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Interpolating Blaschke Sequence and Perturbation

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Abstract
We show that every Blaschke sequence containing at most one zero can be transformed into an interpolating Blaschke sequence by using an arbitrarily small perturbation. More precisely, for a Blaschke sequence $\{z_n\}_{n=1}^{\infty}$ containing at most one zero and any $\varepsilon > 0$, there is an interpolating Blaschke sequence $\{\omega_n\}_{n=1}^{\infty}$ such that for $1 \le n < \infty$, (1) $|z_n| = |\omega_n|$, (2) and $|\arg z_n - \arg \omega_n| < \varepsilon$. This result is a refinement of Naftalevič's Theorem.

Keywords Blaschke sequence · Interpolating sequence · Interpolating Blaschke sequence · Naftalevic's Theorem

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1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk, and $\mathbb{T} = \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle. The pseudo-hyperbolic metric on $\mathbb{D}$ is defined by

$$\rho(z, w) = \left| \frac{z - w}{1 - \overline{w}z} \right|, \qquad z, w \in \mathbb{D}.$$

A sequence $\{z_n\}_{n=1}^{\infty} \subset \mathbb{D}$ is said to be a *Blaschke sequence* if

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty.$$

The corresponding *Blaschke product* $B(z)$ is

$$B(z) = z^m \prod_{z_n \neq 0} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z},$$

where m is the number of z_n equal to zero. The Blaschke product $B(z)$ is well defined and holomorphic in $\mathbb{D}$, with radial limits of modulus one almost everywhere on $\mathbb{T}$.

As in [3], a sequence $\{z_n\}_{n=1}^{\infty} \subset \mathbb{D}$ is said to be *interpolating* or H^∞ *interpolating* if for every sequence $\{\omega_n\}_{n=1}^{\infty} \in l^\infty$, there is a holomorphic function $f \in H^\infty$ such that $f(z_n) = \omega_n$ for $n = 1, 2, \ldots$, where H^∞ is the class of bounded analytic functions on $\mathbb{D}$. A Blaschke product is called an *interpolating Blaschke product* if it has distinct zeros and its zeros form an interpolating sequence.

In 1981, Jones [10] refined Douglas–Rudin theorem and proved that every unimodular function on the unit circle can be uniformly approximated by the ratio of two interpolating Blaschke products, which is [5, Thm. 5.1].

Interpolating Blaschke products also play a crucial role in the research of H^∞. In 1996, Garnett and Nicolau [6] proved that the algebra of $H^\infty(\mathbb{T})$ is generated by the set of interpolating Blaschke products. Marshall and Stray [12] showed that any function in H^∞ which extends to be continuous almost everywhere on $\mathbb{T}$, can be uniformly approximated by finite linear combinations of interpolating Blaschke products. Hjelle and Nicolau [9] showed that the modulus of an inner function in $\mathbb{D}$ can be uniformly approximated by the modulus of an interpolating Blaschke products. In 2012, Gallardo-Gutiêrrez and Gorkin [7] showed that to each inner function, there corresponds at least one interpolating Blaschke product whose angular derivatives have precisely the same behavior as the given inner function.

However, a Blaschke sequence might not be an interpolating sequence. Naturally, one may ask

Problem 1 For a given Blaschke sequence, how close can an interpolating sequence be to it?

In 1956, Naftalevič [13] considered Problem 1 and proved the following theorem.

Theorem A [13] *For every Blaschke sequence* $\{r_n\}_{n=1}^{\infty} \subset (0,1)$, *there exists an interpolating Blaschke sequence* $\{z_n\}_{n=1}^{\infty}$ *such that* $|z_n| = r_n$ *for* $n = 1, 2, \ldots$.

Cochran [3] gave a detailed proof of Theorem A by using the following lemma.

Lemma B [3, Lemma 3.1] *Let* $\delta = 1/2$, *and* $I_k = \left[1-\delta^{k-1}, 1-\delta^k\right)$ *for* $1 \le k < \infty$. *Suppose that* $\{r_n\}_{n=1}^{\infty} \subset (0,1)$ *is a non-decreasing sequence satisfying the following two conditions.*

(i) *For all k sufficiently large*,

$$\sum_{r_n \in I_k} (1 - r_n) > \delta^{k/2}.$$

(ii)

$$\sum_{n=1}^{\infty} (1 - r_n) < \delta^2.$$

Then there exists an interpolating sequence $\{z_n\}_{n=1}^{\infty} \subset \mathbb{D}$ *such that* $|z_n| = r_n$ *for every* n.

Girela et al. [8] mentioned without a proof the following result, which is stronger than Theorem A. Please refer to [1, 8, 11, 14] for more information.

Theorem C [8, 11] *For every Blaschke sequence* $\{\omega_n\}_{n=1}^{\infty}$, *there exists an interpolating Blaschke sequence* $\{z_n\}_{n=1}^{\infty}$ *such that* $|z_n| = |\omega_n|$ *for* $n = 1, 2, \ldots$.

Remark 1 The definition of an interpolating sequence implies that $\{\omega_n\}_{n=1}^{\infty}$ in Theorem C contains at most one zero.

In this article, we continue to consider Problem 1. We first give Lemma 1, which is a generalization and refinement of Lemma B.

Lemma 1 *Let* $0 < \delta \le 1/2$ *and* $I_k = \left[1-\delta^{k-1}, 1-\delta^k\right)$ *for* $1 \le k < \infty$. *Suppose that* $\{r_n\}_{n=1}^{\infty} \subset (0,1)$ *is a non-decreasing sequence satisfying the following two conditions.*

(i) *There exists a positive integer* K_0 *such that for* $k \ge K_0$,

$$\sum_{r_n \in I_k} (1 - r_n) > \delta^{k/2}.$$

(ii)

$$\sum_{n=1}^{\infty} (1 - r_n) < \delta^2.$$

Then there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset \mathbb{D}$ *such that*

(1) $|z_n| = r_n$ *and* $0 < \arg z_n < 2\pi\delta^2$ *for* $n = 1, 2, \ldots$,
(2) *and* $\{z_n\}_{n=1}^{\infty}$ *is an interpolating sequence.*

By using Lemma 1, we prove Lemma 2, which is a generalization and refinement of Theorem A.

