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On universal quotient Blaschke products

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ABSTRACT

In this paper, we introduce a new member to the universal families, called universal quotient Blaschke product, which is a formal quotient of two formal infinite Blaschke products. A formal infinite Blaschke product is of the form

$$B(z) = \prod_{k=1}^{\infty} \frac{z - z_k}{1 - \overline{z_k} z},$$

where $\{z_k\}_{k=1}^{\infty}$ is a sequence of points in the unit disk but may not satisfy the Blaschke condition:

$$\sum_{k=1}^{\infty} (1 - |z_k|) < \infty.$$

A partial quotient of a universal quotient Blaschke product is the quotient of two finite Blaschke products. We show that the set of partial quotients of a universal quotient Blaschke product is dense in the set of continuous self-mappings on the unit circle in the complex plane. Meanwhile, subsequences of the partial quotients of a universal quotient Blaschke product can be used to approximate any holomorphic functions bounded by one on the unit disk. Moreover, we prove that the set of universal quotient Blaschke products is huge in the sense of Baire category.

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1. Introduction

Universality, which was named by Marcinkiewicz [1] in 1935, is a generic phenomenon in analysis. The first case of universality was observed by Fekete [2] in 1914, which is a universal real power series $\sum_{n=1}^{\infty} a_n x^n$ on $[-1, 1]$. He showed that for any continuous function g on $[-1, 1]$ with $g(0) = 0$, there exists a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ of partial sums of the power series such that

$$s_{n_k} = \sum_{n=1}^{n_k} a_n x^n \to g(x)$$

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uniformly as $k \to \infty$. In 2010, Mouze and Nestoridis [3] obtained an extension of Fekete's result concerning the existence of universal power series of C^∞ functions.

In 1951, Seleznev [4] constructed the first universal complex power series, which has radius of convergence 0. For any function h continuous on $\mathbb{K}$ and holomorphic in the interior of $\mathbb{K}$, where $\mathbb{K} \subset \mathbb{C} \setminus \{0\}$ is compact with connected complement, there exists a subsequence of its partial sums converging uniformly to h. In 1971, Chui and Parnes [5] constructed universal complex power series with radius of convergence 1. In 1996, Nestoridis [6] proved that the set of such universal complex power series is huge in the sense of Baire category.

The idea of universality has penetrated into many branches of mathematics. Mathematicians have been studying this phenomenon from different angles. Universality is a classical but has yet been a flourishing topic. For a comprehensive survey, please refer to [7,8], which were published in recent two decades. In real and complex analysis, the research mainly focuses on universal power series and using subsequences of their partial sums to approximate a set of functions. A natural question is whether there exists universal elements generated by infinite products such that the subsequences of its partial products can have certain universality.

In 1955, Heins [9] constructed a convergent Blaschke product $B(z)$ with the following properties: (1) the point $z = 1$ is the sole cluster point of the zeros of $B(z)$ and (2) there exists a strictly increasing sequence of positive numbers $\{x_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = 1$ such that

$$\left\{B\left(\frac{z + x_n}{1 + x_n z}\right)\right\}_{n=1}^{\infty}$$

is locally uniformly dense in the set of all analytic functions bounded by one on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

In 2004, Gorkin and Mortini [10] obtained an extension of Heins' result. They showed that for any sequence of points $\{z_n\}_{n=1}^{\infty}$ in the unit disk $\mathbb{D}$ satisfying $\lim_{n\to\infty} |z_n| = 1$, there exists an infinite Blaschke product $B(z)$ such that the sequence

$$\left\{B\left(\frac{z + z_n}{1 + \overline{z_n} z}\right)\right\}_{n=1}^{\infty}$$

is locally uniformly dense in the set of all analytic functions bounded by one on $\mathbb{D}$. Such Blaschke product is called *universal Blaschke product*.

Although the universality of a universal Blaschke product in [10] is represented through a sequence of composition operators instead of partial products, the universal Blaschke product itself is an infinite product. This motivated us to consider the partial products of formal infinite Blaschke products. In this paper, by using formal infinite Blaschke products which may be divergent, we construct a universal element named as *universal quotient Blaschke product* and use subsequences of its partial quotients to approximate continuous self-mappings of the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, as well as the analytic functions bounded by one in $\mathbb{D}$.

In Section 2, we give preliminaries and our main results Theorems 2.1 and 2.2. Their proofs are given in Section 3. We show that there exist abundant universal quotient Blaschke products in Section 4.

2. Preliminaries and main results

In this article, let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote, respectively, the set of positive integers, integers and real numbers. The set $C(\mathbb{T},\mathbb{T})$ of continuous self-mappings of $\mathbb{T}$ is endowed with a metric d defined by

$$d(u_1,u_2)=\max_{\xi\in\mathbb{T}}\left|u_1(\xi)-u_2(\xi)\right|,$$

where $u_1,\ u_2\in C(\mathbb{T},\mathbb{T})$. It is obvious that $C(\mathbb{T},\mathbb{T})$ with metric d is a complete metric space.

A *finite Blaschke product* is a rational function of the form

$$B(z)=\gamma\prod_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}, \tag{1}$$

where $z_1,\dots,z_n\in\mathbb{D}$ and $\gamma\in\mathbb{T}$. If $\gamma=1$, then $B(z)$ is said to be *monic*; if $\gamma=1, B(0)\neq 0$ and its zeros $z_1,z_2,\dots,z_n$ are distinct, then it is said to be *simple*. Each finite Blaschke product maps $\mathbb{D}$ to $\mathbb{D}$ and $\mathbb{T}$ to $\mathbb{T}$. Moreover, it is analytic on a neighborhood of $\overline{\mathbb{D}}$ and meromorphic on the extended complex plane $\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. The following lemma, called the Helson–Sarason theorem in [11], shows that each $u\in C(\mathbb{T},\mathbb{T})$ can be approximated uniformly on $\mathbb{T}$ by the quotients of two finite Blaschke products.

