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THE SPARSE REPRESENTATION RELATED WITH FRACTIONAL HEAT EQUATIONS

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Abstract This study introduces a pre-orthogonal adaptive Fourier decomposition (POAFD) to obtain approximations and numerical solutions to the fractional Laplacian initial value problem and the extension problem of Caffarelli and Silvestre (generalized Poisson equation). As a first step, the method expands the initial data function into a sparse series of the fundamental solutions with fast convergence, and, as a second step, makes use of the semigroup or the reproducing kernel property of each of the expanding entries. Experiments show the effectiveness and efficiency of the proposed series solutions.

Key words reproducing kernel Hilbert space; dictionary; sparse representation; approximation to the identity; fractional heat equations

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1 Introduction

For $0 < \alpha < 1$, the fractional Laplacian of order α, denoted by $(-\Delta)^\alpha$, can be defined on functions $f : \mathbb{R}^n \to \mathbb{R}$ via the Fourier multiplier given by the formula

$$\widehat{(-\Delta)^\alpha} f(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi),$$

where the Fourier transform $\widehat{f}$ of $f : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-\mathrm{i}x\cdot\xi} \mathrm{d}x.$$

The fractional Laplacian can be written alternatively as the singular integral operator defined by

$$(-\Delta)^\alpha f(x) = c_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+2\alpha}} \mathrm{d}y,$$

where the prefactor $c_{n,\alpha}$ is the constant

$$\frac{4^\alpha \Gamma(n/2+\alpha)}{\pi^{n/2}|\Gamma(-\alpha)|}$$

involving the Gamma function and serving as a normalizing factor. We refer the reader to [10] for other equivalent definitions of the fractional Laplacian.

The fractional Laplacian $(-\Delta)^\alpha$ plays a significant role in many areas of mathematics, such as harmonic analysis and PDEs, and is often applied to describe complicated phenomena via partial differential equations. Anomalous diffusion processes with non-locality in complex media can be well-characterized by using fractional-order diffusion equation models, e.g. the following fractional heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t+x) = (-\Delta)^\alpha u(t+x), & (x,t) \in \mathbb{R}^{n+1}_+; \\ u(0+x) = f(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.1}$$

In addition, the fractional Laplacian has been applied to study a wide class of physical systems and engineering problems, including Lévy flights, stochastic interfaces and anomalous diffusion problems. By an extension problem for the generalized Poisson equation,

$$\begin{cases} \operatorname{div}(t^\sigma \nabla u)(x) = 0, & (x,t) \in \mathbb{R}^{n+1}_+; \\ u = f, & x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

Caffarelli and Silvestre showed in [4] that any fractional power of the Laplacian can be determined as an operator that maps a Dirichlet boundary condition to a Neumann type condition via an extension problem. In the study on the obstacle problem for the fractional Laplace operators, this characterization of $(-\Delta)^\alpha$, via the above local (degenerate) PDE, was first used in [5] to get related estimates of regularities. We also refer the reader to [6, 7, 9, 16] for further information on applications of the fractional Laplacian in PDEs.

The aim of this article is to develop sparse decompositions of the solutions to the initial problems given in (1.1) and (1.2) by using the POAFD (pre-orthogonal adaptive Fourier decomposition) method under the $\mathcal{H} - H_K$ formulation of Qian ([12]), and more precisely, the convolution-type sparse representation of the identity which was further developed by Qu *et al.* in [13]. We also refer the reader to [14, 15] for the closely related AFD adaptive Fourier decompositions methods. In [13], the POAFD methodology was used to give fast approximation to the identity, and hence to expand the initial (boundary) data by using the dictionary elements, the parameterized fractional heat and the Poisson kernels. Then, by the "lifting up" method based on semigroup properties, in the two cases we obtain the sparse representation of the original Dirichlet boundary and Cauchy initial value problems.

It should be pointed out that the sparse representations of the solutions to the equation (1.1) and (1.2) are not simple analogues of those related to the classical heat kernel and Poisson kernels. In [13], an important mechanism, together with the sparse representation of the Dirac-δ generalized function, is the following superimposed effect of the classical heat kernel and Poisson kernels: for any $(x,t),(y,s)\in\mathbb{R}^n$,

$$\begin{cases}\displaystyle\int_{\mathbb{R}^n}\frac{t}{(t^2+|x-z|)^{(n+1)/2}}\frac{s}{(s^2+|y-z|)^{(n+1)/2}}\mathrm{d}z=\frac{t+s}{((t+s)^2+|x-y|)^{(n+1)/2}};\\ \displaystyle\frac{1}{(4\pi)^n(ts)^{n/2}}\int_{\mathbb{R}^n}\mathrm{e}^{-|x-z|^2/4t}\mathrm{e}^{-|y-z|^2/4s}\mathrm{d}z=\frac{1}{(4\pi(t+s))^{n/2}}\mathrm{e}^{-|x-y|^2/4(t+s)},\end{cases}\tag{1.3}$$

which can be deduced from the uniqueness of the solutions of the heat equation and the Poisson equation. Like the heat kernel and the Poisson kernel cases, the integral kernel of the fractional heat equation (1.1), denoted as $K_{\alpha,t+x}(\cdot)$, although without an explicit formula, can also adopt the same argument based on the uniqueness of the solution to (1.1), together with the linearity of the t-differentiation. It holds that

$$\langle K_{\alpha,t+x}(\cdot),\ K_{\alpha,s+y}(\cdot)\rangle_{L^2}=K_{\alpha,t+s}(x-y).$$

In Section 3, we alternatively apply the Fourier multipliers and the Plancherel formula to establish the above semigroup identities for the fractional heat equation (1.1). For the special cases $\alpha=1/2,1$, the identities become those in (1.3) in the given order.

For the case of equation (1.2), the solutions $\{P_t^\sigma f\}_{t>0}$ do not satisfy the semigroup property, in other words, the analogous relations like those in (1.3) are invalid.

Another difference between our results and those obtained in [13] is the technology used for proving the boundary vanishing condition (BVC) of the reproducing kernels. In [13], the authors used (1.3) to verify the BVC properties of the heat and the Poisson type kernels; see [13, Theorems 3.1 & 3.2]. Since the semigroup property does not hold for the fundamental solutions of (1.2), we utilize the fractional Poisson kernel

$$p_t^\sigma(x)=\frac{t^\sigma}{(t^2+|x-y|^2)^{(n+2)/\sigma}}.$$

Through providing a decay estimate of the integral

$$\frac{t^{n/2}}{(c_{n,\sigma})^{1/2}}\int_{\mathbb{R}^n}\frac{t^\sigma}{(t^2+|x-z|^2)^{(n+\sigma)/2}}\frac{s^\sigma}{(s^2+|y-z|^2)^{(n+\sigma)/2}}\mathrm{d}z,$$

we show the BVC property; see Theorem 4.2.