Lemma 2 *Suppose that* $0 \le \alpha < 2\pi$ *and* $\{\omega_n\}_{n=1}^{\infty}$ *is a sequence of the points on a ray* $R_\alpha = \left\{z \in \mathbb{D} : z = re^{i\alpha}, 0 < r < 1\right\}$ *satisfying*

$$\sum_{n=1}^{\infty} (1 - |\omega_n|) < \infty.$$

Then for every $0 < \varepsilon < 2\pi - \alpha$, *there is an interpolating sequence* $\{z_n\}_{n=1}^{\infty}$ *such that* $|z_n| = |\omega_n|$ *and* $|\arg z_n - \alpha| < \varepsilon$ *for* $1 \le n < \infty$.

Remark 2 In the proof of Lemma 2, we may assume that $\alpha = 0$ by rotation. By adding some elements into $\{\omega_n\}_{n=1}^{\infty}$, we obtain a new Blaschke sequence satisfying Condition (i) of Lemma 1. Then there exists a subsequence of the new sequence satisfying the assumptions and conditions in Lemma 1.

For any Blaschke sequence containing at most one zero, we divide the points into two kinds. One kind consists of all the points with modulus not larger than 3/4. We rotate each point of this kind by a small angle and obtain a new subsequence consisting of distinct points. Divide $\mathbb{D}$ into several sectors such that the angle of each sector is small enough. For the other kind, we rotate the points of each sector to its angle bisector and obtain several subsequences such that different subsequences are located on different angle bisectors. By applying Lemma 2 to these subsequences and using the properties of interpolating sequences, we obtain a refinement of Theorem C, which is our main result stated as follows.

Theorem 1 *Let* $\{\omega_n\}_{n=1}^{\infty}$ *be a Blaschke sequence containing at most one zero. Then for every* $\varepsilon > 0$, *there exists an interpolating Blaschke sequence* $\{z_n\}_{n=1}^{\infty}$ *in* $\mathbb{D}$ *such that for* $1 \le n < \infty$,

(1) $|z_n| = |\omega_n|$,
(2) *and* $|\arg z_n - \arg \omega_n| < \varepsilon$.

By using Theorem 1, we have the following.

Corollary 1 *Let* $\{\omega_n\}_{n=1}^{\infty} \subset \mathbb{D}$ *be a Blaschke sequence. Then for any* $\varepsilon > 0$, *there exists an interpolating Blaschke sequence* $\{z_n\}_{n=1}^{\infty} \subset \mathbb{D}$ *such that* $|z_n - \omega_n| < \varepsilon$ *for every* n.

The paper is organized as follows. In Sect. 2, we introduce Carleson's interpolating Theorem and give some properties of interpolating sequences and hyperbolic metric for later usage. The proof of Lemmas 1, 2 and Theorem 1 is in Sects. 3, 4 and 5 respectively. In Sect. 6, we pose an open problem and give some remarks.

2 Preliminary

2.1 Carleson's Interpolation Theorem

Our main tool is a geometric characterization of interpolating sequence, called Carleson's Interpolation Theorem [2]. Please refer to [5, Thm. 1.1] or the proposition on [3, p. 738] for more information. Applying a necessary and sufficient condition of Carleson measure [3, Lem. 2.2], Carleson's interpolation theorem can be restated in the following form.

Theorem D [3] $\{z_n\}_{n=1}^{\infty} \subset \mathbb{D}$ *is an interpolating sequence if and only if both conditions* (1) *and* (2) *are satisfied.*

(1) $\{z_n\}_{n=1}^{\infty}$ *is weakly separated. That is, there exists* $\delta > 0$ *such that*

$$\inf_{k,n,k\neq n} \left| \frac{z_n - z_k}{1 - \overline{z_n} z_k} \right| = \inf_{k,n,k\neq n} \rho(z_n, z_k) \geq \delta.$$

(2)

$$\sup_n \sum_{k=1}^{\infty} \frac{\left(1 - |z_k|^2\right)\left(1 - |z_n|^2\right)}{|1 - \overline{z_n} z_k|^2} < \infty.$$

2.2 Some Propositions

The following three propositions are the basic properties of interpolating sequences.

Proposition 1 *If* $m \geq 2$, *and* $S_1, \ldots, S_m$ *are interpolating sequences, which are disjoint from each other, then the union* $S = \bigcup_{k=1}^{m} S_k$ *is an interpolating sequence if and only if*

$$\min_{1\leq i<j\leq m} \left\{\rho(S_i, S_j)\right\} > 0,$$

where $\rho(S_i, S_j) = \inf\left\{\rho(z, \omega) : z \in S_i, \omega \in S_j\right\}$.

Proof For $m = 2$, refer to [5, Ex. 2, p. 305]. For $m > 2$, we can get the conclusion by induction. □

Proposition 2 (1) *Every finite sequence* $\{z_1, \ldots, z_m\} \subset \mathbb{D}$ *consisting of distinct points is an interpolating sequence.*

(2) *If* $\{z_n\}_{n=1}^{\infty}$ *is an interpolating sequence, then all its subsequences are interpolating sequences. In particular, the truncated subsequences* $\{z_n\}_{n=N}^{\infty}$ *are interpolating sequences, where* N *is any positive integer.*

(3) *Suppose that* $\{z_n\}_{n=1}^{\infty}$ *is an interpolating sequence. If rearrange the elements in the sequence and get a new sequence, then the new sequence is also interpolating.*

Proof It follows from Theorem D. □

Proposition 3 *If $\{z_n\}_{n=1}^{\infty}$ is an interpolating sequence, then*

(1) $\{\overline{z_n}\}_{n=1}^{\infty}$ *is an interpolating sequence.*
(2) $\left\{e^{i\theta} z_n\right\}_{n=1}^{\infty}$ *is an interpolating sequence.*
(3) $\{\tau_\omega(z_n)\}_{n=1}^{\infty}$ *is an interpolating sequence, where $\omega \in \mathbb{D}$ and*

$$\tau_\omega(z) = \frac{\omega - z}{1 - \overline{\omega} z}.$$

Proof (1) follows from Theorem D and the fact

$$\rho(z, w) = \left|\frac{z - w}{1 - \overline{w} z}\right| = \left|\frac{\overline{z} - \overline{w}}{1 - w\overline{z}}\right| = \rho(\overline{z}, \overline{w}).$$

For $f = e^{i\theta}$ and $f = \tau_\omega(z)$, we have

$$\rho\left(f(z), f(\omega)\right) = \rho(z, \omega)$$

for $z, \omega \in \mathbb{D}$. (2) and (3) follow from Theorem D. □

The following proposition asserts two facts that will be used in Sect. 3.

