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Sparse series solutions of random boundary and initial value problems

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The study investigates the utilization of sparse representations through dictionary elements to address the classical Dirichlet and Cauchy types of stochastic boundary value problems (BVPs) and initial value problems (IVPs). A novel approach is introduced based on the recently developed stochastic pre-orthogonal adaptive Fourier decomposition (SPOAFD) technique. By employing SPOAFD, both analytic and numerical solutions for the stochastic BVPs and IVPs are formulated. Furthermore, the scope of the study is extended to include BVPs and IVPs associated with a specific class of fractional heat equations and fractional Poisson equations. In addition to establishing the theoretical framework, important computational aspects are thoroughly discussed to enable the implementation of practical algorithms. The proposed methodology is validated through numerical examples, demonstrating its effectiveness and computational efficiency.

Keywords: Stochastic BVPs; stochastic IVPs; numerical PDEs; sparse series representation; reproducing kernel Hilbert space; stochastic Hardy space; Bochner space; fractional heat equation; fractional Poisson equation; fractional Laplacian operator.

AMS Subject Classification: 41A99; 60H15

1. Introduction

In many physical and engineering problems, it is crucial to give effective numerical solutions of stochastic partial differential equations (SPDEs).[1,16,20,22,26–29,55] The method we use is approximation by linear combination of parameterized kernels in respective function spaces. Throughout the paper we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, where a general sample point is denoted ω. We will use D for the domain of the parameters (open and connected) in the homogeneous $\mathbb{R}^n$ or the in homogeneous $\mathbb{R}^{1+n}_+ = \{t + \underline{x} : t > 0, \underline{x} \in \mathbb{R}^n\}$. The notation $t + \underline{x}$ is borrowed from Clifford algebra in which $t \in \mathbb{R}_+$, the set of positive real numbers, and $\underline{x} = x_1\mathbf{e}_1 + \cdots x_n\mathbf{e}_n$, as a vector expressed as a real-linear combination of the basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$, belonging to the homogeneous n-dimensional Euclidean space $\mathbb{R}^n$. We will, nevertheless, not involve multiplication between vectors. We restrict ourselves to deal with the cases where D is either the unit ball or the upper half space. In $\mathbb{R}^n$ denote by $B_r(a)$ the open ball centered at a with radius r, and $\partial B_r(a)$ its boundary, called the r-sphere. $B_1(0)$ is written as B_1 for short. We will first deal with the stochastic Dirichlet boundary value problem (SDBVP) for the Laplacian with a random boundary condition

$$\begin{cases} \Delta_{\underline{x}} u(\underline{x}, \omega) = 0, \quad \underline{x} \in B_1, \ \text{a.s. } \omega \in \Omega, \\ u(\underline{x}, \omega) = f(\underline{x}, \omega), \quad \underline{x} \in \partial B_1, \ \text{a.s. } \omega \in \Omega, \end{cases} \tag{1.1}$$

and the stochastic Cauchy initial value problem (SCIVB) of the heat equation with a random initial condition

$$\begin{cases} (\partial_t - \Delta_{\underline{x}}) u(t + \underline{x}, \omega) = 0, \quad (t, \underline{x}) \in \mathbb{R}^1_+ \times \mathbb{R}^n, \ \text{a.s. } \omega \in \Omega, \\ u(0 + \underline{x}, \omega) = f(\underline{x}, \omega), \quad \underline{x} \in \mathbb{R}^n, \ \text{a.s. } \omega \in \Omega, \end{cases} \tag{1.2}$$

where $\Delta_{\underline{x}}$ is the standard Laplacian for the n variables $x_1, \ldots, x_n$. The SDBVP can also be asked for the upper half space instead of (1.1) for the unit ball B_1.

