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Locating Poles of Rational Functions in the Half Plane by Using the Backward Shift Operator

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ABSTRACT

In this study, we propose a method for locating poles of rational functions within the half-plane. This work extends the previous research on the backward shift algorithm developed for Hardy spaces in the unit disc. For rational functions, repeated application of the backward shift operator eventually yields an algebraic equation. The solutions of this equation precisely correspond to the poles of the original function, capturing both their locations and multiplicities. Based on this observation, we introduce an approach to determine these poles using discrete boundary data from the half-plane. We analyze the approximation error and prove the convergence of the proposed method.

1 | Introduction

The problem of locating poles of a rational function is attractive and has numerous engineering applications such as system identification in control theory [1–3], electroencephalography [4, 5], electric impedance tomography [5, 6], and so on. In system identification, for instance, by taking Fourier transformation or Laplace transformation, the identification problem becomes a 2D problem in frequency domain, and identifying a linear system is generally to estimate the parameters of a rational function (transfer function of the system) in frequency domain. Hence, these problem can be well treated in a framework of complex analysis.

Let Ω be a simply connected domain and $G(s)$ be a rational function with complex variable s, say $G(s) = \frac{B(s)}{A(s)}$, where $A(s)$ and $B(s)$ are polynomials. There are requirements in real application to identify $A(s)$ and $B(s)$ by using obtained boundary values on $\partial\Omega$. Given $\{E_k\}$ a set of measurements of $G(s)$ on $\partial\Omega$ at s_k, the classical linear/nonlinear least-squares methods are exploited to identify $A(s)$ and $B(s)$ by solving the following problem for their parameters in statistical sense [3, 7]:

$$\min \sum_k |A(s_k)E_k - B(s_k)|^2.$$

Some iterative algorithms to improve the estimation have also been investigated in the literature, such as [8]. In [5], Baratchart et al. presented the best meromorphic application by employing gradient algorithms. These methods usually identify $A(s)$ and $B(s)$ simultaneously; however, in the identification of a Hammerstein model, the numerator $B(s)$ is not unique, while $A(s)$ is fixed. Then the identification of $A(s)$ would be the key and initial step.

On the other hand, to avoid the nonlinear extreme problem, a new research branch that uses generalized rational orthogonal basis functions (GROBF) is developed in system identification [1,

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2, 9–12]. In these methods, generalized rational orthogonal basis functions $\{\mathbb{B}_k\}$ is adopted to constructed a model

$$\hat{G}(s) = \sum_k \theta_k \mathbb{B}_k(s) \tag{1}$$

where, in the right-plane case, $\{\mathbb{B}_k(s)\}$ are defined by a sequence $\{\alpha_k\}$, that is,

$$\begin{aligned} \mathbb{B}_1(s) &= \frac{\sqrt{2\Re\{\alpha_1\}}}{s+\overline{\alpha}_1}, \\ \mathbb{B}_k(s) &= \frac{\sqrt{2\Re\{\alpha_k\}}}{s+\overline{\alpha}_k}\prod_{i=1}^{k-1}\frac{s-\alpha_i}{s+\overline{\alpha}_i}, \ (k=2,3,\ \dots) \end{aligned} \tag{2}$$

with the real part of α_k, $\Re\{\alpha_k\} > 0$ $(k = 1, 2, \dots)$. The structure (1) provides a linear estimator, it not only allows more knowledge of original systems to be used before (such as prior information of poles) but also majorly reduces the number of parameters to be estimated. So far, there are three folds in using bases $\{\mathbb{B}_k\}$: The first choice is to use the true poles of system $G(s)$ as poles of basis functions $\{\mathbb{B}_k(s)\}$ [1, 9], and this choice needs accurate information about poles of systems; the second fold is to use an arbitrary fixed series that makes $\{\mathbb{B}_k\}$ a basis; the third fold is to select poles for basis functions in some principle in the sense of approximation [2]. However, when the order of system is known, the first fold would be the best choice. Then, there still rises a major mathematical challenge: How to determine poles of a rational function from its noisy boundary data?

There are a few works concentrating only on the problem of locating poles for $G(s)$. In [13], the authors proposed a method by taking the derivative of the target function. However, this method only applies when $G(s)$ has single poles and the functions are rational. In [14], we proposed the so called backward shift algorithm for the unit disc $\mathbb{D}$, which applies regardless of whether the poles are single or multiple. In this study, we extend the research to Hardy spaces in the half plane.

There are some assumptions for this study below.

- Let $\Omega = \Pi$ be the plane to the right of the axis $x = \sigma < 0$, that is, $\Pi = s \in \mathbb{C} \mid \Re\{s\} > \sigma$, where $\Re\{s\}$ denotes the real part of a complex number s, and $\mathbb{C}$ is the complex plane.
- For a rational function $G(s)$ with real-valued coefficients, assume $G(z)$ is stable and causal, that is, its poles stay out of the closure of Π. A set of noisy data on the boundary of Π are available, $\tilde{G}_l = G(\sigma + j\omega_l) + \eta_l$ $(l = 1, 2, \dots, L.)$, where η_l represents the noise.

We are going to find the poles of $G(s)$ including the values and multiplicities.

The outline of this paper is given as follows. In Section 2, we present the results by taking backward shift operator. The backward shift algorithm in the half plane is given in Section 3. After that, analysis of convergence aspect is given in Section 4. Some conclusions are drawn in the last section.