Proposition 4 (1) *Let $z = re^{i\theta}$, where $0 < r < 1$ and $|\theta| < \pi/2$. Then $|1 - z| \geq 2|\theta|/\pi$.*
(2) *For $z_1, z_1 \in \mathbb{D}$, we have the identity*

$$1 - \rho(z_1, z_2)^2 = 1 - \left|\frac{z_1 - z_2}{1 - \overline{z_2} z_1}\right|^2 = \frac{(1 - |z_1|^2)(1 - |z_2|^2)}{|1 - \overline{z_2} z_1|^2}. \tag{1}$$

Proof (1) $\theta = 0$ is trivial. If $\theta \neq 0$, geometric observation gives

$$|1 - z| \geq \sin|\theta| > \frac{2}{\pi}|\theta|.$$

(2) follows from a direct calculation. □

3 Proof of Lemma 1

Proof Let $N_k = \#\{n : r_n \in I_k\}$, $M_0 = 0$ and

$$M_k = N_1 + \cdots + N_k, \qquad k = 1, 2, \ldots.$$

Then for $k \geq 1$ and $N_k \neq 0$, we have

$$\{r_n\}_{n=1}^{\infty} \cap I_k = \{r_n\}_{n=M_{(k-1)}+1}^{M_k}. \tag{2}$$

Set $z_1 = r_1$ and

$$\theta_n = 2\pi \sum_{j=1}^{n-1}(1 - r_j), \quad z_n = r_n e^{i\theta_n}, \qquad n \geq 2.$$

Then

$$0 < \arg z_n = 2\pi \sum_{j=1}^{n-1}(1 - r_j) < 2\pi \sum_{j=1}^{\infty}(1 - r_j) < 2\pi\delta^2 \leq \frac{\pi}{2} \qquad \text{for } n \geq 2.$$

Therefore, Conclusion (1) holds.

Since $\{\arg z_n\}_{n=1}^{\infty} \subset [0, \pi/2]$ is increasing, $\{z_n\}_{n=1}^{M_{K_0}}$ consists of distinct points and thus is an interpolating sequence by Proposition 2. We now consider $\{z_n\}_{n=M_{K_0}+1}^{\infty}$. If $k > n > M_{K_0}$, then

$$\overline{z_n} z_k = r_n r_k e^{i(-\theta_n)} e^{i(\theta_k)} = r_n r_k e^{i(\theta_k - \theta_n)}.$$

Note that $\theta_k - \theta_n = 2\pi \sum_{j=n}^{k-1}(1 - r_j)$. The above facts and Proposition 4 imply that

$$\begin{aligned} |1 - \overline{z_n} z_k| &\geq \frac{2}{\pi}(\theta_k - \theta_n) = \frac{2}{\pi} \times 2\pi \sum_{j=n}^{k-1}(1 - r_j) \\ &= 4 \sum_{j=n}^{k-1}(1 - r_j). \end{aligned} \tag{3}$$

Similarly, if $n > k > M_{K_0}$, then we have

$$|1 - \overline{z_n} z_k| > 4 \sum_{j=k}^{n-1}(1 - r_j). \tag{4}$$

Claim $\{z_n\}_{n=M_{K_0}+1}^{\infty}$ satisfies Condition (1) in Theorem D.

If $k > n > M_{K_0}$, then $1 - r_k \leq 1 - r_n$, for $\{r_n\}_{n=1}^{\infty}$ is a non-decreasing sequence. It is obvious that

$$1 + r_k < 2, \qquad 1 + r_n < 2, \qquad \sum_{j=n}^{k-1}(1 - r_j) \geq (1 - r_n).$$

These facts together with (3) and Proposition 4 give that

$$\begin{aligned} 1-\rho(z_k,z_n)^2 &= \frac{(1-|z_k|^2)(1-|z_n|^2)}{|1-\overline{z_n}z_k|^2} \\ &\le \frac{(1+r_k)(1-r_k)(1+r_n)(1-r_n)}{\left(4\sum_{j=n}^{k-1}(1-r_j)\right)^2} \\ &< 4\times\frac{(1-r_k)(1-r_n)}{16(1-r_n)^2} \\ &= \frac{1}{4}\times\frac{1-r_k}{1-r_n}\le\frac{1}{4}, \end{aligned}$$

implying that $\rho(z_k,z_n)^2 \ge 1-1/4 = 3/4$.

If $n > k > M_{K_0}$, then by similar reasoning as above, we also have

$$\rho(z_k,z_n)^2 \ge \frac{3}{4}.$$

Therefore, for $k, n > M_{K_0}$ we have

$$\inf_{k\ne n}\left|\frac{z_k-z_n}{1-\overline{z_n}z_k}\right| \ge \frac{\sqrt{3}}{2}.$$

Claim $\{z_n\}_{n=M_{K_0}+1}^{\infty}$ satisfies Condition (2) in Theorem D.

For each $n > M_{K_0}$, set

$$S_n = \sum_{k=M_{K_0}+1}^{\infty}\frac{(1-|z_k|^2)(1-|z_n|^2)}{|1-\overline{z_n}z_k|^2}.$$

By (2), we know that

$$\{r_n\}_{n=M_{K_0}+1}^{\infty} \subset \bigcup_{k=K_0+1}^{\infty} I_k.$$

If we assume that $r_n \in I_p = [1-\delta^{p-1}, 1-\delta^p)$, then $p \ge K_0+1$ and

$$\delta^p < 1-r_n \le \delta^{p-1}.$$

For the case $p = K_0+1, K_0+2$, we have

$$\{r_n\}_{n=M_{K_0}+1}^{\infty} \cap \left(\bigcup_{k=1}^{p-2} I_k\right) = \emptyset.$$

Let $L = M_{K_0}$ and $S_n^1 = 0$.

