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Advances in Applied Clifford Algebras



Hilbert Boundary Value Problems for Monogenic Functions on the Hyperplane

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The paper is dedicated to the memory of Professor Frank Sommen.

Abstract. This paper systematically studies Hilbert boundary value problems for monogenic functions on the hyperplane for the solutions being of any integer orders at infinity, where the negative order cases are new even when restricted to the complex plane context. The explicit solution formulas are provided and the solvability conditions are specified. The results are proved using the Clifford symmetric extension method, which reduces Hilbert boundary value problems to Riemann boundary value problems, involving many innovative geometric techniques.

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1. Introduction

As is well known, boundary value problems (BVPs) for analytic functions in the classical complex analysis form an important branch of mathematical analysis. Due to their theoretical elegance and numerous applications in physics and other subjects such as elasticity theory, fracture mechanics, hydromechanics, etc., they have been thoroughly studied over a long period. The theory of boundary value problems for analytic functions has been investigated systematically in the literature [14,20,24], and applications have been presented comprehensively in monographs [21,25].

It has been proved that the function theory over Clifford algebra is an appropriate setting to generalize many aspects of the function theory of one complex variable to higher dimensions [3,5,10–12,16,17]. It is natural

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that mathematicians hope to develop theories on boundary value problems for monogenic functions, also called regular functions simply, in the hypercomplex analysis. These theories would be analogous to those for analytic functions in the classical complex analysis. In fact, some results on boundary value problems for analytic functions in the classical complex analysis have been generalized to regular functions in Clifford analysis (e.g., [1,18], some articles listed in [8]).

In [4,22], we discussed Riemann boundary value problems on closed smooth surfaces and on the hyperplanes in detail. In this paper, we consider Hilbert boundary value problems (Hilbert BVPs) on half-hyperplanes, such as the Poincaré upper half space. Hilbert BVPs are very important in both pure mathematics and engineering practice. The discussion is rather difficult and complex. Even in the classical complex analysis, Hilbert BVPs on the real axis have not been completely explored [20,24]. The researchers have been restricted to the case of bounded growth at infinity, while there are some obstacles for general growth orders at infinity . In [27], Xu and Zhou tried to solve Hilbert BVPs on the hyperplane. They generalized the classical results for Hilbert BVPs on the real axis to the hypercomplex analysis setting under the condition that the solutions are bounded at infinity. In [15], Gong and Du continued to discuss Hilbert BVPs under the condition that the solutions have finite non-negative orders at infinity by the symmetric extension method. The arguments of [15,27] contain some mistakes that will be pointed out below. To the authors' knowledge, Hilbert BVPs with negative order at infinity have not been discussed in any paper, because the corresponding Riemann BVPs were not discussed before [4], so Hilbert BVPs with negative order at infinity are always an open problem.

In this paper, we will systematically discuss the Hilbert BVPs on the hyperplane for regular functions in Clifford analysis, including cases of negative orders at infinity. We will use the so-called symmetric extension method to solve the Hilbert BVPs. The paper is organized as follows. In Sect. 2 we review some of the necessary preparations in Clifford analysis. In Sect. 3, we formulate the Hilbert BVPs in the Poincaré upper half space. In Sect. 4, we introduce the symmetric extension of a regular function in the Poincaré upper half space and discuss its regularity. Then the Hilbert BVPs are solved by converting them equivalently to the Riemann BVPs discussed in [4]. In Sect. 5, the solution formulas and the conditions of the Schwarz BVPs and the Hilbert BVPs for all orders at infinity of the solutions are obtained. These results extend both the classical ones in complex analysis [14,20,24] and those in the Clifford analysis setting [15,27].

2. Hypercomplex Functions

We begin by recalling the necessary preliminary knowledge in Clifford algebra and Clifford analysis [3,5], which are used throughout this paper.

2.1. Clifford Analysis

Let $C(V_n)$ be a 2^n-dimensional real linear space. To expediently introduce the product on it, we write its basis as $\{e_A, A = (h_1, \ldots, h_r) \in \mathcal{P}N, 1 \leq h_1 < \cdots < h_r \leq n\}$, where $N = \{1, \ldots, n\}$ and $\mathcal{P}N$ denotes the family of all the ordered subsets of N in the above fixed way. Sometimes, $e_\emptyset$ is written as e_0 and e_A as $e_{h_1 \cdots h_r}$ for $A = \{h_1, \ldots, h_r\} \in \mathcal{P}N$. The product on $C(V_n)$ is defined by

$$\begin{cases} e_A e_B = (-1)^{\#(A\cap B)}(-1)^{P(A,B)} e_{A\mathbf{\Delta}B}, & \text{if } A, B \in \mathcal{P}N, \\ \lambda\mu = \sum_{A\in\mathcal{P}N}\sum_{B\in\mathcal{P}N} \lambda_A\mu_B e_A e_B, & \text{if } \lambda = \sum_{A\in\mathcal{P}N} \lambda_A e_A,\ \ \mu = \sum_{A\in\mathcal{P}N} \mu_A e_A, \end{cases} \tag{2.1}$$

where the notation $\#(A)$ denotes the number of the elements in A and $P(A, B) = \sum_{j\in B} P(A, j)$ with $P(A, j) = \#\{i, i \in A, i > j\}$, the symmetric difference set $A\mathbf{\Delta}B$ is also an ordered one in the above way, and $\lambda_A \in \mathbb{R}$ is the coefficient of the e_A-component of the Clifford number λ. It follows at once from the multiplication rule (2.1) that e_0 is the identity element written as 1 and in particular,

$$\begin{cases} e_i^2 = -1, & \text{if } i = 1, \ldots, n, \\ e_i e_j = -e_j e_i, & \text{if } 1 \leq i < j \leq n, \\ e_{h_1} e_{h_2} \cdots e_{h_r} = e_{h_1 h_2 \cdots h_r}, & \text{if } 1 \leq h_1 < h_2 < \cdots < h_r \leq n. \end{cases} \tag{2.2}$$

It is clear that $C(V_n)$ is a real linear and associative, non-commutative algebra by algebraically spanning the linear subspace $V_n = \text{span}\{e_1, e_2, \ldots, e_n\}$. It is called the Clifford algebra over V_n. The elements $\lambda = \lambda_0 + \lambda_1 e_1 + \cdots + \lambda_n e_n$ for $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ are called paravectors.

We frequently use the following defined involution:

$$\begin{cases} \overline{e_A} = (-1)^{\frac{\#(A)(\#(A)+1)}{2}} e_A, & \text{if } A \in \mathcal{P}N, \\ \overline{\lambda} = \sum_{A\in\mathcal{P}N} \lambda_A \overline{e_A}, & \text{if } \lambda = \sum_{A\in\mathcal{P}N} \lambda_A e_A. \end{cases} \tag{2.3}$$

In the sequel, λ_A is also written as $[\lambda]_A$. In particular, the coefficient $\lambda_\emptyset$ is denoted by λ_0 or $[\lambda]_0$, which is called the scalar part of the Clifford number λ. An inner product $(\cdot, \cdot)$ on $C(V_n)$ is defined by putting for any λ and μ in $C(V_n)$

$$(\lambda, \mu) = \left[\lambda\overline{\mu}\right]_0 = \sum_A \lambda_A \mu_A, \tag{2.4}$$

where $\lambda = \sum_A \lambda_A e_A$, $\mu = \sum_A \mu_A e_A$ and the symbol $\sum_A$ is an abbreviation of $\sum_{A\in\mathcal{P}N}$.

Thus, the corresponding norm on $C(V_n)$ reads,

$$|\lambda| = \sqrt{(\lambda, \lambda)} = \left[\sum_A \lambda_A^2\right]^{\frac{1}{2}}. \tag{2.5}$$

In such way, $C(V_n)$ is a real Hilbert space and at the same time it is a Banach algebra with the equivalent norm

$$|\lambda|_0 = 2^{\frac{n}{2}} |\lambda|, \tag{2.6}$$

that is

$$|\lambda\mu|_0 \leq |\lambda|_0 |\mu|_0, \quad |\lambda\mu| \leq 2^{\frac{n+1}{2}} |\lambda||\mu|. \tag{2.7}$$

In particular, if λ is a paravector and $\mu \in C(V_n)$, then [3]

$$\big|\lambda\mu\big| = \big|\mu\lambda\big| = \big|\lambda\big|\big|\mu\big|. \tag{2.8}$$

Let Ω be a non-empty subset of $\mathbb{R}^{n+1}$. Hypercomplex functions f defined in Ω and with values in $C\left(V_n\right)$ will be considered, *i.e.*, $f : \Omega \longrightarrow C(V_n)$. They are of the form

$$f(w) = \sum_A f_A(w) e_A, \quad w = (w_0, w_1, \ldots, w_n) \in \Omega \subset \mathbb{R}^{n+1}, \tag{2.9}$$

where the $f_A(w)$ is the e_A-component of $f(w)$. Obviously, the f_A's are real-valued functions in Ω, which are called the e_A-component functions of f. Whenever a property such as differentiability and continuity is ascribed to f, it is clear that in fact all the component functions f_A possess the cited property.

