\langle f_\omega, B_n\rangle| &= \frac{1}{2\pi}\left|\int_0^{2\pi} f_\omega(e^{it})\overline{\frac{\sqrt{1-|a_n|^2}}{1-\overline{a_n}e^{it}}\prod_{k=1}^{n-1}\frac{e^{it}-a_k}{1-\overline{a_k}e^{it}}}\,dt\right|\\ &\le \frac{1}{2\pi}\int_0^{2\pi}|f_\omega(e^{it})|\frac{\sqrt{1-|a_n|^2}}{|1-\overline{a_n}e^{it}|}dt\\ &\le \frac{C_\omega}{2\pi}\sqrt{1-r^2}\int_0^{2\pi}\frac{dt}{|1-\overline{a_n}e^{it}|}\quad (r=|a_n|)\\ &= \frac{C_\omega}{2\pi}\sqrt{1-r^2}\int_0^{2\pi}\frac{d\theta}{\sqrt{1+r^2+2r\cos\theta}}\\ &= \frac{2C_\omega}{\pi}\sqrt{\frac{1-r}{1+r}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\frac{4r}{(1+r)^2}\sin^2\theta}}.\end{aligned}$$

Consider the elliptic integral of the first kind

$$K(\rho) = \int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\rho^2\sin^2\theta}},$$

from [6, 16] we know that

$$\frac{\log\rho'}{\rho'-1} < K(\rho) < \frac{\pi}{2}\frac{\log\rho'}{\rho'-1},\quad \text{with } \rho' = \sqrt{1-\rho^2}.$$

Therefore,

$$|\langle f_\omega, B_n\rangle|^2 \le \left(C_\omega\cdot\frac{r+1}{-2r}\cdot\sqrt{\frac{1-r}{1+r}}\log\frac{1-r}{1+r}\right)^2 \to 0$$

for a.s. $\omega \in \Omega$ when $r \to 1$. Note that $|\langle f_\omega, B_n\rangle|^2 \le \|f_\omega\|^2$ and $E_\omega(\|f_\omega\|^2) < \infty$, by the Lebesgue dominated convergence theorem we conclude the proof of (1). The proof of Theorem 2.1 is now complete. □

Remark 2.2 Our method is heavily dependent on the special construction of TM systems: the n-th function in the system is a product of n factors respectively determined by one and only one of the n parameters. In a general dictionary case the GS orthogonalization process does not give rise to such orthogonal systems. The above proof is even a quick one owing to the fact that the energy of the projection onto the span of selected parameterized kernels is invariant with respect to the order of the selected n parameters. The method can also be applied to the context of Hardy space of the upper half complex plane. There, however, exists cases in hypercomplex and several complex variables having analogous TM systems but relevant with the order of parameters (see [7] for the slice quaternionic case and [11] for the polydisc case using product-TM systems). In some of the non-standard cases the method we use in proving Theorem 2.1 is still valid after certain modification. Below we sketch a proof of Theorem 2.1 without involving the use of commutativity.

The first step is still to prove (1) as before. Next, assume that f is not a random Blaschke form of order less than n, then it follows from (1) that the supremum of $\sum_{k=1}^n E_\omega(|\langle f_\omega, B_k\rangle|^2)$ should take for some $a_n \in D$ and some $a_1, \ldots, a_{n-1} \in \overline{D}$. In what follows we will show that for any $a_n \in D$, as a function of $a_1, \ldots, a_{n-1}$, $\sum_{k=1}^n E_\omega(|\langle f_\omega, B_k\rangle|^2)$ could take the maximum in the interior of D^{n-1}. For convenience, we consider the special case $n = 3$, the general case will easily follow by induction. By (1) and the compactness of $\overline{D}$, one can choose a sequence of tuples $\{(a_1^j, a_2^j, a_3^j)\}_{j=1}^\infty$ such that

$$a_1^j \to a_1' \in \overline{D}, a_2^j \to a_2' \in \overline{D}, a_3^j \to a_3' \in D,$$