For the case $p \geq K_0 + 3$, the above facts and Condition (i) imply that

$$\{r_n\}_{n=M_{K_0}+1}^{\infty} \cap I_{(p-2)} \neq \emptyset.$$

We let $L = M_{(p-2)}$ and

$$S_n^1 = \sum_{k=M_{K_0}+1}^{L} \frac{(1-|z_k|^2)(1-|z_n|^2)}{|1-\overline{z_n}z_k|^2}.$$

Set $U = M_{(p+1)} + 1$,

$$S_n^2 = \sum_{k=L+1}^{n-1} \frac{(1-|z_k|^2)(1-|z_n|^2)}{|1-\overline{z_n}z_k|^2},$$
$$S_n^3 = \sum_{k=n+1}^{U-1} \frac{(1-|z_k|^2)(1-|z_n|^2)}{|1-\overline{z_n}z_k|^2},$$
$$S_n^4 = \sum_{k=U}^{\infty} \frac{(1-|z_k|^2)(1-|z_n|^2)}{|1-\overline{z_n}z_k|^2}.$$

Then

$$S_n = S_n^1 + S_n^2 + 1 + S_n^3 + S_n^4,$$

as

$$\frac{(1-|z_k|^2)(1-|z_n|^2)}{|1-\overline{z_n}z_k|^2} = 1 \ \text{ when } \ k = n.$$

For S_n^1, we only need to consider the case $p \geq K_0 + 3$. Since $k \leq L < n$, we have $\{r_i : k \leq i < n\} \supset \{r_i : r_i \in I_{p-1}\}$, which together with (4) implies that

$$|1-\overline{z_n}z_k| \geq 4\sum_{i=k}^{n-1}(1-r_i) \geq 4 \sum_{r_i \in I_{p-1}} (1-r_i).$$

By Condition (i), the last estimate shows that

$$|1-\overline{z_n}z_k|^2 > 16\delta^{p-1},$$

which together with $1 - |z_n| = 1 - r_n \leq \delta^{p-1}$ yields

$$\frac{(1-|z_k|^2)(1-|z_n|^2)}{|1-\overline{z_n}z_k|^2} \leq \frac{(1+|z_k|)(1-|z_k|)(1+|z_n|)(1-|z_n|)}{16\delta^{p-1}}$$
$$< \frac{4}{16} \times \frac{(1-|z_k|)\delta^{p-1}}{\delta^{p-1}} = \frac{1}{4}(1-r_k).$$

Hence by Condition (ii), we have

$$S_n^1 \le \frac{1}{4}\sum_{k=M_{K_0}+1}^{L}(1-r_k) < \frac{1}{4}\sum_{k=1}^{\infty}(1-r_k) < \frac{\delta^2}{4}. \tag{5}$$

For S_n^2, since $L < L+1 \le k \le n-1 < n$, we have

$$1-r_n \le 1-r_i \ \text{ for } \ k \le i \le n-1 \quad \Rightarrow \quad \sum_{i=k}^{n-1}(1-r_i) \ge (n-k)(1-r_n),$$

$$r_k \in I_{p-1} \cup I_p \quad \Rightarrow \quad 1-\delta^{p-2} \le r_k \quad \Rightarrow \quad 1-r_k \le \delta^{p-2} < \delta^{-2}(1-r_n).$$

These facts and (4) show that

$$\begin{aligned}\frac{(1-|z_k|^2)(1-|z_n|^2)}{|1-\overline{z_n}z_k|^2} &\le \frac{(1+|z_k|)(1-|z_k|)(1+|z_n|)(1-|z_n|)}{\left(4\sum_{i=k}^{n-1}(1-r_i)\right)^2}\\ &< \frac{1}{4}\frac{(1-r_k)(1-r_n)}{((n-k)(1-r_n))^2} \le \frac{1}{4\delta^2}\frac{1}{(n-k)^2}.\end{aligned}$$

So

$$S_n^2 < \frac{1}{4\delta^2}\sum_{k=L+1}^{n-1}\frac{1}{(n-k)^2} = \frac{1}{4\delta^2}\sum_{k=1}^{n-L-1}\frac{1}{k^2} < \frac{1}{4\delta^2}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{24\delta^2}. \tag{6}$$

For S_n^3, since $n+1 \le k \le U-1 < U$, we have

$$r_i < 1-\delta^{p+1} \ \text{ for } \ n+1 \le i \le U-1 \quad \Rightarrow \quad 1-r_i > \delta^{p+1} \ge \delta^2(1-r_n),$$

$$|1-\overline{z_n}z_k| \ge 4\sum_{i=n}^{k-1}(1-r_i) \ge 4\delta^2(k-n)(1-r_n).$$

These facts and (3) show that

$$\begin{aligned}\frac{(1-|z_k|^2)(1-|z_n|^2)}{|1-\overline{z_n}z_k|^2} &\le \frac{(1+|z_k|)(1-|z_k|)(1+|z_n|)(1-|z_n|)}{\left(4\sum_{i=n}^{k-1}(1-r_i)\right)^2}\\ &\le \frac{1}{4\delta^4}\frac{(1-r_n)(1-r_n)}{[(k-n)(1-r_n)]^2}\\ &\le \frac{1}{4\delta^4}\frac{1}{(k-n)^2}.\end{aligned}$$

Hence

$$S_n^3 < \frac{1}{4\delta^4}\sum_{k=n+1}^{U-1}\frac{1}{(k-n)^2} = \frac{1}{4\delta^4}\sum_{k=1}^{U-n-1}\frac{1}{k^2} < \frac{1}{4\delta^4}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{24\delta^4}. \tag{7}$$

For S_n^4, since $k \geq U > n$, we have $\{r_i : n \leq i < k\} \supset \{r_i : r_i \in I_{p+1}\}$, which together with (3) and Condition (i) implies that

$$|1-\overline{z_n}z_k| \geq 4\sum_{i=n}^{k-1}(1-r_i) \geq 4\sum_{r_i\in I_{p+1}}(1-r_i) \geq 4\delta^{(p+1)/2}.$$

These facts show that

$$\begin{aligned}\frac{(1-|z_k|^2)(1-|z_n|^2)}{|1-\overline{z_n}z_k|^2} &\leq \frac{(1+r_k)(1-r_k)(1+r_n)(1-r_n)}{16\delta^{p+1}}\\ &< \frac{1}{4}\frac{(1-r_k)\delta^{p-1}}{\delta^{p+1}} = \frac{1}{4\delta^2}(1-r_k).\end{aligned}$$

It follows that

$$S_n^4 < \frac{1}{4\delta^2}\sum_{k=U}^{\infty}(1-r_k) < \frac{1}{4\delta^2}\sum_{k=1}^{\infty}(1-r_k) < \frac{1}{4\delta^2}\delta^2 = \frac{1}{4}. \tag{8}$$

Combining (5), (6), (7) and (8), we conclude

$$S_n < \frac{\delta^2}{4} + \frac{\pi^2}{24\delta^2} + \frac{\pi^2}{24\delta^4} + \frac{5}{4} := M < \infty.$$

Since M is independent of n, we know that Condition (2) in Theorem D is satisfied.