So the meaning of $f \in C^{(r)}(\Omega, C\left(V_n\right))$ is very clear.

Obviously, $C(V_{n-1})$ is a subalgebra of $C(V_n)$ where $V_{n-1} = \text{span}\{e_1, e_2, \ldots, e_{n-1}\}$. Then, $\lambda \in C(V_n)$ has the unique decomposition [15,27]

$$\lambda = x + e_n\, y^l \quad \textit{where } x, y^l \in C(V_{n-1}), \tag{2.10}$$

i.e.,

$$C\left(V_n\right) = C\left(V_{n-1}\right) \oplus e_n\, C\left(V_{n-1}\right). \tag{2.11}$$

We define

$$\text{Re}(\lambda) = x, \quad \text{Im}^l\left(\lambda\right) = y^l. \tag{2.12}$$

Remark 2.1. Similarly, $\lambda \in C(V_n)$ has also the unique decomposition

$$\lambda = x + y^r\, e_n \quad \text{where } x, y^r \in C(V_{n-1}). \tag{2.13}$$

We also define

$$\text{Re}(\lambda) = x, \quad \text{Im}^r\left(\lambda\right) = y^r. \tag{2.14}$$

For clarity, we call, respectively, y^l and y^r in (2.10) and (2.13), the left and right imaginary part of λ. From (2.10) and (2.13), we have

$$e_n\, y^l = y^r\, e_n, \quad i.e., \quad e_n\, \text{Im}^l(\lambda) = \text{Im}^r(\lambda)\, e_n. \tag{2.15}$$

It is clear that the decompositions (2.10) and (2.13) are generalizations of the representation of the classical complex numbers. In other words, (2.12) and (2.14) are the generalizations of operators Re and Im acting on the complex numbers. From (2.11) and (2.13) we obviously have

$$\big|\lambda\big|^2 = \big|x\big|^2 + \big|y^r\big|^2 = \big|x\big|^2 + \big|y^l\big|^2. \tag{2.16}$$

For a hypercomplex function f given by (2.9), we call, respectively,

$$\left(\text{Re}f\right)(w) = \text{Re}(f(w)), \quad \left(\text{Im}^r f\right)(w) = \text{Im}^r(f(w)), \quad \left(\text{Im}^l f\right)(w) = \text{Im}^l(f(w)), \tag{2.17}$$

the real part of f, the right and left imaginary part of f. In particular, if

$$\left(\text{Im}f\right)^r = \left(\text{Im}^l f\right) = 0, \tag{2.18}$$

i.e., $f:\Omega\longrightarrow C(V_{n-1})$, then we say it be a $C(V_{n-1})$-valued function, briefly, a para real-valued function which mimics the case of the real-valued function in the classical complex analysis.

Sometimes, for clarity, we write

$$\lambda=\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}}\lambda_A e_A\quad \textit{when } \lambda\in C\big(V_{n-1}\big),\tag{2.19}$$

where $\mathcal{P}\{1,\ldots,n-1\}$ denotes the family of all ordered subsets of $\{1,\ldots,n-1\}$ in the similar way used in $\mathcal{P}N$. Thus, a hypercomplex function f given in (2.9) may be re-written as

$$\begin{aligned}f(w)&=\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}}u_A(w)\,e_A+\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}}v_A^l(w)e_Ae_n\\&=\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}}u_A(w)e_A+\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}}v_A^r(w)e_n\,e_A,\end{aligned}\tag{2.20}$$

where u_A, v_A^l and v_A^r are the real-valued functions, which are called respectively the component functions on $C\big(V_{n-1}\big)$, $e_n\,C\big(V_{n-1}\big)$ and $C\big(V_{n-1}\big)e_n$. Obviously,

$$\big(\mathrm{Re}f\big)(w)\triangleq u(w)=\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}}u_A(w)\,e_A,\quad x\in\Omega,\tag{2.21}$$

and

$$\begin{aligned}\big(\mathrm{Im}f\big)^l(w)&\triangleq v^l(w)=\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}}v_A^l(w)\,e_A,\quad w\in\Omega,\\\big(\mathrm{Im}f\big)^r(w)&\triangleq v^r(w)=\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}}v_A^r(w)\,e_A,\quad w\in\Omega.\end{aligned}\tag{2.22}$$

Obviously,

$$v_A^r=(-1)^{\#(A)}v_A^l\quad\text{and}\quad e_n\,\big(\mathrm{Im}f\big)^l(w)=\big(\mathrm{Im}f\big)^r(w)\,e_n.\tag{2.23}$$

When $w=(w_0,w_1,\ldots,w_n)\in\mathbb{R}^{n+1}$, we introduce the mapping

$$\text{capital: }w\longmapsto W=\sum_{i=0}^{n}w_ie_i,\tag{2.24}$$

which is a proper isomorphism between $\mathbb{R}^{n+1}$ and the linear subspace $\mathrm{span}\{e_0,e_1,\ldots,e_n\}$ of $C(V_n)$. In the sequel, we simply treat the capital W as w.

This is Vahlen's choice [2,26].

Thus, we have

$$\mathrm{Re}\,(w)=w_0+w_1e_1+\cdots+w_{n-1}e_{n-1}\quad \textit{while}\quad w=(w_0,w_1,\ldots,w_n)\in\mathbb{R}^{n+1},\tag{2.25}$$

and

$$\mathrm{Im}^l(w)=\mathrm{Im}^r(w)=w_n\triangleq\mathrm{Im}(w)\quad \textit{while}\quad w=(w_0,w_1,\ldots,w_n)\in\mathbb{R}^{n+1}.\tag{2.26}$$

In the following, if

$$\mathrm{Im}^l(\lambda) = \mathrm{Im}^r(\lambda), \tag{2.27}$$

we will write both of them, without confusion, just as $\mathrm{Im}(\lambda)$.

We define

$$\mathbb{R}^{n+1}_+ = \{w,\, w \in \mathbb{R}^{n+1},\, \mathrm{Im}(w) > 0\}, \quad \mathbb{R}^{n+1}_- = \{w,\, w \in \mathbb{R}^{n+1},\, \mathrm{Im}(w) < 0\}, \tag{2.28}$$

which are called the Poincaré upper half space and the Poincaré lower half space, respectively, while the hyperplane

$$\mathbb{R}^{n+1}_0 = \{w,\, \mathrm{Im}(w) = 0\} \tag{2.29}$$

is called the para-real plane in $\mathbb{R}^{n+1}$, and the element $w \in \mathbb{R}^{n+1}_0$ is called a para-real (hypercomplex) number.

It is noted that the paravectors given in $\mathbb{R}^{n+1}_0$ play a treble role as elements of $\mathbb{R}^n$ and $\mathbb{R}^{n+1}_0$ as well as $C(V_{n-1}) \subset C(V_n)$.

2.2. Regular Functions

Let Ω be a domain of $\mathbb{R}^{n+1}$. Introduce the following Dirac operator

$$D = \sum_{k=0}^{n} e_k \frac{\partial}{\partial x_k} :\ C^{(r)}(\Omega, C(V_n)) \longrightarrow C^{(r-1)}(\Omega, C(V_n)), \tag{2.30}$$

whose actions on functions from the left and from the right are governed by the rules

$$D[f] = \sum_{k=0}^{n} \sum_{A} e_k e_A \frac{\partial f_A}{\partial x_k}, \quad [f]D = \sum_{k=0}^{n} \sum_{A} e_A e_k \frac{\partial f_A}{\partial x_k}. \tag{2.31}$$

Definition 2.1. We say that a function $f \in C^{(r)}(\Omega, C(V_n))$ $(r \geq 1)$ is left (*right*) regular or monogenic in Ω if $D[f]=0$ ($[f]D=0$) in Ω. f is said to be biregular in Ω if it is both left and right regular.

Remark 2.2. Generally speaking, if

$$f(w) = \sum_{j=1}^{m} f_j(w)\lambda_j \quad \Big(f_j \in C(\Omega, \mathbb{R}) \ \textit{and}\ \lambda_j \in C(V_n)\Big) \tag{2.32}$$

then

$$D[f] = \sum_{k=0}^{n} \sum_{j=1}^{m} e_k \lambda_j \frac{\partial f_j}{\partial w_k}, \quad [f]D = \sum_{k=0}^{n} \sum_{j=1}^{m} \lambda_j e_k \frac{\partial f_j}{\partial w_k}. \tag{2.33}$$

Example 2.1. Let

$$E(w) = \frac{\overline{w}}{|w|^{n+1}} = \frac{w^{-1}}{|w|^{n-1}}, \quad w \in \mathbb{R}^{n+1} \setminus \{0\}. \tag{2.34}$$

Then E is biregular [3,9]. The function E is called the Cauchy kernel function.

Example 2.2. The hypercomplex variables

$$z_j = z_j(w) = w_j e_0 - w_0 e_j, \quad j = 1, \ldots, n \tag{2.35}$$

are biregular [3,6,7,28,29].