and

$$\begin{aligned} &E_\omega(|\langle f_\omega, B_{a_1^j}\rangle|^2) + E_\omega(|\langle f_\omega, B_{a_1^j,a_2^j}\rangle|^2) + E_\omega(|\langle f_\omega, B_{a_1^j,a_2^j,a_3^j}\rangle|^2) \\ \to &\sup_{(a_1,a_2,a_3)\in D^3} \left\{E_\omega(|\langle f_\omega, B_{a_1}\rangle|^2) + E_\omega(|\langle f_\omega, B_{a_1,a_2}\rangle|^2) + E_\omega(|\langle f_\omega, B_{a_1,a_2,a_3}\rangle|^2)\right\} \end{aligned} \tag{2}$$

as $j \to \infty$. We will conclude the result by excluding the following three cases:

Case (i): $a_1' \in \partial D$, $a_2' \in D$. Similar with the reasoning of (1) we have

$$\lim_{j\to\infty} E_\omega(|\langle f_\omega, B_{a_1^j}\rangle|^2) = 0. \tag{3}$$

Since

$$\lim_{j\to\infty} \frac{e^{it} - a_1^j}{1 - \overline{a_1^j} e^{it}} = \frac{e^{it} - a_1'}{1 - \overline{a_1'} e^{it}} = \frac{-a_1'(1 - \overline{a_1'} e^{it})}{1 - \overline{a_1'} e^{it}} = -a_1', \text{ for a.e. } t \in [0, 2\pi),$$

$$\lim_{j\to\infty}\frac{e^{it}-a_2^j}{1-\overline{a_2^j}e^{it}}=\frac{e^{it}-a_2'}{1-\overline{a_2'}e^{it}},$$

and

$$\left|f_\omega(e^{it})\overline{B_{a_1^j,a_2^j}(e^{it})}\right|\le C|f_\omega(e^{it})|,$$

$$\left|f_\omega(e^{it})\overline{B_{a_1^j,a_2^j,a_3^j}(e^{it})}\right|\le C|f_\omega(e^{it})|,$$

with

$$\frac{1}{2\pi}\int_0^{\frac{\pi}{2}}|f_\omega(e^{it})|dt\le\left(\frac{1}{2\pi}\int_0^{\frac{\pi}{2}}|f_\omega(e^{it})|^2dt\right)^{\frac{1}{2}}<\infty.$$

By the Lebesgue dominated convergence theorem for fixed $\omega\in\Omega$ we have

$$\begin{aligned}\lim_{j\to\infty}|\langle f_\omega,B_{a_1^j,a_2^j}\rangle|&=\frac{1}{2\pi}\left|\lim_{j\to\infty}\int_0^{\frac{\pi}{2}}f_\omega(e^{it})\overline{B_{a_1^j,a_2^j}(e^{it})}dt\right|\\&=\frac{1}{2\pi}\left|\int_0^{\frac{\pi}{2}}f_\omega(e^{it})\overline{B_{a_2'}(e^{it})\cdot(-a_1')}dt\right|\\&=|\langle f_\omega,B_{a_2'}\rangle|,\end{aligned}$$

$$\begin{aligned}\lim_{j\to\infty}|\langle f_\omega,B_{a_1^j,a_2^j,a_3^j}\rangle|&=\frac{1}{2\pi}\left|\lim_{j\to\infty}\int_0^{\frac{\pi}{2}}f_\omega(e^{it})\overline{B_{a_1^j,a_2^j,a_3^j}(e^{it})}dt\right|\\&=\frac{1}{2\pi}\left|\int_0^{\frac{\pi}{2}}f_\omega(e^{it})\overline{B_{a_2',a_3'}(e^{it})\cdot(-a_1')}dt\right|\\&=|\langle f_\omega,B_{a_2',a_3'}\rangle|.\end{aligned}$$