Moreover, Theorem 4.2 partly improves the results of the sparse approximation for the general convolution cases, as given in [13, Theorem 3.3]. In the latter, the authors considered the sparse approximation for convolution kernel $\phi_t(x):=t^{-n/2}\phi(x/t)$, where

$$\begin{cases}\displaystyle\int_{\mathbb{R}^n}\phi(x)\mathrm{d}x=1;\\ \displaystyle\sup_{|y|\ge|x|}|\phi(y)|\le\frac{C}{(1+|x|^2)^{(n+\delta)/2}},\ \delta>0.\end{cases}$$

For $\delta \geq 1$, the functions $\phi(\cdot)$ are dominated by the Poisson kernel. Then, as a consequence of [13, Theorem 3.1], the sparse representation in $\{\phi_t(\cdot)\}_{t>0}$ can be established. Our treatment in Theorem 4.2, however, can guarantee the sparse representation for all $\delta > 0$.

Throughout this paper, we use the symbol $U \simeq V$ to denote that there is a constant $c > 0$ such that $c^{-1}V \leq U \leq cV$. The symbol $U \lesssim V$ means that $U \leq cV$. Similarly, we use $V \gtrsim U$ to denote that there exists a constant c such that $V \geq cU$.

2 Preliminaries

First, we state some preliminaries on the $\mathcal{H}$-H_K formulation, which was introduced by Qian in [12] and is based on the theory of reproducing Hilbert spaces. Let $\mathcal{H}$ be a Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Correspondingly, for any $f \in \mathcal{H}$, the norm is $\|f\|^2_{\mathcal{H}} = \langle f,\ f \rangle_{\mathcal{H}}$. Let $\mathcal{E}$ be an open set in the underline topological space, where, in this article, the elements are real or complex numbers, or vectors. In the $\mathcal{H}$-H_K formulation, each $p \in \mathcal{E}$ is treated as as a parameter. More precisely, for $p \in \mathcal{E}$, there exists an element $h_p \in \mathcal{H}$, corresponding to p. Denote by $\mathbb{C}^{\mathcal{E}}$ the set of all functions from $\mathcal{E}$ to the complex number field $\mathbb{C}$. For $f \in \mathcal{H}$, we can define a linear operator from $\mathcal{H}$ to $\mathbb{C}^{\mathcal{E}}$, denoted by $\mathscr{L}$ via the inner product $\langle \cdot,\ \cdot \rangle$ on $\mathcal{H}$, i.e.,

$$\mathscr{L}(f)(p) = \langle f,\ h_p \rangle_{\mathcal{H}},\ p \in \mathcal{E}. \tag{2.1}$$

We use $\mathcal{N}(\mathscr{L})$ to denote the null space of $\mathscr{L}$, i.e., the set of all elements $f \in \mathcal{H}$ such that $\mathscr{L}(f) = 0$. Take a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in $N(\mathscr{L}) \subset \mathcal{H}$ satisfying that

$$\lim_{n\to\infty} \|f - f_n\|_{\mathcal{H}} = 0.$$

Since

$$|\mathscr{L}(f)(p)| = |\mathscr{L}(f - f_n)(p)| = |\langle f - f_n,\ h_p \rangle_{\mathcal{H}}| \leq \|f - f_n\|_{\mathcal{H}} \|h_p\|_{\mathcal{H}},$$

letting $n \to \infty$ reaches $|\mathscr{L}(f)(p)| = 0$, which implies that $f \in N(\mathscr{L})$, and hence, we can see that $N(\mathscr{L}) \subset \mathcal{H}$ is closed. Let $N(\mathscr{L})^{\perp}$ denote the trivial or non-trivial orthogonal complement of $N(\mathscr{L})$ in $\mathcal{H}$. Then

$$\mathcal{H} = N(\mathscr{L}) \oplus N(\mathscr{L})^{\perp}.$$

For any $f \in \mathcal{H}$, there exists a unique $f_- \in N(\mathscr{L})$ and a unique $f_+ \in N(\mathscr{L})^{\perp}$ such that

$$f = f_- + f_+.$$

Let $F(p) = \mathscr{L}(f)(p)$ and denote by $R(\mathscr{L})$ the rang of $\mathscr{L}$:

$$R(\mathscr{L}) := \Big\{F \in \mathbb{C}^{\mathcal{E}} :\ \text{ there exists} f \in \mathcal{H} \text{ such that } F = \mathscr{L}(f)\Big\}.$$

Obviously, for any $f \in \mathcal{H}$, it holds that

$$\mathscr{L}(f) = \mathscr{L}(f_-) + \mathscr{L}(f_+) = \mathscr{L}(f_+),$$

which implies that $\mathscr{L}(N(\mathscr{L})^{\perp}) = R(\mathscr{L})$. Define the space H_K as the set of all $F \in R(\mathscr{L})$ satisfying $\|F\|_{H_K} < \infty$, where

$$\|F\|_{H_K} := \|f_+\|_{\mathcal{H}}, \quad F = \mathscr{L}(f)\ \&\ f \in \mathcal{H}, \tag{2.2}$$

and denote by $\langle \cdot, \cdot \rangle_{H_K}$ the inner product generated by $\| \cdot \|_{H_K}$ via the polarization identity. In [12], Qian proved the following result:

Proposition 2.1 ([12, p3]) (i) The space H_K is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_{H_K}$.

(ii) Via the mapping $\mathscr{L}$, the space H_K is isometric with $N(\mathscr{L})^\perp$.

(iii) Define that

$$K(q,p) := \langle h_q, h_p\rangle_{\mathcal{H}}.$$

The function $K(q,p)$ is the reproducing kernel of H_K.

When studying linear operators in Hilbert spaces, the $\mathcal{H}$-H_K formulation is very helpful. In particular, in practice, this formulation is applied to offer fast converging numerical solutions; see [12, 13]. In our two contexts, $\mathcal{E}$ is $\mathbb{R}^{n+1}_+$ and $\mathcal{H}$ is $L^2(\mathbb{R}^n)$. The kernel functions h_p are the respective integral kernels giving rise to the solutions, the latter being in the image of Hilbert spaces which are the reproducing kernel spaces H_K. Since the dictionary generated by the parameterized kernels h_p is dense, H_K is isometric with $L^2(\mathbb{R}^n)$ in each of our two cases. In the case where $u = \mathscr{L}f$, noting that the reproducing kernel of H_K is K_p, then $u(p) = \langle u,\ K_p\rangle_{H_K}$, which, together with the definition of $\mathscr{L}$, indicates that

$$\langle u,\ K_p\rangle_{H_K} = u(p) = \mathscr{L}(f)(p) = \langle f,\ h_p\rangle_{L^2(\mathbb{R}^n)}.$$

Now we state briefly our algorithm: the pre-orthogonal adaptive Fourier decomposition (POAFD), used in the article. Fundamentally, the POAFD can be seen as a modified greed algorithm of sparse representations. The main idea of the POAFD algorithm combines the features and advantages of both the greedy algorithm and the adaptive Fourier decomposition (AFD). On the one hand, like the greedy principle, the POAFD algorithm makes the locally optimal choice at each step to seek a global optimum. On the other hand, similarly to the AFD decomposition, which induces the rational approximation via Blaschke products with multiple zeros, in order to construct the approximation, the POAFD algorithm selects multiple kernels with the same parameters in the process of Gram-Schmidt orthogonalization.