Example 2.3. Let $(\ell_1,\ldots,\ell_k)\in N^k$, i.e., ℓ_j's are k elements from N, where repetitions are allowed. We put

$$V_{(\ell_1,\ldots,\ell_k)}(w)=\sum_{\pi(\ell_1,\ldots,\ell_k)} z_{\ell_1}(w)\cdots z_{\ell_k}(w),\quad w\in\mathbb{R}^{n+1}, \tag{2.36}$$

where the sum runs over all the permutations of $(\ell_1,\ldots,\ell_k)$, which is also the so-called hypercomplex symmetric power. The hypercomplex symmetric power $V_{(\ell_1,\ldots,\ell_k)}$ is sometimes also called the Fueter polynomial, being biregular [3,13,19,23].

Example 2.4. All derivatives of E

$$W_{\{\ell_1,\ell_2,\cdots\ell_k\}}(w)=(-1)^k\frac{\partial^k E}{\partial w_{\ell_1}\partial w_{\ell_2}\cdots\partial w_{\ell_k}}(w),\quad w\in\mathbb{R}^{n+1}\setminus\{0\} \tag{2.37}$$

are biregular [3,6,7,28,29], being sometimes called negative powers [19], and

$$W_{\{\ell_1,\ell_2,\cdots\ell_k\}}(w)=O\Big(|w|^{-(n+k)}\Big)\quad near\ \infty. \tag{2.38}$$

3. Formulation of the Hilbert BVPs on $\mathbb{R}^{n+1}_+$

To suitably present and formulate Hilbert BVPs on the Poincaré upper half space $\mathbb{R}^{n+1}_+$, we must introduce a suitable statement for the growth condition at the infinity for the regular functions on the Poincaré upper half space $\mathbb{R}^{n+1}_+$.

3.1. Order at Infinity

Assume F is a regular function on the Poincaré hyperplane $\mathbb{R}^{n+1}_+$, denoted as $F\in\mathcal{M}\big(\mathbb{R}^{n+1}_+\big)$. We sometimes need the concept of the order of F at infinity.

Definition 3.1. Let $F\in\mathcal{M}\left(\mathbb{R}^{n+1}_+\right)$, m be an integer. If

$$0<\beta=\limsup_{w\in\mathbb{R}^{n+1}_+,\,w\to\infty}\left|w^{-m}F(w)\right|<+\infty, \tag{3.1}$$

then F is said to be of order m at $w=\infty$, denoted it by $\mathrm{Ord}\,(F,\infty)=m$.

Sometimes it is more convenient to write

$$|F(w)|\overset{sup}{\approx}|G(w)|\quad\text{near }\infty\quad\text{when } a\le\limsup_{w\to\infty}\frac{|F(w)|}{|G(w)|}\le A, \tag{3.2}$$

where F and G ($|G(w)|>0$) are two hypercomplex functions defined near ∞, $A>a>0$ are two constants.

The following lemma is an obvious fact.

Lemma 3.1. $\left|F(w)\right| \overset{sup}{\approx} \left|w^k\right|$ *near* $w=\infty$ *if and only if* $\mathrm{Ord}\,(F,\infty)=k$.

For boundary behavior of function $\Phi \in \mathcal{M}\left(\mathbb{R}^{n+1}_{+}\right)$ at the infinity, there are commonly three types of formulations:

$$(A) \quad \lim_{w\to\infty, w\in\mathbb{R}^{n+1}_{+}} w^{-(m+1)}\Phi(w)=0, \ \ namely \ \ \Phi(w)=o\left(w^{m+1}\right) \ \ near \ w=\infty, \tag{3.3}$$

$$(B) \quad \limsup_{w\in\mathbb{R}^{n+1}_{+},\, w\to\infty} |w|^{-m}|\Phi(w)|=\beta, \ \ namely \ \ |\Phi(w)|=O\left(w^m\right) \ \ near \ \ w=\infty, \tag{3.4}$$

$$(C) \quad \mathrm{Ord}\,(\Phi,\infty)\le m, \ \ namely \quad |\Phi(w)| \overset{sup}{\approx} \left|w^k\right| \ \text{near} \ \ w=\infty \ \ when \ \ k\le m. \tag{3.5}$$

Remark 3.1. Obviously, (C) implies (B), while (B) implies (A). So, the condition (A) is the weakest. In [4], we used the condition (A) in Riemann boundary value problems with the hyperplane as jump surface, which was an innovation. In the present paper, we will still use it in our Hilbert boundary value problems on the Poincaré upper half space.

Some symbols will be used in the following [29]. Let

$$Z(w)=\left(z_1(w),z_2(w),\ldots,z_n(w)\right) \quad \text{and} \quad \alpha=\left[\alpha_1,\alpha_2,\ldots,\alpha_n\right], \tag{3.6}$$

where z_j's are the hypercomplex variables given in Example 2.2 and α_j's are nonnegative integers. Then the symmetry power Z^α in $\mathbb{R}^{n+1}$ is defined as the sum of all possible z_i products of which each contains z_i factor exactly α_i times. For example, for $n=2$,

$$(z_1,z_2)^{[0,0]}=1, \quad (z_1,z_2)^{[1,1]}=z_1z_2+z_2z_1, \quad (z_1,z_2)^{[2,0]}=z_1^2. \tag{3.7}$$

Introduce the mapping

$$\boldsymbol{\alpha}:\left(\ell_1,\ell_2,\ldots,\ell_k\right)\mapsto\alpha=\left[\alpha_1,\ldots\alpha_n\right], \tag{3.8}$$

where α_j is the number of times that j appears in $\left(\ell_1,\ell_2,\ldots,\ell_k\right)\in N^k$ and $N=\{1,\ldots,n\}$. Now we may rewrite $W_{\{\ell_1,\ell_2,\ldots,\ell_k\}}$ in Example 2.4 by

$$\begin{aligned}(-1)^{|\alpha|}W_{\{\ell_1,\ell_2,\ldots,\ell_k\}}(w)&=\frac{\partial^{|\alpha|}E}{\partial w_0^{\alpha_0}\cdots\partial w_n^{\alpha_n}}(w)\equiv\frac{\partial^{|\alpha|}E}{\partial w^\alpha}(w)\\ &\equiv\left(\partial^\alpha[E]\right)(w), \quad w\in\mathbb{R}^{n+1}\backslash\{0\}.\end{aligned} \tag{3.9}$$

So, $w^{n+|\alpha|}\left(\partial^\alpha[E]\right)(w)$ is bounded according to Example 2.4.

Similarly,

$$Z^\alpha(w)=V_{(\ell_1,\ldots,\ell_k)}(w)=\sum_{\pi(\ell_1,\ldots,\ell_k)} z_{\ell_1}\cdots z_{\ell_k}(w) \quad \text{with } k=|\alpha|, \tag{3.10}$$

where $V_{(\ell_1,\ldots,\ell_k)}$ is the Fueter polynomial given in Example 2.3. So Z^α is a biregular function [3].

Definition 3.2. Let $m\ge 0$. We call

$$f(w)=\sum_{|\alpha|=0}^{m} Z^\alpha(w)\,c_\alpha \quad with \quad \sum_{|\alpha|=m}|c_\alpha|\neq 0 \tag{3.11}$$

a hypercomplex symmetric polynomial of degree m. In such case we denote $\mathrm{Deg}(f) = m$.

Lemma 3.2. (See [22]) *Let f be a hypercomplex symmetric polynomial. Then* $\mathrm{Deg}(f) = m$ *if and only if* $\mathrm{Ord}(f, \infty) = m$.

3.2. $\widehat{H}$ Class of Functions

In order to state the condition of the input function for Hilbert BVPs, we need to introduce some classes of hypercomplex functions used frequently in this paper.

Definition 3.3. Assume f is defined on $\Omega \subseteq \mathbb{R}^{n+1}$. If

$$\big|f(t) - f(s)\big| \leq M\big|t - s\big|^{\mu} \quad (0 < \mu \leq 1) \tag{3.12}$$

for arbitrary points t, s on Ω, where M and μ are constants, then f is said to satisfy Hölder condition of order μ, denoted by $f \in H^{\mu}(\Omega)$. The constants μ and M are called, respectively, the Hölder index and the Hölder coefficient of f. If the order μ is not emphasized, it may be denoted briefly as $f \in H(\Omega)$.

Sometimes we will use $\frac{1}{w}$ to represent w^{-1} for $w \in \mathbb{R}^{n+1}$, which suggests the similarity with some results in the classical complex analysis.

Definition 3.4. Assume f is defined on $\Omega \subseteq \mathbb{R}^{n+1}$. If

$$\left|f(\xi) - f(\zeta)\right| \leq M\left|\frac{1}{\xi} - \frac{1}{\zeta}\right|^{\mu} \quad (0 < \mu \leq 1) \tag{3.13}$$

for arbitrary points ξ, ζ on $\Omega \setminus \{0\}$, where M and μ are constants, then f is said to satisfy $H_{\dagger}$ condition of order μ, denoted by $f \in H_{\dagger}^{\mu}(\Omega)$. The constants μ and M are called, respectively, the †-Hölder index and †-Hölder coefficient of f. If the order μ is not emphasized, it may be denoted briefly as $f \in H_{\dagger}(\Omega)$.