Lemma A ([11, Theorem 4.3.1]): *Suppose that $u\in C(\mathbb{T},\mathbb{T})$. For any $\varepsilon>0$, there exist two monic finite Blaschke products $B_1(z), B_2(z)$, and $\gamma\in\mathbb{T}$ such that*

$$\max_{\xi\in\mathbb{T}}\left|\gamma u(\xi)-\frac{B_1(\xi)}{B_2(\xi)}\right|<\varepsilon. \tag{2}$$

An *infinite Blaschke product* is of the form

$$B(z)=\gamma\prod_{k=1}^{\infty}\frac{z-z_k}{1-\overline{z_k}z}, \tag{3}$$

where $\gamma\in\mathbb{T}$, and $\{z_k\}_{k=1}^{\infty}$ is a sequence of points in $\mathbb{D}$ satisfying Blaschke condition

$$\sum_{k=1}^{\infty}(1-|z_k|)<\infty.$$

For more information of infinite Blaschke products, please refer to [11,12].

In this paper, we consider *formal infinite Blaschke product*, which is of the form

$$B(z)=\prod_{k=1}^{\infty}\frac{z-z_k}{1-\overline{z_k}z}, \tag{4}$$

where $\{z_k\}_{k=1}^{\infty}$ is a sequence of points in $\mathbb{D}$ but may not satisfy the above Blaschke condition. For every integer $n\geq 1$, we denote the nth partial product of $B(z)$ by

$$P(B,n)(z)=\prod_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}. \tag{5}$$

If for each $n \in \mathbb{N}$, $P(B,n)(z)$ is a simple finite Blaschke product, then $B(z)$ is called *a simple formal infinite Blaschke product*. By using formal infinite Blaschke products, we obtain the first universal product as follows.

Theorem 2.1: *There exist two formal infinite Blaschke products*

$$B(z) = \prod_{i=1}^{\infty} \frac{z - z_i}{1 - \overline{z_i} z}, \quad C(z) = \prod_{j=1}^{\infty} \frac{z - w_j}{1 - \overline{w_j} z}$$

such that the quotient $B(z)/C(z)$ is a universal quotient Blaschke product in the following sense.

For $\forall u \in C(\mathbb{T}, \mathbb{T})$, there exist a sequence $\{\gamma_l\}_{l=1}^{\infty} \subset \mathbb{T}$, and two strictly increasing sequences of positive integers $\{\mu_l\}_{l=1}^{\infty}$ and $\{\nu_l\}_{l=1}^{\infty}$ such that

$$\lim_{l \to \infty} \max_{\xi \in \mathbb{T}} \left| \gamma_l u(\xi) - \frac{P(B, \mu_l)(\xi)}{P(C, \nu_l)(\xi)} \right| = 0.$$

To avoid the confusion caused by the common zeros, we enforce more requirements on $B(z)$ and $C(z)$ and obtain the following theorem.

Theorem 2.2: *There exist two formal infinite Blaschke products*

$$B(z) = \prod_{i=1}^{\infty} \frac{z - z_i}{1 - \overline{z_i} z}, \quad C(z) = \prod_{j=1}^{\infty} \frac{z - w_j}{1 - \overline{w_j} z}$$

satisfying the following conditions.
(i) *For each $n \in \mathbb{N}$, $P(B,n)(z)$ and $P(C,n)(z)$ are simple finite Blaschke products.*
(ii) *For each $n \in \mathbb{N}$, $P(B,n)(z)$ and $P(C,n)(z)$ have no common zeros.*
(iii) *For $\forall u \in C(\mathbb{T}, \mathbb{T})$, there exist a sequence $\{\gamma_l\}_{l=1}^{\infty} \subset \mathbb{T}$, and two strictly increasing sequences of positive integers $\{\mu_l\}_{l=1}^{\infty}$ and $\{\nu_l\}_{l=1}^{\infty}$ such that*

$$\lim_{l \to \infty} \max_{\xi \in \mathbb{T}} \left| \gamma_l u(\xi) - \frac{P(B, \mu_l)(\xi)}{P(C, \nu_l)(\xi)} \right| = 0.$$

The quotient $B(z)/C(z)$ is named as simple universal quotient Blaschke product.

Let $\mathcal{B}$ denote the set of the functions ω which are analytic in $\mathbb{D}$ and satisfy $|\omega(z)| \leq 1$ for all $z \in \mathbb{D}$. Then the Cauchy integral of a function in $C(\mathbb{T}, \mathbb{T})$, which is an inner function, belongs to $\mathcal{B}$. By Theorems 2.1 and 2.2, we naturally consider whether the Cauchy integrals of its partial products of a universal quotient Blaschke product can approximate analytic functions in $\mathcal{B}$. We obtain the following results.

Theorem 2.3: *Let $B(z)/C(z)$ be a universal quotient Blaschke product as in Theorem* 2.1. *Suppose that $\omega \in \mathcal{B}$ has continuous extension on $\overline{\mathbb{D}}$ and $\omega|_{\mathbb{T}}$ is a continuous self-mapping*

of $\mathbb{T}$. *Then for any* $\varepsilon > 0$, *there exist integers* $m > 1$, $n > 1$ *and* $\gamma \in \mathbb{T}$ *such that the Cauchy integral* $H(m, n, \gamma)(z)$ *defined by*

$$H(m,n,\gamma)(z) = \frac{1}{2\pi i}\int_{|\xi|=1} \frac{\gamma P(B,m)(\xi)/P(C,n)(\xi)}{\xi - z}\,\mathrm{d}\xi, \quad z \in \mathbb{D}$$

satisfying

$$\sup_{z\in\mathbb{D}} |\omega(z) - H(m,n,\gamma)(z)| < \varepsilon.$$

Proof: Let $u(\xi)$ be the continuous extension of ω on $\mathbb{T}$. By Theorem 2.1, there exist $m > 1$, $n > 1$, and $\gamma \in \mathbb{T}$ such that

$$\max_{\xi\in\mathbb{T}} \left| u(\xi) - \gamma \frac{P(B,m)(\xi)}{P(C,n)(\xi)} \right| < \varepsilon.$$

The maximum principle ensures that

$$\sup_{z\in\mathbb{D}} |\omega(z) - H(m,n,\gamma)(z)| < \varepsilon.$$

■

Since each finite Blaschke product can be extended continuously to $\mathbb{T}$ and maps $\mathbb{T}$ to $\mathbb{T}$, Theorem 2.3 yields that