Definition 2.2 Let H_K be a reproducing kernel Hilbert space with the reproducing kernel $K_q,\ q\in\mathcal{E}$. In the $\mathcal{H}-H_K$ formulation, a dictionary can be constituted by the normalization of $\{K_q\}_{q\in\mathcal{E}}$, denoted by $\{E_q\}_{q\in\mathcal{E}}$, which is defined as

$$E_q := \frac{K_q}{\|K_q\|_{H_K}}.$$

The purpose of the so-called POAFD method is to find, consecutively, a sequence $q_1,\cdots,q_k,\cdots$ in $\mathcal{E}$ so that

$$q_k = s_k + y_k = \arg\sup_{q\in\mathcal{E}}\Big\{|\langle f, E^q_k\rangle|\Big\},$$

where $(E_1,\cdots,E_{k-1},E^q_k)$ is the Gram-Schmidt process of the kernels $(\widetilde{K}_{q_1},\cdots,\widetilde{K}_{q_{k-1}},\widetilde{K}_q)$, in which the Gram-Schmidt orthogonalization of $(\widetilde{K}_{q_1},\cdots,\widetilde{K}_{q_{k-1}})$ is denoted by $(E_1,\cdots,E_{k-1})$.

The POAFD type sparse representation of the initial or boundary data f is, with a sequence of suitable constants $\{c_n\}$,

$$f = \sum_{k=1}^{\infty}\langle f, E_k\rangle E_k = \sum_{k=1}^{\infty} c_k\widetilde{K}_{q_k}.$$

The solutions of both problems (1.1) and (1.2) are of the same form:

$$u(p) = \langle f, K_p\rangle = \sum_{k=1}^{\infty} c_k\langle\widetilde{K}_{q_k}, K_p\rangle = \sum_{k=1}^{\infty} c_k\widetilde{K}_{q_k}(p),\ \ p = t + x.$$

In the (1.1) case, owing to the semigroup property, the solution may be rewritten as

$$u(t+x)=\sum_{k=1}^{\infty}c_k\widetilde{K}_{\alpha,t+s}(x-y_k)$$

(see Section 3). In the (1.2) case, there is no semigroup property, and one has to compute that

$$\widetilde{K}_{q_k}(p)=\langle\widetilde{K}_{q_n},K_p\rangle,$$

according to (4.2). We finally note that both of the solutions have the same convergence rate: the error for the N-partial sum in the norm sense is $O(\frac{1}{\sqrt{N}})$ (see [14]).

3 Sparse Representation of Fractional Heat Equations

In this section, we study the sparse representation of approximation to the identity via the fractional heat equations (1.1). Throughout the rest of this paper, we take the Hilbert space $\mathcal{H}=L^2(\mathbb{R}^n)$, defined as the set of all complex-valued measurable functions f over $\mathbb{R}^n$ satisfying that

$$\int_{\mathbb{R}^n}|f(x)|^2\mathrm{d}x<\infty.$$

The inner product on $L^2(\mathbb{R}^n)$ is defined via the Lebesgue integral as

$$\langle f,\ g\rangle:=\int_{\mathbb{R}^n}f(x)\overline{g(x)}\mathrm{d}x,\quad \forall\ f,g\in L^2(\mathbb{R}^n).$$

The corresponding norm of $L^2(\mathbb{R}^n)$ is

$$\|f\|^2_{L^2(\mathbb{R}^n)}=\langle f,\ f\rangle:=\int_{\mathbb{R}^n}|f(x)|^2\mathrm{d}x.$$

For the initial value problem (1.1), the unique solution $u(t+x)$ can be represented by the fractional integral transform

$$u(t+x)=\mathscr{L}(f)(t+x):=\int_{\mathbb{R}^n}K_{\alpha,t}(x-y)f(y)\mathrm{d}y.$$

We note that the notation $t+x$ has the meaning $(t,(x_1,\cdots,x_n))$, with $x=(x_1,\cdots,x_n)\in\mathbb{R}^n$. By the Fourier transform, the fractional Laplacian $(-\Delta)^\alpha$ can be equivalently represented as, for $\xi=(\xi_1,\xi_2,\cdots,\xi_n)$,

$$\widehat{(-\Delta)^\alpha}f(\xi)=|\xi|^{2\alpha}\widehat{f}(\xi).$$

The fractional heat semigroup can be represented as

$$(\mathrm{e}^{\widehat{-t(-\Delta)^\alpha}})f(\xi)=\mathrm{e}^{-t|\xi|^{2\alpha}}\widehat{f}(\xi).$$

Denote by $K_{\alpha,t}(\cdot)$ the integral kernel related with $\mathrm{e}^{-t(-\Delta)^\alpha}$, i.e., that

$$\mathrm{e}^{-t(-\Delta)^\alpha}f(x)=\int_{\mathbb{R}^n}K_{\alpha,t}(x-y)f(y)\mathrm{d}y.$$

By the inverse Fourier transform, the fractional heat kernel can be represented as

$$K_{\alpha,t}(x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\mathrm{e}^{-t|\xi|^{2\alpha}}\mathrm{e}^{\mathrm{i}x\xi}\mathrm{d}\xi. \tag{3.1}$$

One has the following basic estimate:

Proposition 3.1 ([1, 8, 17]) Under $\alpha \in (0,1)$, the fractional heat kernel satisfies the estimates

$$K_{\alpha,t}(x) \simeq \frac{t}{(t^{1/(2\alpha)}+|x|)^{n+2\alpha}}, \quad \forall\ (x,t) \in \mathbb{R}^{n+1}_+.$$

Because $\{\mathrm{e}^{-t(-\Delta)^\alpha}\}_{t\geq 0}$ is a strongly continuous semigroup in $L^2(\mathbb{R}^n)$,

$$\lim_{t\to 0+} u(t+x) = f(x)$$

in both the sense of $L^2(\mathbb{R}^n)$ and the pointwise sense almost everywhere. Based on this fact, we denote that $f(x) = u(0+x)$. Due to the isometric relation it holds that

$$\|u\|^2_{H_K} = \|f\|^2_{L^2(\mathbb{R}^n)} = \|u(0+x)\|^2_{L^2(\mathbb{R}^n)}. \tag{3.2}$$

To construct the approximation, we apply the Plancherel theorem to deduce the following semigroup property: for $q = t+x$ and $p = s+y$,

$$\begin{aligned} K(q,p) &= \langle h_q,\ h_p\rangle_{L^2} = \langle K_{\alpha,t+x}(\cdot),\ K_{\alpha,s+y}(\cdot)\rangle_{L^2} \\ &= \int_{\mathbb{R}^n} \mathrm{e}^{-(t+s)|\xi|^{2\alpha}}\mathrm{e}^{\mathrm{i}(x-y)\xi}\mathrm{d}\xi = K_{\alpha,t+s}(x-y). \end{aligned}$$

Denote that $h_q(y) = K_{\alpha,t}(x-y)$, where $q = t+x$. For $t+s>0$, let $q_1 = t+s+x$ and $q_2 = t+s+y$. Then $h_{q_1}(y) = K_{\alpha,s+t}(x-y)$ and $h_{q_2}(x) = K_{\alpha,s+t}(y-x)$. Hence,

$$K(q,p) = h_{t+s+y}(x) = h_{t+s+x}(y) = h_{t+s+x-y}(0).$$

On the other hand, it follows from the Fourier transform and the change of variable $\xi = -\eta$ that

$$K_{\alpha,s+t}(x-y) = \int_{\mathbb{R}^n} \mathrm{e}^{-(t+s)|\xi|^{2\alpha}}\mathrm{e}^{\mathrm{i}(x-y)\xi}\mathrm{d}\xi = \int_{\mathbb{R}^n} \mathrm{e}^{-(t+s)|\eta|^{2\alpha}}\mathrm{e}^{\mathrm{i}(y-x)\eta}\mathrm{d}\eta = K_{\alpha,s+t}(y-x).$$

Now we give the sparse representation theorem related to the fractional Laplace equations.