2 | Backward Shift for Rational Functions

Denote $H_2(\Pi)$ the Hardy 2 space in the right-half plane Π. For the generalized rational orthogonal basis functions in $H_2(\Pi)$ defined specifically by (2), it can be found that $\{\mathbb{B}_k(s)\}_{k=1}^{\infty}$ is dense in $H_p(\Pi)(1 < p < \infty)$ when the following condition about series $\{\alpha_k\}_{k=1}^{\infty}$ is satisfied [9, 15, 16]:

$$\sum_{k=1}^{\infty}\frac{\Re\{\alpha_k\}}{1+|\alpha_k|^2} = +\infty.$$

This condition is minor because it is easily satisfied when there are not too many α_k close to the imaginary axis.

Consider a function,

$$G(s) = \frac{b_{n\prime}s^{n\prime} + b_{n\prime-1}s^{n\prime-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}, \quad (n\prime < n) \tag{3}$$

with order n. According to Assumptions, $G(s)$ belongs to $H_2(\Pi)$; hence, $G(s)$ can be represented in term of (2), that is,

$$G(s) = \sum_{k=1}^{\infty} c_k \mathbb{B}_k(s)$$

where c_k is coefficient of $\mathbb{B}_k(s)$; precisely, $c_k =< G(s), \mathbb{B}_k(s) >$. Particularly, because $G(s)$ is a finite order rational function, we can find finite basis functions $\mathbb{B}_k(s)$ to represent it, when the basis functions are constructed by using the poles of $G(s)$, that is,

$$G(s) = \sum_{k=1}^{n} c_k \mathbb{B}_k(s).$$

On the other hand, we can also decompose $G(s)$ as

$$G(s) = \sum_{k=1}^{n} d_k e_k(s) \tag{4}$$

including the multiplicities, where d_k is the corresponding coefficient of $e_k(s)$ and $e_k(s)$ is the shifted Cauchy kernel:

$$e_k(s) = \frac{\sqrt{2\Re\{\alpha_k\}}}{s+\overline{\alpha}_k} \tag{5}$$

In fact, it is shown in [17] that the set $\{\mathbb{B}_k(s)\}_{k=1}^{n}(n < \infty)$ can be obtained by $\{e_k(s)\}_{k=1}^{n}$ under Schmidt orthogonalization process, even some α_k are multiple, that is to say,

$$span\{\mathbb{B}_k(s)\}_{k=1}^{n} = span\{e_k(s)\}_{k=1}^{n}.$$

Under the decomposition (4), we can estimate the poles of $G(s)$ by applying the backward shift operator, which is defined by

$$\mathfrak{S}(f)(s) = \frac{f(s) - f(0)}{s}.$$

A simple result is given as follows.

Lemma 2.1. *After taking the mth backward shift operator on $e_k(s)$ (5), there is*

$$\mathfrak{S}^{(m)}(e_k)(s) = \left(-\frac{1}{\overline{\alpha}_k}\right)^m e_k(s) \tag{6}$$

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Proof. Take backward shift on $e_k(s)$, we have

$$\begin{aligned}\mathfrak{S}(e_k)(s) &= \frac{e_k(s)-e_k(0)}{s}\\ &= \frac{\frac{\sqrt{2\Re\{\alpha_k\}}}{s+\overline{\alpha}_k}-\frac{\sqrt{2\Re\{\alpha_k\}}}{\overline{\alpha}_k}}{s}\\ &= \sqrt{2\Re\{\alpha_k\}}\frac{\overline{\alpha}_k-(s+\overline{\alpha}_k)}{\overline{\alpha}_k(s+\overline{\alpha}_k)s}\\ &= -\frac{1}{\overline{\alpha}_k}e_k(s).\end{aligned} \tag{7}$$

Continue to take backward shift operator to (7), there is

$$\begin{aligned}\mathfrak{S}^{(2)}(e_k)(s) &= \mathfrak{S}(\mathfrak{S}(e_k))(s)\\ &= (-\tfrac{1}{\overline{\alpha}_k})^2 e_k(s).\end{aligned}$$

By mathematical induction, after taking backward shift operator consecutively m times, there is

$$\mathfrak{S}^{(m)}(e_k)(s) = \left(-\frac{1}{\overline{\alpha}_k}\right)^m e_k(s) \tag{8}$$

the proof is complete. □

The result (8) also indicates that $-\frac{1}{\overline{\alpha}_k}$ is an eigenvalue of $\mathfrak{S}$.

On the other hand, $\{e_k(s)\}_{k=1}^n$ generates a backward shift invariant subspace in $H_2(\Pi)$ [18], because if $f(s)\in span\{e_k(s)\}$, that is, (set all α_k are distinguished, if there are multiple α_k, the result still holds [14]),

$$f(s) = \sum_{k=1}^n d_k e_k(s)$$

then $\mathfrak{S}(f)(s)\in span\{e_k(s)\}$.

Precisely, by Lemma 2.1, we have

$$\mathfrak{S}(f)(s) = \sum_{k=1}^n \left(-\frac{1}{\overline{\alpha}_k}\right) d_k e_k(s).$$

A result similar to [14] can be obtained as follows.

Theorem 2.1. *Consider a rational function $G(s)\in H_2(\Pi)$ with order n whose poles stay out of the closed Π. Taking backward shift operator n times consecutively on $G(s)$, the poles-related numbers $(-\frac{1}{\overline{\alpha}_k})$ are exactly solutions (including multiplicities) of an algebraic equation*

$$x^n + \beta_n x^{n-1} + \ldots + \beta_2 x + \beta_1 = 0 \tag{9}$$

where coefficients $\{\beta_k\}_1^n$ satisfy

$$\mathfrak{S}^{(n)}(G)(s) + \beta_n\mathfrak{S}^{(n-1)}(G)(s) + \ldots + \beta_1 G(s) = 0.$$

Proof. We can proof this by using case study.