The two claims show that $\{z_n\}_{n=M_{K_0}+1}^{\infty}$ is an interpolating sequence. For all $z_k \in \{z_n\}_{n=1}^{M_{K_0}}$ and all $z_n \in \{z_n\}_{n=M_{K_0}+1}^{\infty}$, we have

$$\rho(z_n, z_k) = \left|\frac{z_n - z_k}{1-\overline{z_n}z_k}\right| \geq \frac{r_n - r_k}{2} \geq \frac{r_{M_{K_0}+1} - r_{M_{K_0}}}{2} > 0.$$

Proposition 1 shows that the sequence $\{z_n\}_{n=1}^{\infty}$ is an interpolating sequence and thus the proof of Lemma 1 is complete. □

4 Proof of Lemma 2

Proof By Proposition 3, we may assume that $\alpha = 0$. Since $\{\omega_n\}_{n=1}^{\infty} \subset (0, 1)$, we may further assume that $\{\omega_n\}_{n=1}^{\infty}$ is a non-decreasing sequence by (3) of Proposition 2.

Choose $0 < \delta \leq 1/2$ such that $2\pi\delta^2 < \varepsilon$. As in Lemma 1, let $M_0 = 0$, and for $k \geq 1$, set

$$I_k = [1-\delta^{k-1}, 1-\delta^k), \qquad N_k = \#\{n : \omega_n \in I_k\} \qquad \text{and} \qquad M_k = \sum_{i=1}^{k} N_i.$$

Obviously, if $N_k = 0$, then $M_{k-1} = M_k$. If $N_k > 0$, then

$$\{\omega_n\}_{n=1}^{\infty} \cap I_k = \left\{\omega_{M_{k-1}+1}, \ldots, \omega_{M_k}\right\}.$$

Step 1. For $k = 1, 2 \ldots$, we will recursively define n_k, and construct a non-decreasing sequence $\{\beta_n\}_{n=1}^{\infty} \subset (0, 1)$ such that

$$\{\beta_n\}_{n=1}^{\infty} \cap I_k = \{\beta_n\}_{n=n_{k-1}+1}^{n_k}, \qquad \sum_{\beta_n \in I_k} (1-\beta_n) > \delta^{k/2}, \tag{9}$$

where $n_0 = 0$.

For $k = 1$, if $\sum_{\omega_n \in I_1}(1-\omega_n) > \delta^{1/2}$, then let $m_1 = 0$, $n_1 = N_1$ and

$$\beta_i = \omega_i \qquad \text{for } i = 1, 2, \ldots, n_1.$$

If $\sum_{\omega_n \in I_1}(1-\omega_n) \leq \delta^{1/2}$, then set $\tau_1 = 0$ if $N_1 = 0$, and $\tau_1 = \omega_{M_1}$ if $N_1 > 0$,

$$\begin{aligned} m_1 &= \min\left\{m : \sum_{\omega_n \in I_1}(1-\omega_n) + m(1-\tau_1) > \delta^{1/2}\right\}, \\ n_1 &= m_1 + N_1, \\ \eta_1 &= \sum_{\omega_n \in I_1}(1-\omega_n) + m_1(1-\tau_1) - \delta^{1/2} > 0. \end{aligned}$$

Choose η_1' such that

$$0 < \eta_1' < \min\left\{\frac{\eta_1}{m_1}, 1-\delta-\tau_1\right\}.$$

We select γ_j^1 for $1 \leq j \leq m_1$ such that

$$\tau_1 < \gamma_1^1 < \cdots < \gamma_{m_1}^1 < \tau_1 + \eta_1'. \tag{10}$$

Let

$$\beta_i = \omega_i \qquad \text{for } i = 1, 2, \ldots, N_1,$$

and

$$\beta_{j+N_1} = \gamma_j^1 \qquad \text{for } j = 1, 2, \ldots, m_1.$$

By (10) and the definition of η_1', we can easily get that $\beta_{j+N_1} = \gamma_j^1 \in I_1$ for $j = 1, 2, \ldots, m_1$, and we have

$$\sum_{j=1}^{m_1}\left(1-\gamma_j^1\right) > \sum_{j=1}^{m_1}\left(1-\tau_1-\eta_1'\right) = m_1(1-\tau_1) - m_1\eta_1' > m_1(1-\tau_1) - \eta_1.$$

Therefore,

$$\begin{aligned}\sum_{\beta_n\in I_1}(1-\beta_n) &= \sum_{\omega_n\in I_1}(1-\omega_n) + \sum_{j=1}^{m_1}\left(1-\gamma_j^1\right)\\ &> \sum_{\omega_n\in I_1}(1-\omega_n) + m_1(1-\tau_1) - \eta_1\\ &= \eta_1 + \delta^{1/2} - \eta_1 = \delta^{1/2}.\end{aligned}$$

Suppose that for $k \geq 2$, we have defined n_{k-1} and m_{k-1}, and obtained $\beta_n \in I_{k-1}$ such that

$$\sum_{\beta_n\in I_{k-1}}(1-\beta_n) > \delta^{(k-1)/2}.$$

If $\sum_{\omega_n\in I_k}(1-\omega_n) > \delta^{k/2}$, then let $m_k = 0$, $n_k = n_{k-1} + N_k$, and set

$$\beta_{n_{k-1}+j} = \omega_{M_{k-1}+j} \quad \text{for} \quad j = 1, 2, \ldots, N_k.$$

If $\sum_{\omega_n\in I_k}(1-\omega_n) \leq \delta^{k/2}$, we will insert some new items in I_k. Set $\tau_k = 1-\delta^{k-1}$ if $N_k = 0$, and $\tau_k = \omega_{M_k}$ if $N_k > 0$. Define

$$\begin{aligned}m_k &= \min\left\{m : \sum_{\omega_n\in I_k}(1-\omega_n) + m(1-\tau_k) > \delta^{k/2}\right\},\\ n_k &= n_{k-1} + m_k + N_k,\\ \eta_k &= \sum_{\omega_n\in I_k}(1-\omega_n) + m_k(1-\tau_k) - \delta^{k/2} > 0.\end{aligned}$$

Choose η_k' such that

$$0 < \eta_k' < \min\left\{\eta_k/m_k, 1-\delta^k-\tau_k\right\}.$$

We select γ_j^k for $1 \leq j \leq m_k$ such that

$$\tau_k < \gamma_1^k < \ldots < \gamma_{m_k}^k < \tau_k + \eta_k'.$$

Let

$$\begin{aligned} \beta_{n_{k-1}+i} &= \omega_{M_{k-1}+i} \qquad \text{for } 1 \le i \le N_k, \\ \beta_{n_{k-1}+N_k+j} &= \gamma_j^k \qquad\quad \text{for } 1 \le j \le m_k. \end{aligned}$$