Theorem 3.2 The dictionary of the fractional heat kernels satisfies BVC. As a consequence, the POAFD algorithm can be performed in this context to obtain the sparse representation of functions in $\mathcal{H} = L^2(\mathbb{R}^n)$.

Proof For $q = t+x$, we can compute the norm $\|K_q\|_{H_K}$ as follow:

$$\|K_q\|^2_{H_K} = \langle K_q,\ K_q\rangle_{H_K} = h_{2t+x-x}(0) = h_{2t+0}(0).$$

By the Fourier transform,

$$\begin{aligned} h_{2t+0}(0) = K_{\alpha,2t}(0) &= \int_{\mathbb{R}^n} \mathrm{e}^{-2t|\xi|^{2\alpha}}\mathrm{e}^{\mathrm{i}0\cdot\xi}\mathrm{d}\xi = C_n \int_0^\infty \mathrm{e}^{-2t|\xi|^{2\alpha}}|\xi|^{n-1}\mathrm{d}|\xi| \\ &= \frac{C_n}{(2t)^{n/2\alpha}} \int_0^\infty \mathrm{e}^{-u^{2\alpha}}u^{n-1}\mathrm{d}u = C_{n,\alpha}(2t)^{-n/(2\alpha)}. \end{aligned}$$

Hence, for $q = t+x$ & $p = s+y$, the normalization of the reproducing kernel K_q can be expressed as

$$\|K_q\|^2_{H_K} = \langle K_q,\ K_q\rangle_{L^2} = \frac{C_{n,\alpha}}{(2t)^{n/(2\alpha)}}.$$

Set that

$$E_q = \frac{K_q}{\|K_q\|_{H_K}} = \left(\frac{(2t)^{n/(2\alpha)}}{C_{n,\alpha}}\right)^{1/2} K_q$$

such that $\|E_q\|_{H_K}=1$. Then

$$E_q(p)=\frac{(2t)^{n/(4\alpha)}}{\sqrt{C_{n,\alpha}}}h_{t+s+x}(y)=\frac{(2t)^{n/(4\alpha)}}{\sqrt{C_{n,\alpha}}}K_{\alpha,t+s}(x-y).$$

Below, we begin to prove BVC, i.e., for $u\in H_K$,

$$\lim_{\mathbb{R}^{n+1}_+\ni q\to\partial^*\mathbb{R}^{n+1}_+}|\langle u,\ E_q\rangle_{H_K}|=0.$$

Here we use the one point compactification topology in which $\partial^*\mathbb{R}^{n+1}_+$ stands for the boundary of $\mathbb{R}^{n+1}_+$ in the topology. Since the span of $\{K_p\}_{p\in\mathbb{R}^{n+1}_+}$ is dense in $L^2(\mathbb{R}^n)$, it suffices to verify that, for any fixed $p=s+y$,

$$\lim_{q\to\partial^*\mathbb{R}^{n+1}_+}|\langle K_p,\ E_q\rangle_{H_K}|=0. \tag{3.3}$$

Applying Proposition 3.1,

$$\langle K_p,\ E_q\rangle_{H_K}=E_{p,q}=\frac{(2t)^{n/(4\alpha)}}{\sqrt{C_{n,\alpha}}}K_{\alpha,s+t+x}(y),$$

we can see that there exist two constants $c_1<c_2$ which are independent of $(x,t)\in\mathbb{R}^{n+1}_+$ such that

$$\frac{c_1(t+s)}{((t+s)^{1/(2\alpha)}+|x-y|)^{n+2\alpha}}\le K_{\alpha,s+t+x}(y)\le\frac{c_2(t+s)}{((t+s)^{1/(2\alpha)}+|x-y|)^{n+2\alpha}}.$$

Hence the statement (3.3) is equivalent to the following:

$$\lim_{q=t+x\to\partial^*\mathbb{R}^{n+1}_+}\frac{(2t)^{n/(4\alpha)}}{\sqrt{C_{n,\alpha}}}\frac{(t+s)}{((t+s)^{1/(2\alpha)}+|x-y|)^{n+2\alpha}}=0. \tag{3.4}$$

Now we begin to prove (3.4). We can obtain that

$$\frac{(2t)^{n/(4\alpha)}}{\sqrt{C_{n,\alpha}}}\frac{t+s}{((t+s)^{1/(2\alpha)}+|x-y|)^{n+2\alpha}}\lesssim\frac{(2t)^{n/(4\alpha)}}{\sqrt{C_{n,\alpha}}}\frac{t+s}{(t+s)^{(n+2\alpha)/(2\alpha)}}\lesssim\left(\frac{t}{t+s}\right)^{n/(4\alpha)},$$

which implies that

$$\lim_{t\to0+}|\langle K_p,\ E_q\rangle_{H_K}|=0.$$

Now we analyze the process $R\to\infty$, where $R=\sqrt{t^{1/\alpha}+|x|^2}$. We split the rest of the proof into two cases.

Case 1: $t^{1/(2\alpha)}<\frac{1}{2}\sqrt{t^{1/\alpha}+|x|^2}=R/2$. For this case, $|x|>\sqrt{3}t^{1/(2\alpha)}$ and $\sqrt{3}R<|x|$. Without loss of generality, we assume that $R>4|y|$, so

$$|x-y|\ge|x|-|y|\ge\sqrt{3}R-R/4.$$

Now we can get that

$$\begin{aligned}\frac{(2t)^{n/(4\alpha)}}{\sqrt{C_{n,\alpha}}}K_{\alpha,t+s+x}(y)&\lesssim\left(\frac{2t}{C_{n,\alpha}}\right)^{n/(4\alpha)}\frac{t+s}{((t+s)^{1/(2\alpha)}+|x-y|)^{n+2\alpha}}\\&\lesssim(R/2)^{n/2}\frac{t+s}{(t+s)|x-y|^n}\lesssim\frac{1}{R^{n/2}},\end{aligned}$$

which gives that

$$\lim_{R\to\infty}|\langle K_p,\ E_q\rangle_{H_K}|=0.$$

Case 2: $t^{1/(2\alpha)} \geq R/2$. For $s > 0$, we can also get that

$$\begin{aligned}\frac{(2t)^{n/(4\alpha)}}{\sqrt{C_{n,\alpha}}} K_{\alpha,t+s+x}(y) &\lesssim \frac{(2t)^{n/(4\alpha)}}{\sqrt{C_{n,\alpha}}} \frac{t+s}{((t+s)^{1/(2\alpha)} + |x-y|)^{n+2\alpha}} \\ &\lesssim \frac{(2t)^{n/(4\alpha)}}{\sqrt{C_{n,\alpha}}} \frac{t+s}{(t+s)^{(n+2\alpha)/(2\alpha)}} \\ &\lesssim \frac{1}{(t+s)^{n/(4\alpha)}} \lesssim \frac{1}{R^{n/(4\alpha)}}.\end{aligned}$$

We can also get that $\lim\limits_{R\to\infty} |\langle K_p,\ E_q\rangle_{H_K}| = 0$. □

By the above theorem, the initial data f of (1.1) has a POAFD sparse representation.

