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RESEARCH

Semi-norms of Higher Dimensional Dirichlet Spaces and Impacts to Frequency Analysis of Multivariate Functions

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Abstract
The present paper studies the Dirichlet spaces in balls and the upper-half Euclidean spaces. As main results, we prove a number of identical forms of the Dirichlet semi-norms in the mentioned contexts as generalizations of the classical results proved by Douglas and Ahlfors with the 2-D disc case. With the scalar-valued phase derivative concept the semi-norm identical forms, in all cases, are extended to include the energy average of the instantaneous frequency. The latter has impacts to frequency analysis for multivariate functions.

Keywords Dirichlet space · Dirichlet semi-norm · Hilbert transform · Phase derivative

Mathematics Subject Classification 30G30 · 30G35 · 31B25

1 Introduction

Denote by $\mathbb{D}$ the unit disk and $T = \partial\mathbb{D}$ the unit circle in the complex plane $\mathbb{C}$.
Let

$$h^2(\mathbb{D}) = \{u(z) \mid u(z) \text{ is real-valued, harmonic in } \mathbb{D}, \text{ and satisfies } \|u\|_{h^2} < +\infty\}$$

be the harmonic Hardy space, where

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$$\|u\|_{h^2} := \sup_{0<r<1} \left(\int_0^{2\pi} |u(re^{\mathbf{i}t})|^2 dt \right)^{\frac{1}{2}}.$$

Define the space of harmonic gradients

$$\mathcal{D}(\mathbb{D}) := \{u(z) \mid u(z) \text{ is real-valued harmonic in } \mathbb{D} \text{ and satisfies } \int_{\mathbb{D}} |\nabla u(z)|^2 d\sigma < +\infty\}$$

as the harmonic Dirichlet space in $\mathbb{D}$, where $d\sigma$ is the area measure.

Define the semi-norm of $\mathcal{D}(\mathbb{D})$ by

$$\|u\|^2_{\mathcal{D}(\mathbb{D}),*} := \int_{\mathbb{D}} |\nabla u(z)|^2 d\sigma < +\infty$$

and the norm of $\mathcal{D}(\mathbb{D})$ by

$$\|u\|^2_{\mathbb{D}} := \|u\|^2_{\mathcal{D}(\mathbb{D}),*} + \|u\|^2_{h^2}.$$

We will call the just defined semi-norm as *the gradient form semi-norm*.

If $u \in h^2(\mathbb{D})$, Douglas established, in [8], the identical relation

$$\int_{\mathbb{D}} |\nabla u|^2 d\sigma = \frac{1}{2\pi} \int_T \int_T \frac{|f(z_1) - f(z_2)|^2}{|z_1 - z_2|^2} |dz_1||dz_2| \tag{1.1}$$

that was used as the main technical tool by him to solve the minimum surface problem of Plateau in 1930's. He proved the identical relation through a third identical quantity in terms of the Fourier coefficients of the boundary function f (see below Theorem 1.1), named as *the Fourier series (integral) form* in the sequel. We will call the right-hand-side quantity of (1.1) as the *double (singular) integral of the difference-quotient form*, or simply, *the double integral form* quantity.

It is known that if u is harmonic in Ω, there exists, uniquely, an associated canonical conjugate harmonic function v in a planar region Ω (that is with $v(0) = 0$) such that $F = u + \mathbf{i}v$ is holomorphic in Ω. Here we take Ω to be the open unite disc $\mathbb{D}$.

Using the Cauchy-Riemann equation, we have

$$F'(z) = \frac{\partial u}{\partial x} + \mathbf{i}\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - \mathbf{i}\frac{\partial u}{\partial y}.$$

Therefore, we have

$$\int_{\mathbb{D}} |\nabla u|^2 dxdy = \int_{\mathbb{D}} |F'(z)|^2 dxdy.$$

Using the identity in terms of the canonical holomorphic function

$$\int_{\mathbb{D}} |F'(z)|^2 d\sigma = \lim_{r\to 1^-} \frac{1}{2} \int_{|z|=r} \overline{F(z)} F'(z) \frac{dz}{\mathbf{i}}, \tag{1.2}$$

Ahlfors gave, in 1970's, another proof of (1.1). See [2]. In this paper we will call the left-hand-side of (1.2) as *the holomorphic-derivative form quantity*, and the right-hand-side *the Ahlfors form* quantity.

In summary, there exist the following equivalent characterizations of the semi-norm $\|u\|_{\mathcal{D}(\mathbb{D}),*}$:

Theorem 1.1 [2, 22] *Let $u \in \mathcal{D}(\mathbb{D})$ and F be the canonical holomorphic function associated with u. Then $u \in h^2(\mathbb{D})$. Denote by f the function on the unit circle defined through the non-tangential boundary limits of u, there hold*

$$\begin{aligned}\|u\|^2_{\mathcal{D}(\mathbb{D}),*} &:= \int_{\mathbb{D}} |\nabla u|^2 d\sigma \\ &= \int_{\mathbb{D}} |F'(z)|^2 d\sigma \\ &= \pi \sum_{k=1}^{+\infty} k(a_k^2 + b_k^2) \\ &= \frac{1}{2\pi} \int_T \int_T \frac{|f(z_1) - f(z_2)|^2}{|z_1 - z_2|^2} |dz_1||dz_2|,\end{aligned}$$

where $a_k = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f(e^{\mathbf{i}\theta}) \cos(k\theta) d\theta$ *and* $b_k = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f(e^{\mathbf{i}\theta}) \sin(k\theta) d\theta$.

We will call the above set of identical relations as *the Douglas-Ahlfors identical relations*. We note that the described quantity comes directly from the real valued harmonic function u, while the one in terms of $F'(z)$ establishes a bridge leading to the complex analysis method. See (1.2), also (5.21).

Remark 1.2 Let $u(z)$ be harmonic in $\mathbb{D}$ and $F(z) = u(z) + \mathbf{i}v(z)$ the associated canonical holomorphic function in $\mathbb{D}$. Using the Taylor series expansion, we have $F(z) = \sum_{k=0}^{\infty} c_k z^k$. Denote $c_k = a_k - \mathbf{i}b_k$ and $z = re^{\mathbf{i}\theta}$, we obtain

$$u(z) = \sum_{k=0}^{\infty} r^k [a_k \cos(k\theta) + b_k \sin(k\theta)], \ r < 1.$$

If $u(z) \in \mathcal{D}(\mathbb{D})$, we have, according to Theorem 1.1,

$$\pi \sum_{k=0}^{\infty} |c_k|^2 \le \pi(|c_0|^2 + \sum_{k=1}^{\infty} k|c_k|^2) = \pi|c_0|^2 + \|u\|^2_{\mathcal{D}(\mathbb{D}),*} < \infty.$$

This implies $u \in h^2(\mathbb{D})$. The Dirichlet space $\mathcal{D}(\mathbb{D})$, therefore, is contained in the harmonic Hardy space $h^2(\mathbb{D})$, and, as a consequence, functions belonging to the Dirichlet space all have non-tangential boundary limits a.e. on the boundary of $\mathbb{D}$, and the numbers c_k, a_k, b_k turn out to be the Fourier coefficients of the corresponding functions. In

summary, with a little abuse of notation, the relations between the involved function spaces may be described as

$$\mathcal{D}(\mathbb{D}) = h^2(\mathbb{D}) \cap \mathcal{H}^{1/2}(T),$$

where $\mathcal{H}^{1/2}(T)$ is the fractional Sobolev space constituted by functions $f \in L^2(T)$ having the "1/2" order derivative in $L^2(T)$ ([13]).

Studies of the Dirichlet space in the unit disk $\mathbb{D}$ have achieved fruitful results, see, for instance, [3] and [17], and the references thereby. Recently, the authors of [22] studied Dirichlet spaces over bounded Jordan domains in one complex variable. They, in particular, proved that the Douglas-Ahlfors identical relations stand as characteristic properties of the bounded chord-arc domains. The identical relations of the quantities are identical forms of the semi-norm of the Dirichlet spaces on bounded chord-arc domains. To our knowledge Douglas' program has not been carried out to contexts other than one complex variable. In the present paper, Theorem 1.1 for $\mathcal{D}(\mathbb{D})$ is extended to higher dimensional spheres B_n (Theorem 2.3). There with the harmonic function setting we use the Fourier-Laplace spherical harmonics expansion to replace the classical Fourier series expansion used in Douglas' original proof. Besides computations involving Poisson kernels and Funk-Hecke formula, a new identity (2.6) involving Gegenbauer polynomials is established. Next, we prove the analogous identical relations for harmonic functions in the half Euclidean spaces. For unbounded domains, only for the complex plane case a conformal mapping proof is available, as cited in the paper. For higher dimensions we use mainly the g-function argument ([19, 20]). We bypass also give a mild generalization of the g-function theory. When involving Ahlfors' complex form quantities, in both the sphere and half space cases, Clifford algebra comes into play. The main tools we use including Stokes' formula, spherical Laplacian and Dirac operators, decomposition of spherical harmonics into spherical monogenics, Cauchy-Riemann equations, etc. It is noted that on the spheres the Ahlfors' form quantity is not strictly identical with the others but with an error term.

The present work mainly concentrates in the Hilbert space context. For the Banach spaces case, there exist equivalent but not identical conditions: In [19], for instance, starting from the double singular integral form, the equivalent Fourier integral and the gradient forms are deduced. They correspond to the 3-indices Besov space $\Lambda_\alpha^{p,q}$ ([19] Chapter V, section 5). The study of this paper contributes new understanding to the particular case $p = q = 2$ and $\alpha = 1/2$ for the 3 indices spaces.

There have been, recently, increasing interests in various types of Dirichlet spaces of which what is concerned in this paper is one related to instantaneous frequency (IF). Among different definitions of IF we take the one using phase derivative. This subject is closely related to Ahlfors' formulation. Let $f^+(e^{it}) = \rho(t)e^{i\theta(t)}$ be the boundary value of an analytic function in $\mathbb{D}$ with $v(0) = 0$. On one hand, we let

$$\|f\|^2_{\mathcal{D},*} = \frac{1}{\pi}\int_{\mathbb{D}} |F'(z)|^2 d\sigma = \sum_{k=1}^{\infty} k|c_k|^2.$$

On the other hand, since $z = e^{it}$,

$$\begin{aligned}\|f\|^2_{\mathcal{D},*} &= \frac{1}{2\pi}\int_{|z|=1} f^{+\prime}(z)\overline{f^+(z)}\frac{dz}{i}\\ &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Im}(\overline{f^+}df^+)\\ &= \frac{1}{2\pi}\int_0^{2\pi} \theta'(t)\rho^2(t)dt,\end{aligned}$$

because $\mathrm{Im}(\overline{f^+}df^+) = \rho^2(t)d\theta(t)$. As a consequence,

$$\frac{1}{2\pi}\int_0^{2\pi} \theta'(t)\rho^2(t)dt = \sum_{k=1}^{\infty} k|c_k|^2. \tag{1.3}$$

In the 1-D case, we therefore realize new identical relations as consequences of the identical Douglas-Ahlfors forms. An amazing fact is the scalar-valued phase derivative concept can be precisely generalized to multivariate signals. This is done, as in the 1-D case, through adding the Hilbert transform of the original (real-valued) signal as the non-scalar part, and so to construct the associated higher dimensional monogenic (analytic) signals (also regarded as NBLs of the left-monogenic functions of the region). Out of which a phase signal part can be extracted out ([23, 24]). For the multidimensional cases again phase derivatives are closely related to the Ahlfors form. The results of this paper, in turn, provide new identical relations beyond, but complementary, to those proved in [23, 24]. We will include the details in the last section of the paper.