Then by similar reasoning as in the case $k = 1$, we know that $\beta_{n_{k-1}+i} \in I_k$ for $i = 1, 2, \ldots, N_k + m_k$, and we have

$$\begin{aligned} \sum_{j=1}^{m_k} \left(1 - \gamma_j^k\right) &> \sum_{j=1}^{m_k} \left(1 - \tau_k - \eta_k'\right) \\ &= m_k(1 - \tau_k) - m_k\eta_k' > m_k(1 - \tau_k) - \eta_k, \\ \sum_{\omega_n \in I_k} (1 - \omega_n) + \sum_{j=1}^{m_k} \left(1 - \gamma_j^k\right) &> \sum_{\omega_n \in I_k} (1 - \omega_n) + m_k(1 - \tau_k) - \eta_k \\ &= \eta_k + \delta^{k/2} - \eta_k = \delta^{k/2}, \\ \sum_{\beta_n \in I_k} (1 - \beta_n) &> \delta^{k/2}. \end{aligned}$$

From the process of constructing β_n, it is obvious that $\{\beta_n\}_{n=1}^{\infty}$ is non-decreasing and

$$\{\beta_n\}_{n=1}^{\infty} \cap I_k = \{\beta_n\}_{n=n_{k-1}+1}^{n_k}.$$

Step 2. Prove that there exits an integer $K_0 > 0$ such that

$$\sum_{k=K_0+1}^{\infty} \sum_{\beta_n \in I_k} (1 - \beta_n) = \sum_{n=n_{K_0}+1}^{\infty} (1 - \beta_n) < \delta^2.$$

Let $k \ge 2$. Then $\delta^{k-1} \le \delta^{k/2}$. By the minimum property of m_k, we have

$$\sum_{j=1}^{m_k} \left(1 - \gamma_j^k\right) \le 2\delta^{k/2}$$

if $m_k > 0$. Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} (1 - \beta_n) &= \sum_{\beta_n \in I_1} (1 - \beta_n) + \sum_{k=2}^{\infty} \sum_{\omega_n \in I_k} (1 - \omega_n) + \sum_{k=2, m_k > 0}^{\infty} \sum_{j=1}^{m_k} \left(1 - \gamma_j^k\right) \\ &\le \sum_{\beta_n \in I_1} (1 - \beta_n) + \sum_{k=2}^{\infty} \sum_{\omega_n \in I_k} (1 - \omega_n) + \sum_{k=2}^{\infty} 2\delta^{k/2} < \infty. \end{aligned}$$

By the last estimate, there exits an integer $K_0 > 0$ such that

$$\sum_{k=K_0+1}^{\infty} \sum_{\beta_n \in I_k} (1-\beta_n) = \sum_{n=n_{K_0}+1}^{\infty} (1-\beta_n) < \delta^2.$$

Step 3. Obtain an interpolating sequence.

Recall that

$$\{\beta_n\}_{n=1}^{\infty} \cap I_{K_0+1} = \{\beta_n\}_{n=n_{K_0}+1}^{n_{(K_0+1)}}.$$

Let $\beta_n' = \beta_{n+n_{K_0}}$ for $n \geq 1$. Then $\left\{\beta_n'\right\}_{n=1}^{\infty}$ satisfies the assumptions and conditions in Lemma 1. Hence there is an interpolating sequence $\left\{\alpha_n'\right\}_{n=1}^{\infty}$ such that

$$\left|\alpha_n'\right| = \beta_n', \qquad 0 < \arg \alpha_n' < 2\pi\delta^2 < \varepsilon \qquad \text{for } 1 \leq n < \infty.$$

Let $\alpha_{n+n_{K_0}} = \alpha_n'$ for $n \geq 1$. Then $\{\alpha_n\}_{n_{K_0}+1}^{\infty}$ is an interpolating sequence. For $\alpha_n \in \{\alpha_n\}_{n_{K_0}+1}^{\infty}$ but $|\alpha_n| \notin \{\omega_n\}_{n=1}^{\infty}$, we delete all such α_n from $\{\alpha_n\}_{n_{K_0}+1}^{\infty}$ and get a subsequence S_2. We may rewrite S_2 such that

$$S_2 = \{z_n\}_{n=M_{K_0}+1}^{\infty}$$

and $\{|z_n|\}_{n=M_{K_0}+1}^{\infty}$ is non-decreasing. As a subsequence of an interpolating sequence $\{\alpha_n\}_{n_{K_0}+1}^{\infty}$, S_2 is interpolating.

Choose $0 < \theta_1 < \theta_2 < \ldots < \theta_{M_{k_0}} < 2\pi\delta^2 < \varepsilon$ and let

$$z_n = \omega_n e^{i\theta_n} \qquad \text{for } 1 \leq n \leq M_{k_0}, \qquad S_1 = \{z_n\}_{n=1}^{M_{k_0}}.$$

Then $S_1 = \{z_n\}_{n=1}^{M_{k_0}}$ is interpolating for it consists of distinct points.

Let $r = \omega_{M_{k_0}}$ and $R = \omega_{M_{k_0}+1}$. For any $z_1 \in S_1$ and $z_2 \in S_2$, we have $|z_2| \geq R > r \geq |z_1|$.

$$\rho(z_1, z_2) = \left|\frac{z_2 - z_1}{1 - \overline{z_1} z_2}\right| \geq \frac{|z_2 - z_1|}{2} \geq \frac{R-r}{2}.$$

Hence $\rho(S_1, S_2) \geq (R-r)/2 > 0$. Proposition 1 shows that

$$\{z_n\}_{n=1}^{\infty} = S_1 \cup S_2$$

is interpolating. The proof of Lemma 2 is complete. □

5 Proof of Theorem 1

Proof Let $S = \{\omega_n\}_{n=1}^{\infty}$. We may assume that $\{|\omega_n|\}_{n=1}^{\infty}$is non-decreasing. Since $\sum_{n=1}^{\infty}(1-|\omega_n|) < \infty$, we know that

$$S^l = S \cap \left\{ z \in \mathbb{D} : |z| \le \frac{3}{4} \right\}$$

contains at most finitely many points. We can rotate each point in S^l by an angle less than ε and obtain a new sequence Z^l consisting of distinct points. Hence Z^l is an interpolating sequence corresponding to S^l and satisfies (1) and (2) of Theorem D.