Definition 3.5. If $f \in H^{\mu}(\Omega) \cap H_{\dagger}^{\mu}(\Omega)$, then f is said to satisfy the $\widehat{H}$ condition of order μ on Ω, denoted by $f \in \widehat{H}^{\mu}(\Omega)$ or briefly $f \in \widehat{H}(\Omega)$.

The conditions (3.12) and (3.13) are, respectively, called the Hölder condition and †-Hölder condition of the $\widehat{H}(\Omega)$ class function f. More discussions of the two classes of functions may be found in [4].

Definition 3.6. Let f be a function defined on Ω with ∞ as its cluster point. If

$$f(\infty) = \lim_{w \in \Omega,\, w \to \infty} f(w) \tag{3.14}$$

exists and

$$\Big|f(w) - f(\infty)\Big| \leq \frac{M}{|w|^{\mu}} \ \Big(0 < \mu \leq 1\Big), \quad w \in \Omega \backslash \{0\}, \tag{3.15}$$

where M is a constant, then we say that f satisfies the pointwise †-Hölder condition at infinity in Ω, denoted by $f \in H_{\dagger}^{\mu}(\Omega, \infty)$, or also briefly $f \in H_{\dagger}^{\mu}(\infty)$ and $f \in H_{\dagger}(\infty)$.

The following notations will be used. Let

$$f_{\mathbb{m}}(w) = w^m f(w). \tag{3.16}$$

(1) If $f_{\mathbb{m}} \in H^{\mu}\left(\mathbb{R}_0^{n+1}\right)$, then we write $f \in H_m^{\mu}\left(\mathbb{R}_0^{n+1}\right)$, or briefly $f \in H_m\left(\mathbb{R}_0^{n+1}\right)$.
(2) If $f_{\mathbb{m}} \in H_{\dagger}^{\mu}\left(\mathbb{R}_0^{n+1}\right)$, then we write $f \in H_{m,\dagger}^{\mu}\left(\mathbb{R}_0^{n+1}\right)$, or briefly $f \in H_{m,\dagger}\left(\mathbb{R}_0^{n+1}\right)$.
(3) If $f_{\mathbb{m}} \in H_{\dagger}^{\mu}\left(\mathbb{R}_0^{n+1}, \infty\right)$, then we write $f \in H_{m,\dagger}^{\mu}\left(\mathbb{R}_0^{n+1}, \infty\right)$, or briefly $f \in H_{m,\dagger}\left(\infty\right)$.

The following classes of functions will also be used in Hilbert BVPs, of which the details can be found in [4]:

$$\widehat{H}_m\left(\mathbb{R}_0^{n+1}\right) = H_m\left(\mathbb{R}_0^{n+1}\right) \bigcap H_{m,\dagger}\left(\mathbb{R}_0^{n+1}\right), \tag{3.17}$$

$$\widehat{H}_{m,0}\left(\mathbb{R}_0^{n+1}\right) = \widehat{H}_m\left(\mathbb{R}_0^{n+1}\right) \bigcap \left\{f, f_{\mathbb{m}}(\infty) = 0\right\}. \tag{3.18}$$

3.3. Formulation of Hilbert BVPs

Let Φ be a regular function defined on the Poincaré upper half space $\mathbb{R}_+^{n+1}$ which can extend continuously to the hyperplane $\mathbb{R}_0^{n+1}$. For clarity, we denote its boundary value by

$$\Phi^+(t) = \lim_{w \to t,\, w \in \mathbb{R}_+^{n+1}} \Phi(w), \quad t \in \mathbb{R}_0^{n+1}. \tag{3.19}$$

The Hilbert boundary value problem (or simply H_m problem). Find a left (right) regular function in $\mathbb{R}_+^{n+1}$ which can extend continuously to $\mathbb{R}_0^{n+1}$ such that

$$\begin{cases} \left(D[\Phi]\right)(w) = 0 & \text{for } w \in \mathbb{R}_+^{n+1} \quad (\textit{regularity}), \\ \mathrm{Re}\left\{\Phi^+(x)\,\lambda\right\} = c(x) & \text{for } x \in \mathbb{R}_0^{n+1} \quad (\textit{boundary condition}), \\ \Phi(w) = o\left(w^{m+1}\right) & \textit{near } \infty \quad (\textit{growth condition}), \end{cases} \tag{3.20}$$

where λ is a given constant whose inverse exists, c is a given para-real-valued function (the values of a para-real-valued function lie in the para-real plane) and

$$c \in \begin{cases} \widehat{H}\left(\mathbb{R}_0^{n+1}\right), & \text{when } m \geq 0 \ (\textit{non-negative order}), \\ \widehat{H}_{r,0}\left(\mathbb{R}_0^{n+1}\right), & \text{when } m < 0 \textit{ with } r = -(m+1). \end{cases} \tag{3.21}$$

Obviously, the above BVPs are the direct generalizations of the classical Hilbert BVPs [20,24] from complex plane to the space $C(V_n)$. In the classical complex analysis, λ is allowed to be a complex function when Φ is finite at infinity, *i.e.*, when $m = 0$.

A large amount of literature has discussed the solution methods of BVPs for some special case in Clifford analysis. One methodology is the so-called symmetric extension, where the main idea is to translate Hilbert BVPs (3.20) to the equivalent Riemann problems with some additional restriction condition. In the complex analysis, this approach has been quite successful [20,24], but in the higher dimensional spaces there exist many obstacles to implement

this approach. Not all BVPs can be solved by the symmetric extension, especially for the case when λ in (3.20) is a hypercomplex function. For the case when λ is a hypercomplex function, it can still be translated to a Riemann problem, but an open problem. This method was tried by Xu and Zhou in [27] with $m = 0$. Gong and Du solved the H_m problems with $m \geq 0$ in [15] by using the symmetric extension method that generalizes the results in [20,24] directly to the Clifford setting.

The simplest H_m problem is the Schwarz problem, *i.e.*, $\lambda = 1$. Let us start with some discussions on the Schwarz problem.

The Schwarz boundary value problem S_m. Find a left (right) regular function Φ in $\mathbb{R}^{n+1}_+$ that can extend continuously to $\mathbb{R}^{n+1}_0$such that

$$\begin{cases} \big(D[\Phi]\big)(w) = 0 \ \ for \ \ w \in \mathbb{R}^{n+1}_+ \ \ (regularity), \\ \mathrm{Re}\Big\{\Phi^+(x)\Big\} = c(x) \ \ for \ \ x \in \mathbb{R}^{n+1}_0 \ \ (boundary\ condition), \\ \Phi(w) = o\big(w^{m+1}\big) \ \ near \ \ \infty \ \ (growth\ condition), \end{cases} \tag{3.22}$$

where c is a given para real-valued function and satisfies (3.21).

Remark 3.2. In the following we generally assume that Φ is left regular.

4. Symmetric Extension and Self-Reflex Action

In order to solve the H_m problem, we try to transfer it into a jump problem R_m discussed in §6.3 of [4]. To do so, we introduce the symmetric extension and the self-reflex action.

4.1. Symmetric Extension

The reflection operator $* : C(V_n) \longrightarrow C(V_n)$ with respect to the hyperplane $\mathbb{R}^{n+1}_0$ is defined by

$$\lambda^* = \mathrm{Re}(\lambda) - e_n \,\mathrm{Im}^l(\lambda), \quad \lambda \in C(V_n), \tag{4.1}$$

or

$$\lambda^* = x - e_n\, y^l, \quad x, y^l \in C\big(V_{n-1}\big), \tag{4.2}$$

where

$$x = \mathrm{Re}(\lambda), \quad y^l = \mathrm{Im}^l(\lambda). \tag{4.3}$$

Obviously, by (2.15)

$$\lambda^* = x - y^r\, e_n, \quad \lambda \in C(V_n), \tag{4.4}$$

where

$$x = \mathrm{Re}(\lambda), \quad y^r = \mathrm{Im}^r(\lambda). \tag{4.5}$$

In particular,

$$w^* = \sum_{j=0}^{n-1} w_j e_j - w_n e_n \quad \text{when } w = \sum_{j=0}^{n} w_j e_j = \mathrm{Re}(w) + \mathrm{Im}(w)\, e_n \in \mathbb{R}^{n+1}. \tag{4.6}$$

We know that λ^* and λ are a pair of points symmetric to the hyperplane $\mathbb{R}_0^{n+1}$ and

$$\left(\lambda^*\right)^* = \lambda, \quad \lambda \in C(V_n). \tag{4.7}$$

Moreover, by (2.3) and (4.1),

$$\left(\overline{w}\right)^* = \overline{w^*}, \quad w \in \mathbb{R}^{n+1}. \tag{4.8}$$