The writing plan of the paper is as follows: §2 and §3 devote real analysis methods for the harmonic Dirichlet spaces. In §2, we define the real- (or harmonic) Dirichlet space $\mathcal{D}(B_n)$ of the unit balls B_n. The harmonic gradient energy is used to define the semi-norm of the Dirichlet space. Then we give two more identical quantities of, respectively, the Fourier series form and the double integral form. §3 is devoted to studying harmonic Dirichlet spaces of upper-half Euclidean spaces. In such unbounded domain context we establish, using harmonic analysis method, the identical relations between the gradient form and the Fourier integral form and the double singular integral form. We add a little extra by defining a sequence of g^λ functions and show in which g^0 and g^1 correspond to, respectively, the gradient form quantity and the classical $g(f)$ functions. What is missed out by the real methods in §3 is the monogenic-derivative, or Ahlfors' form. That will be proved by using the Clifford analysis method in §4. In §4 we study upper-half spaces, as particular unbounded regions. As a generalization of one complex variable, we introduce the Clifford analysis (hyper complex) settings. It is with the Clifford settings the Ahlfors form comes into play. We prove that all the identical relations are true for the unbounded domains. §5 is devoted to the balls case. We derive that, on the unit balls $B_{n+1}(n > 1)$ in the non-homogeneous $(n+1)$-dimensional setting, with a small difference the Ahlfors form quantity is different from the quantity of the Douglas forms. In the case the Ahlfors form quantities are

directly equivalent with the underlying Dirichlet space norms, but not the semi-norms. In §6 we indicate the impacts of the main results to the existing phase and frequency analysis studies.

2 The Dirichlet Space in the Unit Ball B_n

In the homogeneous setting of $\mathbb{R}^n$ denote by $B_n = \{\underline{x} \mid |\underline{x}| < 1,\ \underline{x} \in \mathbb{R}^n\}$ and $B_n(r) = \{\underline{x} \mid |\underline{x}| < r,\ \underline{x} \in \mathbb{R}^n\}$ the unit ball and the ball with radius r in $\mathbb{R}^n$, respectively.

Denote by

$$S^{n-1} = \{\underline{\xi} \mid |\underline{\xi}| = 1,\ \underline{\xi} \in \mathbb{R}^n\}$$

the unit sphere in $\mathbb{R}^n$.

Let the harmonic Hardy space

$$h^2(B_n) = \{u(\underline{x}) \mid\ u(\underline{x}) \text{ is real-vaued harmonic in } B_n \text{ and satisfies } \|u\|_{h^2} < +\infty\},$$

where

$$\|u\|_{h^2} = \sup_{0<r<1} \left(\int_{S^{n-1}} |u(r\underline{\xi})|^2 dS_{\underline{\xi}} \right)^{\frac{1}{2}}.$$

It is a fundamental result of the harmonic Hardy space that $u \in h^2(B_n)$ implies that u has non-tangential boundary limits a.e. If denoting by $f(\underline{\xi})$ the boundary limit function, then $f \in L^2(S^{n-1})$, and in the L^2-sense f has a spherical Fourier-Laplace series expansion:

$$f(\underline{\xi}) = \sum_{k=0}^{+\infty} Y_k(f)(\underline{\xi}),$$

where

$$Y_k(f)(\underline{\xi}) = c_{n,k} \int_{S^{n-1}} f(\underline{\eta}) P_k^n(\underline{\xi} \cdot \underline{\eta}) dS_{\underline{\eta}}$$

stands for the projection of f onto $\mathcal{H}_k^n$, where P_k^n are the Gegenbauer polynomials and

$$c_{n,k} = \frac{1}{\omega_{n-1}} \frac{(n+2k-2)\Gamma(n+k-1)}{(n+k-2)k!\Gamma(n-1)},$$

$\omega_{n-1} = \dfrac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ is the surface area of S^{n-1}.

For $0 < r < 1$, $\underline{x} = r\underline{\xi}$, there hold the relations

$$\begin{aligned} u(\underline{x}) &= \sum_{k=0}^{+\infty} r^k Y_k(f)(\underline{\xi}) \\ &= \sum_{k=0}^{+\infty} r^k c_{n,k} \int_{S^{n-1}} f(\underline{\eta}) P_k^n(\underline{\xi}\cdot\underline{\eta}) dS_{\underline{\eta}} \\ &= \int_{S^{n-1}} f(\underline{\eta}) \sum_{k=0}^{+\infty} r^k c_{n,k} P_k^n(\underline{\xi}\cdot\underline{\eta}) dS_{\underline{\eta}} \\ &= \frac{1}{\omega_{n-1}} \int_{S^{n-1}} P_r(\underline{\eta},\underline{\xi}) f(\underline{\eta}) dS_{\underline{\eta}}, \\ &:= (f * P_r)(\underline{\xi}), \end{aligned} \tag{2.4}$$

where

$$P_r(\underline{\eta},\underline{\xi}) = \frac{1}{\omega_{n-1}} \frac{1-r^2}{|\underline{\eta} - r\underline{\xi}|^n} = \sum_{k=0}^{+\infty} r^k c_{n,k} P_k^n(\underline{\xi}\cdot\underline{\eta})$$

is the Poisson kernel of B_n at $x = r\underline{\xi}$, which, as a harmonic function by itself, gives rise to the unique harmonic extension u of f into the unit ball B_n.

On the other hand, for any $u(\underline{x}) \in h^2(B_n)$, there exist $f \in L^2(S^{n-1})$, such that $u(\underline{x}) = (f * P_r)(\underline{\xi})$ and $\|u\|_{h^2} = \|f\|_{L^2(S^{n-1})}$.

Lemma 2.1 [21] *Let $n \geq 2$, $k \in \mathcal{N}^+$. For any orthonormal base $\{y_1, y_2, \cdots, y_{a_k^n}\}$ of $\mathcal{H}_k^n$ and any $\underline{\xi}, \underline{\eta} \in S^{n-1}$, we have*

$$P_k^n(\underline{\xi}\cdot\underline{\eta}) = \frac{1}{c_{n,k}} \sum_{j=1}^{a_k^n} \overline{y_j(\underline{\xi})} y_j(\underline{\eta}) = \frac{1}{c_{n,k}} \sum_{j=1}^{a_k^n} \overline{y_j(\underline{\eta})} y_j(\underline{\xi}).$$

In virtue of Lemma 2.1, there exists another form of the Fourier-Laplace series:

$$\begin{aligned} f(\underline{\xi}) &= \sum_{k=0}^{+\infty} c_{n,k} \int_{S^{n-1}} f(\underline{\eta}) P_k^n(\underline{\xi}\cdot\underline{\eta}) dS_{\underline{\eta}} \\ &= \sum_{k=0}^{+\infty} \int_{S^{n-1}} f(\underline{\eta}) \sum_{j=1}^{a_k^n} \overline{y_j(\underline{\eta})} y_j(\underline{\xi}) dS_{\underline{\eta}} \\ &= \sum_{k=0}^{+\infty} \sum_{j=1}^{a_k^n} b_j y_j(\underline{\xi}), \end{aligned}$$

where

$$b_j = \int_{S^{n-1}} f(\underline{\eta}) \overline{y_j(\underline{\eta})} dS_{\underline{\eta}}. \tag{2.5}$$

Clearly, if $f \in L^2(S^{n-1})$, then $\|f\|_2^2 = \sum_{k=0}^{+\infty} \sum_{j=1}^{a_k^n} |b_j|^2$.

Definition 2.2 The Dirichlet space in B_n is defined as

$$\mathcal{D}(B_n) := \{u(\underline{x}) \mid u \text{ is real-valued, harmonic, and satisfies } \int_{B_n} |\nabla u(\underline{x})|^2 d\underline{x} < +\infty\}.$$

The semi-norm of $u \in \mathcal{D}(B_n)$ is defined as

$$\|u\|_{\mathcal{D}(B_n),*}^2 := \int_{B_n} |\nabla u(\underline{x})|^2 d\underline{x} < +\infty.$$

The Dirichlet space norm of $u \in \mathcal{D}(B_n)$ is defined as

$$\|u\|_{\mathcal{D}(B_n)}^2 := \|u\|_{\mathcal{D}(B_n),*}^2 + \|u\|_{h^2_{B_n}}^2 .$$

In the following, several identical representations of the quantity $\|u\|_{\mathcal{D}(B_n),*}$ are obtained:

Theorem 2.3 *Let $u(\underline{x})$ be in the Dirichlet space $\mathcal{D}(B_n)$. Then $u \in h^2(B_n)$. Denote by $f(\underline{\eta})$ the boundary limit function of u, then there follow*

$$\begin{aligned}
\|u\|_{\mathcal{D}(B_n),*}^2 &:= \int_{B_n} |\nabla u(\underline{x})|^2 d\underline{x} \\
&= \sum_{k=1}^{+\infty} k \sum_{j=1}^{a_k^n} |b_j|^2 \\
&= \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \int_{S^{n-1}} \frac{|f(\underline{\eta_1}) - f(\underline{\eta_2})|^2}{|\underline{\eta_1} - \underline{\eta_2}|^n} dS_{\underline{\eta_1}} dS_{\underline{\eta_2}},
\end{aligned}$$

where b_j's are the coefficients of a Fourier-Laplace series of f given by the formula (2.5).

Before proving Theorem 2.3, we need several technical Lemmas.

Lemma 2.1 immediately implies

Lemma 2.4 *[21] (Funk-Hecke formula)*
If $g \in \mathcal{H}_k^n$, for any $\underline{\xi} \in S^{n-1}$, we have

$$c_{n,k} \int_{S^{n-1}} P_k^n(\underline{\xi} \cdot \underline{\eta}) g(\underline{\eta}) dS_{\underline{\eta}} = g(\underline{\xi}).$$

Lemma 2.5 *[21]*
Let $-1 < t < 1$. *When* $n > 2$,

$$\sum_{k=0}^{+\infty} \frac{\Gamma(n+k-2)}{k!\Gamma(n-2)} x^k P_k^n(t) = \frac{1}{(1-2tx+x^2)^{\frac{n-2}{2}}}.$$

When $n = 2$,

$$\sum_{k=1}^{+\infty} \frac{1}{k} x^k P_k^2(t) = -\frac{1}{2}\ln(1-2xt+x^2).$$

Lemma 2.6 *Let* $-1 < t < 1$, $n \geq 2$, *then*

$$\begin{aligned} &\sum_{k=1}^{+\infty} k r^{2k+n-2} c_{n,k} P_k^n(t) \\ &\quad = \frac{1}{\omega_{n-1}} r^n \frac{nt-(n-4)tr^4-(n+2)r^2+(n-2)r^6}{(1-2tr^2+r^4)^{\frac{n}{2}+1}}. \end{aligned} \tag{2.6}$$

We denote

$$J(r,t) = -r^n \frac{nt-(n-4)tr^4-(n+2)r^2+(n-2)r^6}{(1-2tr^2+r^4)^{\frac{n}{2}+1}}.$$

Proof When $n > 2$, taking derivative with respect to x on both sides of the equality in Lemma 2.5, we have

$$\sum_{k=1}^{+\infty} \frac{\Gamma(n+k-2)}{k!\Gamma(n-2)} k x^{k-1} P_k^n(t) = \frac{(n-2)(t-x)}{(1-2tx+x^2)^{\frac{n}{2}}}.$$

Then

$$\sum_{k=1}^{+\infty} \frac{\Gamma(n+k-2)}{(k-1)!\Gamma(n-1)} x^{k-1} P_k^n(t) = \frac{t-x}{(1-2tx+x^2)^{\frac{n}{2}}}.$$

Letting $x = r^2$ and multiplying r^n to both sides, we obtain

$$\sum_{k=1}^{+\infty} \frac{\Gamma(n+k-2)}{(k-1)!\Gamma(n-1)} r^{n+2k-2} P_k^n(t) = \frac{tr^n - r^{n+2}}{(1-2tr^2+r^2)^{\frac{n}{2}}}.$$

Taking derivative with respect to r on both sides, we have

$$\sum_{k=1}^{+\infty} \frac{\Gamma(n+k-2)}{(k-1)!\Gamma(n-1)} (n+2k-2) r^{n+2k-3} P_k^n(t) = \frac{d}{dr}\left(\frac{tr^n - r^{n+2}}{(1-2tr^2+r^2)^{\frac{n}{2}}}\right).$$