After treating the classical cases we will use the SPOAFD method to solve two more general types of random initial value problems (IVPs), the first being the fractional heat equation involving fractional Laplacian operator,

$$\begin{cases} \dfrac{\partial u}{\partial t}(t+\underline{x},\omega) = (-\Delta)^{\alpha} u(t+\underline{x},\omega), & (t,\underline{x}) \in \mathbb{R}^{1+n}_{+}, \text{ a.s. } \omega \in \Omega, \\ u(0+\underline{x},\omega) = f(\underline{x},\omega), & \underline{x} \in \mathbb{R}^n, \text{ a.s. } \omega \in \Omega, \end{cases} \tag{1.3}$$

and the second being related to the stochastic fractional Poisson equation

$$\begin{cases} \operatorname{div}(t^{\sigma}\nabla u)(t+\underline{x},\omega) = 0, & (t,\underline{x}) \in \mathbb{R}^{1+n}_{+}, \text{ a.s. } \omega \in \Omega, \\ u(0+\underline{x},\omega) = f(\underline{x},\omega), & \underline{x} \in \mathbb{R}^n, \text{ a.s. } \omega \in \Omega. \end{cases} \tag{1.4}$$

Caffarelli–Silvestre in Ref. 10 introduced the following equations: with $\alpha \in (0,2)$,

$$\begin{cases} \operatorname{div}(t^{1-\alpha}\nabla u) = 0, & (t,\underline{x}) \in \mathbb{R}^{1+n}_{+}; \\ u(0+\underline{x}) = f(\underline{x}), & \underline{x} \in \mathbb{R}^n. \end{cases} \tag{1.5}$$

They proved the following identity: if u is a solution to (1.5) then there exists a constant C_α such that

$$(-\Delta)^{\alpha/2} f(\underline{x}) = -C_\alpha \lim_{t\to t+0} t^{1-\alpha}\partial_t u(t+\underline{x}),$$

where the solution $u(\cdot,\cdot)$ can be represented as

$$u(t+\underline{x}) = \int_{\mathbb{R}^n} \frac{t^\alpha}{(|\underline{x}-\underline{y}|^2+t^2)^{(n+\alpha)/2}} f(\underline{y})d\underline{y}.$$

Here, the convolution kernel

$$p_t^\alpha(\underline{x}) := \frac{c_{n,\alpha}t^\alpha}{(|\underline{x}|^2+t^2)^{(n+\alpha)/2}},$$

can be seen as the fractional counterpart of the classical Poisson kernel, i.e.

$$p_t^1(\underline{x}) = \frac{t}{(t^2+|\underline{x}|^2)^{(n+1)/2}}.$$

Hence, formally, in this paper, we call Eqs. (1.5) the fractional Poisson equations.

Becus and Cozzarelli introduced the concept of generalized random solutions for SPDE problems of the types (1.1) and (1.2). They provided proofs for the existence and uniqueness of these solutions in their works.[6,8,9] Additionally, Becus proposed an iterative scheme for solving the random heat equation and established a corresponding convergence result in Ref. 7. Tasaka presented finite element solutions for the one-dimensional (1D) heat equation with random initial conditions in Ref. 53. Furthermore, the deterministic equations (1.3), (1.4), and a time-space fractional heat equation have been extensively studied in Refs. 46 and 23, along with the relevant references mentioned therein.

In addition to the four types mentioned earlier, there have been extensive studies on more general types of elliptic PDEs and parabolic PDEs with random coefficients or random forcing terms.[25,57] Notably, these studies encompass the investigation of incompressible Navier–Stokes equations as well.[50] The solutions to various types of SPDEs also manifest as random processes or random fields. With the influence of contemporary signal analysis, there is a growing trend to explore sparse decomposition techniques for random processes and random fields. Sparse representations offer demonstrative convergence properties, making them a promising avenue for research in this field.

Numerous numerical methods have been developed for solving SPDEs, such as the Monte Carlo finite element method (FEM), Karhunen–Loéve (KL) expansions, Wiener chaos expansion (WCE) (Fourier–Hermite expansion), FEMs, stochastic Galerkin FEM, and stochastic collocation FEM, among others.[5,25,30–32,58] In this study, we refer to our methodology as sparse representation for we effectively express the signal of interest through linear combinations of adaptively selected dictionary elements, which do not necessarily form a basis. It is important to note that the concept of sparsity used here is broader and distinct from the notion found in compressed sensing literature, where the sparsity of a signal is known a priori, and the task is to reconstruct it. The sparse representation employed in our methodology is primarily of the adaptive Fourier decomposition (AFD) type, with a particular emphasis on the stochastic pre-orthogonal adaptive Fourier decomposition (SPOAFD) introduced in Ref. 38. The previously established deterministic variants of AFD, such as core adaptive Fourier decomposition (Core AFD) or pre-orthogonal Fourier decompositions (POAFD), have found extensive applications in 1D and multi-dimensional signal processing, system identification, and control theory (see Refs. 11,14,15 and references therein). However, this study represents the first attempt to explore the use of AFD methods in solving PDEs. In this paper, our focus will be on solving the four types of SPDEs mentioned earlier, namely, (1.1)–(1.4). We discuss the related algorithms, convergence properties, and applicability issues in detail. Given the fundamental nature of the methodology, the proposed methods have the potential to be extended to solve more general types of SPDEs, including time-space fractional heat equations.[23]