Case 1. Suppose $G(s)$ has only single poles, for example,

$$G(s) = \sum_{k=1}^n d_k e_k(s),$$

where d_k are coefficients of $e_k(s)$, $\{-\alpha_k\}_{k=1}^n$ are poles of $G(s)$, and they are distinguished with each other. Taking backward shift operator to $G(s)$, there is

$$\begin{aligned}\mathfrak{S}(G)(s) =& \left(-\frac{1}{\overline{\alpha}_1}\right)d_1e_1(s) + \left(-\frac{1}{\overline{\alpha}_2}\right)d_2e_2(s)\\ &+ \ldots + \left(-\frac{1}{\overline{\alpha}_n}\right)d_ne_n(s),\end{aligned}$$

continue to take backward shift operator to $\mathfrak{S}(G)(s)$, we have

$$\begin{aligned}\mathfrak{S}^2(G)(s) =& \left(-\frac{1}{\overline{\alpha}_1}\right)^2 d_1e_1(s) + \left(-\frac{1}{\overline{\alpha}_2}\right)^2 d_2e_2(s)\\ &+ \ldots + \left(-\frac{1}{\overline{\alpha}_n}\right)^2 d_ne_n(s),\end{aligned}$$

going on and we stop at the nth time,

$$\begin{aligned}\mathfrak{S}^n(G)(s) =& \left(-\frac{1}{\overline{\alpha}_1}\right)^n d_1e_1(s)\\ &+ \left(-\frac{1}{\overline{\alpha}_2}\right)^n d_2e_2(s) + \ldots + \left(-\frac{1}{\overline{\alpha}_n}\right)^n d_ne_n(s).\end{aligned}$$

When $\{-\alpha_k\}_{k=1}^n$ are distinguished with each other, $\{e_k(s)\}_{k=1}^n$ are linearly independent. Since there are $n+1$ functions, namely $G(s), \mathfrak{S}(G)(s), \ldots, \mathfrak{S}^n(G)(s)$ consisted of n components $\{e_k(s)\}_{k=1}^n$, then there is a unique sequence $\{\beta_n, \ldots, \beta_1\}$, whose elements are not all zeros, such that

$$\mathfrak{S}^{(n)}(G)(s) + \beta_n\mathfrak{S}^{(n-1)}(G)(s) + \ldots + \beta_1 G(s) = 0 \tag{10}$$

Substitute $\{\mathfrak{S}^k(G)(s)\}_{k=0}^n$ into (10), we have

$$\begin{cases}0 = \left[\beta_1 + \beta_2\left(-\frac{1}{\overline{\alpha}_1}\right) + \beta_3\left(-\frac{1}{\overline{\alpha}_1}\right)^2\right.\\ \left.+\ldots+\beta_{n-2}\left(-\frac{1}{\overline{\alpha}_1}\right)^{n-1} + \left(-\frac{1}{\overline{\alpha}_1}\right)^n\right]d_1e_1(s)\\ \left[\beta_1 + \beta_2\left(-\frac{1}{\overline{\alpha}_2}\right) + \beta_3\left(-\frac{1}{\overline{\alpha}_2}\right)^2\right.\\ \left.+\ldots+\beta_{n-2}\left(-\frac{1}{\overline{\alpha}_2}\right)^{n-1} + \left(-\frac{1}{\overline{\alpha}_2}\right)^n\right]d_2e_2(s)\\ \vdots\\ \left[\beta_1 + \beta_2\left(-\frac{1}{\overline{\alpha}_n}\right) + \beta_3\left(-\frac{1}{\overline{\alpha}_n}\right)^2\right.\\ \left.+\ldots+\beta_{n-2}\left(-\frac{1}{\overline{\alpha}_n}\right)^{n-1} + \left(-\frac{1}{\overline{\alpha}_n}\right)^n\right]d_ne_n(s),\end{cases}$$

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Again, since $\{e_k(s)\}_{k=1}^n$ are linearly independent, above equation implies

$$\begin{cases} \beta_1 + \beta_2\left(-\frac{1}{\overline{\alpha}_1}\right) + \beta_3\left(-\frac{1}{\overline{\alpha}_1}\right)^2 \\ + \ldots + \beta_{n-2}\left(-\frac{1}{\overline{\alpha}_1}\right)^{n-1} + \left(-\frac{1}{\overline{\alpha}_1}\right)^n = 0, \\ \beta_1 + \beta_2\left(-\frac{1}{\overline{\alpha}_2}\right) + \beta_3\left(-\frac{1}{\overline{\alpha}_2}\right)^2 \\ + \ldots + \beta_{n-2}\left(-\frac{1}{\overline{\alpha}_2}\right)^{n-1} + \left(-\frac{1}{\overline{\alpha}_2}\right)^n = 0, \\ \vdots \\ \beta_1 + \beta_2\left(-\frac{1}{\overline{\alpha}_n}\right) + \beta_3\left(-\frac{1}{\overline{\alpha}_n}\right)^2 \\ + \ldots + \beta_{n-2}\left(-\frac{1}{\overline{\alpha}_n}\right)^{n-1} + \left(-\frac{1}{\overline{\alpha}_n}\right)^n = 0. \end{cases} \tag{11}$$

Equation (11) means $\{-\frac{1}{\overline{\alpha}_k}\}_{k=1}^n$ are exactly the solutions of the following algebraic equation:

$$x^n + \beta_n x^{n-1} + \ldots + \beta_2 x + \beta_1 = 0.$$

Case 2. Suppose $G(s)$ has one multiple pole $\frac{-1}{\alpha_1}$ with multiplicity m. Analysis to the case with more than one multiple poles is the same. Then $G(s)$ has the following form:

$$G(s) = \sum_{k=1}^{m} d_k e_1^k(s) + \sum_{k=m+1}^{n} d_k e_k(s);$$

in this formula, the coefficients d_k are all non-zero for $k \geq m$. At first, we show, for any integer m:

$$\mathfrak{S}(e_\alpha^m(s)) = \left(-\frac{1}{\overline{\alpha}^m}\right)\left[\frac{1}{(s+\overline{\alpha})} + \frac{\overline{\alpha}}{(s+\overline{\alpha})^2} \ldots + \frac{\overline{\alpha}^{m-1}}{(s+\overline{\alpha})^m}\right].$$