By replacing the expression of $c_{n,k}$, we get

$$\begin{aligned}
&\sum_{k=1}^{+\infty} k r^{n+2k-2} c_{n,k} P_k^n(t) \\
&= \frac{1}{\omega_{n-1}} \sum_{k=1}^{+\infty} \frac{\Gamma(n+k-1)(n+2k-2)}{(n+k-2)(k-1)!\Gamma(n-1)} r^{n+2k-2} P_k^n(t) \\
&= \frac{1}{\omega_{n-1}} r \frac{d}{dr} \left(\frac{tr^n - r^{n+2}}{(1-2tr^2+r^2)^{\frac{n}{2}}} \right) \\
&= \frac{1}{\omega_{n-1}} r^n \frac{nt - (n-4)tr^4 - (n+2)r^2 + (n-2)r^6}{(1-2tr^2+r^4)^{\frac{n}{2}+1}}.
\end{aligned}$$

When $n = 2$.

$$\sum_{k=1}^{+\infty} \frac{1}{k} x^k P_k^2(t) = -\frac{1}{2} \ln(1 - 2xt + x^2).$$

Using a similar method, we have

$$\sum_{k=1}^{+\infty} k x^{2k} P_k^2(t) = r^2 \frac{t + tr^4 - 2r^2}{(1-2tr^2+r^4)^2}.$$

In short, when $n \geq 2$, we have formula (2.6). The proof is complete. □

Proof of Theorem 2.3 By Levi's Theorem and stokes formula, we have

$$\begin{aligned}
\int_{B_n} |\nabla u|^2 d\underline{x} &= \lim_{r \to 1^-} \int_{B_n(r)} |\nabla u|^2 d\underline{x} \\
&= \lim_{r \to 1^-} \int_{S_r^{n-1}} u \frac{\partial u}{\partial \mathbf{n}} dS \\
&= \lim_{r \to 1^-} \int_{S^{n-1}} u \frac{\partial u}{\partial r} r^{n-1} dS. \qquad (2.7)
\end{aligned}$$

Using formula (2.4), we obtain

$$u = c_{n,0} \int_{S^{n-1}} f(\underline{\eta}) dS_{\underline{\eta}} + \sum_{k=1}^{+\infty} r^k c_{n,k} \int_{S^{n-1}} f(\underline{\eta}) P_k^n(\underline{\xi} \cdot \underline{\eta}) dS_{\underline{\eta}}.$$

Then

$$\frac{\partial u}{\partial r} r^{n-1} = \sum_{k=1}^{+\infty} k r^{k+n-2} c_{n,k} \int_{S^{n-1}} f(\underline{\eta}) P_k^n(\underline{\xi} \cdot \underline{\eta}) dS_{\underline{\eta}}.$$

Therefore, we get

$$\int_{S^{n-1}} u\frac{\partial u}{\partial r}r^{n-1}dS = \sum_{k=1}^{+\infty} kr^{2k+n-2}c_{n,k}^2 \int_{S^{n-1}}\int_{S^{n-1}} f(\underline{\eta_1})f(\underline{\eta_2}) \left[\int_{S^{n-1}} P_k^n(\underline{\xi}\cdot\underline{\eta_1})P_k^n(\underline{\xi}\cdot\underline{\eta_2})dS_{\underline{\xi}}\right] dS_{\underline{\eta_1}}dS_{\underline{\eta_2}},$$

where we used orthogonality between $P_k^n(\underline{\xi}\cdot\underline{\eta})$ and $P_j^n(\underline{\xi}\cdot\underline{\eta})$ when $k\neq j$.

By the Funk-Hecke formula, we have

$$c_{n,k}\int_{S^{n-1}} P_k^n(\underline{\xi}\cdot\underline{\eta_1})P_k^n(\underline{\xi}\cdot\underline{\eta_2})dS_{\underline{\xi}} = P_k^n(\underline{\eta_1}\cdot\underline{\eta_2}).$$

Then, we obtain

$$\begin{aligned}&\int_{S^{n-1}} u\frac{\partial u}{\partial r}r^{n-1}dS\\ &= \sum_{k=1}^{+\infty} kr^{2k+n-2}c_{n,k}\int_{S^{n-1}}\int_{S^{n-1}} f(\underline{\eta_1})f(\underline{\eta_2})P_k^n(\underline{\eta_1}\cdot\underline{\eta_2})dS_{\underline{\eta_1}}dS_{\underline{\eta_2}}. \end{aligned}\tag{2.8}$$

On the one hand, from formula (2.8), Lemma 2.1 and formula (2.5), we have

$$\begin{aligned}&\int_{S^{n-1}} u\frac{\partial u}{\partial r}r^{n-1}dS\\ &= \sum_{k=1}^{+\infty} kr^{2k+n-2}c_{n,k}\int_{S^{n-1}}\int_{S^{n-1}} f(\underline{\eta_1})f(\underline{\eta_2})P_k^n(\underline{\eta_1}\cdot\underline{\eta_2})dS_{\underline{\eta_1}}dS_{\underline{\eta_2}}\\ &= \sum_{k=1}^{+\infty} kr^{2k+n-2}\int_{S^{n-1}}\int_{S^{n-1}} f(\underline{\eta_1})f(\underline{\eta_2})\sum_{j=1}^{a_k^n}\overline{y_j(\underline{\eta_1})}y_j(\underline{\eta_2})dS_{\underline{\eta_1}}dS_{\underline{\eta_2}}\\ &= \sum_{k=1}^{+\infty} kr^{2k+n-2}\sum_{j=1}^{a_k^n}\int_{S^{n-1}} f(\underline{\eta_1})\overline{y_j(\underline{\eta_1})}dS_{\underline{\eta_1}}\int_{S^{n-1}}\overline{f(\underline{\eta_2})\overline{y_j(\underline{\eta_2})}}dS_{\underline{\eta_2}}\\ &= \sum_{k=1}^{+\infty} kr^{2k+n-2}\sum_{j=1}^{a_k^n}|b_j|^2. \end{aligned}\tag{2.9}$$

Therefore, by formula (2.7) and (2.9), we obtain

$$\begin{aligned}\int_{B_n}|\nabla u|^2 d\underline{x} &= \lim_{r\to 1^-}\int_{B_n(r)}|\nabla u|^2 d\underline{x}\\ &= \lim_{r\to 1^-}\int_{S^{n-1}} u\frac{\partial u}{\partial r}r^{n-1}dS\\ &= \lim_{r\to 1^-}\sum_{k=1}^{+\infty} kr^{2k+n-2}\sum_{j=1}^{a_k^n}|b_j|^2.\end{aligned}$$

If

$$\sum_{k=1}^{+\infty} k\sum_{j=1}^{a_k^n}|b_j|^2 < +\infty,$$

using the second theorem of Abel, we have

$$\lim_{r\to 1^-}\sum_{k=1}^{+\infty} kr^{2k+n-2}\sum_{j=1}^{a_k^n}|b_j|^2 = \sum_{k=1}^{+\infty} k\sum_{j=1}^{a_k^n}|b_j|^2.$$

Then

$$\int_{B_n}|\nabla u|^2 d\underline{x} = \sum_{k=1}^{+\infty} k\sum_{j=1}^{a_k^n}|b_j|^2.$$

On the other hand,

$$\begin{aligned}&\int_{S^{n-1}} u\frac{\partial u}{\partial r}r^{n-1}dS\\ &= -\frac{1}{2}\sum_{k=1}^{+\infty} kr^{2k+n-2}c_{n,k}\int_{S^{n-1}}\int_{S^{n-1}}[f(\underline{\eta_1})-f(\underline{\eta_2})]^2 P_k^n(\underline{\eta_1}\cdot\underline{\eta_2})dS_{\underline{\eta_1}}dS_{\underline{\eta_2}}\\ &= \int_{S^{n-1}}\int_{S^{n-1}}[f(\underline{\eta_1})-f(\underline{\eta_2})]^2[-\frac{1}{2}\sum_{k=1}^{+\infty} kr^{2k+n-2}c_{n,k}P_k^n(\underline{\eta_1}\cdot\underline{\eta_2})]dS_{\underline{\eta_1}}dS_{\underline{\eta_2}}.\end{aligned}$$

Using Lemma 2.6, we have

$$\begin{aligned}&\int_{S^{n-1}} u\frac{\partial u}{\partial r}r^{n-1}dS\\ &= \frac{1}{2\omega_{n-1}}\int_{S^{n-1}}\int_{S^{n-1}}[f(\underline{\eta_1})-f(\underline{\eta_2})]^2 J(r,\cos\theta)dS_{\underline{\eta_1}}dS_{\underline{\eta_2}},\end{aligned}\tag{2.10}$$

where θ is the angle between $\underline{\eta_1}$ and $\underline{\eta_2}$, then $\underline{\eta_1}\cdot\underline{\eta_2}=\cos\theta$ and $|\underline{\eta_1}-\underline{\eta_2}|^2=4\sin^2\frac{\theta}{2}$. By calculating directly, we have

$$\begin{aligned}&J(r,\cos\theta)\\&=r^n\frac{n\left[(1+r^2)^2\sin^2\frac{\theta}{2}-(1-r^2)^2\cos^2\frac{\theta}{2}\right]-(n-2)r^2\left[(1+r^2)^2\sin^2\frac{\theta}{2}+(1-r^2)^2\cos^2\frac{\theta}{2}\right]}{[(1+r^2)^2\sin^2\frac{\theta}{2}+(1-r^2)^2\cos^2\frac{\theta}{2}]^{\frac{n}{2}+1}}\\&\to\frac{2}{2^n(\sin^2\frac{\theta}{2})^{\frac{n}{2}}}=J(1,\cos\theta)\ \ (r\to1^-).\end{aligned}$$

In particular, when r is close to 1, $J(r,\cos\theta)$ is positive.

If

$$\begin{aligned}&\lim_{r\to1^-}\int_{S^{n-1}}\int_{S^{n-1}}[f(\underline{\eta_1})-f(\underline{\eta_2})]^2J(r,\cos\theta)dS_{\underline{\eta_1}}dS_{\underline{\eta_2}}\\&=\int_{S^{n-1}}\int_{S^{n-1}}[f(\underline{\eta_1})-f(\underline{\eta_2})]^2J(1,\cos\theta)dS_{\underline{\eta_1}}dS_{\underline{\eta_2}},\end{aligned}\tag{2.11}$$

then by formula (2.7), (2.10) and (2.11), we get

$$\begin{aligned}&\int_{B_n}|\nabla u(\underline{x})|^2d\underline{x}\\&=\lim_{r\to1^-}\frac{1}{2\omega_{n-1}}\int_{S^{n-1}}\int_{S^{n-1}}[f(\underline{\eta_1})-f(\underline{\eta_2})]^2J(r,\cos\theta)dS_{\underline{\eta_1}}dS_{\underline{\eta_2}}\\&=\frac{1}{2\omega_{n-1}}\int_{S^{n-1}}\int_{S^{n-1}}[f(\underline{\eta_1})-f(\underline{\eta_2})]^2J(1,\cos\theta)dS_{\underline{\eta_1}}dS_{\underline{\eta_2}}\\&=\frac{1}{\omega_{n-1}}\int_{S^{n-1}}\int_{S^{n-1}}\frac{|f(\underline{\eta_1})-f(\underline{\eta_2})|^2}{|\underline{\eta_1}-\underline{\eta_2}|^n}dS_{\underline{\eta_1}}dS_{\underline{\eta_2}}.\end{aligned}$$

In fact,

$$\begin{aligned}&\frac{|J(r,\cos\theta)|}{J(1,\cos\theta)}\\&=\frac{1}{2}\frac{2^n(\sin^2\frac{\theta}{2})^{\frac{n}{2}}r^n}{[(1+r^2)^2\sin^2\frac{\theta}{2}+(1-r^2)^2\cos^2\frac{\theta}{2}]^{\frac{n}{2}}}\\&\quad\times\frac{\left|-n\left[(1+r^2)^2\sin^2\frac{\theta}{2}-(1-r^2)^2\cos^2\frac{\theta}{2}\right]+(n-2)r^2\left[(1+r^2)^2\sin^2\frac{\theta}{2}+(1-r^2)^2\cos^2\frac{\theta}{2}\right]\right|}{(1+r^2)^2\sin^2\frac{\theta}{2}+(1-r^2)^2\cos^2\frac{\theta}{2}}\\&\le\frac{1}{2}[n+(n-2)r^2]\\&\le\frac{1}{2}[n+(n-2)]=n-1.\end{aligned}$$

If

$$\int_{S^{n-1}}\int_{S^{n-1}}[f(\underline{\eta_1})-f(\underline{\eta_2})]^2J(1,\cos\theta)dS_{\underline{\eta_1}}dS_{\underline{\eta_2}}<+\infty,$$

then using the Lebesgue dominated convergence theorem, formula (2.11) holds. The proof of Theorem 2.3 is complete.