For a function $\Phi(w)$ in $\mathbb{R}_+^{n+1}$, we define a function on $\mathbb{R}_-^{n+1}$ by

$$\Phi^*(w) = \left[\Phi\left(w^*\right)\right]^*, \quad w \in \mathbb{R}_-^{n+1}, \tag{4.9}$$

which is called its accompanying function. More specifically, if

$$\Phi(w) = \left(\mathrm{Re}\,\Phi\right)(w) + e_n\left(\mathrm{Im}^l\,\Phi\right)(w), \quad w \in \mathbb{R}_+^{n+1}, \tag{4.10}$$

with

$$\left(\mathrm{Re}\,\Phi\right)(w) = \sum_{A\in\mathcal{P}\{1,\ldots,n-1\}} u_A(w_0,\ldots,w_{n-1},w_n)e_A \tag{4.11}$$

and

$$\left(\mathrm{Im}^l\,\Phi\right)(w) = \sum_{A\in\mathcal{P}\{1,\ldots,n-1\}} v_A(w_0,\ldots,w_{n-1},w_n)e_A, \tag{4.12}$$

where u_A and v_A are para real-valued functions, then

$$\Phi^*(w) = \left(\mathrm{Re}\,\Phi^*\right)(w) + e_n\left(\mathrm{Im}^l\,\Phi^*\right)(w), \quad w \in \mathbb{R}_-^{n+1}, \tag{4.13}$$

where

$$\left(\mathrm{Re}\,\Phi^*\right)(w) = \sum_{A\in\mathcal{P}\{1,\ldots,n-1\}} u_A(w^*)e_A = \sum_{A\in\mathcal{P}\{1,\ldots,n-1\}} u_A(w_0,\ldots,w_{n-1},-w_n)e_A, \tag{4.14}$$

and

$$\left(\mathrm{Im}^l\,\Phi^*\right)(w) = -\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}} v_A(w^*)e_A = -\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}} v_A(w_0,\ldots,w_{n-1},-w_n)e_A. \tag{4.15}$$

Similarly, if the original function $\Phi(w)$ is defined in $\mathbb{R}_-^{n+1}$, then its accompanying function $\Phi^*(w)$ is given by

$$\Phi^*(w) = \left[\Phi\left(w^*\right)\right]^*, \quad w \in \mathbb{R}_+^{n+1}. \tag{4.16}$$

We easily see that

$$\left(\Phi^*\right)^*(w) = \Phi(w), \quad w \in \mathbb{R}_+^{n+1}\ \left(\mathbb{R}_-^{n+1}\right), \tag{4.17}$$

that is to say, if $\Phi(w)$ is acted by the reflection operator with respect to $\mathbb{R}_0^{n+1}$ twice, then it returns to $\Phi(w)$ itself, which means the reflection operator $*$ is idempotent.

Obviously, when (4.11) and (4.12) hold for $w \in \mathbb{R}_-^{n+1}$, then (4.14) and (4.15) also hold for $w \in \mathbb{R}_+^{n+1}$.

Remark 4.1. When the original function Φ is defined on $\mathbb{R}_+^{n+1}$ or $\mathbb{R}_-^{n+1}$, we have that

$$\begin{aligned}\Phi(w) \quad & \Big(w=(w_0,w_1,\ldots,w_n)\in\mathbb{R}_+^{n+1}\ \Big(\mathbb{R}_-^{n+1}\Big)\Big)\\ &=\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}} u_A(w_0,\ldots,w_{n-1},w_n)e_A\\ &\quad +e_n\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}} v_A(w_0,\ldots,w_{n-1},w_n)e_A\end{aligned}\tag{4.18}$$

is equivalent to

$$\begin{aligned}\Phi^*(w) \quad & \Big(w=(w_0,w_1,\ldots,w_n)\in\mathbb{R}_-^{n+1}\ \Big(\mathbb{R}_+^{n+1}\Big)\Big)\\ &=\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}} u_A(w^*)e_A-e_n\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}} v_A(w^*)e_A\\ &=\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}} u_A(w_0,\ldots,w_{n-1},-w_n)e_A\\ &\quad -\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}} v_A(w_0,\ldots,w_{n-1},-w_n)e_Ae_n.\end{aligned}\tag{4.19}$$

Lemma 4.1. [4] *Let $f(w)$ and $g(w)$ be hypercomplex functions defined on $\mathbb{R}_+^{n+1}$ $\Big(\mathbb{R}_-^{n+1}\Big)$, then*

$$\Big|f^*(w^*)\Big|=\Big|f(w)\Big|,\quad x\in\mathbb{R}_+^{n+1}\quad\Big(\mathbb{R}_-^{n+1}\Big),\tag{4.20}$$

and

$$\Big[f(w)g(w)\Big]^*=f^*(w^*)g^*(w^*),\quad w\in\mathbb{R}_+^{n+1}\quad\Big(\mathbb{R}_-^{n+1}\Big).\tag{4.21}$$

If the inverse function f^{-1} exists, then the inverse function of f^* also exists, and

$$\Big[f^*\Big]^{-1}=\Big[f^{-1}\Big]^*,\tag{4.22}$$

more precisely,

$$\Big[f^*\Big]^{-1}(s^*)=\Big[f^{-1}\Big]^*(s^*),\quad s\in f\Big(\mathbb{R}_+^{n+1}\Big)\quad\Big(f\Big(\mathbb{R}_-^{n+1}\Big)\Big).\tag{4.23}$$

Proof. (4.20) comes directly from (4.18) and (4.19) by (2.16). Just to be clear, in (2.20) we rewrite u_A and v_A as, respectively, $u_{f,A}$ and $v^l_{f,A}$. Then,

$$\begin{aligned}
&\Big[f(w)\,g(w)\Big]^* \quad \Big(say\ \ w\in\mathbb{R}^{n+1}_+\Big)\\
&= \Bigg[\sum_{A,B\in\mathcal{P}\{1,\dots,n-1\}} \Big\{u_{f,A}(w)e_A + e_n\, v^l_{f,A}(w)e_A\Big\}\Big\{u_{g,B}e_B + e_n\, v^l_{g,B}(w)e_B\Big\}\Bigg]^*\\
&= \sum_{A,B\in\mathcal{P}\{1,\dots,n-1\}} \Big[u_{f,A}(w)u_{g,B}(w) - (-1)^{\#(A)}v^l_{f,A}(w)v^l_{g,B}(w)\Big]\, e_Ae_B\\
&\quad - e_n \sum_{A,B\in\mathcal{P}\{1,\dots,n-1\}} \Big[v^l_{f,A}(w)u_{g,B}(w) - (-1)^{\#(A)}u_{f,A}(w)v^l_{g,B}(w)\Big]\, e_Ae_B.
\end{aligned} \tag{4.24}$$

On the other hand, by (4.11), (4.12), (4.14), (4.15) and (4.2), we have

$$\mathrm{Re}\Big[f^*g^*\Big](w^*) = \sum_{A,B\in\mathcal{P}\{1,\dots,n-1\}} \Big[u_{f,A}(w)u_{g,B}(w) - (-1)^{\#(A)}v^l_{f,A}(w)v^l_{g,B}(w)\Big]\, e_Ae_B, \tag{4.25}$$

and

$$\mathrm{Im}^l\Big[f^*g^*\Big](w^*) = -\sum_{A,B\in\mathcal{P}\{1,\dots,n-1\}} \Big[(-1)^{\#(A)}u_{f,A}(w)v^l_{g,B}(w) + v^l_{f,A}(x)u_{g,B}(w)\Big]\, e_Ae_B. \tag{4.26}$$

(4.24), (4.25) and (4.26) imply (4.21).

Noting that

$$w = \Big(f^{-1}\circ f\Big)(w) = f^{-1}\big(f(w)\big), \quad w\in\mathbb{R}^{n+1}_+\ \Big(\mathbb{R}^{n+1}_-\Big), \tag{4.27}$$

we have, by (4.7) and (4.9),

$$\begin{aligned}
&\Big(\big[f^{-1}\big]^*\circ f^*\Big)(w^*) = \Big[f^{-1}\Big(\big[f^*\big]^*\big([w^*]^*\big)\Big)\Big]^* = \Big[f^{-1}\big(f(w)\big)\Big]^*\\
&= w^*,\ \ w^*\in\mathbb{R}^{n+1}_-\ \Big(\mathbb{R}^{n+1}_+\Big),
\end{aligned} \tag{4.28}$$

which is (4.22). □

Introduce the operators

$$\frac{\partial}{\partial x} = \sum_{j=0}^{n-1} e_j\,\frac{\partial}{\partial w_j} \quad with\ x=\mathrm{Re}(w) \tag{4.29}$$

and

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial w_n} \quad with\ y=\mathrm{Im}(w). \tag{4.30}$$

Their actions on functions from the left and from the right are governed by the rules

$$\frac{\partial[f]}{\partial x} = \sum_{k=0}^{n-1}\sum_A e_k\,e_A\frac{\partial f_A}{\partial w_k}, \quad \frac{[f]\partial}{\partial x} = \sum_{k=0}^{n-1}\sum_A e_A\,e_k\frac{\partial f_A}{\partial w_k}. \tag{4.31}$$

Lemma 4.2. (Cauchy–Riemann equations [15,27]) (1) Φ *is left regular on* $\mathbb{R}_+^{n+1}$ $\left(\mathbb{R}_-^{n+1}\right)$ *if and only if*

$$\frac{\partial[U]}{\partial x}(w)=\frac{\partial[V^l]}{\partial y}(w) \quad \text{and} \quad \frac{\partial[U]}{\partial y}(w)=-\frac{\partial[V^l]}{\partial x}(w), \tag{4.32}$$

where

$$U(w)=\big(\mathrm{Re}\,\Phi\big)(w) \quad \textit{and} \quad V^l(w)=\big(\mathrm{Im}^l\,\Phi\big)(w) \tag{4.33}$$

are, respectively, the real and the left imaginary parts of Φ.