Corollary 2.1: *Let* $B(z)/C(z)$ *be a universal quotient Blaschke product. Then for every finite Blaschke product* $F(z)$ *and* $0 < \varepsilon < 1$, *there exist integers* $m > 1$ *and* $n > 1$ *as well as* $\gamma \in \mathbb{T}$ *such that*

$$\sup_{z\in\mathbb{D}} |F(z) - H(m,n,\gamma)(z)| < \varepsilon,$$

where $H(m, n, \gamma)(z)$ *is defined as in Theorem 2.3.*

Theorem 2.4: *Let* $f \in \mathcal{B}$ *and* $B(z)/C(z)$ *be a universal quotient Blaschke product as in Theorem* 2.1. *For every* $\varepsilon > 0$ *and every compact subset* $\mathbb{K} \subset \mathbb{D}$, *there exist integers* $m > 1$ *and* $n > 1$ *as well as* $\gamma \in \mathbb{T}$ *such that*

$$\sup_{z\in\mathbb{K}} \left| f(z) - H(m,n,\gamma)(z) \right| < \varepsilon,$$

where $H(m, n, \gamma)(z)$ *is defined as in Theorem* 2.3

Proof: By the Carathéodory Theorem ([11], Theorem 4.1.1), there is a finite Blaschke product $B(f, \varepsilon, \mathbb{K})$ such that

$$\sup_{z\in\mathbb{K}} \left| f(z) - B(f,\varepsilon,\mathbb{K})(z) \right| < \varepsilon/2.$$

By Corollary 2.1, there exist $m > 1$, $n > 1$, and $\gamma \in \mathbb{T}$ such that

$$\sup_{z\in\mathbb{D}} \left| B(f,\varepsilon,\mathbb{K})(z) - H(m,n,\gamma)(z) \right| < \varepsilon/2.$$

By combining these two inequalities, we obtain the conclusion. ■

3. Proof of Theorems 2.1 and 2.2

For the proof of Theorem 2.1, we will apply Lemma A repeatedly while for the proof of Theorem 2.2, we need a modified Helson–Sarason theorem.

Lemma 3.1: *Let $Z = \{a_1, a_2, \ldots, a_n\}$ be a finite subset of $\mathbb{D}$, $u \in C(\mathbb{T}, \mathbb{T})$ and $\varepsilon > 0$. Then there exist simple finite Blaschke products $B_1(z)$, $B_2(z)$, and $\gamma \in \mathbb{T}$ such that*

$$Z_{B_1} \bigcap Z = \emptyset, \quad Z_{B_2} \bigcap Z = \emptyset, \quad Z_{B_1} \bigcap Z_{B_2} = \emptyset, \quad \max_{\xi \in \mathbb{T}} \left| \gamma u(\xi) - \frac{B_1(\xi)}{B_2(\xi)} \right| < \varepsilon. \tag{6}$$

where Z_{B_i} consists of the zeros of $B_i(z)$ for $i = 1, 2$.

Proof: By Lemma A, there exist monic Blaschke products C_1 and C_2 such that

$$\max_{\xi \in \mathbb{T}} \left| \gamma_1 u(\xi) - \frac{C_1(\xi)}{C_2(\xi)} \right| < \varepsilon/3. \tag{7}$$

By similar technique as in the proof of the Blaschke product approximation theorem (Theorem 4.4.1 in [11]), we can find a simple Blaschke products B_1 and $\gamma_2 \in \mathbb{T}$ such that $Z_{B_1} \bigcap Z = \emptyset$ and

$$\sup_{z \in \overline{\mathbb{D}}} |\gamma_2 B_1(z) - C_1(z)| < \varepsilon/3. \tag{8}$$

Similarly, we can find a simple Blaschke product B_2 and $\gamma_3 \in \mathbb{T}$ such that

$$Z_{B_2} \bigcap \left(Z \bigcup Z_{B_1} \right) = \emptyset, \quad \max_{z \in \overline{\mathbb{D}}} |\gamma_3 B_2(z) - C_2(z)| < \varepsilon/3. \tag{9}$$

Set $\gamma = \gamma_1 \gamma_3 / \gamma_2$ and note that $|C_1(\xi)| = |C_2(\xi)| = |B_1(\xi)| = |B_2(\xi)| = 1$ for $\xi \in \mathbb{T}$. By the above facts, and using (7), (8) and (9), we have

$$\begin{aligned}
&\max_{\xi \in \mathbb{T}} \left| \gamma u(\xi) - \frac{B_1(\xi)}{B_2(\xi)} \right| = \max_{\xi \in \mathbb{T}} \left| \gamma_1 u(\xi) - \frac{\gamma_2 B_1(\xi)}{\gamma_3 B_2(\xi)} \right| \\
&\leq \max_{\xi \in \mathbb{T}} \left| \gamma_1 u(\xi) - \frac{C_1(\xi)}{C_2(\xi)} \right| + \max_{\xi \in \mathbb{T}} \left| \frac{C_1(\xi)}{C_2(\xi)} - \frac{\gamma_2 B_1(\xi)}{\gamma_3 B_2(\xi)} \right| \\
&= \max_{\xi \in \mathbb{T}} \left| \gamma_1 u(\xi) - \frac{C_1(\xi)}{C_2(\xi)} \right| + \max_{\xi \in \mathbb{T}} |\gamma_3 C_1(\xi) B_2(\xi) - \gamma_2 B_1(\xi) C_2(\xi)| \\
&< \varepsilon/3 + \max_{\xi \in \mathbb{T}} |\gamma_3 C_1(\xi) B_2(\xi) - \gamma_3 \gamma_2 B_1(\xi) B_2(\xi) + \gamma_3 \gamma_2 B_1(\xi) B_2(\xi) - \gamma_2 B_1(\xi) C_2(\xi)| \\
&\leq \varepsilon/3 + \max_{\xi \in \mathbb{T}} |C_1(\xi) - \gamma_2 B_1(\xi)| + \max_{\xi \in \mathbb{T}} |C_2(\xi) - \gamma_3 B_2(\xi)| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}$$

Hence (6) is proved. ■

Proof of Theorem 2.1: As a subspace of separable space $C(\mathbb{T}, \mathbb{C})$, the space $C(\mathbb{T}, \mathbb{T})$ is separable too. Let $S = \{u_j\}_{j=1}^{\infty}$ be a countable dense subset of $C(\mathbb{T}, \mathbb{T})$. Let $\mathcal{L}$ be a list which

contains all the $u_j \in S$ but no other elements such that every u_j appears infinitely times in $\mathcal{L}$. Rewrite $\mathcal{L}$ by $\mathcal{L} = \{g_k\}_{k=1}^{\infty}$. We will construct a universal quotient Blaschke product by induction on g_k. ■