Remark 2.7 The relation between the involved function spaces may be described by

$$\mathcal{D}(B_n) = h^2(B_n) \cap \mathcal{H}^{1/2}(S^{n-1}),$$

where $\mathcal{H}^{1/2}(S^{n-1})$ is the fractional Sobolev space containing functions $f \in L^2(S^{n-1})$ having "1/2" derivative in $L^2(S^{n-1})$.

3 Dirichlet Spaces in the Upper-Half Euclidean Spaces: The Real Harmonic Analysis Methods

Denote by $\mathbb{R}^{n+1}_+ = \{(\underline{x}, y) | \ \underline{x} \in \mathbb{R}^n, \ y > 0\}$ the upper-half space in the $(n+1)$-dimensional Euclidean space.

Denote the harmonic Hardy h^2 space of the half Euclidean space by

$$h^2(\mathbb{R}^{n+1}_+) = \{u(\underline{x}, y) \mid u(\underline{x}, y) \text{ is harmonic in } \mathbb{R}^{n+1}_+ \text{ and satisfies } \|u\|_{h^2(\mathbb{R}^{n+1}_+)} < +\infty\},$$

where

$$\|u\|_{h_2(\mathbb{R}^{n+1}_+)} = \sup_{0<y<+\infty} \left(\int_{\mathbb{R}^n} |u(\underline{x}, y)|^2 d\underline{x} \right)^{\frac{1}{2}}.$$

It is well-known that a harmonic function $u \in h^2(\mathbb{R}^{n+1}_+)$ if and only if there exists $f(\underline{x}) \in L^2(\mathbb{R}^n)$ such that

$$u(\underline{x}, y) = \int_{\mathbb{R}^n} P_y(\underline{x} - \underline{t}) f(\underline{t}) d\underline{t}, \tag{3.12}$$

where $P_y(\underline{x} - \underline{t})$ is the Poisson kernel. Moreover,

$$\lim_{y \to 0^+} u(\underline{x}, y) = f(\underline{x}) \ a.e.,$$

in the the harmonic Hardy space norm, and $\|u\|_{h^2} = \|f\|_{L^2}$. The Poisson kernel in $\mathbb{R}^{n+1}_+$ has the expressions

$$P_y(\underline{x} - \underline{t}) = \frac{2}{\omega_n} \frac{y}{(y^2 + |\underline{x} - \underline{t}|^2)^{\frac{n+1}{2}}} = \int_{\mathbb{R}^n} e^{-2\pi \mathbf{i} \underline{u} \cdot (\underline{x} - \underline{t})} e^{-2\pi |\underline{u}| y} d\underline{u}$$

with

$$\frac{2}{\omega_n} = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}.$$

Using the Plancherel Theorem to the right-hand-side of (3.12), we have

$$u(\underline{x}, y) = \int_{\mathbb{R}^n} \hat{f}(\underline{t}) e^{-2\pi \mathbf{i} \underline{t} \cdot \underline{x}} e^{-2\pi |\underline{t}| y} d\underline{t}, \ y > 0, \tag{3.13}$$

where $\hat{f}$ is the Fourier transform of f defined by

$$\hat{f}(\underline{t}) = \int_{\mathbb{R}^n} e^{2\pi \mathbf{i}\underline{x}\cdot\underline{t}} f(\underline{x}) d\underline{x}.$$

Next, we define the Dirichlet space in $\mathbb{R}^{n+1}_+$.

Definition 3.1 Define the Dirichlet space

$$\mathcal{D}(\mathbb{R}^{n+1}_+) := \{u(\underline{x}, y) |\ u(\underline{x}, y) \in h^2(\mathbb{R}^{n+1}_+) \text{ and satisfies } \int_{\mathbb{R}^{n+1}_+} |\nabla u(\underline{x}, y)|^2 d\underline{x}dy < +\infty\}.$$

Define the semi-norm in $\mathcal{D}(\mathbb{R}^{n+1}_+)$

$$\|u\|^2_{\mathcal{D}(\mathbb{R}^{n+1}_+),*} := \int_{\mathbb{R}^{n+1}_+} |\nabla u(\underline{x}, y)|^2 d\underline{x}dy < +\infty$$

and the norm

$$\|u\|^2_{\mathcal{D}(\mathbb{R}^{n+1}_+)} := \|u\|^2_{\mathcal{D}(\mathbb{R}^{n+1}_+),*} + \|u\|^2_{h^2}.$$

Next, we prove the identical relation between $\|u\|_{\mathcal{D}(\mathbb{R}^{n+1}_+),*}$ and its Fourier integral, as well as its double singular integral form on the boundary.

Theorem 3.2 *Let $u(\underline{x}, y) \in \mathcal{D}(\mathbb{R}^{n+1}_+)$, then we have*

$$\begin{aligned}\|u\|^2_{\mathcal{D}(\mathbb{R}^{n+1}_+),*} &:= \int_{\mathbb{R}^{n+1}_+} |\nabla u(\underline{x}, y)|^2 d\underline{x}dy \\ &= 2\pi \int_{\mathbb{R}^n} |\underline{t}||\hat{f}(\underline{t})|^2 d\underline{t} \\ &= \frac{1}{\omega_n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(\underline{t_1}) - f(\underline{t_2})|^2}{|\underline{t_1} - \underline{t_2}|^{n+1}} d\underline{t_1}d\underline{t_2}.\end{aligned}$$

Proof: First, we will prove

$$\int_{\mathbb{R}^{n+1}_+} |\nabla u(\underline{x}, y)|^2 d\underline{x}dy = 2\pi \int_{\mathbb{R}^n} |\underline{t}||\hat{f}(\underline{t})|^2 d\underline{t}.$$

By the Fubini Theorem, we have

$$\int_{\mathbb{R}^{n+1}_+} |\nabla u(\underline{x}, y)|^2 d\underline{x}dy = \int_0^{+\infty} dy \int_{\mathbb{R}^n} |\nabla u(\underline{x}, y)|^2 d\underline{x}.$$

From (3.13), we have

$$\frac{\partial u}{\partial y} = \int_{\mathbb{R}^n} -2\pi|\underline{t}|\hat{f}(\underline{t}) e^{-2\pi \mathbf{i}\underline{t}\cdot\underline{x}} e^{-2\pi|\underline{t}|y} d\underline{t}$$

and

$$\frac{\partial u}{\partial x_j} = \int_{\mathbb{R}^n} -2\pi \mathbf{i} t_j \hat{f}(\underline{t}) e^{-2\pi \mathbf{i}\underline{t}\cdot\underline{x}} e^{-2\pi|\underline{t}|y} d\underline{t}.$$

Therefore, we have

$$\int_{\mathbb{R}^n} |\nabla u(\underline{x}, y)|^2 d\underline{x} = \int_{\mathbb{R}^n} 8\pi^2 |\underline{t}|^2 |\hat{f}(\underline{t})|^2 e^{-4\pi|\underline{t}|y} d\underline{t}, \ y > 0.$$

Then,

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}_+} |\nabla u(\underline{x}, y)|^2 d\underline{x} dy &= \int_0^{+\infty} dy \int_{\mathbb{R}^n} |\nabla u(\underline{x}, y)|^2 d\underline{x} \\
&= \int_0^{+\infty} dy \int_{\mathbb{R}^n} 8\pi^2 |\underline{t}|^2 |\hat{f}(\underline{t})|^2 e^{-4\pi|\underline{t}|y} d\underline{t} \\
&= \int_{\mathbb{R}^n} 8\pi^2 |\underline{t}|^2 |\hat{f}(\underline{t})|^2 d\underline{t} \int_0^{+\infty} e^{-4\pi|\underline{t}|y} dy \\
&= 2\pi \int_{\mathbb{R}^n} |\underline{t}| |\hat{f}(\underline{t})|^2 d\underline{t}.
\end{aligned}$$

Next, we will prove

$$2\pi \int_{\mathbb{R}^n} |\underline{t}| |\hat{f}(\underline{t})|^2 d\underline{t} = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(\underline{t_1}) - f(\underline{t_2})|^2}{|\underline{t_1} - \underline{t_2}|^{n+1}} d\underline{t_1} d\underline{t_2}.$$

In fact, we have

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(\underline{t_1}) - f(\underline{t_2})|^2}{|\underline{t_1} - \underline{t_2}|^{n+1}} d\underline{t_1} d\underline{t_2} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(\underline{t_1} + \underline{\alpha}) - f(\underline{t_1})|^2}{|\underline{\alpha}|^{n+1}} d\underline{t_1} d\underline{\alpha}.$$

Using the Plancherel Theorem, we have

$$\int_{\mathbb{R}^n} |f(\underline{t_1} + \underline{\alpha}) - f(\underline{t_1})|^2 d\underline{t_1} = \int_{\mathbb{R}^n} |\hat{f}(\underline{t})|^2 |e^{-2\pi i \underline{t}\cdot\underline{\alpha}} - 1|^2 d\underline{t}.$$

Therefore, we have

$$\begin{aligned}
\frac{1}{\omega_n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(\underline{t_1} + \underline{\alpha}) - f(\underline{t_1})|^2}{|\underline{\alpha}|^{n+1}} d\underline{t_1} d\underline{\alpha} &= \frac{1}{\omega_n} \int_{\mathbb{R}^n} |\hat{f}(\underline{t})|^2 d\underline{t} \left(\int_{\mathbb{R}^n} \frac{|e^{-2\pi i \underline{t}\cdot\underline{\alpha}} - 1|^2}{|\alpha|^{n+1}} d\underline{\alpha} \right) \\
&= \frac{1}{\omega_n} \int_{\mathbb{R}^n} |\underline{t}| |\hat{f}(\underline{t})|^2 d\underline{t} \left(\int_{\mathbb{R}^n} \frac{|e^{-2\pi i \underline{\eta}\cdot\underline{\alpha}} - 1|^2}{|\alpha|^{n+1}} d\underline{\alpha} \right),
\end{aligned}$$

where $\underline{t} = |\underline{t}|\underline{\eta}$. Let $\underline{\alpha} = r\underline{\omega}$, then

$$\begin{aligned}\frac{1}{\omega_n}\int_{\mathbb{R}^n}\frac{|e^{-2\pi i\underline{\eta}\cdot\underline{\alpha}}-1|^2}{|\alpha|^{n+1}}d\underline{\alpha} &= \frac{1}{\omega_n}\int_0^{+\infty}\frac{1}{r^2}dr\int_{S^{n-1}}|e^{-2\pi ir\underline{\eta}\cdot\underline{\omega}}-1|^2dS_\omega\\ &= \frac{4}{\omega_n}\int_{S^{n-1}}dS_\omega\int_0^{+\infty}\frac{\sin^2\pi r\underline{\eta}\cdot\underline{\omega}}{r^2}dr\\ &= \frac{4}{\omega_n}\int_{S^{n-1}}\frac{\pi}{2}\pi|\underline{\eta}\cdot\underline{\omega}|\,dS_\omega\\ &= \frac{2\pi^2}{\omega_n}\frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})}\\ &= 2\pi.\end{aligned}$$

Thus, we complete the proof. □

Remark 3.3 From Theorem 3.2, we have

$$\mathcal{D}(\mathbb{R}^{n+1}_+) = h^2(\mathbb{R}^{n+1}_+)\cap\mathcal{H}^{1/2}(\mathbb{R}^n).$$

where $\mathcal{H}^{1/2}(\mathbb{R}^n)$ is the fractional Sobolev space containing functions $f\in L^2(\mathbb{R}^n)$ having "1/2" derivative in $L^2(\mathbb{R}^n)$. More precisely, that is

$$\|f\|^2_{\mathcal{H}^{1/2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n}|\underline{t}||\hat{f}(\underline{t})|^2d\underline{t}.$$

In the following, we will present the relationship between the Dirichlet space and the g-function.