Because, when $m = 2$,

$$\begin{aligned} \mathfrak{S}(e_\alpha^2(s)) &= \frac{\frac{1}{(s+\overline{\alpha})^2} - \frac{1}{\overline{\alpha}^2}}{s} \\ &= \frac{\overline{\alpha}^2 - (s+\overline{\alpha})^2}{s\overline{\alpha}^2(s+\overline{\alpha})^2} \\ &= \frac{-s^2 - 2\overline{\alpha}s}{s\overline{\alpha}^2(s+\overline{\alpha})^2} \\ &= \frac{-s - \overline{\alpha} - \overline{\alpha}}{\overline{\alpha}^2(s+\overline{\alpha})^2} \\ &= \frac{-1}{\overline{\alpha}^2}\left(\frac{1}{(s+\overline{\alpha})} + \frac{\overline{\alpha}}{(s+\overline{\alpha})^2}\right). \end{aligned}$$

For any integer m,

$$\begin{aligned} \mathfrak{S}(e_\alpha^m(s)) &= \frac{\frac{1}{(s+\overline{\alpha})^m} - \frac{1}{\overline{\alpha}^m}}{s} \\ &= \frac{\overline{\alpha}^m - (s+\overline{\alpha})^m}{s\overline{\alpha}^m(s+\overline{\alpha})^m} \\ &= \frac{\overline{\alpha}^m - s(s+\overline{\alpha})^{m-1} - \overline{\alpha}(s+\overline{\alpha})^{m-1}}{s\overline{\alpha}^m(s+\overline{\alpha})^m} \\ &= (\frac{1}{\overline{\alpha}^m})\left[\frac{-1}{(s+\overline{\alpha})} + \frac{\overline{\alpha}^m - \overline{\alpha}(s+\overline{\alpha})^{m-1}}{s(s+\overline{\alpha})^m}\right] \\ &= (-\frac{1}{\overline{\alpha}^m})\left[\frac{1}{(s+\overline{\alpha})} + \frac{\overline{\alpha}}{(s+\overline{\alpha})^2} \ldots + \frac{\overline{\alpha}^{m-1}}{(s+\overline{\alpha})^m}\right]. \end{aligned}$$

Based on the above truth, we now go to investigate what is the result when taking backward shift operator to

$$f(s) = \sum_{k=1}^{m} d_k e_1^k(s).$$

We have

$$\begin{aligned} &\mathfrak{S}(f(s)) \\ &= \left(-\frac{1}{\overline{\alpha}_1}\right) d_1 \frac{1}{(s+\overline{\alpha_1})} \\ &\quad + \left(-\frac{1}{\overline{\alpha}_1}\right)^2 d_2\left[\frac{1}{(s+\overline{\alpha_1})} + \frac{\overline{\alpha_1}}{(s+\overline{\alpha_1})^2}\right] \\ &\ldots + \left(-\frac{1}{\overline{\alpha_1}^m}\right) d_m\left[\frac{1}{(s+\overline{\alpha_1})} + \frac{\overline{\alpha_1}}{(s+\overline{\alpha_1})^2} \ldots + \frac{\overline{\alpha_1}^{m-1}}{(s+\overline{\alpha_1})^m}\right]. \end{aligned} \tag{12}$$

From (12), we can see $\mathfrak{S}(f(s))$ is also consisted of the components $e_1(s)$, $e_1^2(s)$, … , $e_1^m(s)$. Continue to take backward shift operator to this result until m times, each resulted function is consisted of the components $e_1(s), e_2^2(s)$, … , $e_2^m(s)$, too. Added up with other component $e_{m+1}(s), e_{m+2}(s)$, … , $e_n(s)$. We can get the same results, by the similar computation to case I and Lemma 1 in [14], as follows:

There exists a unique sequence $\{\beta_n, \ldots, \beta_1\}$, whose elements are not all zeros, such that

$$\mathfrak{S}^{(n)}(G)(s) + \beta_n \mathfrak{S}^{(n-1)}(G)(s) + \ldots + \beta_1 G(s) = 0 \tag{13}$$

and $\{-\frac{1}{\overline{\alpha}_k}\}(k = 1, m+1, \ldots, n)$ are exactly the solutions of the following algebraic equation:

$$x^n + \beta_n x^{n-1} + \ldots + \beta_2 x + \beta_1 = 0 \tag{14}$$

More importantly, the multiplicity for solution $-\frac{1}{\overline{\alpha}_1}$ in Equation (14) is exactly m.

Then the proof is complete. □

Remark 2.1. It is quite clear that any n poles are solutions of an algebraic equation. By our result, we emphasize the relationship between the equation and taking backward shift. It can be seen that they share the same coefficients $\{\beta_k\}$ in (14) and (13). With this relationship, it is possible to find poles using the boundary data.

3 | Estimation of Poles

In this section, we are going to estimate the poles of $G(s)$ based on a set of boundary data $\{\omega_l, \tilde{G}_l(\sigma + j\omega_l)\}_{l=1}^L$ ($\omega_l = l\omega_0$, ω_0 is the

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step size.). The motivation of using the backward shift operator is from Equation (10). By submitting the data into (10), one can get a group of algebraic equations and then coefficients $\{\beta_n, \dots, \beta_1\}$ can be estimated. The obtained $\{\beta_n, \dots, \beta_1\}$ finally are used in (9) to find the poles. The details are given in the following text.