Theorem 3.3 Let, under the POAFD scheme, f possess a sparse representation

$$f(x) = \sum_{k=1}^{\infty} \langle f, E_k\rangle E_k,$$

where, for each N, the N-orthonormal system $(E_1, \cdots, E_N)$ corresponds to the POAFD maximally selected $(q_1, \cdots, q_n), q_k = s_k + y_k, k = 1, \cdots, N$. Then there exist constants $c_1, \cdots, c_N, \cdots$, such that

$$f(x) = \sum_{k=1}^{\infty} c_k \widetilde{K}_{\alpha,s_k}(y_k - x),$$

and

$$u_f(t+x) = \sum_{k=1}^{\infty} c_k \widetilde{K}_{\alpha,t+s_k}(y_k - x)$$

is the solution of (1.1).

4 Sparse Representation of Fractional Poisson Equations

Given a regular function f on $\mathbb{R}^n$, if u is a solution to (1.2) with the initial data f, $u(x,t) = P_\alpha f(x,t)$ is said to be the Caffarelli-Silvestre extension of f to the upper half-space $\mathbb{R}^{n+1}_+$. For a smooth function f, the Caffarelli-Silvestre extension of f can be expressed as the convolution of f and the generalized (fractional) Poisson kernel can be defined as

$$p_t^\alpha(x) := \frac{c(n,\alpha)t^\alpha}{(|x|^2 + t^2)^{(n+\alpha)/2}}.$$

More precisely,

$$P_\alpha f(x,t) = p_t^\alpha * f(x,t) = c(n,\alpha) \int_{\mathbb{R}^n} \frac{t^\alpha}{(|x-y|^2 + t^2)^{(n+\alpha)/2}} f(y)\mathrm{d}y.$$

Here $f * g$ means the convolution of f and g, and the constant

$$c(n,\alpha) = \frac{\Gamma((n+\alpha)/2)}{\pi^{n/2}\Gamma(\alpha/2)}$$

is chosen such that $\int_{\mathbb{R}^n} p_t^\alpha(x)\mathrm{d}x = 1$.

Now we investigate the sparse representation via the generalized Poisson equations. The fractional Poisson kernel satisfies the following estimates:

Proposition 4.1 ([11, Proposition 2.1]) (i) It holds that

$$\widehat{p_t^\sigma}(\xi) = \frac{c(n,\sigma)}{\Gamma((n+\sigma)/2)} \int_0^\infty \lambda^{\sigma/2} \mathrm{e}^{-\lambda - |t\xi|^2/(4\lambda)} \frac{\mathrm{d}\lambda}{\lambda} := c(n,s) G_\sigma(t|\xi|),$$

where

$$G_\sigma(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi\mathrm{i}\xi\cdot x} \frac{\mathrm{d}x}{(1+|x|^2)^{(n+\sigma)/2}}.$$

(ii) For some positive constant c,

$$G_\sigma(\xi) = \begin{cases} 1+|\xi|^\sigma, \ \xi \text{ near the origin;} \\ O(\mathrm{e}^{-c\xi}), \ |\xi| \to \infty. \end{cases}$$

By the change of variable, it is easy to verify that

$$\begin{aligned} \int_{\mathbb{R}^n} p_t^\sigma(x)\mathrm{d}x &= c(n,\sigma) \int_{\mathbb{R}^n} \frac{t^\sigma}{(t^2+|x|^2)^{(n+\sigma)/2}} \mathrm{d}x \\ &= c(n,\sigma) \int_0^\infty \frac{1}{(1+u^2)^{(n+\sigma)/2}} u^{n-1} \mathrm{d}u = 1. \end{aligned}$$

Let $\mathcal{H} = L^2(\mathbb{R}^n)$ and $E = \mathbb{R}^{n+1}_+$. For $t, s > 0$ and $x, y \in \mathbb{R}^n$, set that $q = t+x$ and $p = s+y$. Define $h_p(y)$ as

$$h_p(y) = \frac{c(n,\sigma) t^\sigma}{(t^2+|x-y|^2)^{(n+\sigma)/2}}. \tag{4.1}$$

The space H_K is defined as

$$H_K := \Big\{ u : \mathbb{R}^{n+1}_+ \to \mathbb{R}, \ u(p) = \langle f, \ h_p \rangle_{L^2(\mathbb{R}^n)} \Big\}.$$

The reproducing kernel is computed as

$$\begin{aligned} K_q(p) = K(q,p) &= \langle h_q, \ h_p \rangle_{L^2(\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^n} \frac{t^\sigma}{(t^2+|x-z|^2)^{(n+\sigma)/2}} \frac{s^\sigma}{(s^2+|y-z|^2)^{(n+\sigma)/2}} \mathrm{d}z. \end{aligned} \tag{4.2}$$

Once we prove the next theorem, then the solution of (1.2) will follow from the scheme set in the end of Section 2.

Theorem 4.2 The fractional Poisson kernel $\mathcal{H}$-H_K structure satisfies the BVC. As a consequence, the POAFD algorithm can be performed in this context to obtain the sparse representation of functions in $\mathcal{H} = L^2(\mathbb{R}^n)$.

Proof We first compute the norm $\|K_q\|_{H_K}$. In fact,

$$\begin{aligned} \|K_q\|^2_{H_K} &= \langle K_q, \ K_q \rangle_{H_K} = K(q,q) = \langle p_t^\sigma(\cdot), \ p_t^\sigma(\cdot) \rangle \\ &= \int_{\mathbb{R}^n} \frac{t^{2\sigma}}{(t^2+|x-z|^2)^{n+\sigma}} \mathrm{d}z \\ &= \frac{1}{t^n} \int_0^\infty \frac{u^{n-1}}{(1+u^2)^{n+\sigma}} \mathrm{d}u = \frac{c_{n,\sigma}}{(2t)^n}. \end{aligned}$$

Then we define that

$$E_q = \frac{K_q}{\|K_q\|_{H_K}} = \left(\frac{(2t)^n}{c_{n,\sigma}} \right)^{1/2} K_q,$$

so that $\|E_q\|_{H_K} = 1$. Now we verify that the BVC property holds, i.e., that

$$\lim_{q \to \partial E} |\langle u, \ E_q \rangle_{H_K}| = 0.$$

Due to the density of the span of the kernels K_p, this is equivalent to showing that, for any but a fixed $p = s + y$, under the process $q = t + x \to \partial\mathbb{R}^{n+1}_+$,

$$\lim_{q\to\partial E} |\langle K_p,\ E_q\rangle_{H_K}| = 0.$$

By the Plancherel formula,

$$\begin{aligned}\langle K_p,\ E_q\rangle_{H_K} &= \frac{t^{n/2}}{(c_{n,\sigma})^{1/2}}\langle K_p,\ K_q\rangle \\ &= \frac{t^{n/2}}{(c_{n,\sigma})^{1/2}}\int_{\mathbb{R}^n}\frac{t^\sigma}{(t^2+|x-z|^2)^{(n+\sigma)/2}}\frac{s^\sigma}{(s^2+|y-z|^2)^{(n+\sigma)/2}}\mathrm{d}z \\ &= \frac{t^{n/2}}{(c_{n,\sigma})^{1/2}}\int_{\mathbb{R}^n} G_\sigma(t|\xi|)G_\sigma(s|\xi|)\mathrm{e}^{\mathrm{i}(x-y)\cdot\xi}\mathrm{d}\xi \\ &= M_1 + M_2,\end{aligned}$$