First, let us recall the definition of the g-function.

Definition 3.4 [19] Let $f\in L^2(\mathbb{R}^n)$ be the boundary function of harmonic function u in the upper half space $\mathbb{R}^{n+1}_+$. Define the g-function

$$g(f)(\underline{x}) := \left(\int_0^{+\infty}|\nabla u(\underline{x},y)|^2ydy\right)^{\frac{1}{2}}.$$

It is known that ([19])

$$\|g(f)\|_2 = \frac{\sqrt{2}}{2}\|f\|_2.$$

We introduce a parameterized g^λ-functions ($0\le\lambda\le 1$), where $g^1(f)(\underline{x}) = g(f)(\underline{x})$, the classical g function of f, and

$$g^0(f)(\underline{x}) = \left(\int_0^{+\infty}|\nabla u(\underline{x},y)|^2dy\right)^{\frac{1}{2}}.$$

Definition 3.5 Let $f \in L^2(\mathbb{R}^n)$. For $0 \le \lambda \le 1$, define the g^λ-function

$$g^\lambda(f)(\underline{x}) := \left(\int_0^{+\infty} |\nabla u(\underline{x}, y)|^2 y^\lambda dy\right)^{\frac{1}{2}}.$$

Then we have

Theorem 3.6 *For $0 \le \lambda \le 1$, if $f \in L^2(\mathbb{R}^n) \cap \mathcal{H}^{(1-\lambda)/2}(\mathbb{R}^n)$, then we have*

$$\|g^\lambda(f)\|_2 = \sqrt{\Gamma(\lambda+1)2^{1-2\lambda}\pi^{1-\lambda}}\|f\|_{\mathcal{H}^{(1-\lambda)/2}(\mathbb{R}^n)},$$

where

$$\|f\|^2_{\mathcal{H}^{(1-\lambda)/2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\underline{t}|^{(1-\lambda)} |\hat{f}(\underline{t})|^2 d\underline{t}.$$

Proof

$$\begin{aligned}
\|g^\lambda(f)\|_2^2 &= \int_{\mathbb{R}^n} \left(\int_0^{+\infty} |\nabla u(\underline{x}, y)|^2 y^\lambda dy\right) d\underline{x} \\
&= \int_0^{+\infty} y^\lambda dy \int_{\mathbb{R}^n} |\nabla u(\underline{x}, y)|^2 d\underline{x} \\
&= \int_0^{+\infty} y^\lambda dy \int_{\mathbb{R}^n} 8\pi^2 |\underline{t}|^2 |\hat{f}(\underline{t})|^2 e^{-4\pi|\underline{t}|y} d\underline{t} \\
&= \int_{\mathbb{R}^n} 8\pi^2 |\underline{t}|^2 |\hat{f}(\underline{t})|^2 d\underline{t} \int_0^{+\infty} e^{-4\pi|\underline{t}|y} y^\lambda dy \\
&= \Gamma(\lambda+1)2^{1-2\lambda}\pi^{1-\lambda} \int_{\mathbb{R}^n} |\underline{t}|^{1-\lambda} |\hat{f}(\underline{t})|^2 d\underline{t}.
\end{aligned}$$

This completes the proof. □

From Theorem 3.6, we obtain

Corollary 3.7 *When $\lambda = 1$, $g^1(f)(\underline{x}) = g(f)(\underline{x})$, which is the g-function, then*

$$\|g^1(f)\|_2^2 = \|g(f)\|_2^2 = \frac{1}{2}\|\hat{f}\|_2^2 = \frac{1}{2}\|f\|_2^2.$$

When $\lambda = 0$, we have $g^0(f)(\underline{x}) = \left(\int_0^{+\infty} |\nabla u(\underline{x}, y)|^2 dy\right)^{\frac{1}{2}}$, then

$$\|g^0(f)\|_2^2 = 2\pi\|U\|^2_{\mathcal{D}(\mathbb{R}^{n+1}_+),*} = 2\pi\|\sqrt{|\underline{t}|}\hat{f}(\underline{t})\|_2^2 = 2\pi\|f\|^2_{\mathcal{H}^{1/2}(\mathbb{R}^n)}.$$

4 The Complex and Hyper-Complex Settings for the Ahlfors form Quantity

§4 will be divided into three subsections, respectively, to establish the theories on the upper-half complex plane, on the upper-half Euclidean spaces $\mathbb{R}^{n+1}_+$. In the last subsection we give a revision of the balls cases in $\mathbb{R}^{n+1}$, $n > 1$.

4.1 The Dirichlet Space in the Upper-Half Complex Plane

It is standard knowledge that there exist Moebius transformations that conformally map the unit disk $\mathbb{D}$ to the upper half complex plane $\mathbb{C}_+$.

Let

$$z = L(\zeta) = \frac{\zeta - \mathbf{i}}{\zeta + \mathbf{i}},$$

that conformally maps the upper half plane $\mathbb{C}_+$ to the unit disk $\mathbb{D}$, and maps their boundaries in one-to-one and onto manner keeping the orientation.

Denote $L(x_1) = z_1$ and $L(x_2) = z_2$. By direct computation we have

$$|L(x_1) - L(x_2)| = \left| \frac{2\mathbf{i}(x_1 - x_2)}{(x_1 + \mathbf{i})(x_2 + \mathbf{i})} \right|$$

and

$$|dL(x)| = \frac{2}{|x + \mathbf{i}|^2} dx.$$

Let $u \in h^2(\mathbb{D})$, f is its boundary function, then we obtain

$$\begin{aligned} &\frac{1}{2\pi} \int_T \int_T \frac{|f(z_1) - f(z_2)|^2}{|z_1 - z_2|^2} |dz_1||dz_2| \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{|f(L(x_1)) - f(L(x_2))|^2}{|L(x_1) - L(x_2)|^2} |dL(x_1)||dL(x_2)| \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{|\tilde{f}(x_1) - \tilde{f}(x_2)|^2}{|x_1 - x_2|^2} dx_1 dx_2, \end{aligned}$$

where $\tilde{f} = f(L) = f \circ L$.

Since $L(\zeta)$ is holomorphic, through simple computation, we also have

$$\begin{aligned} \int_{\mathbb{D}} |\nabla_{(x,y)} u(z)|^2 dxdy &= \int_{\mathbb{C}_+} |\nabla_{(s,t)} u(L(\zeta))|^2 dsdt \\ &= \int_{\mathbb{C}_+} |\nabla_{(s,t)} \tilde{u}(\zeta)|^2 dsdt, \end{aligned}$$

where $\tilde{u}(\zeta) = u(L(\zeta)) = u \circ L(\zeta)$ is harmonic in $\mathbb{C}_+$ when $u(z)$ is harmonic in $\mathbb{D}$.

Define

$$\mathcal{D}(\mathbb{C}_+) := \{u(z) \mid u(z) \in h^2(\mathbb{C}_+) \text{ and satisfies } \int_{\mathbb{C}_+} |\nabla u(z)|^2 d\sigma < +\infty\}$$

the Dirichlet space in $\mathbb{C}_+$.

Define the semi-norm in $\mathcal{D}(\mathbb{C}_+)$

$$\|u\|^2_{\mathcal{D}(\mathbb{C}_+),*} := \int_{\mathbb{C}_+} |\nabla u(z)|^2 d\sigma < +\infty$$

and the norm

$$\|u\|^2_{\mathbb{C}_+} := \|u\|^2_{\mathcal{D}(\mathbb{C}_+),*} + \|u\|^2_{h^2}.$$

Summarizing what just obtained and Corrolary 4.11, we have

Theorem 4.1 *Let $u \in \mathcal{D}(\mathbb{C}_+)$, f is its boundary function, and F is the canonical holomorphic extension into $\mathbb{C}_+$ with real part u. Then*

$$\begin{aligned}\|u\|^2_{\mathcal{D}(\mathbb{C}_+),*} &:= \int_{\mathbb{C}_+} |\nabla u|^2 d\sigma \\ &= 2\pi \int_{-\infty}^{+\infty} |t||\hat{f}(t)|^2 dt \\ &= \int_{\mathbb{C}_+} |F'(z)|^2 d\sigma \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{\infty} \frac{|f(x_1) - f(x_2)|^2}{|x_1 - x_2|^2} dx_1 dx_2,\end{aligned}$$

where $\hat{f}$ is the Fourier transform of f defined by

$$\hat{f}(t) = \int_{-\infty}^{\infty} e^{2\pi \mathbf{i} x t} f(x) dx.$$

Remark 4.2 It is necessary that functions of the Dirichlet spaces belong to the corresponding harmonic Hardy spaces. The latter implies that they have non-tangential boundary limits. In the bounded domain cases the harmonic Hardy space attribution is a consequence of fineness of the semi-norm. With infinite domains it is not the case. This can be observed from the fact that Moebius a transformation does not, in general, map L^2 functions on the unit circle to L^2 on the line. This suggests that to define a Dirichlet space in an unbounded domain one needs to separately and firstly assume u is in the relevant harmonic Hardy space.

Remark 4.3 We are to establish the theory in the upper-half spaces $\mathbb{R}^{n+1}_+$, $n \geq 1$. It is observed that the conformal mapping method to transform a higher-dimensional unit balls into an upper-half space would hardly work for higher dimensions. There are at least two obstacles. First, in the higher dimensions, the composition of a Moebius transform and a harmonic function is no longer a harmonic function. Secondly. Moebius transforms in higher dimensions are, by themselves, not monogenic (Clifford holomorphic) ([1]). We, therefore, seek for a direct approach in higher dimensions to enhance Alhfors' approach.

4.2 The Identical Relation Involving the Ahlfors form Quantity of the Dirichlet Space in $\mathbb{R}^{n+1}_+$ with in the Clifford Algebra Setting

We will give a short revision on Clifford analysis. The basic references include [5, 7, 10]. Clifford analysis is an extension of analysis of one complex variable to high

dimensions. Essentially speaking, Clifford analysis is to add one new real variable x_0 to the existing n real variables $x_1, \cdots, x_n$, so to create a Cauchy structure. In contrast, analysis of several complex variables corresponds to adding n complex parts $y_j\mathbf{i}$ to each existing real variable x_j, $j = 1, \cdots, n$, to form n complex variables $z_j = x_j + y_j\mathbf{i}$, and so to create a Cauchy structure. The latter is, in fact, real $2n$-dimension. We will study functions defined in one and multi-dimensional Euclidean spaces taking values in real (or complex) Clifford algebra. Similar to complex analysis, the Cauchy integral formula, Taylor and Laurent series expansions etc., the most basic objects in one complex variable, are all available in Clifford analysis. Just as the relationship between holomorphic functions of one complex variable and 2-D harmonic functions, Clifford analysis has close connections with multi-variable harmonic analysis. As examples, a conjugate harmonic system ([20]) is just the components of a Clifford monogenic function (holomorphic function in Clifford analysis), and Hilbert transform of a function in Euclidean space is the $\mathbf{e}_j$-multiple-sum of the corresponding Riesz transforms ([6]). Clifford algebraic structure has been well adopted in contemporary harmonic analysis ([10], [16]), and helps to solve deep real-analysis problems ([11, 12, 14]). We prove the identical relations between the double integral form and the other forms in the Douglas-Ahlfors format through the Ahlfors form quantity which gives one more example of the crucial role of Clifford analysis.