Step 1. $k = 1$

By Lemma A, there exist two finite Blaschke products

$$B^{(1)}(z) = \prod_{i=1}^{M_1} \frac{z - a_i^{(1)}}{1 - \overline{a_i^{(1)}} z}, \quad C^{(1)}(z) = \prod_{j=1}^{N_1} \frac{z - b_j^{(1)}}{1 - \overline{b_j^{(1)}} z},$$

and $\alpha_1 \in \mathbb{T}$ such that

$$\max_{\xi \in \mathbb{T}} \left| \alpha_1 g_1(\xi) - \frac{B^{(1)}(\xi)}{C^{(1)}(\xi)} \right| < 1.$$

Let $m_1 = M_1$, $n_1 = N_1$, $z_i = a_i^{(1)}$ and $w_j = b_j^{(1)}$, where $i = 1, 2, \cdots, m_1$ and $j = 1, 2, \cdots, n_1$. Set

$$B^1(z) = \prod_{i=1}^{m_1} \frac{z - z_i}{1 - \overline{z_i} z}, \quad C^1(z) = \prod_{j=1}^{n_1} \frac{z - w_j}{1 - \overline{w_j} z}.$$

Then

$$\max_{\xi \in \mathbb{T}} \left| \alpha_1 g_1(\xi) - \frac{B^1(\xi)}{C^1(\xi)} \right| < 1.$$

Step 2. $k = 2$

By Lemma A, there exist two finite Blaschke products

$$B^{(2)}(z) = \prod_{i=1}^{M_2} \frac{z - a_i^{(2)}}{1 - \overline{a_i^{(2)}} z}, \quad C^{(2)}(z) = \prod_{j=1}^{N_2} \frac{z - b_j^{(2)}}{1 - \overline{b_j^{(2)}} z},$$

and $\alpha_2 \in \mathbb{T}$ such that

$$\max_{\xi \in \mathbb{T}} \left| \alpha_2 \frac{g_2(\xi)}{B^1(\xi)/C^1(\xi)} - \frac{B^{(2)}(\xi)}{C^{(2)}(\xi)} \right| < \frac{1}{2},$$

which is equivalent to

$$\max_{\xi \in \mathbb{T}} \left| \alpha_2 g_2(\xi) - \frac{B^1(\xi) B^{(2)}(\xi)}{C^1(\xi) C^{(2)}(\xi)} \right| < \frac{1}{2}.$$

Let $m_2 = m_1 + M_2$, $n_2 = n_1 + N_2$, $z_{i+m_1} = a_i^{(2)}$ and $w_{j+n_1} = b_j^{(2)}$, where $i = 1, 2, \cdots, M_2$ and $j = 1, 2, \cdots, N_2$. Set $B^2(z) = B^1(z) B^{(2)}(z)$ and $C^2(z) = C^1(z) C^{(2)}(z)$. Then these facts yield that

$$B^2(z) = \prod_{i=1}^{m_2} \frac{z - z_i}{1 - \overline{z_i} z}, \quad C^2(z) = \prod_{j=1}^{n_2} \frac{z - w_j}{1 - \overline{w_j} z}, \quad \max_{\xi \in \mathbb{T}} \left| \alpha_2 g_2(\xi) - \frac{B^2(\xi)}{C^2(\xi)} \right| < \frac{1}{2}.$$

Step 3. *Repetition*

Repeat the above process and we will obtain four sequences

$$\{B^{(k)}(z)\}_{k=1}^{\infty}, \quad \{C^{(k)}(z)\}_{k=1}^{\infty}, \quad \{B^{k}(z)\}_{k=1}^{\infty}, \quad \{C^{k}(z)\}_{k=1}^{\infty},$$

where

$$B^k(z)=\prod_{p=1}^{k}B^{(p)}(z)=\prod_{i=1}^{m_k}\frac{z-z_i}{1-\overline{z_i}z}, \quad C^k(z)=\prod_{p=1}^{k}C^{(p)}(z)=\prod_{j=1}^{n_k}\frac{z-w_j}{1-\overline{w_j}z},$$

$$\max_{\xi\in\mathbb{T}}\left|\alpha_k g_k(\xi)-\frac{B^k(\xi)}{C^k(\xi)}\right|<\frac{1}{k}. \tag{10}$$

Let

$$B(z)=\prod_{k=1}^{\infty}B^{(k)}(z), \quad C(z)=\prod_{k=1}^{\infty}C^{(k)}(z).$$

Then

$$B(z)=\prod_{i=1}^{\infty}\frac{z-z_i}{1-\overline{z_i}z}, \quad C(z)=\prod_{j=1}^{\infty}\frac{z-w_j}{1-\overline{w_j}z}.$$

Step 4. *Proof of B(z)/C(z) being a universal quotient Blaschke product*

For any $u(\xi)\in C(\mathbb{T},\mathbb{T})$ and $0<\epsilon<1$, there exists $u^{(1)}\in S$ such that

$$\max_{\xi\in\mathbb{T}}\left|u(\xi)-u^{(1)}(\xi)\right|<\varepsilon/2. \tag{11}$$

Recall that $u^{(1)}$ appears in $\mathcal{L}$ infinite times. We may assume that

$$u^{(1)}=g_{1^{(1)}}=g_{2^{(1)}}=\cdots,$$

where $1^{(1)}<2^{(1)}<\cdots$. Let $j_1^{(1)}=\min\{j^{(1)}:\frac{1}{j^{(1)}}<\varepsilon/2\}$. The inequality (10) implies that

$$\max_{\xi\in\mathbb{T}}\left|\alpha_{j_1^{(1)}}g_{j_1^{(1)}}(\xi)-\frac{B^{j_1^{(1)}}(\xi)}{C^{j_1^{(1)}}(\xi)}\right|<\frac{1}{j_1^{(1)}}<\varepsilon/2. \tag{12}$$

By using (11) and (12), we have

$$\begin{aligned}\max_{\xi\in\mathbb{T}}\left|\alpha_{j_1^{(1)}}u(\xi)-\frac{B^{j_1^{(1)}}(\xi)}{C^{j_1^{(1)}}(\xi)}\right|&\le\max_{\xi\in\mathbb{T}}\left|\alpha_{j_1^{(1)}}u(\xi)-\alpha_{j_1^{(1)}}u^{(1)}(\xi)\right|+\max_{\xi\in\mathbb{T}}\left|\alpha_{j_1^{(1)}}g_{j_1^{(1)}}(\xi)-\frac{B^{j_1^{(1)}}(\xi)}{C^{j_1^{(1)}}(\xi)}\right|\\&<\varepsilon/2+\varepsilon/2=\varepsilon.\end{aligned}$$

Set $\gamma_1=\alpha_{j_1^{(1)}}$, $\mu_1=m_{j_1^{(1)}}$, and $\nu_1=n_{j_1^{(1)}}$.