where

$$\begin{cases} M_1 := \dfrac{t^{n/2}}{(c_{n,\sigma})^{1/2}}\displaystyle\int_{|\xi|\le 1} G_\sigma(t|\xi|)G_\sigma(s|\xi|)\mathrm{e}^{\mathrm{i}(x-y)\cdot\xi}\mathrm{d}\xi; \\ M_2 := \dfrac{t^{n/2}}{(c_{n,\sigma})^{1/2}}\displaystyle\int_{|\xi|>1} G_\sigma(t|\xi|)G_\sigma(s|\xi|)\mathrm{e}^{\mathrm{i}(x-y)\cdot\xi}\mathrm{d}\xi. \end{cases}$$

By (ii) of Proposition 4.1, we can get that

$$|M_1| \lesssim t^{n/2}\int_{|\xi|\le 1}(1+t^\sigma|\xi|^\sigma)|\mathrm{e}^{\mathrm{i}(x-y)\xi}|\mathrm{d}\xi \lesssim t^{n/2}(1+t^\sigma)\int_{|\xi|\le 1} 1\mathrm{d}\xi \lesssim t^{n/2}(1+t^\sigma).$$

For M_2, we deduce from (ii) of Proposition 4.1 that

$$\begin{aligned}|M_2| &\lesssim t^{n/2}\int_{|\xi|\ge 1}\mathrm{e}^{-(t+s)|\xi|}\mathrm{d}\xi \lesssim t^{n/2}\int_{|\xi|\ge 1}\frac{1}{(1+(t+s)|\xi|)^{n+1}}\mathrm{d}\xi \\ &\lesssim \frac{t^{n/2}}{(t+s)^{n+1}}\int_{|\xi|\ge 1}\frac{|\xi|^{n-1}}{|\xi|^{n+1}}\mathrm{d}|\xi| \lesssim \frac{t^{n/2}}{(t+s)^{n+1}}.\end{aligned}$$

The estimates for M_1 and M_2 imply that

$$|\langle K_p,\ E_q\rangle_{H_K}| \lesssim t^{n/2}(1+t^\sigma) + \frac{t^{n/2}}{(t+s)^{n+1}},$$

which indicates that

$$\lim_{t\to 0+} |\langle K_p,\ E_q\rangle_{H_K}| = 0.$$

Next we deal with the limit as $R = \sqrt{t^2+|x|^2} \to \infty$. Without loss of generality, assume that $R \ge 4|y| + 2s + 1$.

Case 1: $t \ge R/2$. We can get that

$$\begin{aligned}\langle K_p,\ E_q\rangle_{H_K} &= \frac{t^{n/2}}{(c_{n,\sigma})^{1/2}}\int_{\mathbb{R}^n}\frac{t^\sigma}{(t^2+|x-z|^2)^{(n+\sigma)/2}}\frac{s^\sigma}{(s^2+|y-z|^2)^{(n+\sigma)/2}}\mathrm{d}z \\ &\lesssim \frac{t^\sigma}{t^{n+\sigma}}\int_{\mathbb{R}^n}\frac{s^\sigma}{(s^2+|y-z|^2)^{(n+\sigma)/2}}\mathrm{d}z \\ &\lesssim \frac{1}{t^{n/2}}\int_0^\infty\frac{u^{n-1}}{(1+u^2)^{(n+\sigma)/2}}\mathrm{d}u \lesssim \frac{1}{R^{n/2}},\end{aligned}$$

which gives that

$$\lim_{R\to\infty} |\langle K_p,\ E_q\rangle_{H_K}| = 0.$$

Case 2: $0 < t < R/2$. We split

$$\begin{aligned}\langle K_p,\ E_q\rangle_{H_K} &= \frac{t^{n/2}}{(c_{n,\sigma})^{1/2}}\int_{\mathbb{R}^n}\frac{t^\sigma}{(t^2+|x-z|^2)^{(n+\sigma)/2}}\frac{s^\sigma}{(s^2+|y-z|^2)^{(n+\sigma)/2}}\mathrm{d}z\\ &\simeq L_1+L_2,\end{aligned}$$

where

$$\begin{cases} L_1 := \dfrac{t^{n/2}}{(c_{n,\sigma})^{1/2}}\displaystyle\int_{|x-z|\geq|x|/2}\frac{t^\sigma}{(t^2+|x-z|^2)^{(n+\sigma)/2}}\frac{s^\sigma}{(s^2+|y-z|^2)^{(n+\sigma)/2}}\mathrm{d}z;\\ L_2 := \dfrac{t^{n/2}}{(c_{n,\sigma})^{1/2}}\displaystyle\int_{|x-z|\leq|x|/2}\frac{t^\sigma}{(t^2+|x-z|^2)^{(n+\sigma)/2}}\frac{s^\sigma}{(s^2+|y-z|^2)^{(n+\sigma)/2}}\mathrm{d}z.\end{cases}$$

Notice that $0 < t < \frac{1}{2}\sqrt{t^2+|x|^2}$. We can see that $|x| > \sqrt{3}t$, i.e., that $|x| > \sqrt{3}R/2$. For L_1, since $|x-z| > |x|/2 > \sqrt{3}R/4$, we obtain that

$$\begin{aligned}L_1 &\lesssim \frac{t^{n/2+\sigma}}{R^{n+\sigma}}\int_{|x-z|\geq|x|/2}\frac{s^\sigma}{(s^2+|y-z|^2)^{(n+\sigma)/2}}\mathrm{d}z\\ &\lesssim \frac{R^{n/2+\sigma}}{R^{n+\sigma}}\int_{\mathbb{R}^n}\frac{s^\sigma}{(s^2+|y-z|^2)^{(n+\sigma)/2}}\mathrm{d}z \lesssim \frac{1}{R^{n/2}}.\end{aligned}$$

For L_2, since $|x-z| < |x|/2$ and $R > 4|y|$, by the triangle inequality, we have that

$$|y-z| \geq |z|-|y| \geq |x|-|x-z|-|y| \geq |x|-|x|/2-|y| \geq (\sqrt{3}-1)R/4.$$

Then

$$\begin{aligned}L_2 &\lesssim \int_{|x-z|\leq|x|/2}\frac{t^{\sigma+n/2}}{(t^2+|x-z|^2)^{(n+\sigma)/2}}\frac{s^\sigma}{(s^2+|y-z|^2)^{(n+\sigma)/2}}\mathrm{d}z\\ &\lesssim \int_{|x-z|\leq|x|/2}\frac{1}{|y-z|^n}\frac{t^{\sigma+n/2}}{(t^2+|x-z|^2)^{(n+\sigma)/2}}\mathrm{d}z\\ &\lesssim \frac{t^{n/2}}{((\sqrt{3}-1)R/4)^n}\int_{\mathbb{R}^n}\frac{t^\sigma}{(t^2+|x-z|^2)^{(n+\sigma)/2}}\mathrm{d}z \lesssim \frac{1}{R^{n/2}}.\end{aligned}$$

The estimates for I_1 and I_2 indicate that

$$\lim_{R\to\infty}|\langle K_p,\ E_q\rangle_{H_K}| = 0.$$

This completes the proof of Theorem 4.2. □

5 Experiments

In this section, we present two examples to illustrate the effectiveness and validity of the POAFD method. The decomposition results are shown in Figures 1 and 3. In Figures 2 and 4, we give the corresponding parameters and relative errors.