This paper serves as a comprehensive introduction and summary of the AFD-type methods found in various sources. AFD or Core AFD, is formulated based on the complex Hardy $H^2(D)$ space and has close connections to classical rational function approximation and Fourier theory.[41] By employing the backward shift technique, AFD emphasizes the attainment of maximal energy matching pursuit, resulting in positive analytic instantaneous frequency decomposition. Detailed analysis in Ref. 24 establishes that, in terms of reconstruction efficiency, commonly used matching pursuit algorithms can be ranked as follows: POAFD, orthogonal greedy algorithm (OGA) and greedy algorithm (GA). The convergence efficiency can be explicitly computed based on the type of remainder and dictionary used. GA is the weakest as it only uses the standard remainder, while OGA performs better by utilizing an orthogonal remainder, which possesses smaller energy due to its Hilbert space properties. POAFD, through a one-by-one parameter matching pursuit, offers the strongest reconstruction capability, as it reduces both the remainders and dictionary elements into consecutively obtained orthogonal complement subspaces. POAFD, without requiring intricate complex analysis, generalizes AFD with equal power in Hilbert spaces with a dictionary satisfying a boundary vanishing condition (BVC).[11,35,39,44,45] Applying POAFD in the Hardy $H^2(D)$ space with the Szegö kernel dictionary yields the AFD algorithm. In addition to POAFD, several other variations and extensions of AFD have been developed. These include the unwinding Blaschke expansion (independently developed by Coifman's group in Yale and Qian's group in Macao, see Refs. 12,13,33 and 40), n-best reproducing kernel approximation (n-best AFD, equivalent to the best approximation by rational functions of order not exceeding n, see Refs. 42 and 47), and extensions to higher dimensions for matrix-valued functions.[2,3] The multi-dimensional theory involves several complex variables and Clifford analysis. Stochastic adaptive Fourier decomposition (SAFD) and SPOAFD represent the latest generalizations of AFD. Among the sparse representations mentioned, the KL expansion exhibits similarities with SAFD and SPOAFD in nature. Due to the principal component expansion, KL possesses optimality over all orthonormal systems. However, SPOAFD achieves the same optimal convergence rate as KL for none smooth random signals. Moreover, SPOAFD eliminates the need to compute eigen-pairs of the integral operator defined by the covariance kernel function of the underlying random signal. Additionally, SPOAFD, as a general matching pursuit methodology, can be based on a variety of dictionaries. The flexibility to use a suitable dictionary is an art of the method and sometimes crucial for its efficiency. When solving a specific equation, for example, utilizing the potential kernel dictionary, if available, can provide a sparse series expansion of the ultimate solution. In contrast, none of the existing stochastic signal decomposition methods offer the same convenience.

We will study second-order processes $f(\omega, x), \omega \in \Omega, x \in \partial D$, in the Bochner-type space $L^2(\Omega, L^2(\partial D))$, where D is a domain in $\mathbb{R}^{n+1}$, and ∂D its boundary. Note that when we write $x \in D$ or $x \in \partial D$, x can be $x = t + \underline{x}$ or just $x = \underline{x}$, depending on further specification of the domain D and its boundary ∂D.