(2) Φ *is right regular on* $\mathbb{R}_+^{n+1}$ $\left(\mathbb{R}_-^{n+1}\right)$ *if and only if*

$$\frac{[U]\,\partial}{\partial x}(w)=\frac{[V^r]\partial}{\partial y}(w) \quad \textit{and} \quad \frac{[U]\,\partial}{\partial y}(w)=-\frac{[V^r]\partial}{\partial x}(w), \tag{4.34}$$

where

$$U(w)=\big(\mathrm{Re}\,\Phi\big)(w) \quad \textit{and} \quad V^r(w)=\big(\mathrm{Im}^r\,\Phi\big)(w) \tag{4.35}$$

are, respectively, the real and the right imaginary parts of Φ.

Proof. If Φ is left regular on $\mathbb{R}_+^{n+1}$, it is easy to see, by Remark 4.1, that

$$\begin{aligned}
&\big(D[\Phi]\big)(w)\ \left(w\in\mathbb{R}_+^{n+1}\right)\\
&=\sum_{k=0}^{n-1}e_k\left[\sum_{A\in\mathcal{P}\{1,\dots,n-1\}}e_k\,e_A\frac{\partial u_A}{\partial w_k}(w)+e_n\sum_{A\in\mathcal{P}\{1,\dots,n-1\}}e_A\frac{\partial v_A^l}{\partial w_k}(w)\right]\\
&\quad+e_n\left[\sum_{A\in\mathcal{P}\{1,\dots,n-1\}}e_A\frac{\partial u_A}{\partial w_n}(w)+e_n\sum_{A\in\mathcal{P}\{1,\dots,n-1\}}e_A\frac{\partial v_A^l}{\partial w_n}(w)\right]\quad\big(by\ (4.10)\big)\\
&=\sum_{k=0}^{n-1}\sum_{A\in\mathcal{P}\{1,\dots,n-1\}}e_k\,e_A\frac{\partial u_A}{\partial w_k}(w)-\sum_{k=0}^{n-1}\sum_{A\in\mathcal{P}\{1,\dots,n-1\}}e_A\frac{\partial v_A^l}{\partial w_k}(w)\\
&\quad+\sum_{A\in\mathcal{P}\{1,\dots,n-1\}}e_n\,e_A\frac{\partial u_A}{\partial w_n}(w)+\sum_{A\in\mathcal{P}\{1,\dots,n-1\}}e_n\,e_A\frac{\partial v_A^l}{\partial w_n}(w)\\
&=\frac{\partial U}{\partial x}(w)-\frac{\partial[V^l]}{\partial y}(w)+e_n\left[\frac{\partial U}{\partial y}(w)+\frac{\partial[V^l]}{\partial x}(w)\right]\quad\big(by\ (4.29)\ and\ (4.30)\big),
\end{aligned} \tag{4.36}$$

which implies (1). The proof of (2) is similar. □

Theorem 4.1. (Symmetry principle for regular functions) *If* Φ *is regular on* $\mathbb{R}_+^{n+1}$ $\left(\mathbb{R}_-^{n+1}\right)$ *then* Φ^* *is also regular on* $\mathbb{R}_-^{n+1}$ $\left(\mathbb{R}_+^{n+1}\right)$.

Proof. By (4.18) and (4.19), we have

$$U^*(w)=\mathrm{Re}\,\Phi^*(w)=\mathrm{Re}\,\Phi(w^*)=U\big(w^*\big),\quad w\in\mathbb{R}_-^{n+1}\left(\mathbb{R}_+^{n+1}\right), \tag{4.37}$$

and

$$\big(V^*\big)^l\big(w\big)=\mathrm{Im}\,\Phi^*(w)=-\mathrm{Im}\,\Phi\big(w^*\big)=-V^l\big(w^*\big),\quad w\in\mathbb{R}_-^{n+1}\left(\mathbb{R}_+^{n+1}\right). \tag{4.38}$$

So,

$$\frac{\partial[U^*]}{\partial x}(w)=\sum_{k=0}^{n-1}e_k\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}}e_A\frac{\partial u_A}{\partial w_k}(w_0,\ldots,-w_n)=\frac{\partial[U]}{\partial x}(w_0,\ldots,-w_n),\tag{4.39}$$

$$\frac{\partial[U^*]}{\partial y}(w)=\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}}e_A\frac{\partial u_A}{\partial w_n}(w_0,\ldots,-w_n)=\frac{\partial[U]}{\partial y}(w_0,\ldots,-w_n),\tag{4.40}$$

and

$$\frac{\partial\big[(V^*)^l\big]}{\partial y}(w)=\sum_{k=0}^{n-1}e_k\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}}e_A\frac{\partial v_A^l}{\partial w_k}(w_0,\ldots,-w_n)=\frac{\partial\big[V^l\big]}{\partial y}(w),\tag{4.41}$$

$$\frac{\partial\big[(V^*)^l\big]}{\partial x}(w)=\sum_{k=0}^{n-1}e_k\sum_{A\in\mathcal{P}\{1,\ldots,n-1\}}e_A\frac{\partial v_A^l}{\partial w_k}(w_0,\ldots,-w_n)=\frac{\partial\big[V^l\big]}{\partial x}(w).\tag{4.42}$$

By Theorem 4.1, (4.39), (4.40), (4.41) and (4.42), we know that

$$\Phi\text{ is (left) regular on }\mathbb{R}_+^{n+1}\left(\mathbb{R}_-^{n+1}\right)\Longleftrightarrow\Phi^*\text{ is (left) regular on }\mathbb{R}_-^{n+1}\left(\mathbb{R}_+^{n+1}\right).\tag{4.43}$$

Hence the symmetry principle for regular functions is proved. □

By Remark 4.1, we may get the following results for boundary values.

Lemma 4.3. *If Φ is defined in $\mathbb{R}_+^{n+1}$ with the boundary value $\Phi^+(t)$, then Φ^* has the boundary value $\big[\Phi^*\big]^-(t)$ and*

$$\big[\Phi^+\big]^*(t)=\big[\Phi^*\big]^-(t),\quad t\in\mathbb{R}_0^{n+1}.\tag{4.44}$$

In other words,

$$\Phi^+(t)+[\Phi^*]^-(t)=2\,\mathrm{Re}\big(\Phi^+(t)\big),\quad t\in\mathbb{R}_0^{n+1}.\tag{4.45}$$

Lemma 4.4. *If Φ is defined in $\mathbb{R}_-^{n+1}$ with the boundary value $\Phi^-(t)$, then Φ^* also has the boundary value $\big[\Phi^*\big]^+(t)$ and*

$$\big[\Phi^-\big]^*(t)=\big[\Phi^*\big]^+(t),\quad t\in\mathbb{R}_0^{n+1}.\tag{4.46}$$

In other words,

$$\Phi^-(t)+[\Phi^*]^+(t)=2\,\mathrm{Re}\big(\Phi^-(t)\big),\quad t\in\mathbb{R}_0^{n+1}.\tag{4.47}$$

By Lemmas 4.1 and 4.3, we immediately have the following theorem, which is the basis for the Hilbert BVP being transformed into the Riemann BVP.

If Φ is defined in $\mathbb{R}_+^{n+1}\left(\mathbb{R}_-^{n+1}\right)$, we call the function

$$\big(\mathcal{E}[\Phi]\big)(w)=\begin{cases}\Phi(w), & when\ \ w\in\mathbb{R}_+^{n+1}\left(\mathbb{R}_-^{n+1}\right),\\ \Phi^*(w), & when\ \ w\in\mathbb{R}_-^{n+1}\left(\mathbb{R}_+^{n+1}\right),\end{cases}\tag{4.48}$$

the symmetric extension of Φ defined in $\mathbb{R}_+^{n+1}$ $\left(\mathbb{R}_-^{n+1}\right)$.

Theorem 4.2. *If Φ is regular on $\mathbb{R}_+^{n+1}$ $\left(\mathbb{R}_-^{n+1}\right)$ and can be extended continuously to $\mathbb{R}_0^{n+1}$, then its symmetric extension $\mathcal{E}[\Phi]$ is the sectionally holomorphic function with $\mathbb{R}_0^{n+1}$ as the jump surface (see [4]).*

With the help of this theorem, the Hilbert BVP (3.20) will be equivalently transformed to the Riemann BVP discussed in [4], along with an additional restriction.