For $\epsilon/2$, there exists $u^{(2)} \in S - \{u^{(1)}\}$ such that

$$\max_{\xi\in\mathbb{T}} \left|u(\xi) - u^{(2)}(\xi)\right| < \varepsilon/4. \tag{13}$$

We may assume that

$$u^{(2)} = g_{1^{(2)}} = g_{2^{(2)}} = \cdots,$$

where $1^{(2)} < 2^{(2)} < \cdots$. Let $j_2^{(2)} = \min\{j^{(2)} : j^{(2)} > j_1^{(1)}, \frac{1}{j^{(2)}} < \varepsilon/4\}$. The similar reasoning as above gives

$$\max_{\xi\in\mathbb{T}} \left|\alpha_{j_2^{(2)}} u(\xi) - \frac{B^{j_2^{(2)}}(\xi)}{C^{j_2^{(2)}}(\xi)}\right| < \varepsilon/2.$$

Set $\gamma_2 = \alpha_{j_2^{(2)}}$, $\mu_2 = m_{j_2^{(2)}}$ and $\nu_2 = n_{j_2^{(2)}}$.

Keep going, we obtain the required sequence $\{\gamma_l\}_{l=1}^{\infty}$, $\{\mu_l\}_{l=1}^{\infty}$ and $\{\nu_l\}_{l=1}^{\infty}$. The proof is complete.

Proof of Theorem 2.2 is similar to that of Theorem 2.1. The main difference is that, in the approximation process, instead of using the Helson–Sarason theorem (Lemma A) in the proof of Theorem 2.1, we use Lemma 3.1 in the proof of Theorem 2.2. We provide the details.

Proof of Theorem 2.2: Let S and $\mathcal{L}$ be as in the proof of Theorem 1. We will construct a simple universal quotient Blaschke product by induction on g_k. ■

Step 1. $k = 1$

By Lemma 3.1, there exist $\gamma_1 \in \mathbb{T}$, and two simple finite Blaschke products having no common zeros in $\mathbb{D}$

$$B^{(1)}(z) = \prod_{i=1}^{M_1} \frac{z - a_i^{(1)}}{1 - \overline{a_i^{(1)}} z}, \quad C^{(1)}(z) = \prod_{j=1}^{N_1} \frac{z - b_j^{(1)}}{1 - \overline{b_j^{(1)}} z}$$

such that

$$\max_{\xi\in\mathbb{T}} \left|\gamma_1 g_1(\xi) - \frac{B^{(1)}(\xi)}{C^{(1)}(\xi)}\right| < 1.$$

Let $m_1 = M_1$, $n_1 = N_1$, $z_i = a_i^{(1)}$ and $w_j = b_j^{(1)}$, where $i = 1, 2, \cdots, m_1$ and $j = 1, 2, \cdots, n_1$. Set

$$B^1(z) = \prod_{i=1}^{m_1} \frac{z - z_i}{1 - \overline{z_i} z}, \quad C^1(z) = \prod_{j=1}^{n_1} \frac{z - w_j}{1 - \overline{w_j} z}.$$

Then

$$\max_{\xi\in\mathbb{T}} \left|\gamma_1 g_1(\xi) - \frac{B^1(\xi)}{C^1(\xi)}\right| < 1. \tag{14}$$

Step 2. $k = 2$

Let Z_B be the set of zeros of the Blaschke product B. By Lemma 3.1, there exist $\gamma_2 \in \mathbb{T}$, and two simple finite Blaschke products having no common zeros

$$B^{(2)}(z) = \prod_{i=1}^{M_2} \frac{z - a_i^{(2)}}{1 - \overline{a_i^{(2)}} z}, \quad C^{(2)}(z) = \prod_{j=1}^{N_2} \frac{z - b_j^{(2)}}{1 - \overline{b_j^{(2)}} z}$$

satisfying the following conditions:

$$Z_{B^{(2)}} \bigcap \left(Z_{B^1} \bigcup Z_{C^1} \right) = \emptyset, \quad Z_{C^{(2)}} \bigcap \left(Z_{B^1} \bigcup Z_{C^1} \right) = \emptyset,$$
$$\max_{\xi \in \mathbb{T}} \left| \gamma_2 \frac{g_2(\xi)}{B^1(\xi)/C^1(\xi)} - \frac{B^{(2)}(\xi)}{C^{(2)}(\xi)} \right| < \frac{1}{2}.$$

The last inequality is equivalent to

$$\max_{\xi \in \mathbb{T}} \left| \gamma_2 g_2(\xi) - \frac{B^1(\xi) B^{(2)}(\xi)}{C^1(\xi) C^{(2)}(\xi)} \right| < \frac{1}{2}.$$

Let $m_2 = m_1 + M_2$, $n_2 = n_1 + N_2$, $z_{i+m_1} = a_i^{(2)}$ and $w_{j+n_1} = b_j^{(2)}$, where $i = 1, 2, \cdots, M_2$ and $j = 1, 2, \cdots, N_2$. Set $B^2(z) = B^1(z) B^{(2)}(z)$ and $C^2(z) = C^1(z) C^{(2)}(z)$. Then these facts yield that B^2 and C^2 are simple finite Blaschke products having no common zeros. Moreover,

$$B^2(z) = \prod_{i=1}^{m_2} \frac{z - z_i}{1 - \overline{z_i} z}, \quad C^2(z) = \prod_{j=1}^{n_2} \frac{z - w_j}{1 - \overline{w_j} z}, \max_{\xi \in \mathbb{T}} \left| \gamma_2 g_2(\xi) - \frac{B^2(\xi)}{C^2(\xi)} \right| < 1/2.$$