Definition 1 (Refs. 25 and 38). The Bochner space $L^2(\Omega, L^2(\partial D))$ is defined to be the Hilbert space consisting of all $L^2(\partial D)$-valued random variables $v : \partial D \times \Omega \to \mathbb{R}$ satisfying

$$\|f\|^2_{L^2(\Omega,L^2(\partial D))} \triangleq \int_\Omega \int_{\partial D} |f(x,\omega)|^2 dx d\mathbb{P} < \infty. \tag{1.6}$$

For notational brevity, we also write $\mathcal{N}(L^2(\partial D)) = L^2(\Omega, L^2(\partial D))$, or $\mathcal{N}(L^2)$, or just $\mathcal{N}$. In the unit circle context thanks to the Plancherel theorem[38] adopts an equivalent setting in terms of random Fourier coefficients. A Bochner space is a Hilbert space with an inner product defined through a Bochner-type integral. See Refs. 21,51,59 and the references therein. There is an analogous notation $\mathcal{N}(H_K(D))$, or $\mathcal{N}(H_K)$, if in the above definition the space $L^2(\partial D)$ is replaced by a RKHS $H_K(D)$.

The paper is organized as follows. In Sec. 2, we review the relevant knowledge and define notation and terminology that will be used throughout the paper. We formulate the Dirichlet problem of Laplace equation with a random boundary condition and the Cauchy problem of heat equation with a random initial condition. We will introduce the stochastic harmonic Hardy space and the stochastic heat Hardy space as the appropriate function spaces of the solutions. In Sec. 3, we revise the SPOAFD methodology in relation to Eqs. (1.1) and (1.2). In Sec. 4, we solve the stochastic fractional heat equations and the fractional Poisson equations by the proposed method. It is noted that the kernels of the first three types of equations, viz., (1.1), (1.2), (1.3), possess the semigroup property, that provides explicit formulas as the "lifting up" terms in the sparse series form solutions. The potential kernel of Eq. (1.4) is lack of the semigroup property. In the case the lifted terms in the sparse series are no longer with a potential kernel form, but with an integral form involving the potential and the generated multiple kernels. Some basic numerical experiments are presented in Sec. 5 to verify the theoretical results.

2. Preliminaries

In order to introduce our axiomatic methodology it would be necessary to give a short revision on the notation and fundamental theory of the Dirichlet BVPs and the Cauchy IVP on the classical domains, as well as the related random functional analysis. The concerned classical domains include the n-dimensional unit ball $B_1(0)$ in $\mathbb{R}^n$ and the upper half space $\mathbb{R}^{1+n}_+$ as an open region in $\mathbb{R}^{1+n}$.

The classical (deterministic) Dirichlet BVP problem in the unit ball B_1 for boundary value $f \in L^2(\partial B_1)$ is to find u defined inside the open ball that satisfies

$$\begin{cases} \Delta u(\underline{x}) = 0, & \underline{x} \in B_1, \\ u(\underline{x}) = f(\underline{x}), & \text{a.e. } \underline{x} \in \partial B_1. \end{cases} \tag{2.1}$$

The second requirement in (2.1) takes the non-tangential boundary limit (NBL) sense.[52] Throughout the paper we restrict ourselves to the fundamental case $f \in L^2(\partial D)$. In the case the solution u to (2.1) is given by the Poisson integral[19]

$$P[f](\underline{x}) = \int_{\partial B_1} P_{\underline{x}}(\underline{y}')f(\underline{y}')d\sigma(\underline{y}'), \tag{2.2}$$

where $P_{\underline{x}}(\cdot)$ is the Poisson kernel of the unit ball at $\underline{x} \in B_1$,

$$P_{\underline{x}}(\underline{y}') = c_n \frac{1 - |\underline{x}|^2}{|\underline{x} - \underline{y}'|^n}, \quad \underline{x} \in B_1, \; \underline{y}' \in \partial B_1, \tag{2.3}$$

and c_n is the surface area of ∂B_1. Below, c_n will always mean a constant depending on the dimension n that can be different from one occurrence to another. $P_{\underline{x}}$ is also called the *potential kernel*, and (2.2) called the *potential integral formula* of the solution of the BVP.

We will also denote $u_f(\underline{x}) = P[f](\underline{x})$, and use the phrase: *$u_f$ is the harmonic lifting up of the boundary data f*. The lifting up mechanism will also be used to the other types BVPs and IVPs considered in this paper.