Let $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ be the basic elements satisfying $\mathbf{e}_j\mathbf{e}_k + \mathbf{e}_k\mathbf{e}_j = -2\delta_{jk}$, where $\delta_{jk} = 1$ if $j = k$, and $\delta_{jk} = 0$ otherwise, $j, k = 1, 2, \cdots, n$. Let

$$\mathbb{R}^n = \{\underline{x} \mid \underline{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n : x_k \in \mathbb{R}, k = 1, 2, \cdots, n\}$$

be the n-dimensional homogeneous Euclidean space, and

$$\mathbb{R}^{n+1} = \{(\underline{x}, x_0) := x_0 + \underline{x} \mid \underline{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n : x_k \in \mathbb{R}, k = 0, 1, \cdots, n\}$$

the non-homogeneous Euclidean $(n + 1)$-dimensional space we concern.

Let $\mathcal{CL}_{0,n}$ denote the real (or complex) Clifford algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$. The linear basis for the Clifford algebra is given by $\mathbf{e}_A$, where A runs over all the ordered subsets of $\{0, 1, \cdots, n\}$, namely,

$$A = \{1 \le i_1 < i_2 < \cdots < i_l \le n\},\ 1 \le l \le n,$$

with $\mathbf{e}_0 = \mathbf{e}_\emptyset = 1$.

A general element x of $\mathcal{CL}_{0,n}$ can be represented in the form $x = \sum_{k=0}^{n-1}[x]_k$, where $[x]_k = \sum_A x_A\mathbf{e}_A$, $\mathbf{e}_A = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_k}$, $1 \le i_1 < i_2 < \cdots < i_k \le n$. $x_A \in \mathbb{R}$ (or $\mathbb{C}$) and $[x]_0 \in \mathbb{R}$ (or $\mathbb{C}$).

Let $x = \sum_{k=0}^{n}[x]_k \in \mathcal{CL}_{0,n}$, then x consists of a scalar part and a non-scalar part, denoted, respectively, by

$$x_0 = \mathrm{Sc}(x), \quad \sum_{k=1}^{n}[x]_k = \mathrm{NSc}(x).$$

We define the norm of $x \in \mathcal{CL}_{0,n}$ to be $|x| = \sqrt{\mathrm{Sc}(x\bar{x})} = \sqrt{\mathrm{Sc}(\bar{x}x)} = \sqrt{\sum_{k=0}^{n}\sum_{|A|=k}|x_A|^2}$.

Define the Dirac operator by

$$\underline{D} = \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\mathbf{e}_k$$

and the generalized Cauchy-Riemann operator by

$$D = \frac{1}{2}(\frac{\partial}{\partial x_0} + \underline{D}) = \frac{1}{2}(\frac{\partial}{\partial x_0} + \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\mathbf{e}_k).$$

The related conjugate operator of D is defined by

$$\bar{D} = \frac{1}{2}(\frac{\partial}{\partial x_0} - \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\mathbf{e}_k).$$

Clearly, the Laplace operator Δ in the $(n+1)$-D space satisfies $\Delta_x = 4D\bar{D} = 4\bar{D}D$.

We will involve the radial decomposition of D. Let, as in [7], $x = r\xi$ with $r = |x|$ and $\xi \in S^n$. Then D can be expressed as

$$D = \frac{1}{2}(\xi\,\partial_r + \frac{1}{r}\partial_\xi) = \frac{1}{2}\xi(\partial_r + \Gamma_\xi),$$

where $\Gamma_\xi = \bar{\xi}\,\partial_\xi$.

$$\bar{D} = \frac{1}{2}(\partial_r + \Gamma_\xi^*)\bar{\xi}, \tag{4.14}$$

where $\Gamma_\xi^* = \bar{\partial}_\xi \xi$. We note that the action of Γ_ξ^* on a function f is proceeded as $\Gamma_\xi^* f = \bar{\partial}_\xi(\xi f)$.

As is well known, the Laplacian

$$\Delta_x = \partial_r^2 + \frac{n}{r}\partial_r + \frac{1}{r^2}\Delta_\xi, \tag{4.15}$$

where Δ_ξ is the Laplace-Beltrami operaor on S^n.

While we also have

$$\begin{aligned}
\Delta_x &= 4\bar{D}D \\
&= (\partial_r + \frac{1}{r}\Gamma_\xi^*)\bar{\xi}\xi(\partial_r + \frac{1}{r}\Gamma_\xi) \\
&= (\partial_r + \frac{1}{r}\Gamma_\xi^*)(\partial_r + \frac{1}{r}\Gamma_\xi) \\
&= \partial_r^2 + \frac{\partial_r}{r}(\Gamma_\xi^* + \Gamma_\xi) + \frac{1}{r^2}(\Gamma_\xi^*\Gamma_\xi - \Gamma_\xi).
\end{aligned}$$

With formula (4.15), we obtain

$$\Gamma_\xi^* + \Gamma_\xi = n\mathbf{I}, \tag{4.16}$$

where $\mathbf{I}$ is the identity operator.

Using the radial form of D and formula (4.16), we know that if $P_k(r\xi)$ is a $k-$homogeneous left-monogenic function in $\mathbb{R}^{n+1}$, then $\Gamma_\xi P_k(\xi) = -kP_k(\xi)$ and $\Gamma_\xi^* P_k(\xi) = (n+k)P_k(\xi)$.

Remark 4.4 For the one complex variable case treated in [8] and [2] the corresponding objects are $z = x + iy = r\xi$ with $\xi = e^{i\theta}$. In the case we have

$$D = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}) = \frac{1}{2}\xi(\partial_r + \Gamma_\xi) = \frac{1}{2}e^{i\theta}(\frac{\partial}{\partial r} + i\frac{\partial}{\partial \theta})$$

and

$$\bar{D} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) = \frac{1}{2}\bar{\xi}(\partial_r + \Gamma_\xi^*) = \frac{1}{2}e^{-i\theta}(\frac{\partial}{\partial r} - i\frac{\partial}{\partial \theta}).$$

We hence have $\Gamma_\xi = i\frac{\partial}{\partial\theta}$ and $\Gamma_\xi^* = -i\frac{\partial}{\partial\theta}$, and $\Gamma_\xi + \Gamma_\xi^* = 0$. The above can be interpreted as the Clifford algebra setting for $n = 1$. The formula (4.16), however, cannot be applied to the case due to its restriction $n > 1$. In the Clifford analysis setting, when $n > 1$ the non-commutativity nature occurs, and, as a consequence, $\Gamma_\xi^* + \Gamma_\xi \neq 0$.

Any function $g : \mathbb{R}^{n+1} \to \mathcal{CL}_{0,n}$ is called a Clifford-valued function.

Definition 4.5 g is said to be left-monogenic in domain $\Omega \subseteq \mathbb{R}^{n+1}$, if $g \in C^1(\Omega)$ and $Dg = 0$. If g is left-monogenic, then we call $\bar{D}g$ the left-derivative of g.

Remark 4.6 For Clifford monogenic functions we have the following simple conclusions. 1. If g is left-monogenic, then $\bar{D}g = \dfrac{\partial}{\partial x_0}g$; and 2. If g is left-monogenic, then g and all its components are harmonic.

In the Clifford analysis setting, we first prove the identical relation between the gradient form and the *monogenic derivative* form quantity.

Theorem 4.7 *Let $u(\underline{x}, y) \in \mathcal{D}(\mathbb{R}_+^{n+1}), n \geq 1$, F be the associated canonical left-monogenic function whose scalar part is u, and the non-scalar part has zero value at $\underline{x} = 0$. Then we have*

$$\begin{aligned} \|u\|^2_{\mathcal{D}(\mathbb{R}_+^{n+1}),*} &:= \int_{\mathbb{R}_+^{n+1}} |\nabla u(\underline{x}, y)|^2 d\underline{x}dy \\ &= \int_{\mathbb{R}_+^{n+1}} |\bar{D}F(\underline{x}, y)|^2 d\underline{x}dy. \end{aligned} \tag{4.17}$$

To prove the theorem we need, as preparation, the following two lemmas.

Lemma 4.8 [6] *Let $x_0 = y > 0$ and $F(\underline{x}, y) = u(\underline{x}, y) + \sum_{k=1}^{n} v_k(\underline{x}, y)\mathbf{e_k}$ be left-monogenic in $\mathbb{R}_+^{n+1}$, then we have the Cauchy-Riemann equations*

$$DF(\underline{x}, y) = 0 \Longleftrightarrow \begin{cases} \dfrac{\partial u}{\partial y} = \displaystyle\sum_{k=1}^{n} \dfrac{\partial v_k}{\partial x_k} \\ \dfrac{\partial u}{\partial x_k} = -\dfrac{\partial v_k}{\partial y}, k = 1, \cdots, n \\ \dfrac{\partial v_k}{\partial x_j} = \dfrac{\partial v_j}{\partial x_k}, j \neq k. \end{cases} \tag{4.18}$$

Lemma 4.9 [6, 19] *Let $u(\underline{x}, y) \in h^2(\mathbb{R}_+^{n+1})$ with the non-tangential boundary value $f(\underline{x})$, then $F(\underline{x}, y) = u(\underline{x}, y) + \sum_{k=1}^{n} v_k(\underline{x}, y)\mathbf{e_k}$ is left-monogenic in $\mathbb{R}_+^{n+1}$, where $v_k(\underline{x}, y) = R_k(u)$ is the k-th Rieze transformation of f with the expressions*

$$\begin{aligned} v_k(\underline{x}, y) &= \int_{\mathbb{R}^n} \frac{it_k}{|\underline{t}|} e^{-2\pi \mathbf{i}\underline{t}\cdot\underline{x}} e^{-2\pi|\underline{t}|y} \hat{f}(\underline{t}) d\underline{t} \\ &= \frac{2}{\omega} \int_{\mathbb{R}^n} \frac{(t_k - x_k)}{(y^2 + |\underline{t} - \underline{x}|^2)^{\frac{n+1}{2}}} f(\underline{t}) d\underline{t}. \end{aligned}$$

Moreover,

$$F(\underline{x}, y) = u(\underline{x}, y) + \sum_{k=1}^{n} v_k(\underline{x}, y)\mathbf{e_k} = \frac{1}{\omega_n} E(\cdot, y) * f(\underline{x}),$$

where

$$E(\underline{t}, y) = \frac{\overline{y + \underline{t}}}{|y + \underline{t}|^{n+1}}$$

is the Clifford Cauchy kernel being monogenic from both the left and right sides.