Step 3. *Repetition*

Repeat the process above and we will obtain four sequences

$$\{B^{(k)}(z)\}_{k=1}^{\infty}, \quad \{C^{(k)}(z)\}_{k=1}^{\infty}, \quad \{B^k(z)\}_{k=1}^{\infty}, \quad \{C^k(z)\}_{k=1}^{\infty},$$

where B^k and C^k are simple finite Blaschke products having no common zeros,

$$B^k(z) = \prod_{p=1}^{k} B^{(p)}(z) = \prod_{i=1}^{m_k} \frac{z - z_i}{1 - \overline{z_i} z}, \; C^k(z) = \prod_{p=1}^{k} C^{(p)}(z) = \prod_{j=1}^{n_k} \frac{z - w_j}{1 - \overline{w_j} z},$$
$$\max_{\xi \in \mathbb{T}} \left| \gamma_k g_k(\xi) - \frac{B^k(\xi)}{C^k(\xi)} \right| < \frac{1}{k}.$$

Let

$$B(z) = \prod_{k=1}^{\infty} B^{(k)}(z), \quad C(z) = \prod_{k=1}^{\infty} C^{(k)}(z).$$

Then by similar reasoning as in the proof of Theorem 2.1, we know that $B(z)/C(z)$ is a simple universal quotient Blaschke product.

4. Denseness of universal quotient Blaschke products

Let $\mathbb{D}^{\mathbb{N}}$ be the space of infinite sequences of points in $\mathbb{D}$ endowed with the product topology. We have seen that every formal infinite Blaschke product

$$B(z) = \prod_{k=1}^{\infty} \frac{z - a_k}{1 - \overline{a_k} z}$$

is uniquely determined by a sequence $\mathbf{a} = (a_1, a_2, \ldots) \in \mathbb{D}^{\mathbb{N}}$, and vice versa. To highlight the sequence $\mathbf{a} = (a_1, a_2, \ldots)$ of the zeros of $B(z)$, we rewrite $B(z) = B^{\mathbf{a}}(z)$. So we can identify a universal quotient Blaschke product $B^{\mathbf{a}}(z)/B^{\mathbf{b}}(z)$ with the pair $(\mathbf{a}, \mathbf{b}) \in \mathbb{D}^{\mathbb{N}} \times \mathbb{D}^{\mathbb{N}}$, where

$$B^{\mathbf{a}}(z) = \prod_{i=1}^{\infty} \frac{z - a_i}{1 - \overline{a_i} z}, \quad B^{\mathbf{b}}(z) = \prod_{j=1}^{\infty} \frac{z - b_j}{1 - \overline{b_j} z},$$

$$\mathbf{a} = (a_1, a_2, \ldots) \in \mathbb{D}^{\mathbb{N}}, \ \ \mathbf{b} = (b_1, b_2, \ldots) \in \mathbb{D}^{\mathbb{N}},$$

and $\mathbb{D}^{\mathbb{N}} \times \mathbb{D}^{\mathbb{N}}$ is endowed with the product topology. Therefore, the research of universal quotient Blaschke products is deduced to the research of the paired sequences. This is a typical approach in the research of universality, see [13,14].

Let $\mathbb{U}$ be the set of the pairs $(\mathbf{a}, \mathbf{b}) \in \mathbb{D}^{\mathbb{N}} \times \mathbb{D}^{\mathbb{N}}$ such that $B^{\mathbf{a}}(z)/B^{\mathbf{b}}(z)$ is a universal quotient Blaschke product. Theorem 2.1 implies that $\mathbb{U} \neq \emptyset$. Experience has shown that whenever there exists a universal element, the set of such universal elements may be huge in the sense of Baire categories. One may ask whether $\mathbb{U}$ is dense in $\mathbb{D}^{\mathbb{N}} \times \mathbb{D}^{\mathbb{N}}$. The answer is as follows.

Theorem 4.1: *$\mathbb{U}$ is a G_δ dense subset of the space $\mathbb{D}^{\mathbb{N}} \times \mathbb{D}^{\mathbb{N}}$.*

As in the proof of Theorem 2.1, let $S = \{u_j\}_{j \geq 1}$ be a countable dense subset of the space $C(\mathbb{T}, \mathbb{T})$. For each $j \in \mathbb{N}$, we define

$$E(j) = \Big\{(\mathbf{a}, \mathbf{b}) \in \mathbb{D}^{\mathbb{N}} \times \mathbb{D}^{\mathbb{N}} : \text{ for any } \epsilon > 0, \text{ there exist } \gamma \in \mathbb{T} \text{ and } M, N \in \mathbb{N}$$
$$\text{such that } \max_{\xi \in \mathbb{T}} \left| \gamma u_j(\xi) - \frac{P(B^{\mathbf{a}}, M)(\xi)}{P(B^{\mathbf{b}}, N)(\xi)} \right| < \epsilon \Big\}.$$

The following lemma shows the relationship between $\mathbb{U}$ and $E(j)$.

Lemma 4.1: $\mathbb{U} = \bigcap_{j=1}^{\infty} E(j)$.

Proof: It is obvious that $\mathbb{U} \subset \bigcap_{j=1}^{\infty} E(j)$. Let $(\mathbf{a}, \mathbf{b}) \in \bigcap_{j=1}^{\infty} E(j)$. For any $u \in C(\mathbb{T}, \mathbb{T})$ and any $\epsilon > 0$, there exists $u_{j_0} \in S$ such that

$$\max_{\xi \in \mathbb{T}} \left| u_{j_0}(\xi) - u(\xi) \right| < \frac{\epsilon}{2}.$$

Since $(\mathbf{a}, \mathbf{b}) \in E(j_0)$, there exist $\gamma \in \mathbb{T}$ and $M, N \in \mathbb{N}$ such that

$$\max_{\xi \in \mathbb{T}} \left| \gamma u_{j_0}(\xi) - \frac{P(B^{\mathbf{a}}, M)(\xi)}{P(B^{\mathbf{b}}, N)(\xi)} \right| < \frac{\epsilon}{2}.$$