3.1 | Estimate of $\mathfrak{S}^{(k)}(G)(\sigma + j\omega_l)$

According to the Cauchy integral formula in $H_2(\Pi)$, for $G(s) \in H_2(\Pi)$, there is

$$G(s) = \frac{1}{2\pi j}\int_{\sigma+jR}\frac{G(z)}{s-z}dz \tag{15}$$

As $G(0)$ is used in the backward shift operator, σ should be negative, and

$$\begin{aligned} G(0) &= \frac{1}{2\pi j}\int_{-\infty}^{+\infty}\frac{G(\sigma+j\omega)}{-\sigma-j\omega}jd\omega \\ &= -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{G(\sigma+j\omega)}{\sigma+j\omega}d\omega. \end{aligned}$$

Then the data after taking backward shift operator can be estimated as

Lemma 3.1. *Denote $\mathfrak{S}^{(k)}(G)(s)$ the function obtained by taking backward shift operator k ($k = 0, 1, \dots, n$) times on $G(s)$, the value of $\mathfrak{S}^{(k)}(G)(s)$ at point $\sigma + j\omega_l$ ($l = 1, 2, \dots, L$) is recursively estimated as*

$$\begin{aligned} &\mathfrak{S}^{(k)}(G)(\sigma+j\omega_l) \\ &\approx \mathfrak{S}^{(k)}(\tilde{G})(\sigma+j\omega_l) \\ &= \frac{\left[\mathfrak{S}^{(k-1)}\tilde{G}(\sigma+j\omega_l) + \frac{\omega_0}{\pi}\Re\left(\sum_{i=1}^{L}\frac{\mathfrak{S}^{(k-1)}\tilde{G}(\sigma+j\omega_i)}{\sigma+j\omega_i}\right)\right]}{\sigma+j\omega_l}. \end{aligned} \tag{16}$$

with $\mathfrak{S}^{(0)}(\tilde{G})(\sigma+j\omega_l) = \tilde{G}_l = G(\sigma+j\omega_l) + \eta_l$ ($l = 1, 2, \dots L$).

Proof. At first, the given data on the axis $x = \sigma$ can be used to estimate $G(0)$ as

$$\begin{aligned} G(0) &= -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{G(\sigma+j\omega)}{\sigma+j\omega}d\omega. \\ &\approx -\frac{1}{2\pi}\int_{-L\omega_0}^{L\omega_0}\frac{G(\sigma+j\omega)}{\sigma+j\omega}d\omega \\ &= -\frac{1}{2\pi}\left[\int_0^{L\omega_0}\frac{G(\sigma+j\omega)}{\sigma+j\omega}d\omega + \int_{-L\omega_0}^{0}\frac{G(\sigma+j\omega)}{\sigma+j\omega}d\omega\right] \\ &\approx -\frac{1}{2\pi}\left[\frac{L\omega_0}{L}\sum_{l=1}^{L}\frac{G(\sigma+j\omega_l)}{\sigma+j\omega_l} + \frac{L\omega_0}{L}\sum_{l=1}^{L}\frac{G(\sigma+j\omega_l)}{\sigma+j\omega_l}\right] \\ &= -\frac{\omega_0}{2\pi}\left[\sum_{l=1}^{L}\frac{G(\sigma+jl\omega_0)}{\sigma+jl\omega_0} + \sum_{l=1}^{L}\frac{G(\sigma-jl\omega_0)}{\sigma-jl\omega_0}\right]. \end{aligned} \tag{17}$$

At negative frequencies, we set $\tilde{G}(\sigma - jl\omega_0) = \overline{\tilde{G}(\sigma+jl\omega_0)}$ as the coefficients of $G(s)$ are real-valued; thus,

$$\begin{aligned} G(0) &\approx -\frac{\omega_0}{2\pi}\left[\sum_{l=1}^{L}\frac{\tilde{G}(\sigma+jl\omega_0)}{\sigma+jl\omega_0} + \sum_{l=1}^{L}\overline{\left(\frac{\tilde{G}(\sigma+jl\omega_0)}{\sigma+jl\omega_0}\right)}\right] \\ &= -\frac{\omega_0}{2\pi}2\Re\left(\sum_{l=1}^{L}\frac{\tilde{G}(\sigma+jl\omega_0)}{\sigma+jl\omega_0}\right) \\ &= -\frac{\omega_0}{\pi}\Re\left(\sum_{l=1}^{L}\frac{\tilde{G}(\sigma+jl\omega_0)}{\sigma+jl\omega_0}\right). \end{aligned}$$

When $\tilde{G}(\sigma+j\omega_l) = G(\sigma+j\omega_l) + \eta_l$, we have

$$\begin{aligned} G(0) &\approx -\frac{\omega_0}{\pi}\Re\left(\sum_{l=1}^{L}\frac{\tilde{G}(\sigma+jl\omega_0)}{\sigma+jl\omega_0}\right) \\ &= -\frac{\omega_0}{\pi}\Re\left(\sum_{l=1}^{L}\frac{G(\sigma+jl\omega_0)}{\sigma+jl\omega_0}\right) - \frac{\omega_0}{\pi}\Re\left(\sum_{l=1}^{L}\frac{\eta_l}{\sigma+jl\omega_0}\right). \end{aligned}$$