Likewise, we also have a potential integral formula for the solution of (2.1) in the upper half-space $\mathbb{R}^{1+n}_+$ and $f \in L^2(\mathbb{R}^n)$,

$$u_f(t+\underline{x}) = \int_{\mathbb{R}^n} P(t+\underline{x}, \underline{y}) f(\underline{y}) d\underline{y}, \tag{2.4}$$

where $P(t+\underline{x}, \underline{y})$ is the Poisson kernel, or the potential kernel of the Dirichlet BVP, in the upper half-space $\mathbb{R}^{1+n}_+$,

$$P(t+\underline{x}, \underline{y}) = c_n \frac{t}{(t^2 + |\underline{x}-\underline{y}|^2)^{\frac{n+1}{2}}}.$$

To be consistent with a more general notation used in the later part of the paper we also write

$$h_p(\underline{y}) = P(t+\underline{x}, \underline{y}), \quad p = t+\underline{x}, \tag{2.5}$$

(see (2.19) and (4.7)).

For D being either the unit ball B_1 in $\mathbb{R}^n$ or the upper half space $\mathbb{R}^{1+n}_+$, there exists a harmonic Hardy h^p space theory. We give an account for the unit ball at the $p = 2$ case. The harmonic Hardy space $h^2(B_1)$ is defined as

Definition 2 (Ref. 4, the Harmonic Hardy h^2 Space on the Ball).

$$h^2(B_1) \triangleq \{u : \Delta u|_{B_1} = 0, \quad \|u\|_{h^2} = \sup_{0<r<1} \|u(r\cdot)\|_2 < \infty\}. \tag{2.6}$$

It is a fundamental result that every $u \in h^2(B_1)$ has a NBL function[52] and there holds

$$\sup_{0<r<1} \|u(r\cdot)\|_2 = \lim_{r\to 1} \|u(r\cdot)\|_2 = \|\lim_{r\to 1} u(r\cdot)\|_2, \quad \forall\, u \in h^2(B_1).$$

$h^2(B_1)$ is a Hilbert space equipped with an inner product in terms of their NBLs: For u and v in $h^2(B_1)$,

$$\langle u, v\rangle \triangleq \int_{\partial B_1} f(s) g(s) d\sigma(s), \tag{2.7}$$

where $f(s),\ g(s)$ are, respectively, the NBLs of u and v, $d\sigma(s)$ the normalized measure on the sphere. On the other hand, $u = P[f]$ and $v = P[g]$. As a consequence, $h^2(B_1)$ is isometric to $L^2(\partial B_1)$, denoted

$$L^2(\partial B_1) \cong h^2(B_1). \tag{2.8}$$

The harmonic Hardy space $h^2(B_1)$ is a special case of the Hardy spaces introduced by Stein and Fefferman.[18]

Of the same category, the heat-Hardy space, H^2_{heat}, is defined in relation to the heat equation in $\mathbb{R}^{1+n}_+$. Denote by

$$H_t(\underline{x}) \triangleq \left(\frac{1}{4t\pi}\right)^{n/2} e^{-\frac{1}{4t}|\underline{x}|^2}, \tag{2.9}$$

the heat potential kernel. Then the solution of the corresponding heat IVP can be formed as the lifting up by the heat (potential) kernel:

$$u_f(t+\underline{x}) \triangleq \int_{\mathbb{R}^n} H_t(\underline{x}-\underline{y})f(\underline{y})d\underline{y} = (H_t * f)(\underline{x}), \quad f \in L^2(\mathbb{R}^n). \tag{2.10}$$

The functions u on $\mathbb{R}^{1+n}_+$ of the form $u = u_f$, and only those, constitute the heat-Hardy space $H^2_{\text{heat}}(\mathbb{R}^{1+n}_+)$. The formulation is similar to (2.6) but with the Laplacian being replaced by the heat operator. Precisely, the space $H^2_{\text{heat}}(\mathbb{R}^{1+n}_+)$ is equivalent with one formulated via their NBLs:

$$\left\{u : \exists f \in L^2(\mathbb{R}^n),\ u = H_t * f,\ h^+(f) \triangleq \sup_{t>0} |(H_t * f)| \in L^2(\mathbb{R}^n)\right\}, \tag{2.11}$$

with the norm $\|u\|_{H^2} \triangleq \|h^+(f)\|_{L^2} \simeq \|f\|_{L^2}$. We note that the same L^2 space on the boundary ∂D can have different representations in Hardy spaces in the region D, viz., the harmonic Hardy space $h^2(\mathbb{R}^{1+n}_+)$ and the heat-Hardy space $H^2_{\text{heat}}(\mathbb{R}^{1+n}_+)$, both being reproducing kernel Hilbert spaces (RKHSs).