The above facts imply that

$$\max_{\xi\in\mathbb{T}}\left|\gamma u(\xi)-\frac{P(B^{\mathbf{a}},M)(\xi)}{P(B^{\mathbf{b}},N)(\xi)}\right|<\epsilon.$$

Therefore, $(\mathbf{a},\mathbf{b})\in\mathbb{U}$ and the proof is complete. ■

Lemma 4.2: *For each $j\in\mathbb{N}$, $E(j)$ is an open set of the space $\mathbb{D}^{\mathbb{N}}\times\mathbb{D}^{\mathbb{N}}$.*

Proof: Let $(\mathbf{a},\mathbf{b})\in E(j)$, where $\mathbf{a}=(a_1,a_2,\ldots)$ and $\mathbf{b}=(b_1,b_2,\ldots)$. Then for any $\epsilon>0$, there exist $\gamma\in\mathbb{T}$ and $M,N\in\mathbb{N}$ such that

$$\max_{\xi\in\mathbb{T}}\left|\gamma u_j(\xi)-\frac{P(B^{\mathbf{a}},M)(\xi)}{P(B^{\mathbf{b}},N)(\xi)}\right|<\epsilon. \tag{15}$$

Set

$$\epsilon'=\epsilon-\max_{\xi\in\mathbb{T}}\left|\gamma u_j(\xi)-\frac{P(B^{\mathbf{a}},M)(\xi)}{P(B^{\mathbf{b}},N)(\xi)}\right|.$$

By similar technique as in the proof of the Blaschke product approximation theorem (Theorem 4.4.1 in [11]), we can find $\delta>0$ such that

$$U(a_i,\delta)\subset\mathbb{D},\ U(b_j,\delta)\subset\mathbb{D},\quad i=1,2,\ldots,M \text{ and } j=1,2,\ldots,N,$$

and for

$$c_i\in U(a_i,\delta),\ d_j\in U(b_j,\delta),\quad i=1,2,\ldots,M \text{ and } j=1,2,\ldots,N,$$

we have

$$\max_{z\in\overline{\mathbb{D}}}\left|P(B^{\mathbf{a}},M)(z)-\Pi_{i=1}^{M}\frac{z-c_i}{1-\overline{c_i}z}\right|<\frac{\epsilon'}{2},\quad \max_{z\in\overline{\mathbb{D}}}\left|P(B^{\mathbf{b}},N)(z)-\Pi_{j=1}^{N}\frac{z-d_j}{1-\overline{d_j}z}\right|<\frac{\epsilon'}{2}, \tag{16}$$

where $U(a_i,\delta)$ (resp. $U(b_j,\delta)$) denotes the open disk centered at a_i (resp. b_j) with radius δ. We define

$$V(\mathbf{a},\mathbf{b},M,N,\delta)=\left(\prod_{i=1}^{M}U(a_i,\delta)\prod_{i>M}X_i,\prod_{j=1}^{N}U(b_j,\delta)\prod_{j>N}X_j\right),$$

where $X_i=X_j=\mathbb{D}$ for $i>M, j>N$. Then $V(\mathbf{a},\mathbf{b},M,N,\delta)$ is a neighborhood of $(\mathbf{a},\mathbf{b})$.

It comes to prove that $V(\mathbf{a},\mathbf{b},M,N,\delta)\subset E(j)$. For any $(\mathbf{p},\mathbf{q})\in V(\mathbf{a},\mathbf{b},M,N,\delta)$, where $\mathbf{p}=(p_1,p_2,\ldots)$ and $\mathbf{q}=(q_1,q_2,\ldots)$, we have

$$\max_{\xi\in\mathbb{T}}\left|\gamma u_j(\xi)-\frac{P(B^{\mathbf{p}},M)(\xi)}{P(B^{\mathbf{q}},N)(\xi)}\right|\le\max_{\xi\in\mathbb{T}}\left|\gamma u_j(\xi)-\frac{P(B^{\mathbf{a}},M)(\xi)}{P(B^{\mathbf{b}},N)(\xi)}\right|+A, \tag{17}$$

where

$$A=\max_{\xi\in\mathbb{T}}\left|\frac{P(B^{\mathbf{a}},M)(\xi)}{P(B^{\mathbf{b}},N)(\xi)}-\frac{P(B^{\mathbf{p}},M)(\xi)}{P(B^{\mathbf{q}},N)(\xi)}\right|.$$

By calculation, we have

$$A=\max_{\xi\in\mathbb{T}}\left|P(B^{\mathbf{a}},M)(\xi)P(B^{\mathbf{q}},N)(\xi)-P(B^{\mathbf{p}},M)(\xi)P(B^{\mathbf{b}},N)(\xi)\right|$$

$$
\begin{aligned}
&\le \max_{\xi\in\mathbb{T}} \left|P(B^{\mathbf{a}},M)(\xi)P(B^{\mathbf{q}},N)(\xi) - P(B^{\mathbf{p}},M)(\xi)P(B^{\mathbf{q}},N)(\xi)\right| \\
&\quad + \max_{\xi\in\mathbb{T}} \left|P(B^{\mathbf{p}},M)(\xi)P(B^{\mathbf{q}},N)(\xi) - P(B^{\mathbf{p}},M)(\xi)P(B^{\mathbf{b}},N)(\xi)\right| \\
&= \max_{\xi\in\mathbb{T}} \left|P(B^{\mathbf{a}},M)(\xi) - P(B^{\mathbf{p}},M)(\xi)\right| + \max_{\xi\in\mathbb{T}} \left|P(B^{\mathbf{q}},N)(\xi) - P(B^{\mathbf{b}},N)(\xi)\right| \\
&= \max_{\xi\in\mathbb{T}} \left|P(B^{\mathbf{a}},M)(\xi) - \Pi_{i=1}^{M}\frac{\xi-p_i}{1-\overline{p_i}\xi}\right| \\
&\quad + \max_{\xi\in\mathbb{T}} \left|P(B^{\mathbf{b}},N)(\xi) - \Pi_{j=1}^{N}\frac{\xi-q_j}{1-\overline{q_j}\xi}\right| < \epsilon' \text{(by using (16))}.
\end{aligned}
$$