Example 5.1 This example is for the problem (1.1) with the initial value

$$f = \frac{0.5}{0.25+(0.5-x)^2}+\frac{2\pi}{1+(0.5+x)^2}+\frac{0.8}{0.64+(1+x)^2}.$$

For a fixed α, we use the dictionary $K_q(\cdot) = K_{\alpha,t}(x-\cdot), q = t+x \in \mathbb{R}^{n+1}_+$, where $K_{\alpha,t}(x)$ is as in (3.1).

Let $K^\top = (K_{q_1}, \cdots, K_{q_{13}})$ and $E^\top = (E_1, \cdots, E_{13})$. We have that $K = A^\top E$, where

$$A = \begin{pmatrix}
1 & 0.37 & 0.99 & 0.49 & 0.75 & 0.79 & 0.30 & 0.91 & 0.42 & 0.66 & 0.65 & 0.23 & 0.42 & 0.91 \\
0 & 0.93 & -0.06 & 0.84 & 0.31 & 0.00 & 0.50 & 0.14 & 0.18 & -0.04 & 0.22 & 0.86 & 0.59 & 0.03 \\
0 & 0 & 0.14 & -0.11 & -0.34 & -0.34 & -0.12 & -0.24 & -0.35 & -0.32 & -0.34 & 0.08 & -0.16 & -0.26 \\
0 & 0 & 0 & 0.20 & 0.16 & -0.12 & 0.49 & 0.03 & 0.19 & -0.16 & 0.12 & -0.14 & 0.46 & -0.07 \\
0 & 0 & 0 & 0 & 0.45 & -0.26 & -0.18 & 0.19 & -0.18 & -0.29 & 0.62 & 0.00 & -0.16 & -0.19 \\
0 & 0 & 0 & 0 & 0 & 0.42 & -0.03 & -0.14 & -0.11 & 0.56 & 0.02 & 0.01 & -0.03 & 0.22 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.61 & 0.02 & -0.08 & 0.01 & 0.03 & -0.10 & 0.44 & 0.00 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.17 & -0.16 & 0.08 & -0.02 & 0.01 & -0.01 & -0.09 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.74 & -0.02 & -0.01 & -0.01 & 0.05 & -0.01 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.16 & 0.01 & 0.00 & -0.01 & -0.01 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.12 & 0.03 & 0.02 & 0.01 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.42 & -0.04 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.10 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.10
\end{pmatrix}_{14\times 14}.$$

$$\left\| f - \sum_{k=1}^{13} c_k \widetilde{K}_{q_k} \right\| \le 1.1668 \times 10^{-4}.$$

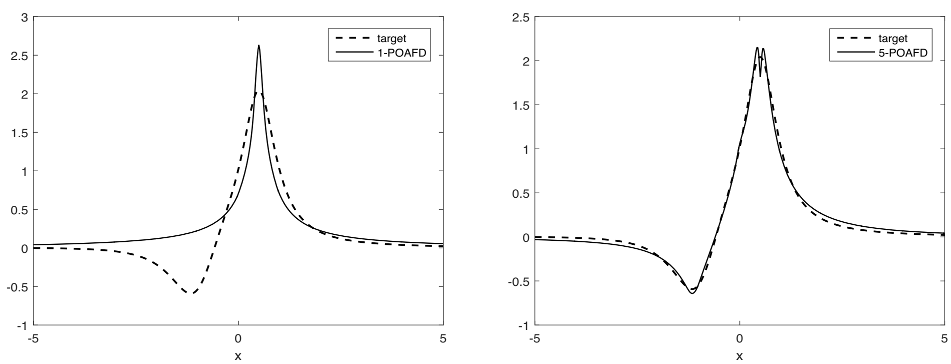
The isometric relation then gives that

$$\left\| u_f - \sum_{k=1}^{13} c_k \widetilde{K}_{q_k} \right\|_{H_K} \le 1.1668 \times 10^{-4},$$

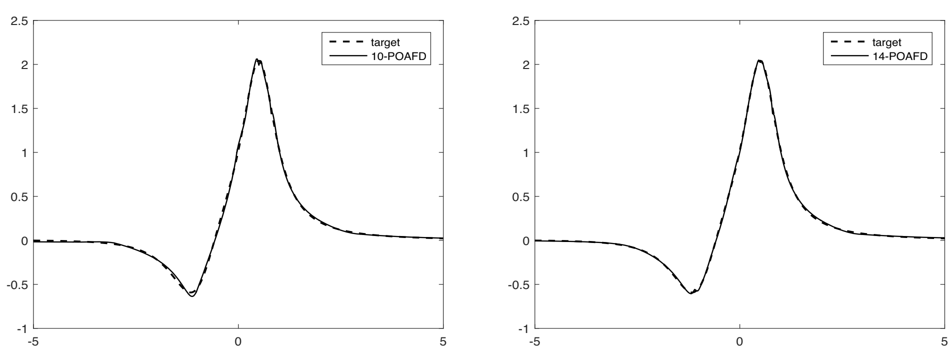
where the coefficient c_k and $\langle f, E_k \rangle$ are given in Table 1.

Table 1 The coefficients of K_{q_k} and E_k

c_k	-33.8652	0.7736	-6.1651	0.0093	-0.2163	-0.2053	-0.1342	0.1869	-0.0153	-0.0916	-0.0593	0.0238	0.0642	0.0551
$\langle f, E_k \rangle$	0.5183	-0.2303	-0.1278	-0.0515	0.0462	0.0254	0.0249	0.0187	-0.0128	-0.0139	-0.0102	0.0072	0.0066	0.0056

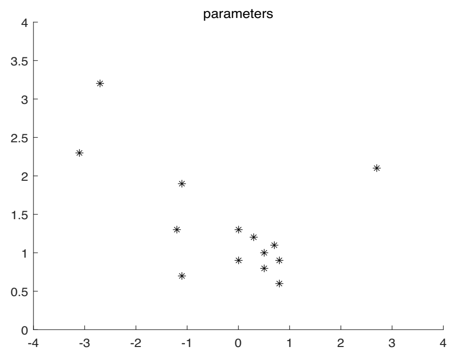


POAFD: 1 partial sum POAFD: 5 partial sum

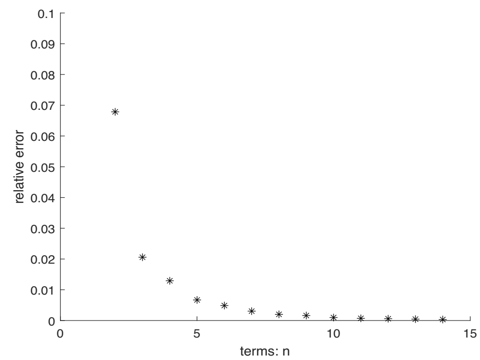


POAFD: 10 partial sum POAFD: 14 partial sum

Figure 1 Approximate f by POAFD with heat integral kernel on the upper half plane



parameters



relative error

Figure 2 Parameters and relative error

Example 5.2 The next example is for the problem (1.2) with the initial value

$$f = \sum_{j=1}^{5} a_j \mathrm{e}^{-\frac{|b_j - x|^2}{a_j}} \cos x,$$

where $a_j = 0.5, 3.1, 2.4, 0.2, 1.6$ and $b_j = 1, 0.6, -0.04, 2.3, -2$. Now let the dictionary consist of $K_q(y) = h_p(y)$ given by (4.1).