Proof of (4.17) If $F(\underline{x}, y) = u(\underline{x}, y) + \sum_{k=1}^{n} v_k(\underline{x}, y)\mathbf{e_k}$ is left-monogenic in $\mathbb{R}_+^{n+1}$, using (4.18) we have

$$\begin{aligned} \bar{D}F(\underline{x}, y) &= \frac{\partial}{\partial y} F(\underline{x}, y) \\ &= \frac{\partial}{\partial y} [u(\underline{x}, y) + \sum_{k=1}^{n} v_k(\underline{x}, y)\mathbf{e_k}] \\ &= \frac{\partial u}{\partial y} + \sum_{k=1}^{n} \frac{\partial v_k}{\partial y}\mathbf{e_k} \\ &= \frac{\partial u}{\partial y} - \sum_{k=1}^{n} \frac{\partial u}{\partial x_k}\mathbf{e_k}. \end{aligned}$$

Thus we have

$$\int_{\mathbb{R}^{n+1}_+} |\nabla u(\underline{x}, y)|^2 d\underline{x}dy = \int_{\mathbb{R}^{n+1}_+} [(\frac{\partial u}{\partial y})^2 + \sum_{k=1}^{n} (\frac{\partial u}{\partial x_k})^2] d\underline{x}dy$$
$$= \int_{\mathbb{R}^{n+1}_+} |\bar{D}F(\underline{x}, y)|^2 d\underline{x}dy.$$

This completes the proof of (4.17). □

In all, using Theorem 3.2 and Theorem 4.7, we have

Theorem 4.10 *Let $u(\underline{x}, y) \in \mathcal{D}(\mathbb{R}^{n+1}_+)$ with the boundary function $f(\underline{x})$ and $F(\underline{x}, y)$ be the canonical left-monogenic function in $\mathbb{R}^{n+1}_+$ with the scalar part $u(\underline{x}, y)$, then we have*

$$\|u\|^2_{\mathcal{D}(\mathbb{R}^{n+1}_+),*} := \int_{\mathbb{R}^{n+1}_+} |\nabla u(\underline{x}, y)|^2 d\underline{x}dy$$
$$= \int_{\mathbb{R}^{n+1}_+} |\bar{D}F(\underline{x}, y)|^2 d\underline{x}dy$$
$$= 2\pi \int_{\mathbb{R}^n} |\underline{t}||\hat{f}(\underline{t})|^2 d\underline{t}$$
$$= \frac{1}{\omega_n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(\underline{t_1}) - f(\underline{t_2})|^2}{|\underline{t_1} - \underline{t_2}|^{n+1}} d\underline{t_1} d\underline{t_2}.$$

In particular, when $n = 1$, we obtain

Corollary 4.11 *Let $u(z) \in \mathcal{D}(\mathbb{C}_+)$ with the boundary function $f(x)$ and $F(z)$ be the associated holomorphic function in $\mathbb{C}_+$ with the real part $u(z)$, then we have*

$$\|u\|^2_{\mathcal{D}(\mathbb{C}_+),*} := \int_{\mathbb{C}_+} |\nabla u(z)|^2 dxdy$$
$$= \int_{\mathbb{C}_+} |F'(z)|^2 dxdy$$
$$= 2\pi \int_{-\infty}^{+\infty} |t||\hat{f}(t)|^2 dt$$
$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{|f(x_1) - f(x_2)|^2}{|x_1 - x_2|^2} dx_1 dx_2.$$

Remark 4.12 It is observed that the with the harmonic function case the identical relations with the double singular integrals is the hardest to prove. In the Ahlfors' original proof he uses a residue computation to directly prove the identity between his form and the double singular integral form. The proof of the identical relation between the Ahlfors' form and the double integral form with the Clifford $\mathbb{R}^{n+1}_+$ setting is indirect, but via Theorem 4.7. We note that there do exist direct proofs of the identical

relation. The proofs are not based on residue argument, for the latter may not exist, but have to be based on the monogenicity of the involved functions. Incidentally, it is a consequence, of the zero-integral value of any monogenic Hardy-H^1 function over $\mathbb{R}^{n+1}_+$. We suppress the proof here.

5 The Balls Cases Revised

In summary, for all the cases consisting of upper-half Euclidean space of any dimension, the unit disc case in the complex plane, there exists, at least morally (with, perhaps, the Fourier series is replaced by Fourier transformation), the same set of identical energy quantities, namely the gradient, the holomorphic or monogenic derivative, the Fourier series or integration, and the double singular integral, as well as the Ahlfors' forms to describe the semi-Dirichlet norm of each context. The only exceptional cases are the unit balls $B_{n+1}, n > 1$.

Lemma 5.1 [14] *Let U be a real-valued function defined on a star-shaped open set $\Omega \subseteq \mathbb{R}^{n+1}$. If U is harmonic, then there uniquely exists a left-monogenic function F defined on Ω such that $ReF = U$ and $F(0) = U(0)$, given by*

$$F(x) = U(x) + 2\mathrm{NSc}\int_0^1 s^{n-1}\bar{D}U(sx)xds.$$

Using Lemma 5.1, the conjugate Poisson kernel in S^n was first given in [4]. It was separately deduced in [16] based on the concept of conjugate harmonic function of the Cauchy type. That is:

Lemma 5.2 *For $\xi, \eta \in S^n$ and $r < 1$. The conjugate Poisson kernel in S^n is given by*

$$Q_r(\eta,\xi) = \frac{1}{\omega_n}\left[\frac{2}{|\eta - r\xi|^{n+1}} - \frac{n-1}{r^n}\int_0^r \frac{\rho^{n-1}}{|\eta-\rho\xi|^n}d\rho\right] r(\bar{\eta}\xi - \xi\cdot\eta).$$

Accordingly, the Schwarz kernel in B_{n+1} is defined as

$$\begin{aligned} S_r(\eta,\xi) &= P_r(\eta,\xi) + Q_r(\eta,\xi) \\ &= \frac{1}{\omega_n}\left[\frac{1-r^2}{|\eta-r\xi|^{n+1}} + \left(\frac{2}{|\eta-r\xi|^{n+1}} - \frac{n-1}{r^n}\int_0^r \frac{\rho^{n-1}}{|\eta-\rho\xi|^{n+1}}d\rho\right) r(\bar{\eta}\xi - \xi\cdot\eta)\right]. \end{aligned}$$

As in the one complex variable case, using the Schwarz kernel we can express a left-monogenic function F in B_n using the boundary value u of the real part U of F. That is:

$$F(x) = F(r,\xi) = \int_{S^n} u(\eta)S_r(\eta,\xi)dS_\eta. \tag{5.19}$$

Remark 5.3 The conjugate Poisson kernel and hence the Schwarz kernel are mixed with 2-forms.

We are to concern generalizations to Euclidean spaces of Fueter's Theorem on inducing quaternionic regular functions ([18]). Obeying the same philosophy as for the one complex variable any scalar-valued function u in $L^2(S^n)(n > 1)$ may be expanded into a series of positive and negative spherical homogeneous monogenics (cf. page 391 [14]) converging in the L^2-sense:

$$u(\xi) = \sum_{k=-\infty}^{+\infty} \frac{1}{\omega_n} \int_{S^n} P^{(k)}(\eta^{-1}\xi) u(\eta) dS_\eta,$$

where for the negative k's the functions $P^{(k)}$ are the images of the Fueter-Sce-Qian mapping of the analytic functions z^k in one complex variable, and for the positive k's the functions $P^{(k-1)}$ are defined to be the Kelvin inversions of the corresponding $P^{(-k)}$ ([14]). There hold $P^{(k)}(\eta^{-1}\xi) = C^+_{n+1,k}(\xi, \eta), k = 0, 1, \cdots$ and $P^{(k)}(\eta^{-}\xi) = C^-_{n+1,|k|-1}(\xi, \eta), k = -1, -2, \cdots$, where

$$C^+_{n+1,k}(\xi, \eta) = \frac{n+k-1}{n-1} C_k^{\frac{n-1}{2}}(\xi \cdot \eta) + C_{k-1}^{\frac{n+1}{2}}(\xi \cdot \eta)(\bar{\eta}\xi - \xi \cdot \eta)$$

and

$$C^-_{n+1,|k|-1}(\xi, \eta) = \frac{|k|}{n-1} C_{|k|}^{\frac{n-1}{2}}(\xi \cdot \eta) - C_{|k|-1}^{\frac{n+1}{2}}(\xi \cdot \eta)(\bar{\eta}\xi - \xi \cdot \eta).$$

In particular, the Gegenbauer polynomials $C_k^{\frac{n-1}{2}}$ satisfy

$$\frac{1}{\omega_n} \frac{n+2k-1}{n-1} C_k^{\frac{n-1}{2}} = c_{n+1,k} P_k^{n+1}.$$

In [16] we obtain the Abelian summation representations of the Poisson and the conjugate Poisson kernels in B_{n+1}.

Lemma 5.4 [16] *The Abel sum expansions of the Poisson and the conjugate Poisson kernels in the unit ball B_{n+1} are, respectively,*

$$P_r(\xi, \eta) = \frac{1}{\omega_n} \sum_{k=-\infty}^{+\infty} r^{|k|} P^{(k)}(\eta^{-1}\xi),$$

and

$$Q_r(\xi, \eta) = \frac{1}{\omega_n} \left[\sum_{k=1}^{+\infty} \frac{k}{n+k-1} r^k P^{(k)}(\eta^{-1}\xi) - \sum_{k=-\infty}^{-1} r^{|k|} P^{(k)}(\eta^{-1}\xi) \right].$$

Using Lemma 5.4 and formula (5.19), for $r < 1$, we get cancelation for the negative index summation part and have

$$\begin{aligned} F(x) &= \frac{1}{\omega_n}\sum_{k=0}^{+\infty}\frac{n+2k-1}{n+k-1}r^k\int_{S^n}P^{(k)}(\eta^{-1}\xi)u(\eta)dS_\eta \\ &= \frac{1}{\omega_n}\sum_{k=0}^{+\infty}\frac{n+2k-1}{n+k-1}r^k\int_{S^n}C^+_{n+1,k}(\xi,\eta)u(\eta)dS_\eta. \end{aligned} \tag{5.20}$$

Ahlfors' involvement is to use the quantity defined in one complex variable

$$\frac{1}{2}\int_{\partial\mathcal{D}}\overline{F}F'\frac{dz}{\mathbf{i}} \tag{5.21}$$

together with the residue computation in proving the identical relation between the double integral form with the other forms of the same quantity ([2]). With the double integral form the Douglas' original proof uses the Fourier series expansion of the gradient of the harmonic function ([8]). Validity of the proofs lays on the identical relation between the holomorphic derivative and the Ahlfors' quantity (1.2). In Clifford analysis setting, we have

Theorem 5.5 *Let $F(x) \in C^1(\bar{\Omega})$. If $F(x)$ is left-monogenic in $\Omega \subseteq \mathbb{R}^{n+1}$, then we have*

$$\int_\Omega |\bar{D}F|^2 dx = \frac{1}{2}\mathrm{Sc}\int_{\partial\Omega}\bar{F}d\sigma_x\bar{D}F. \tag{5.22}$$

where $d\sigma_x = ndS$, dS stands for the surface element and n the outwardly pointed unit normal.