The last inequality together with (15) and (17) yields that

$$\max_{\xi\in\mathbb{T}} \left|\gamma u_j(\xi) - \frac{P(B^{\mathbf{p}},M)(\xi)}{P(B^{q},N)(\xi)}\right| < \epsilon,$$

and thus $(\mathbf{p},\mathbf{q}) \in E(j)$. The proof is complete. ■

Lemma 4.3: *For each $j \in \mathbb{N}$, $E(j)$ is dense in the space $\mathbb{D}^{\mathbb{N}} \times \mathbb{D}^{\mathbb{N}}$.*

Proof: For any $(\mathbf{c},\mathbf{d}) \in \mathbb{D}^{\mathbb{N}} \times \mathbb{D}^{\mathbb{N}}$ and any neighborhood V of $(\mathbf{c},\mathbf{d})$, there exist open sets U_1, U_2 of $\mathbb{D}^{\mathbb{N}}$ such that

$$\mathbf{c} \in U_1,\ \mathbf{d} \in U_2,\ (\mathbf{c},\mathbf{d}) \in U_1 \times U_2 \subset V.$$

By the definition of product topology, there exist n positive integers $1 \le i_1 < i_2 < \cdots < i_n$, and non-empty open sets $V_{i_1}, V_{i_2}, \ldots, V_{i_n}$ of $\mathbb{D}$, such that

$$\mathbf{c} \in O^{\{i_1,i_2,\ldots,i_n\}}_{V_{i_1},V_{i_2},\ldots,V_{i_n}} =: \prod_{i=1}^{\infty} V_i \subset U_1,$$

where $V_i = V_{i_k}$ if $i = i_k \in \{i_1, i_2, \ldots, i_n\}$, and $V_i = \mathbb{D}$ if $i \notin \{i_1, i_2, \ldots, i_n\}$.

Similarly, there exists $O^{\{j_1,j_2,\ldots,j_n\}}_{V_{j_1},V_{j_2},\ldots,V_{j_n}}$ such that

$$\mathbf{d} \in O^{\{j_1,j_2,\ldots,j_n\}}_{V_{j_1},V_{j_2},\ldots,V_{j_n}} \subset U_2.$$

Let $\mathbf{c} = (c_1, c_2, \ldots)$ and $\mathbf{d} = (d_1, d_2, \ldots)$. We consider the finite Blaschke products

$$B_1(z) = \prod_{k=1}^{m} \frac{z - c_{i_k}}{1 - \overline{c_{i_k}}z},\ B_2(z) = \prod_{k=1}^{n} \frac{z - d_{j_k}}{1 - \overline{d_{j_k}}z}.$$

Choose $(\mathbf{a},\mathbf{b}) \in \mathbb{U}$. For any $\epsilon > 0$, there exist $\gamma \in \mathbb{T}$ and $M, N \in \mathbb{N}$ such that

$$\max_{\xi\in\mathbb{T}} \left|\gamma u_j(\xi)\frac{B_2(\xi)}{B_1(\xi)} - \frac{P(B^{\mathbf{a}},M)(\xi)}{P(B^{\mathbf{b}},N)(\xi)}\right| < \epsilon. \tag{18}$$

Without loss of generality, we may assume that $M \ge i_m$ and $N \ge j_n$.

Because $\mathbb{N}-\{i_1,i_2,\ldots,i_m\}$ and $\mathbb{N}-\{j_1,j_2,\ldots,j_n\}$ are countable, we can list the elements of the two sets as follows:

$$\mathbb{N}-\{i_1,i_2,\ldots,i_m\}=\{\mu_l\}_{l=1}^{\infty},\quad \mu_1<\mu_2<\cdots,$$
$$\mathbb{N}-\{j_1,j_2,\ldots,j_n\}=\{\nu_l\}_{l=1}^{\infty},\ \nu_1<\nu_2<\cdots.$$

We let

$$\mathbf{p}=(p_1,p_2,\ldots),\ \mathbf{q}=(q_1,q_2,\ldots),$$

where

$$p_i=\begin{cases}c_{i_k}, & i=i_k\in\{i_1,i_2,\ldots,i_m\}\\ a_l, & i=\mu_l\in\mathbb{N}-\{i_1,i_2,\ldots,i_m\}\end{cases}$$
$$q_j=\begin{cases}d_{j_k}, & j=j_k\in\{j_1,j_2,\ldots,j_n\}\\ b_l, & j=\nu_l\in\mathbb{N}-\{j_1,j_2,\ldots,j_n\}.\end{cases}$$

It is obvious that $\mathbf{p}\in O^{\{i_1,i_2,\ldots,i_m\}}_{V_{i_1},V_{i_2},\ldots,V_{i_m}}$ and $\mathbf{q}\in O^{\{j_1,j_2,\ldots,j_n\}}_{V_{j_1},V_{j_2},\ldots,V_{j_n}}$, which implies that $(\mathbf{p},\mathbf{q})\in U_1\times U_2\subset V$.

On the other hand, we have

$$P(B^{\mathbf{a}},M)(\xi)B_1(\xi)=P(B^{\mathbf{p}},M+i_m)(\xi),\quad P(B^{\mathbf{b}},N)(\xi)B_2(\xi)=P(B^{\mathbf{q}},N+j_n)(\xi).$$

The above facts together with (18) show that

$$\max_{\xi\in\mathbb{T}}\left|\gamma u_j(\xi)-\frac{P(B^{\mathbf{p}},M+i_m)(\xi)}{P(B^{\mathbf{q}},N+j_n)(\xi)}\right|<\epsilon,$$

and thus $(\mathbf{p},\mathbf{q})\in E(j)$. Note that $(\mathbf{p},\mathbf{q})\in U_1\times U_2\subset V$. Therefore, $(\mathbf{p},\mathbf{q})\in E(j)\bigcap V$. The proof is complete. ■

Proof of Theorem 4.1: Lemmas 4.1, 4.2 and 4.3 show that $\mathbb{U}$ can be expressed as an intersection of countably many dense open subsets of $\mathbb{D}^{\mathbb{N}}\times\mathbb{D}^{\mathbb{N}}$. So $\mathbb{U}$ is a G_δ set of $\mathbb{D}^{\mathbb{N}}\times\mathbb{D}^{\mathbb{N}}$. By Baire's theorem, $\mathbb{U}$ is dense in $\mathbb{D}^{\mathbb{N}}\times\mathbb{D}^{\mathbb{N}}$. ■

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