Theorem 4.3. *The Schwarz BVP (3.22) is equivalent to the Riemann BVP (4.51) under the relationship*

$$\Psi(w) = \Big(\mathcal{E}[\Phi]\Big)(w), \quad w \in \mathbb{R}_+^{n+1} \cup \mathbb{R}_-^{n+1} \tag{4.49}$$

or

$$\Phi(w) = \Psi\Big|_{\mathbb{R}_+^{n+1}}(w) = \Psi^+(w), \quad w \in \mathbb{R}_+^{n+1}. \tag{4.50}$$

R_m^*problem with the reflection condition. Find a sectionally holomorphic function Ψ, with $\mathbb{R}_0^{n+1}$ as its jump plane such that

$$\begin{cases} \Psi^+(x) + \Psi^-(x) = 2c(x), \ x \in \mathbb{R}_0^{n+1} \ \ (\textit{boundary value condition}), \\ \Psi(w) = o\big(w^{m+1}\big) \ \textit{near} \ \infty \ (\textit{growth condition at infinity}), \\ \big[\Psi^+\big]^*(x) = \big[\Psi^*\big]^-(x), \quad x \in \mathbb{R}_0^{n+1} \ \ (\textit{reflection condition}), \end{cases} \tag{4.51}$$

where m is some integer.

Proof of Theorem 4.3. Let Φ be a solution of the Schwarz problem (3.22), by using Theorem 4.1, (4.20) and (4.45), we know that $\Psi(w) = \big(\mathcal{E}[\Phi]\big)(w)$ is the solution of the R_m^* problem (4.51). And, in turn, if Ψ is the solution of the R_m^* problem (4.51), then

$$\Phi(w) = \Phi^+(w) = \Psi^+(w), \quad w \in \mathbb{R}_+^{n+1} \tag{4.52}$$

is the solution of the Schwarz problem (3.22). In fact, by the reflection condition in (4.51) we have (4.47) which results in (3.22). □

Now, the remaining question is how to solve the R_m^* problem (4.51). To do so, we introduce the self-reflex action of Φ defined on $\mathbb{R}_+^{n+1} \bigcup \mathbb{R}_-^{n+1}$.

4.2. Self-Reflex Action

If $\mho$ is defined on $\mathbb{R}_+^{n+1} \bigcup \mathbb{R}_-^{n+1}$, say,

$$\mho(w) = \begin{cases} \mho^+(w), & w \in \mathbb{R}_+^{n+1}, \\ \mho^-(w), & w \in \mathbb{R}_-^{n+1}, \end{cases} \tag{4.53}$$

then

$$\mho_\updownarrow(w) = \begin{cases} \big[\mho^-\big]^*(w), & w \in \mathbb{R}_+^{n+1}, \\ \big[\mho^+\big]^*(w), & w \in \mathbb{R}_-^{n+1}, \end{cases} \tag{4.54}$$

is called the reflection function of $\mho$. Obviously, by (4.17), we have

$$\mho(w) = \big(\mho_\updownarrow\big)_\updownarrow(w), \quad w \in \mathbb{R}_\pm^{n+1}. \tag{4.55}$$

In particular, if

$$\mho_{\updownarrow}(w) = \mho(w), \quad w \in \mathbb{R}^{n+1}_{\pm}, \tag{4.56}$$

then we call $\mho$ a self-reflection function. Obviously, by (4.17),

$$\big(\mathscr{R}[\mho]\big)(w) = \frac{\mho(w) + \mho_{\updownarrow}(w)}{2} \tag{4.57}$$

is a self-reflection function, which is called the self-reflection function of $\mho$. For the sake of convenience, we call the above steps from Φ, defined on $\mathbb{R}^{n+1}_{+}$ $\big(\mathbb{R}^{n+1}_{+}\big)$, to $\mathscr{R}\big[\mathcal{E}[\Phi]\big]$ to be the so-called self-reflex action of Φ. From (4.56) and (4.57) we know that Φ is a self-reflection function if and only if

$$\Phi(w) = \big(\mathscr{R}[\Phi]\big)(w), \quad w \in \mathbb{R}^{n+1}_{+} \cup \mathbb{R}^{n+1}_{-}. \tag{4.58}$$

Example 4.1. From Example 2.1, we know that the restriction $E\big|_{\mathbb{R}^{n+1}_{\pm}}$ of the Cauchy kernel given in (2.34) is a self-reflection function.

Example 4.2. Assume f is a para real-valued function. Let

$$\Psi(w) = \begin{cases} \Big(\mathcal{S}[f]\Big)(w) = \frac{1}{\bigvee_{n+1}} \int_{\mathbb{R}^{n+1}_0} E(x-w)\, \mathrm{d}\sigma f(x), & w \in \mathbb{R}^{n+1}_{+}, \\ -\Big(\mathcal{S}[f]\Big)(w) = -\frac{1}{\bigvee_{n+1}} \int_{\mathbb{R}^{n+1}_0} E(x-w)\, \mathrm{d}\sigma f(x), & w \in \mathbb{R}^{n+1}_{-}, \end{cases} \tag{4.59}$$

where $\mathcal{S}[f]$ is Cauchy type integral on the hyperplane $\mathbb{R}^{n+1}_0$, and

$$\mathrm{d}S = \mathrm{d}x_0 \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1} = e_n\, \mathrm{d}\sigma \tag{4.60}$$

is the elementary surface measure on the hyperplane $\mathbb{R}^{n+1}_0$ (see [4]). Then, by Example 4.1 and Lemma 4.1, its self-reflection function is

$$\Big(\mathscr{R}\big[\Psi\big]\Big)(w) = \Big(\mathcal{S}[f]\Big)(w) = \frac{1}{\bigvee_{n+1}} \int_{\mathbb{R}^{n+1}_0} E(x-w)\, \mathrm{d}\sigma f(x), \quad w \in \mathbb{R}^{n+1}_{\pm}. \tag{4.61}$$

Example 4.3. Let

$$\Psi(w) = \begin{cases} Z^{\alpha}(w)\, c_{\alpha}, & w \in \mathbb{R}^{n+1}_{+}, \\ -Z^{\alpha}(w)\, c_{\alpha} & w \in \mathbb{R}^{n+1}_{-}, \end{cases} \tag{4.62}$$

where c_{α} is a hypercomplex constant and Z^{α} is a Fueter polynomial given in (3.10). Then,

$$\Big(\mathscr{R}\big[\Psi\big]\Big)(w) = Z^{\alpha}(w)\, \mathrm{Im}\, c_{\alpha}\, e_n, \quad w \in \mathbb{R}^{n+1}_{\pm}, \tag{4.63}$$

which is called the para-imaginary coefficient polynomial.

We point out an obvious fact that, if Ψ is the solution of R^{*}_{m} problem (4.51) with a reflection condition, then it naturally serves as the solution of the following Riemann BVP R_m discussed in [4].

R_m**problem with no reflection condition.** Find a sectionally holomorphic function Ψ, with $\mathbb{R}^{n+1}_0$ as its jump plane, such that

$$\begin{cases} \Psi^{+}(x) + \Psi^{-}(x) = 2c(x), \ x \in \mathbb{R}^{n+1}_0 \ \ (\textit{boundary value condition}), \\ \Psi(w) = o\big(w^{m+1}\big) \ \ \textit{near} \ \ \infty \ \ (\textit{growth condition at infinity}). \end{cases} \tag{4.64}$$

If Ψ is both a solution of (4.64) and a reflexive function, then it is called a reflexive solution of (4.64). Under such case, by (4.55) the reflection condition in (4.51) automatically holds. So, we have the following result.

Lemma 4.5. *The reflexive solution of the R_m problem (4.64) with no reflection condition surely is the solution of the R_m^* problem (4.51) with the reflection condition.*

In turn, we also have the following result.

Lemma 4.6. *If Ψ is the solution of the R_m problem (4.64) with no reflection condition, then its reflection function $\Psi_\updownarrow$ and the self-reflection function $\mathscr{R}[\Psi]$ are the solutions of (4.51).*

Thus, $\mathscr{R}[\Psi]$ is the self-reflection solution of the R_m^* problem (4.51) with the reflection condition.

Proof. First, the reflection function $\Psi_\updownarrow$ is regular on $\mathbb{R}_{\pm}^{n+1}$ by Lemma 4.2. Secondly, using the reflection operator to both sides of the formulas in (4.64) we have, by (4.20) and (4.9),

$$\begin{cases} \left(\Psi_\updownarrow\right)^+(x) + \left(\Psi_\updownarrow\right)^-(x) = 2c(x), \ x \in \mathbb{R}_0^{n+1} \quad (\textit{boundary value condition}), \\ \Psi_\updownarrow(w) = o\left(w^{m+1}\right) \ \textit{near} \ \infty \ (\textit{growth condition at infinity}). \end{cases} \tag{4.65}$$

This is to say that $\Psi_\updownarrow$ is also a solution of (4.64). So is the self-reflection function $\mathscr{R}[\Psi]$ given in (4.57) with $\mho = \Psi$, since Ψ and $\Psi_\updownarrow$ are the solutions of (4.64). □

Theorem 4.4. *The general solution of the Schwarz BVP (3.22) should be*

$$\Phi(w) = \Big(\mathscr{R}[\Psi]\Big)(w) = \frac{\Psi(w) + \Psi_\updownarrow(w)}{2}, \quad w \in \mathbb{R}_+^{n+1}, \tag{4.66}$$

where Ψ is the solution of the R_m problem (4.64).