Proof Using Stokes' formula, we have

$$\int_{\partial\Omega}\bar{F}d\sigma_x\bar{D}F = \int_\Omega d(\bar{F}d\sigma_x\bar{D}F).$$

There holds

$$\begin{aligned} d(\bar{F}d\sigma_x\bar{D}F) &= d(\bar{F}d\sigma_x)\bar{D}F + \bar{F}d(d\sigma_x\bar{D}F) \\ &= d\bar{F}\wedge d\sigma_x\bar{D}F + (-1)^{n-1}\bar{F}d\sigma_x\wedge d(\bar{D}F) \\ &= \left[2(\bar{F}D)\bar{D}F + 2\bar{F}(D\bar{D}F)\right]dx \\ &= 2\left[(\bar{F}D)\bar{D}F + \bar{F}(\frac{1}{4}\Delta_x F)\right]dx \\ &= 2\left[(\bar{F}D)\bar{D}F\right]dx. \end{aligned}$$

By taking scalar part on both sides, we get (5.22). This completes the proof. □

If, in particular, $F(x)$ is left-monogenic in a ball $B_{n+1}(r)$, then $F(x)$ has the form (5.20). Using the radial decomposition (4.14) of $\bar{D}$, we have

$$\begin{aligned}\bar{D}F(x) &= \frac{1}{2\omega_n}\sum_{k=1}^{+\infty}\frac{n+2k-1}{n+k-1}r^{k-1}\Big[k\int_{S^n}C^+_{n+1,k}(\xi,\eta)u(\eta)dS_\eta \\ &\quad+\int_{S^n}\Gamma^*_\xi C^+_{n+1,k}(\xi,\eta)u(\eta)dS_\eta\Big]\bar{\xi} \\ &= \frac{1}{\omega_n}\sum_{k=1}^{+\infty}\frac{n+2k-1}{n+k-1}(k+\frac{n}{2})r^{k-1}\int_{S^n}C^+_{n+1,k}(\xi,\eta)u(\eta)dS_\eta\bar{\xi}.\end{aligned}$$

Using the orthogonality between $C^+_{n+1,k}(\xi,\eta)$and $C^+_{n+1,j}(\xi,\eta)$ when $k\neq j$, we have

$$\begin{aligned}\int_{B_{n+1}(r)}|\bar{D}F|^2dx &= \frac{1}{2}\mathrm{Sc}\int_{S^n_r}\bar{F}d\sigma_x\bar{D}F \\ &= \frac{1}{2\omega_n{}^2}\sum_{k=1}^{+\infty}(\frac{n+2k-1}{n+k-1})^2(k+\frac{n}{2})kr^{n+2k-1}\int_{S^n}\int_{S^n}f(\eta_1)u(\eta_2)dS_{\eta_1}dS_{\eta_2} \\ &\quad\times\mathrm{Sc}\int_{S^n}\overline{C^+_{n+1,k}(\xi,\eta_1)}\xi C^+_{n+1,k}(\xi,\eta_2)\bar{\xi}dS_\xi \\ &= \frac{1}{2}\sum_{k=1}^{+\infty}(k+\frac{n}{2})r^{2k+n-1}c_{n+1,k}\int_{S^n}\int_{S^n}f(\eta_1)f(\eta_2)P_k^{n+1}(\eta_1\cdot\eta_2)dS_{\eta_1}dS_{\eta_2} \\ &= \frac{1}{2}\sum_{k=1}^{+\infty}(k+\frac{n}{2})r^{2k+n-1}\sum_{j=1}^{a_k^n}|b_j|^2. \qquad (5.23)\end{aligned}$$

Taking limit $r\to 1^-$ on both sides of (5.23), we have

$$\int_{B_{n+1}}|\bar{D}F|^2dx=\frac{1}{2}\mathrm{Sc}\int_{S^n}\bar{F}d\sigma\bar{D}F=\frac{1}{2}\sum_{k=1}^{+\infty}(k+\frac{n}{2})\sum_{j=1}^{a_k^{n+1}}|b_j|^2\neq\int_{B_{n+1}}|\nabla u|^2dx.$$

Therefore, we have for $n>1$,

$$\int_{B_{n+1}}|\nabla u(x)|^2d\underline{x}\neq\int_{B_{n+1}}|\bar{D}F(x)|^2dx.$$

With the replacement of n by $n+1$ (consequently, $n>2$ by $n>1$) in Theorem 2.3 there hold, in the present notation,

$$\begin{aligned}\int_{B_{n+1}}|\nabla u(x)|^2dx &= \sum_{k=1}^{\infty}k\sum_{j=1}^{a_k^{n+1}}|b_j|^2 \\ &= \frac{1}{\omega_n}\int_{S^n}\int_{S^n}\frac{|f(t_1)-f(t_2)|^2}{|t_1-t_2|^{n+1}}dt_1dt_2,\end{aligned}$$

and, in contrast,

$$\int_{B_{n+1}} |\bar{D}F(x)|^2 dx = \frac{1}{2}\mathrm{Sc}\int_{S^n} \overline{F} d\sigma \bar{D}F$$
$$= \frac{1}{2}\sum_{k=1}(k+\frac{n}{2})\sum_{j=1}^{a_k^{n+1}} |b_j|^2.$$

Although in the case the Ahlfors quantity is not identical with the Douglas quantities, there is, however, an equivalence relation:

$$\begin{aligned}\int_{B_{n+1}} |\bar{D}F(x)|^2 dx &= \frac{1}{2}\int_{B_{n+1}} |\nabla u(x)|^2 dx + \frac{n}{4}(\|u\|_{\dot{h}^2}^2 - |u(0)|^2)\\ &\leq \frac{1}{2}\|u\|_{\mathcal{D}(B_{n+1}),*}^2 + \frac{n}{4}\|u\|_{\mathcal{D}(B_{n+1})}^2\\ &\leqq \|u\|_{\mathcal{D}(B_{n+1})}^2.\end{aligned}$$

So, in the balls case merely the Ahlfors quantity is sufficient to describe the norm of the Dirichlet space.

6 Applications to Phase Derivative Analysis

The concept analytic signal was introduced by Gabor in [9] (1946). Given $f \in L^2(\mathbb{R})$. Define $f^+ = \frac{1}{2}(f + iHf)$ to be the analytic signal associated with f, where H is the Hilbert transform on the line. Since f^+ has only non-negative Fourier spectra, i.e., $\mathrm{supp}\widehat{f^+} \in [0, +\infty)$, writing $f^+(t) = \rho(t)e^{i\theta(t)}$, it would be reasonable to conjecture that the phase derivative

$$\theta'(t) \geq 0, \quad a.e. \tag{6.24}$$

If this were true, the quantity phase derivative $\theta'(t)$ would be qualified to be defined to be the *instantaneous frequency* (IF). Unfortunately, the expected positivity does not hold: If f^+ is the non-tangential boundary limit (NBL) of an outer function, then in a set of positive Lebesgue measure the negative sign appears [15]. Sufficient conditions for (6.24) to hold include the cases where f^+ is the NBL of an inner function (a finite or infinite Blaschke product, or a singular inner function), or a generalized starlike or multi-starlike type function, etc. By defining those satisfying (6.24) to be *mono-components*, in general, the issue motivates the research and practice of sparse representations of analytic signals into intrinsic components with meaningful IFs. See related work of Qian et al.

Although the positivity (6.24) does not hold in general, the good news is that the identity (1.3) holds, that is between the energy-average of Fourier frequency and that of the phase derivative. It is noted that the identity is only one of the whole set of

Douglas-Ahfose identities: they contribute new understanding of the quantity. As far as the authors concerned, of which the double singular integral one is non-trivially new to describe the average of the boundary phase derivatives.

Phase derivatives of functions on the spheres and on the Euclidean planes are studied, respectively, in [24] and [23]. In both the cases we proved the counterpart of higher dimensional identities, and addressed the related major signal analysis topics, including IFs, series expansions in terms of positive phase derivatives, and a strong form of uncertainty principle, etc. Here in this paper we only concern identical relations between different forms with the energy-average of the frequency, and only take the example of Euclidean planes. On the spheres S^n there exist similar formulations and results, of which some are with more complexity due to particulars of the spherical Hilbert transforms and the spherical Dirac operators. We will not duplicate the main idea for the spheres but refer the reader to [24].

Let u be a real-valued square-integrable on $\mathbb{R}^n$ with NBL f. From the definition of the Hilbert transform H as the $\mathbf{e}_i$-multiple combination of the Riesz transforms, we have $f(\underline{x}) + H[f](\underline{x}) \in H^2(\mathbb{R}^n)$, where the last notation is for the boundary Hardy space consisting of NBLs of those in the left-monogenic Hardy space $H^2(\mathbb{R}^{n+1}_+)$. We call $f^+(\underline{x}) = \frac{1}{2}[f(\underline{x}) + H[f](\underline{x})]$ as the monogenic signal associated with $f(\underline{x})$. It is then further written as

$$f^+(\underline{x}) = \frac{1}{2}[f(\underline{x}) + H[f](\underline{x})] = A(f)(\underline{x})\left[\frac{f(\underline{x})}{A(f)(\underline{x})} + \frac{H[f](\underline{x})}{|H[f](\underline{x})|}\frac{|H[f](\underline{x})|}{A(f)(\underline{x})}\right]$$
$$= A(f)(\underline{x})e^{\frac{H[f](\underline{x})}{|H[f](\underline{x})|}\theta(\underline{x})},$$

where $A(f) = \frac{1}{2}\sqrt{f^2 + H^2[f]}$ is the amplitude, $\theta(\underline{x}) = \arctan\frac{|H[f]|}{f}$ the phase that is between 0 to $\frac{\pi}{2}$, $\frac{H[f](\underline{x})}{|H[f](\underline{x})|}\theta(\underline{x})$ the phase vector, $e^{\frac{H[f](\underline{x})}{|H[f](\underline{x})|}\theta(\underline{x})}$ the phase directive. We define the directional phase derivative as $\mathrm{Sc}\{[\underline{D}\theta(\underline{x})]\frac{H[f](\underline{x})}{|H[f](\underline{x})|}\}$. The phase derivative or instantaneous frequency is defined to be

$$\mathrm{Sc}\{[\underline{D}f^+(\underline{x})][f^+(\underline{x})]^{-1}\}.$$

The last definition is to be compared with the real line case where

$$\theta'(t) = \mathrm{Re}\left\{\left[\frac{1}{i}\frac{d}{dt}f^+(t)\right][f^+(t)]^{-1}\right\}.$$

Remark 6.1 Under the concept scalar-valued instantaneous frequency one can show that the parameterized Poisson kernels are, in various contexts, of positive instantaneous frequency a.e. Since the normalized parameterized Poisson kernels constitute a dictionary, standard methodology, including greedy algorithm, implies that every signal in L^2 has a ONB-, or a sparse series representation, of which each term is of meaningful instantaneous frequency.

Putting together the results on identical relations obtained in §4 and the Ahlfors type result in terms of phase derivative proved in [24], we have

Theorem 6.2 *Let $u(\underline{x}, y) \in \mathcal{D}(\mathbb{R}^{n+1}_+)$ with the boundary function $f(\underline{x})$, and $F(\underline{x}, y)$ the corresponding canonical left-monogenic function in $\mathbb{R}^{n+1}_+$ whose NBL is the boundary monogenic signal f^+, there then hold*

$$\begin{aligned}
\|u\|^2_{\mathcal{D}(\mathbb{R}^{n+1}_+),*} &:= \int_{\mathbb{R}^{n+1}_+} |\nabla u(\underline{x}, y)|^2 d\underline{x} dy \\
&= \int_{\mathbb{R}^{n+1}_+} |\bar{D} F(\underline{x}, y)|^2 d\underline{x} dy \\
&= 2\pi \int_{\mathbb{R}^n} |\underline{\xi}| |\widehat{f^+}(\underline{\xi})|^2 d\underline{\xi} \\
&= 2\pi \int_{\mathbb{R}^n} \mathrm{Sc}\{[\underline{D} f^+(\underline{x})][f^+(\underline{x})]^{-1}\} |f^+(\underline{x})|^2 d\underline{x} \\
&= \frac{1}{\omega_n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(\underline{t_1}) - f(\underline{t_2})|^2}{|\underline{t_1} - \underline{t_2}|^{n+1}} d\underline{t_1} d\underline{t_2}.
\end{aligned}$$

By letting $n = 1$ in Theorem 6.2 we have

Corollary 6.3 *Let u be a real-valued harmonic function, f its NBL in $L^2(\mathbb{R})$, F the corresponding canonical holomorphic function in $\mathbb{R}^2_+$ with NBL f^+, then the condition*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(t_1) - f(t_2)|^2}{|t_1 - t_2|^2} dt_1 dt_2 < +\infty,$$

implies

$$0 < \int_{\mathbb{R}} \theta'(t) |f^+(t)|^2(t) dt < +\infty,$$

and the above quantities both are equal to the energy-average of the Fourier frequency

$$\langle \xi \rangle^+ = \int_{\mathbb{R}} \xi |\widehat{f^+}(\xi)|^2 d\xi.$$

Remark 6.4 In the quaternionic analysis setting, using similar methods as in this paper, the result for the quaternionic ball corresponds to $n = 4$ in Theorem 2.3; and the result in the quaternionic upper half space corresponds to $n = 3$ of Theorem 3.2. While in the upper half space, the conclusion in Theorem 4.7 cannot be applied to quaternions. The reason is that the proof of Theorem 4.7 uses Lemma 4.8, which does not hold for quaternions.

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Declarations

Competing interests The authors declare no competing interests.

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