This approximation of $G(s)$ at origin can be used to further estimate the data $\mathfrak{S}(\tilde{G})(\sigma+j\omega_l)$ at ω_l; that is, taking backward shift operator to data set $\{\tilde{G}(\sigma+j\omega_l)\}_{l=1}^{L}$, new data set is obtained as

$$\begin{aligned} &\mathfrak{S}(G)(\sigma+j\omega_l) \\ &= \frac{G(\sigma+j\omega_l) - G(0)}{\sigma+j\omega_l} \\ &\approx \frac{1}{\sigma+j\omega_l}\left[\tilde{G}(\sigma+j\omega_l) + \frac{\omega_0}{\pi}\Re\left(\sum_{k=1}^{L}\frac{\tilde{G}(\sigma+j\omega_k)}{\sigma+j\omega_k}\right)\right] \\ &= \mathfrak{S}(\tilde{G})(\sigma+j\omega_l). \end{aligned}$$

Repeat this process, we get the result (16). The proof is complete□

The following result shows the convergence of these estimated data:

Lemma 3.2. *For functions given by (3) and given data, suppose the noise η_l is bounded; that is, $|\eta_l| < \epsilon$ ($\epsilon > 0$). Then the estimated data by taking backward shift operator $\{\mathfrak{S}(\tilde{G})(\sigma + j\omega_l)\}, \dots, \{\mathfrak{S}^n(\tilde{G})(\sigma+j\omega_l)\}$ converge uniformly in l, precisely, for each l*

$$\lim_{\substack{L\to\infty \\ \epsilon\to 0}}\mathfrak{S}^{(k)}(\tilde{G})(\sigma+j\omega_l) = \mathfrak{S}^{(k)}(G)(\sigma+j\omega_l).\ (k = 1, 2, 3\ \dots, n.)$$

Proof. We use mathematical induction to prove this result. First, for each l,

$$\begin{aligned} &\lim_{\substack{L\to\infty \\ \epsilon\to 0}}\mathfrak{S}(\tilde{G})(\sigma+j\omega_l) \\ &= \lim_{\substack{L\to\infty \\ \epsilon\to 0}}\frac{1}{\sigma+j\omega_l}\left[\tilde{G}(\sigma+j\omega_l) + \frac{\omega_0}{\pi}\Re\left(\sum_{l=1}^{L}\frac{\tilde{G}(\sigma+jl\omega_0)}{\sigma+jl\omega_0}\right)\right] \\ &= \lim_{\substack{L\to\infty \\ \epsilon\to 0}}\frac{1}{\sigma+j\omega_l} \\ &\left[G(\sigma+j\omega_l) + \eta_l + \frac{\omega_0}{\pi}\Re\left(\sum_{l=1}^{L}\frac{G(\sigma+jl\omega_0)+\eta_l}{\sigma+jl\omega_0}\right)\right] \\ &= \frac{1}{\sigma j\omega_l}\left[G(\sigma+j\omega_l) - \frac{1}{2\pi j}\int_{-\infty}^{+\infty}\frac{G(\sigma+j\omega)}{\sigma+j\omega}dj\omega\right] \end{aligned}$$

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$$= \frac{1}{\sigma + j\omega_l}[G(\sigma + j\omega_l) - G(0)]$$
$$= \mathfrak{S}(G)(\sigma + j\omega_l).$$

Second, suppose for $k > 1$, there holds

$$\lim_{\substack{L\to\infty \\ \epsilon\to 0}} \mathfrak{S}^{(k)}(\tilde{G})(\sigma + j\omega_l) = \mathfrak{S}^{(k)}(G)(\sigma + j\omega_l),$$

then for $k + 1$, we have

$$\lim_{\substack{L\to\infty \\ \epsilon\to 0}} \mathfrak{S}^{(k+1)}(\tilde{G})(\sigma + j\omega_l)$$
$$= \lim_{\substack{L\to\infty \\ \epsilon\to 0}} \frac{1}{\sigma + j\omega_l}$$
$$\left[\mathfrak{S}^{(k)}(\tilde{G})(\sigma + j\omega_l) + \frac{\omega_0}{\pi}\Re\left(\sum_{l=1}^{L}\frac{\mathfrak{S}^{(k)}(\tilde{G})(\sigma + jl\omega_0)}{\sigma + jl\omega_0}\right)\right]$$
$$= \frac{1}{\sigma + j\omega_l}\left[\mathfrak{S}^{(k)}(G)(\sigma + j\omega_l) + \frac{1}{2\pi j}\int_{-\infty}^{+\infty}\frac{\mathfrak{S}^{(k)}(G)(\sigma + j\omega)}{\sigma + j\omega}d j\omega\right]$$
$$= \frac{1}{\sigma + j\omega_l}\left[\mathfrak{S}^{(k)}(G)(\sigma + j\omega_l) - \mathfrak{S}^{(k)}(G)(0)\right]$$
$$= \mathfrak{S}^{(k+1)}(G)(\sigma + j\omega_l),$$

uniformly in l. By using the mathematical induction, we complete the proof. □

3.2 | Computation of $\{\beta_1, \ldots, \beta_n\}$

Set a column vector $\beta = (\beta_1\ \beta_2\ \ldots\ \beta_n)'$, where " $'$ " denotes transposition of a vector (matrix). For simplicity, we set $\xi_l = \sigma + j\omega_l$ $(l = 1, \ldots, L)$ in the remaining text. Based on the estimated data sets $\{\mathfrak{S}(\tilde{G})(\xi_l)\}, \ldots, \{\mathfrak{S}^n(\tilde{G})(\xi_l)\}$, we get a typical system of linear equations from (10):