Let $K^\top = (K_{q_1}, \cdots, K_{q_{13}})$ and $E^\top = (E_1, \cdots, E_{13})$. We have that $K = A^\top E$, where

$$A = \begin{pmatrix}
1 & 0.24 & 0.26 & 0.99 & 0.34 & 0.21 & 0.92 & 0.33 & 0.65 & 0.44 & 0.24 & 0.83 & 0.14 & 0.50 & 0.08 \\
0 & 0.97 & 0.05 & -0.04 & 0.07 & 0.57 & 0.16 & 0.92 & 0.13 & 0.52 & 0.04 & -0.01 & 0.52 & 0.02 & 0.03 \\
0 & 0 & 0.97 & -0.04 & 0.92 & -0.01 & 0.20 & 0.03 & 0.60 & 0.02 & 0.91 & -0.07 & 0.02 & 0.30 & 0.25 \\
0 & 0 & 0 & 0.12 & -0.07 & -0.11 & -0.25 & -0.07 & -0.26 & -0.28 & -0.02 & -0.11 & -0.02 & -0.38 & -0.04 \\
0 & 0 & 0 & 0 & 0.17 & -0.03 & 0.09 & 0.00 & 0.32 & -0.07 & 0.13 & -0.14 & 0.02 & 0.38 & 0.28 \\
0 & 0 & 0 & 0 & 0 & 0.79 & 0.03 & 0.12 & 0.00 & 0.45 & 0.00 & -0.02 & -0.13 & -0.05 & 0.00 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.10 & 0.06 & 0.12 & 0.23 & -0.06 & -0.25 & 0.18 & 0.32 & 0.00 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.12 & -0.06 & 0.18 & 0.02 & 0.13 & 0.62 & -0.27 & 0.06 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.10 & -0.26 & -0.05 & -0.06 & 0.24 & 0.36 & -0.07 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.31 & 0.00 & 0.08 & -0.24 & -0.02 & -0.05 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.29 & -0.04 & 0.03 & 0.09 & -0.35 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.42 & -0.03 & 0.07 & 0.02 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.40 & -0.01 & -0.05 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.22 & -0.29 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.79
\end{pmatrix}_{15\times 15}.$$

$$\left\| f - \sum_{k=1}^{13} c_k \widetilde{K}_{q_k} \right\| \le 1.3486 \times 10^{-4}.$$

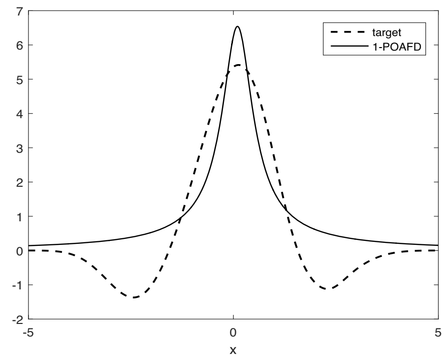
We also conclude that

$$\left\| u_f - \sum_{k=1}^{13} c_k \widetilde{K}_{q_k} \right\|_{H_K} \le 1.3486 \times 10^{-4},$$

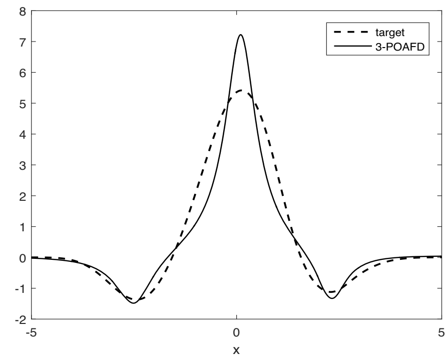
where the coefficient c_k and $\langle f, E_k \rangle$ are given in Table 2.

Table 2 The coefficients of K_{q_k} and E_k

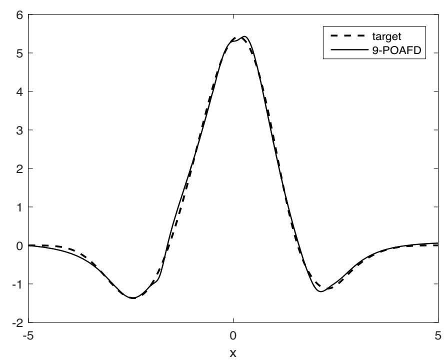
c_k	18.3	-1.39	1.11	-0.64	-0.49	-0.04	-3.03	0.26	0.12	-0.39	0.09	0.04	0.06	0.07	0.01
$\langle f, E_k \rangle$	1.85	-0.6	-0.5	-0.51	-0.15	-0.13	0.16	-0.09	-0.08	-0.07	0.03	0.03	0.02	0.01	0.01



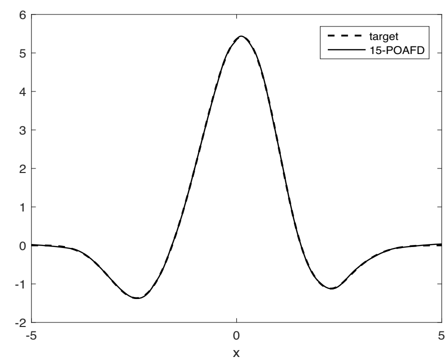
POAFD: 1 partial sum



POAFD: 3 partial sum

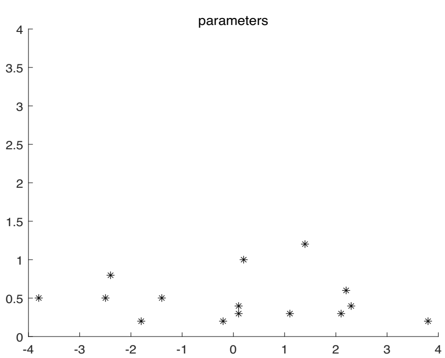


POAFD: 9 partial sum

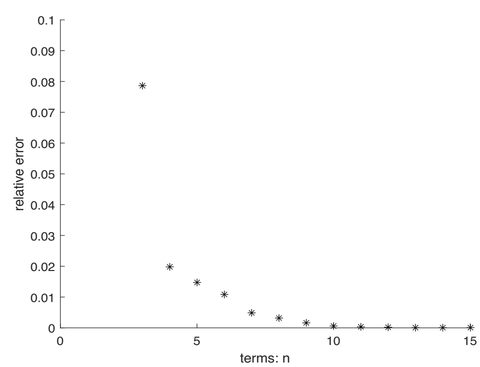


POAFD: 15 partial sum

Figure 3 Approximate f by POAFD with Poisson kernel on the upper half plane



parameters



relative error

Figure 4 Parameters and relative error

Conflict of Interest The authors declare no conflict of interest.

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