Choose a positive integer m such that $2\pi/m = \delta_1 < \varepsilon$. Set

$$\theta_j = \frac{2\pi j}{m}, \qquad 0 \le j \le m.$$

Define m sectors

$$\Omega_j = \left\{ z \in \mathbb{D} : z = re^{i\theta}, \theta_j \le \theta < \theta_{j+1}, \frac{3}{4} < r < 1 \right\}, \qquad 0 \le j \le m-1.$$

Thus we divide $S - S^l$ into $S_0, \ldots, S_{m-1}$, where $S_j = \big(S - S_j\big) \cap \Omega_j$. Some S_j may be empty.

We consider the case $S_j \neq \emptyset$. Let $S_j = \{\omega_n^{(j)}\}_{n=1}^{n_j}$. We rotate each $\omega_n^{(j)} \in S_j$ to a ray and result in a new sequence

$$S_j' = \left\{ \left|\omega_n^{(j)}\right| e^{i(\theta_j+\theta_{j+1})/2} \right\}_{n=1}^{n_j},$$

where $\{|\omega_n^{(j)}|\}_{n=1}^{n_j}$ is non-decreasing.

If $n_j < \infty$, then we rotate each point in S_j' by an angle less than $\frac{\delta_1}{4}$ and obtain a new sequence Z_j^b consisting of distinct points. Therefore,

$$Z_j^b \subset \left\{ z \in \mathbb{D} : \left| \arg z - \frac{\theta_j + \theta_{j+1}}{2} \right| < \frac{\delta_1}{4}, \ |z| > \frac{3}{4} \right\}, \tag{11}$$

where Z_j^b is an interpolating sequence corresponding to S_j and satisfies (1) and (2) of Theorem D.

If $n_j = \infty$, then we can apply Lemma 2 to S_j'. There exists an interpolating sequence which we also denote by $Z_j^b = \{z_n^{(j)}\}_{n=1}^{\infty}$ such that

$$\left|z_n^{(j)}\right| = \left|\omega_n^{(j)}\right|, \quad \left| \arg z_n^{(j)} - \frac{\theta_j + \theta_{j+1}}{2} \right| < \frac{\delta_1}{4}$$

for $1 \le n < \infty$.

The above facts and (11) imply that for each $S_j \neq \emptyset$, the corresponding interpolating sequence

$$Z_j^b \subset \left\{ z \in \mathbb{D} : \left| \arg z - \frac{\theta_j + \theta_{j+1}}{2} \right| < \frac{\delta_1}{4}, \ |z| > \frac{3}{4} \right\}. \tag{12}$$

Therefore, for each point $\omega_n^{(j)} \in S_j$ and its corresponding point $z_n^{(j)} \in Z_j^b$, we have $|z_n^{(j)}| = |\omega_n^{(j)}|$ and

$$\begin{aligned} \left| \arg z_n^{(j)} - \arg \omega_n^{(j)} \right| &\leq \left| \arg z_n^{(j)} - \frac{\theta_j + \theta_{j+1}}{2} \right| + \left| \arg \omega_n^{(j)} - \frac{\theta_j + \theta_{j+1}}{2} \right| \\ &< \frac{\delta_1}{4} + \frac{\delta_1}{2} < \varepsilon. \end{aligned}$$

For $0 \leq j_1 < j_2 \leq m-1$, (12) implies that $Z_{j_1}^b$ and $Z_{j_2}^b$ are bounded away from each other with respect to the Euclidean metric. Hence

$$\rho\left(Z_{j_1}^b, Z_{j_2}^b\right) \geq \frac{\operatorname{dist}\left(Z_{j_1}^b, Z_{j_2}^b\right)}{2} > 0,$$

where dist denotes the Euclidean metric. Proposition 1 shows that

$$Z^b = Z_0^b \cup \cdots \cup Z_{m-1}^b$$

is an interpolating sequence corresponding to $S - S^l$.

Let $r^l = \max\{|z| : z \in Z^l\}$ and $r^b = \inf\left\{|z| : z \in Z^b\right\}$. Then $r^l \leq 3/4$, $r^b > 3/4$ and

$$\rho(Z^l, Z^b) \geq \frac{r^b - r^l}{2} > 0,$$

and thus $Z = Z^l \cup Z^b$ is an interpolating sequence. Re-order Z such that Z is an interpolating sequence corresponding to S. The proof of Theorem 1 is complete. □

6 Open Problem and Remarks

Let $\{\omega_n\}_{n=1}^{\infty}$ be a Blaschke sequence containing at most one zero. For every $\varepsilon > 0$, Theorem 1 have shown that there exists an interpolating Blaschke sequence $\{z_n\}_{n=1}^{\infty}$, where z_n is an rotation of ω_n by no more than angle ε for $1 \leq n < \infty$. The micro-rotating method in Theorem 1 motivates us to consider whether we can make the sum of the rotation angles be arbitrarily small. So we have the following problem.

Problem 2 Let $\{\omega_n\}_{n=1}^{\infty}$ be a Blaschke sequence containing at most one zero. For every $\varepsilon > 0$, can we obtain an interpolating Blaschke sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ such that

(1) $|z_n| = |\omega_n|$ for $1 \leq n < \infty$,
(2) and $\sum_{n=1}^{\infty} |\arg z_n - \arg \omega_n| < \varepsilon$?

Lemma 3 *Let* $|z_1| = |z_2| = r \in (0, 1)$ *and* $|\arg z_1 - \arg z_2| < \pi$. *Then*

$$\left| \frac{|z_1|}{z_1} \frac{z_1 - z}{1 - \overline{z_1} z} - \frac{|z_2|}{z_2} \frac{z_2 - z}{1 - \overline{z_2} z} \right| \leq \frac{2}{(1 - |z|)} |\arg z_1 - \arg z_2|, \qquad z \in \mathbb{D}.$$

Proof Let $\arg z_1 = \theta$ and $\arg z_2 = \alpha$. Then for any $z \in \mathbb{D}$, we have

$$\begin{aligned}
&\left| \frac{|z_1|}{z_1} \frac{z_1 - z}{1 - \overline{z_1} z} - \frac{|z_2|}{z_2} \frac{z_2 - z}{1 - \overline{z_2} z} \right| = \left| \frac{r - z e^{-i\theta}}{1 - r z e^{-i\theta}} - \frac{r - z e^{-i\alpha}}{1 - r z e^{-i\alpha}} \right| \\
&= \left| \frac{(1 - r^2) z (e^{-i\theta} - e^{-i\alpha})}{(1 - r z e^{-i\theta})(1 - r z e^{-i\alpha})} \right| \\
&\leq \frac{2(1 - r^2)|z| \sin \frac{|\theta - \alpha|}{2}}{(1 - r)(1 - |z|)} \\
&< \frac{4}{(1 - |z|)} \sin \frac{|\theta - \alpha|}{2} \leq \frac{2}{(1 - |z|)} |\theta - \alpha|.
\end{aligned}$$