$$\begin{cases} \beta_1\tilde{G}(\xi_1) + \beta_2\mathfrak{S}(\tilde{G})(\xi_1) + \ldots + \beta_n\mathfrak{S}^{(n-1)}(\tilde{G})(\xi_1) = -\mathfrak{S}^{(n)}(\tilde{G})(\xi_1), \\ \beta_1\tilde{G}(\xi_2) + \beta_2\mathfrak{S}(\tilde{G})(\xi_2) + \ldots + \beta_n\mathfrak{S}^{(n-1)}(\tilde{G})(\xi_2) = -\mathfrak{S}^{(n)}(\tilde{G})(\xi_2), \\ \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ \beta_1\tilde{G}(\xi_L) + \beta_2\mathfrak{S}(\tilde{G})(\xi_L) + \ldots + \beta_n\mathfrak{S}^{(n-1)}(\tilde{G})(\xi_L) = -\mathfrak{S}^{(n)}(\tilde{G})(\xi_L), \end{cases} \tag{18}$$

which can be re-formulated as

$$\Lambda\beta = \lambda \tag{19}$$

In (19), Λ is an $L \times n$ matrix with $\Lambda(p, q) = \mathfrak{S}^{(q-1)}(\tilde{G})(\xi_p)$, and λ is a column vector

$$\lambda = -(\mathfrak{S}^n(\tilde{G})(\xi_1)\ \mathfrak{S}^{(n)}(\tilde{G})(\xi_2)\ \ldots\ -\mathfrak{S}^{(n)}(\tilde{G})(\xi_L))'.$$

Note that the true solution of β exists according to Equation (13), denote $\beta^\star$ the true solution of β. From Equation (18), its solution β_L by using least-squares is equivalent to

$$\beta_L = \arg\min_{\beta}\frac{1}{L}\sum_{l=1}^{L}|\sum_{k=1}^{n}\beta_k\mathfrak{S}^{(k-1)}(\tilde{G})(\xi_l) + \mathfrak{S}^{(n)}(\tilde{G})(\xi_l)|^2 \tag{20}$$

The problem (20) has a normal equations as

$$(\Lambda'\Lambda)\beta_L = \Lambda'\lambda \tag{21}$$

Λ' is the conjugate transposed matrix of Λ as its elements are complex-valued. Then the solution by using least-squares method is given by

$$\beta_L = (\Lambda'\Lambda)^{-1}\Lambda'\lambda \tag{22}$$

The computational complexity of (22) lies on the inverse of the matrix $\Lambda'\Lambda$, and some recursive methods can be used to get the solution [19]. At last, submit β_L to Equation (9), we get solutions $\{x_k\}$, then poles $\{\alpha_k\}$ can be obtained by

$$\alpha_k = -\frac{1}{\bar{x}_k},$$

including both locations and multiplicities. Obviously, the efficient estimation of α_k depends on the estimated coefficients β_L. The algorithm is summarized as Algorithm 1.

Algorithm 1 Estimation of poles.

1. Estimate $\mathfrak{S}^{(k)}(G)(\sigma + j\omega_l)$ by (16) with backward shift;
2. Find the solution of Problem (20) for β_L;
3. Find the solution $\{x_k\}$ of Equation (9) with obtained β_L;
4. Compute poles $\alpha_k = -\frac{1}{\bar{x}_k}$.

Below, we present a result showing the robustness of this estimate.

Theorem 3.1. *For a function $G(s)$ in form of (3) with a data set $\{\tilde{G}_l = G(\sigma + j\omega_l) + \eta_l\}_{l=1}^{L}$. Assume the noise is bounded; that is, there is a non-negative number ϵ $(0 < \epsilon < \infty)$ such that $|\eta_l| < \epsilon$ for all $l = 1, \ldots, L$. Let $\beta^\star$ denote the true solution of Equation (10), then for the solution β_L (22), we have*

$$\|\beta_L - \beta^\star\| \le \|\beta^\star\|\frac{\|R_L\|\kappa(R_L)}{\|R_L\| - \kappa(R_L)\|\Delta R_L\|}\left(\frac{\|\Delta R_L\|}{\|R_L\|} + \frac{\|\Delta\delta_L\|}{\|\delta_L\|}\right) \tag{23}$$

where $R_L, \delta_L, \Delta R_L$, and $\Delta\delta_L$ are given by (26), (27), (28), and (29), respectively. $\kappa(R_L)$ denotes the condition number of R_L.

Proof. We first analyze the elements of matrix $\Lambda_{L\times n}$, for all l, and there is

$$\tilde{G}(\xi_l) = G(\xi_l) + \eta_l,\ (l = 1, 2, \ldots, L)$$

and $\tilde{G}(\xi_l)$ can be decomposed as

$$\mathfrak{S}\tilde{G}(\xi_l)$$
$$= \frac{1}{\xi_l}\left[\tilde{G}(\xi_l) + \frac{\omega_0}{\pi}\Re\left(\sum_{l=1}^{L}\frac{\tilde{G}(\xi_l)}{\xi_l}\right)\right]$$
$$= \frac{1}{\xi_l}\left[G(\xi_l) + \eta_l + \frac{\omega_0}{\pi}\Re(\sum_{l=1}^{L}\frac{G(\xi_l) + \eta_l}{\xi_l})\right]$$
$$= \frac{1}{\xi_l}[G(\xi_l) - G(0)] + \frac{1}{\xi_l}\left[G(0) + \eta_l + \frac{\omega_0}{\pi}\Re\left(\sum_{l=1}^{L}\frac{G(\xi_l) + \eta_l}{\xi_l}\right)\right]$$