Thus, we establish the principle of self-reflex action as demonstrated by Theorem 4.2, Lemmas 4.5 and 4.6. In summary, the solutions of R_m problem (4.46) with no reflection condition are derived from the solutions of the Schwarz BVP (3.22) through the self-reflex action. Conversely, the solutions of the Schwarz BVP (3.22) may be obtained from the solutions of R_m problem (4.64) without reflection condition through two steps: first taking the self-reflex action and then taking the restriction on $\mathbb{R}_+^{n+1}$.

5. Solutions of $\mathbf{H}_m$ Problem

Based on the reflexive principle, in order to solve S_m problem (3.22) we only need to solve the R_m problem (4.64), which is discussed in detail in [4].

5.1. Solutions of R_m Problem

For the sake of convenience, we restate the results for R_m problem as follows.

Theorem 5.1. (See [4]) *For the Riemann boundary value problem R_m (4.64) the following four cases are a complete classification.*

(1) *Let $m \geq 0$, $c \in \widehat{H}^{\mu}\left(\mathbb{R}_0^{n+1}\right)$, then its general solution is*

$$\Psi(w) = \begin{cases} \Phi(w), & w \in \mathbb{R}_+^{n+1}, \\ -\Phi(w), & w \in \mathbb{R}_-^{n+1}, \end{cases} \tag{5.1}$$

where

$$\Phi(w) = \Big(\mathcal{S}[c]\Big)(w) + P_m(w) = \frac{1}{\bigvee_{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-w) \mathrm{d}\sigma c(x) + \sum_{|\alpha|=0}^{m} \frac{1}{|\alpha|!} Z^{\alpha}(w)\, c_{\alpha}, \quad w \in \mathbb{R}_{\pm}^{n+1}, \tag{5.2}$$

where P_m is arbitrary hypercomplex symmetric polynomial of degree not exceeding m with C_{n+m}^{m} free hypercomplex constants c_{α}.

(2) Let $m = -1$ with $c \in \widehat{H}\left(\mathbb{R}_0^{n+1}\right)$, it has the unique solution

$$\Phi(w) = \begin{cases} \Big(\mathcal{S}[c]\Big)(w) = \frac{1}{\bigvee_{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-w)\, \mathrm{d}\sigma\, c(x), & w \in \mathbb{R}_+^{n+1}, \\ -\Big(\mathcal{S}[c]\Big)(w) = -\frac{1}{\bigvee_{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-w)\, \mathrm{d}\sigma\, c(x), & w \in \mathbb{R}_-^{n+1}, \end{cases} \tag{5.3}$$

if and only if

$$c(\infty) =: \lim_{x \in \mathbb{R}_0^{n+1},\ x \to \infty} c(x) = 0. \tag{5.4}$$

(3) Let $-n < m < -1$ and $r = -(m+1)$, with $c \in \widehat{H}_{r,0}\left(\mathbb{R}_0^{n+1}\right)$, it has the unique solution (5.3).

(4) Let $m \leq -n$ and $r = -(m+1)$, with $c \in \widehat{H}_{r,0}\left(\mathbb{R}_0^{n+1}\right)$, it has the unique solution (5.3) if the C_{-m-1}^{n} conditions

$$\int_{\mathbb{R}_0^{n+1}} Z^{\alpha}(x)\, \mathrm{d}\sigma\, c(x) = 0, \quad |\alpha| = 0, 1, \ldots, -(n+1+m) \tag{5.5}$$

are fulfilled.

Remark 5.1. The R_m problem discussed in [4] is the jump problem for Φ. Here the R_m problem (4.64) for Ψ is called the Szegö problem in some literature. They are slightly different and governed by the relation (5.1).

5.2. Solutions of $\mathbf{S}_m$ Problem

By Theorems 4.4, 5.1 and Examples 4.2 4.3, we get the following result.

Theorem 5.2. *The general solution of the Schwarz boundary value problem (3.22) is presented in the following four cases.*

Case 1. *When $m \geq 0$ and $c \in \widehat{H}^{\mu}\left(\mathbb{R}_0^{n+1}\right)$, then the solution is given by*

$$\Phi(w) = \Big(\mathcal{S}[c]\Big)(w) = \frac{1}{\bigvee_{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-w) \mathrm{d}\sigma c(x) + \sum_{|\alpha|=0}^{m} \frac{1}{|\alpha|!} Z^{\alpha}(w)\, R_{\alpha}\, e_n \quad w \in \mathbb{R}_+^{n+1}, \tag{5.6}$$

where R_α are C^m_{n+m} free para-real hypercomplex constants.

Case 2. *When $m=-1$ and $c\in \widehat{H}\left(\mathbb{R}^{n+1}_0\right)$, the solution is unique and given by*

$$\Phi(w)=\Big(\mathcal{S}[c]\Big)(w)=\frac{1}{\bigvee_{n+1}}\int_{\mathbb{R}^{n+1}_0}E(x-w)\,\mathrm{d}\sigma\, c(x),\quad w\in\mathbb{R}^{n+1}_+,\quad (5.7)$$

if and only if

$$c(\infty)=\lim_{x\in\mathbb{R}^{n+1}_0,\quad x\to\infty} c(x)=0. \quad (5.8)$$

Case 3. *When $-n<m<-1$, $r=-(m+1)$ and $c\in \widehat{H}_{r,0}\left(\mathbb{R}^{n+1}_0\right)$, the unique solution is given by (5.7).*

Case 4. *When $m\le -n$, $r=-(m+1)$ and $c\in \widehat{H}_{r,0}\left(\mathbb{R}^{n+1}_0\right)$, the unique solution (5.7) exists if the following C^n_{-m-1} conditions are satisfied:*

$$\int_{\mathbb{R}^{n+1}_0} Z^\alpha(x)\,\mathrm{d}\sigma\, c(x)=0,\quad |\alpha|=0,1,\ldots,-(n+1+m) \quad (5.9)$$

or

$$\int_{\mathbb{R}^{n+1}_0} Z^\alpha(x)\,c(x)\,\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n=0,\quad |\alpha|=0,1,\ldots,-(n+1+m). \quad (5.10)$$

5.3. Solutions of $\mathbf{H}_m$ Problem

The Hilbert problem (3.20) for the function Φ may be directly translated into the Schwarz problem for the function Ψ by

$$\Phi(w)=\begin{cases}\Psi(w), & w\in\mathbb{R}^{n+1}_+,\\ \Psi(w)\lambda, & w\in\mathbb{R}^{n+1}_-.\end{cases} \quad (5.11)$$

Theorem 5.3. *The general solution of the Hilbert boundary value problem (3.20) is categorized into the following four cases.*

Case 1. *When $m\ge 0$ and $c\in \widehat{H}^\mu\left(\mathbb{R}^{n+1}_0\right)$, the solution is given by*

$$\Phi(w)=\frac{1}{\bigvee_{n+1}}\int_{\mathbb{R}^{n+1}_0}E(x-w)\mathrm{d}\sigma c(x)\lambda^{-1}+\sum_{|\alpha|=0}^{m}\frac{1}{|\alpha|!}Z^\alpha(w)\,R_\alpha\,\lambda^{-1},\quad w\in\mathbb{R}^{n+1}_+, \quad (5.12)$$

where R_α are C^m_{n+m} free para-real hypercomplex constants.

Case 2. *When $m=-1$ and $c\in \widehat{H}\left(\mathbb{R}^{n+1}_0\right)$, the unique solution is given by*

$$\Phi(w)=\Big(\mathcal{S}[c]\Big)(w)\lambda^{-1}=\frac{1}{\bigvee_{n+1}}\int_{\mathbb{R}^{n+1}_0}E(x-w)\,\mathrm{d}\sigma\, c(x)\,\lambda^{-1},\quad w\in\mathbb{R}^{n+1}_+, \quad (5.13)$$

if and only if

$$c(\infty)=\lim_{x\in\mathbb{R}^{n+1}_0,\quad x\to\infty} c(x)=0. \quad (5.14)$$

Case 3. *When* $-n < m < -1$, $r = -(m+1)$ *and* $c \in \widehat{H}_{r,0}\left(\mathbb{R}_0^{n+1}\right)$, *the unique solution is given by (5.13).*

Case 4. *When* $m \leq -n$, $r = -(m+1)$ *and* $c \in \widehat{H}_{r,0}\left(\mathbb{R}_0^{n+1}\right)$, *it has the unique solution (5.13) provided that the* C_{-m-1}^n *conditions*

$$\int_{\mathbb{R}_0^{n+1}} Z^\alpha(x)\,\mathrm{d}\sigma\, c(x) = 0, \quad |\alpha| = 0, 1, \ldots, -(n+1+m) \tag{5.15}$$

or

$$\int_{\mathbb{R}_0^{n+1}} Z^\alpha(x)\, c(x)\,\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n = 0, \quad |\alpha| = 0, 1, \ldots, -(n+1+m) \tag{5.16}$$

are fulfilled.

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