Again by the Lebesgue dominated convergence theorem we get

$$\lim_{j\to\infty}E_\omega(|\langle f_\omega,B_{a_1^j,a_2^j}\rangle|^2)=E_\omega(|\langle f_\omega,B_{a_2'}\rangle|^2),\tag{4}$$

$$\lim_{j\to\infty}E_\omega(|\langle f_\omega,B_{a_1^j,a_2^j,a_3^j}\rangle|^2)=E_\omega(|\langle f_\omega,B_{a_2',a_3'}\rangle|^2).\tag{5}$$

By combining (2)–(5) we conclude that there exist $a_2',a_3'\in D$ such that

$$\begin{aligned}&\sup_{(a_1,a_2,a_3)\in D^3}\left\{E_\omega(|\langle f_\omega,B_{a_1}\rangle|^2)+E_\omega(|\langle f_\omega,B_{a_1,a_2}\rangle|^2)+E_\omega(|\langle f_\omega,B_{a_1,a_2,a_3}\rangle|^2)\right\}\\&=E_\omega(|\langle f_\omega,B_{a_2'}\rangle|^2)+E_\omega(|\langle f_\omega,B_{a_2',a_3'}\rangle|^2),\end{aligned}$$

which means that f is a random Blaschke form of order 2, a contradiction.

Case (ii): $a_1' \in D, a_2' \in \partial D$. Similarly we get

$$\begin{aligned}
&\lim_{j\to\infty} E_\omega(|\langle f_\omega, B_{a_1^j}\rangle|^2) = E_\omega(|\langle f_\omega, B_{a_1'}\rangle|^2),\\
&\lim_{j\to\infty} E_\omega(|\langle f_\omega, B_{a_1^j,a_2^j}\rangle|^2) = 0,\\
&\lim_{j\to\infty} E_\omega(|\langle f_\omega, B_{a_1^j,a_2^j,a_3^j}\rangle|^2) = E_\omega(|\langle f_\omega, B_{a_1',a_3'}\rangle|^2).
\end{aligned}$$

So there there exist $a_1', a_3' \in D$ such that

$$\begin{aligned}
&\sup_{(a_1,a_2,a_3)\in D^3} \left\{E_\omega(|\langle f_\omega, B_{a_1}\rangle|^2) + E_\omega(|\langle f_\omega, B_{a_1,a_2}\rangle|^2) + E_\omega(|\langle f_\omega, B_{a_1,a_2,a_3}\rangle|^2)\right\}\\
&\quad = E_\omega(|\langle f_\omega, B_{a_1'}\rangle|^2) + E_\omega(|\langle f_\omega, B_{a_1',a_3'}\rangle|^2),
\end{aligned}$$

then f must be a random Blaschke form of order 2, also a contradiction.

Case (iii): $a_1' \in \partial D$ and $a_2' \in \partial D$. Similarly we obtain

$$\begin{aligned}
&\lim_{j\to\infty} E_\omega(|\langle f_\omega, B_{a_1^j}\rangle|^2) = \lim_{j\to\infty} E_\omega(|\langle f_\omega, B_{a_1^j,a_2^j}\rangle|^2) = 0,\\
&\lim_{j\to\infty} E_\omega(|\langle f_\omega, B_{a_1^j,a_2^j,a_3^j}\rangle|^2) = E_\omega(|\langle f_\omega, B_{a_3'}\rangle|^2),
\end{aligned}$$

which leads to

$$\begin{aligned}
&\sup_{(a_1,a_2,a_3)\in D^3} \left\{E_\omega(|\langle f_\omega, B_{a_1}\rangle|^2) + E_\omega(|\langle f_\omega, B_{a_1,a_2}\rangle|^2) + E_\omega(|\langle f_\omega, B_{a_1,a_2,a_3}\rangle|^2)\right\}\\
&\quad = E_\omega(|\langle f_\omega, B_{a_3'}\rangle|^2),
\end{aligned}$$

consequently f must be a random Blaschke form of order 1, a contradiction too. We again conclude the proof of Theorem 2.1.

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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