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$$= \mathfrak{S}G(j\omega_l) + \eta_l^{(1)},$$

where

$$\eta_l^{(1)} = \frac{1}{\xi_l}\left[G(0) + \eta_l + \frac{\omega_0}{\pi}\Re\left(\sum_{l=1}^{L}\frac{G(\xi_l)+\eta_l}{\xi_l}\right)\right].$$

$$\begin{aligned}
&\mathfrak{S}^{(2)}\tilde{G}(\xi_l)\\
&= \frac{1}{\xi_l}\left[\mathfrak{S}\tilde{G}(\xi_l) + \frac{\omega_0}{\pi}\Re\left(\sum_{l=1}^{L}\frac{\mathfrak{S}\tilde{G}(\xi_l)}{\xi_l}\right)\right]\\
&= \frac{1}{\xi_l}\left[\mathfrak{S}G(\xi_l) + \eta_l^{(1)} + \frac{\omega_0}{\pi}\Re\left(\sum_{l=1}^{L}\frac{\mathfrak{S}G(\xi_l)+\eta_l^{(1)}}{\xi_l}\right)\right]\\
&= \frac{1}{\xi_l}[\mathfrak{S}G(\xi_l) - \mathfrak{S}G(0)]\\
&\quad + \frac{1}{\xi_l}\left[\mathfrak{S}G(0) + \eta_l^{(1)} + \frac{\omega_0}{\pi}\Re\left(\sum_{l=1}^{L}\frac{\mathfrak{S}G(\xi_l)+\eta_l^{(1)}}{\xi_l}\right)\right]\\
&= \mathfrak{S}^{(2)}G(j\omega_l) + \eta_l^{(2)}
\end{aligned}$$

where

$$\eta_l^{(2)} = \frac{1}{\xi_l}\left[\mathfrak{S}G(0) + \eta_l^{(1)} + \frac{\omega_0}{\pi}\Re\left(\sum_{l=1}^{L}\frac{\mathfrak{S}G(\xi_l)+\eta_l^{(1)}}{\xi_l}\right)\right].$$

Similarly, for any $k > 2$, computation gives that

$$\mathfrak{S}^{(k)}(\tilde{G})(\xi_l) = \mathfrak{S}^{(k)}(G)(\xi_l) + \eta_l^{(k)}$$

with

$$\begin{aligned}
&\eta_l^{(k)} = \frac{1}{\xi_l}\\
&\left[\mathfrak{S}^{(k-1)}(G)(0) + \eta_l^{(k-1)} + \frac{\omega_0}{\pi}\Re\left(\sum_{l=1}^{L}\frac{\mathfrak{S}^{k-1}G(\xi_l)+\eta_l^{(k-1)}}{\xi_l}\right)\right].
\end{aligned} \tag{24}$$

Then the system (19) can be stated as

$$(\Lambda_0 + \Delta\Lambda)\beta = \lambda_0 + \Delta\lambda,$$

where Λ_0 and λ_0 are the true matrix without errors and

$$\begin{aligned}
\Lambda_0(p,q) &= \mathfrak{S}^{(q-1)}(G)(\xi_p), \quad \Delta\Lambda(p,q) = \eta_p^{(q-1)},\\
\lambda_0(p) &= \mathfrak{S}^{(n)}(G)(\xi_p), \qquad \Delta\lambda(p) = -\eta_p^{(n)}.
\end{aligned}$$

So the normal equations are given by

$$\begin{aligned}
&\frac{1}{L}(\Lambda_0'\Lambda_0 + \Lambda_0'\Delta\Lambda + \Delta\Lambda\Lambda_0 + \Delta\Lambda'\Delta\Lambda)\beta_L\\
&= \frac{1}{L}(\Lambda_0'\lambda_0 + \Lambda_0'\Delta\lambda + \Delta\Lambda'\lambda_0 + \Delta\Lambda'\Delta\lambda),
\end{aligned} \tag{25}$$

in which we denote the true matrix as

$$R_L = \frac{1}{L}\Lambda_0'\Lambda_0 \tag{26}$$

$$\delta_L = \frac{1}{L}\Lambda_0'\lambda_0 \tag{27}$$

and matrices of perturbations are denoted as

$$\Delta R_L = \frac{1}{L}(\Lambda_0'\Delta\Lambda + \Delta\Lambda\Lambda_0 + \Delta\Lambda'\Delta\Lambda) \tag{28}$$

$$\Delta\delta_L = \frac{1}{L}(\Lambda_0'\Delta\lambda + \Delta\Lambda'\lambda_0 + \Delta\Lambda'\Delta\lambda) \tag{29}$$

(25) then becomes

$$(R_L + \Delta R_L)\beta_L = \delta_L + \Delta\delta_L.$$

According to the result of matrix theory [19], an estimation about the error between true solution $\beta^\star$ and β_L is given by

$$\|\beta_L - \beta^\star\| \le \|\beta^\star\|\frac{\|R_L\|\kappa(R_L)}{\|R_L\| - \kappa(R_L)\|\Delta R_L\|}\left(\frac{\|\Delta R_L\|}{\|R_L\|} + \frac{\|\Delta\delta_L\|}{\|\delta_L\|}\right),$$

where $\kappa(R_L)$ is the condition number of R_L. The proof is complete. □

Theorem 3.1 also improves the analysis in [14], and the middle term between β_L and $\beta^\star$ is not necessary in (23). From the recursive formula (24) for perturbations, we also can see that if the modules of ξ_l are bigger than one, it is helpful in removing errors when taking backward shift operator.

4 | Conclusion

In this article, we study the problem of estimating poles of rational functions in the half-plane. By applying the backward shift operator sufficiently often, an algebraic equation can be derived whose solutions yield the poles of the original function. We also propose a method to identify these poles using boundary values on the half-plane. The convergence rate of this method is also established.

Author Contributions

Wen Mi: investigation, funding acquisition, writing – original draft, writing – review and editing, conceptualization, methodology, validation, visualization, project administration, formal analysis. **Tao Qian:** funding acquisition, writing – review and editing, conceptualization, investigation, methodology, validation, visualization, project administration, formal analysis.

Conflicts of Interest

The authors declare no conflicts of interest.

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