□

Lemma 4 *Let* $z_2 = \delta \in (0, 1)$. *Then*

$$\left| z - \frac{z - z_2}{1 - \overline{z_2} z} \right| \leq \frac{2\delta}{(1 - |z|)}, \qquad z \in \mathbb{D}.$$

Proof By assumption, we have

$$\begin{aligned}
\left| z - \frac{z - z_2}{1 - \overline{z_2} z} \right| &= \left| z - \frac{z - \delta}{1 - \delta z} \right| \\
&= \left| \frac{\delta (1 - z^2)}{(1 - \delta z)} \right| \\
&< \frac{2\delta}{(1 - |z|)}, \qquad z \in \mathbb{D}.
\end{aligned}$$

□

Lemma 5 *Let*

$$B_1(z) = \prod_{n=1}^{\infty} \frac{|z_n^{(1)}|}{z_n^{(1)}} \frac{z_n^{(1)} - z}{1 - \overline{z_n^{(1)}} z} \quad \text{and} \quad B_2(z) = \prod_{n=1}^{\infty} \frac{|z_n^{(2)}|}{z_n^{(2)}} \frac{z_n^{(2)} - z}{1 - \overline{z_n^{(2)}} z}$$

be two Blaschke products satisfying $|z_n^{(1)}| = |z_n^{(2)}| > 0$ *for* $n = 1, 2, \ldots$ *and*

$$\sum_{n=1}^{\infty} \left| \arg z_n^{(1)} - \arg z_n^{(2)} \right| < \delta \tag{13}$$

for some $0 < \delta < \pi$. Then for every compact set $K \subset \mathbb{D}$, there exists a constant $M > 0$, which only depends on K, such that for every $z \in K$,

$$|B_1(z) - B_2(z)| \le M\delta.$$

Proof Write $z_n^{(1)} = r_n e^{i\theta_n}$, then $z_n^{(2)} = r_n e^{i\alpha_n}$ for $n = 1, 2, \ldots$. Then (13) turns to be

$$\sum_{n=1}^{\infty} |\theta_n - \alpha_n| < \delta. \tag{14}$$

By Lemma 3, we have

$$D_n(z) := \left| \frac{|z_n^{(1)}|}{z_n^{(1)}} \frac{z_n^{(1)} - z}{1 - \overline{z_n^{(1)}} z} - \frac{|z_n^{(2)}|}{z_n^{(2)}} \frac{z_n^{(2)} - z}{1 - \overline{z_n^{(2)}} z} \right| \le \frac{2}{(1 - |z|)} |\theta_n - \alpha_n|, \qquad z \in \mathbb{D}.$$

Since K is compact, the last estimate shows that there exists a constant $M > 0$ which only depends on K such that

$$D_n(z) \le M|\theta_n - \alpha_n|, \qquad z \in K. \tag{15}$$

By using (15) and (14) and the similar argument in [4, p. 67], we conclude that for every $z \in K$,

$$\begin{aligned}
|B_1(z) - B_2(z)| &= \left| \lim_{k\to\infty} \prod_{n=1}^{k} \frac{|z_n^{(1)}|}{z_n^{(1)}} \frac{z_n^{(1)} - z}{1 - \overline{z_n^{(1)}} z} - \lim_{k\to\infty} \prod_{n=1}^{k} \frac{|z_n^{(2)}|}{z_n^{(2)}} \frac{z_n^{(2)} - z}{1 - \overline{z_n^{(2)}} z} \right| \\
&= \left| \lim_{k\to\infty} \left(\prod_{n=1}^{k} \frac{|z_n^{(1)}|}{z_n^{(1)}} \frac{z_n^{(1)} - z}{1 - \overline{z_n^{(1)}} z} - \prod_{n=1}^{k} \frac{|z_n^{(2)}|}{z_n^{(2)}} \frac{z_n^{(2)} - z}{1 - \overline{z_n^{(2)}} z} \right) \right| \\
&\le \lim_{k\to\infty} \sum_{n=1}^{k} D_n(z) \\
&= \sum_{n=1}^{\infty} D_n(z) \le M \sum_{n=1}^{\infty} |\theta_n - \alpha_n| \le M\delta.
\end{aligned}$$

□

If the answer to Problem 2 is positive, then by using Lemmas 4 and 5, we have the following remarks.

Remark 3 Let $K \subset \mathbb{D}$ be a compact set and

$$B_1(z) = \prod_{n=1}^{\infty} \frac{|\omega_n|}{\omega_n} \frac{\omega_n - z}{1 - \overline{\omega_n} z}$$

be a Blaschke product, where each $\omega_n \neq 0$. If the answer to Problem 2 is positive, then for every $\varepsilon > 0$, there exists an interpolating Blaschke sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ such that

(1) $|z_n| = |\omega_n|$ for $1 \leq n < \infty$,
(2) and $|B_1(z) - B_2(z)| \leq \varepsilon$, $z \in K$,

where

$$B_2(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z}.$$

Remark 4 Let $K \subset \mathbb{D}$ be a compact set, $m \geq 1$ and

$$B_1(z) = z^m \prod_{n=m+1}^{\infty} \frac{|\omega_n|}{\omega_n} \frac{\omega_n - z}{1 - \overline{\omega_n} z}$$

be a Blaschke product, where each $\omega_n \neq 0$ for $n > m$. If the answer to Problem 2 is positive, then for every $\varepsilon > 0$, there exists an interpolating Blaschke sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ such that

(1) $0 < z_1 < z_2 < \cdots < z_m < \max\{\omega_n\}_{n=m+1}^{\infty}$, $|z_n| = |\omega_n|$ for $m + 1 \leq n < \infty$,
(2) and $|B_1(z) - B_2(z)| \leq \varepsilon$ for $z \in K$,

where

$$B_2(z) = \prod_{n=1}^{m} \frac{z - z_n}{1 - z_n z} \prod_{n=m+1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z}.$$

In 1981, Garnett [5] posted the following open questions.

Problem E [5, Prob. 5.4] Can every Blaschke product be uniformly approximated by interpolating Blaschke products?

It aroused the interest of many mathematicians but still remains open. Remarks 3 and 4 show that if the answer to Problem 2 is positive, then every Blaschke product can be locally uniformly approximated by an interpolating Blaschke product.

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