We recall the fundamental concept of RKHS.

Definition 3 (Ref. 49, Reproducing Kernel Hilbert Space). Let $D \neq \emptyset$. A RKHS on a region D is a Hilbert space $H_K(D)$ with a function

$$K : D \times D \to \mathbb{C},$$

possessing the reproducing property

$$\begin{cases} K_q \triangleq K(q,\cdot) \in H_K(D), & \forall\ q \in D, \\ \langle f, K_q\rangle = f(q), \quad \forall\ q \in D, & \forall\ f \in H_K(D). \end{cases} \tag{2.12}$$

The function $K_q(\cdot)$ in (2.12) is called the reproducing kernel of $H_K(D)$ with the parameter q.

As an example, denoting by $P_{\underline{t}}$ the Poisson potential kernel defined in (2.3), where $\underline{t} \triangleq |\underline{t}|\underline{t}', \underline{s} \triangleq |\underline{s}|\underline{s}',\ \underline{t}, \underline{s} \in D = B_1, \underline{t}', \underline{s}' \in \partial B_1$, the harmonic Hardy space $h^2(B_1)$ is a RKHS[43] with the reproducing kernel

$$K(\underline{s}, \underline{t}) \triangleq \langle P_{\underline{s}}, P_{\underline{t}}\rangle = P_{|\underline{s}||\underline{t}|\underline{s}'}(\underline{t}') = P_{|\underline{s}||\underline{t}|\underline{t}'}(\underline{s}'). \tag{2.13}$$

Also, the reproducing kernel for the heat equation is, for $p = t+\underline{x}$, $h_p(\underline{u}) = H_t(\underline{x}-\underline{u})$ and $q = s + \underline{y}$,

$$K(t+\underline{x}, s+\underline{y}) \triangleq \langle h_{t+\underline{x}}, h_{s+\underline{y}}\rangle = h_{t+s+\underline{x}}(\underline{y}) = H_{s+t}(\underline{x}-\underline{y}). \tag{2.14}$$

The computations can be found in Ref. 43. For their RKHS properties and the basic connections with operator theory the reader is referred to Refs. 49 and 36.

Remark 2.1. This paper has $\mathcal{H}$-H_K structure as its background formulation. For a detailed study we refer to Refs. 49 and 36. Below we give a brief description. Let $\mathcal{H}$ be a Hilbert space containing a class of elements $h_p \in \mathcal{H}$ parameterized by $p \in \mathcal{E}$, an open and connected set in a real- or complex-Euclidean space. Define the operator $L : \mathcal{H} \to \mathbb{C}^{\mathcal{E}}, Lf(p) \triangleq \langle f, h_p \rangle_{\mathcal{H}} = F(p)$. Let $N(L) = \{f \in \mathcal{H} : L(f) = 0\}$ and $N^{\perp}(L)$ the orthogonal complement of $N(L)$. $\mathcal{H}$ therefore has a direct sum decomposition $\mathcal{H} = N^{\perp}(L) \oplus N(L)$, corresponding to $f = f^+ + f^-, f^+ \in N^{\perp}(L), f^- \in N(L), \forall f \in \mathcal{H}$. Define the function $K(q,p) \triangleq \langle h_q, h_p \rangle = K_q(p)$, which is conjugate symmetric and non-negative. Denote by $E_q = \frac{K_q}{\|K_q\|}$ as the norm-1 normalization of K_q. Denote the range of operator L as $R(L)$. Then $R(L)$ can be assigned with the Hilbert inner product: for F, G in $R(L), \langle F, G \rangle \triangleq \langle f^+, g^+ \rangle_{\mathcal{H}}$. Under the induced inner product $R(L)$ coincides with the RKHS generated by the kernel K, denoted as H_K. The space H_K is isometric with $N^{\perp}(L)$, although the latter is not a RKHS. Besides the deterministic setting, the KL decomposition of stochastic processes (and random fields), too, fit well with the $\mathcal{H}$-H_K formulation, where $\mathcal{H}$ is taken to be one consisting of the second-order processes defined in (1.6). With the $\mathcal{H}$-H_K formulation it is natural to raise three basic questions: (i) representation of elements in H_K; (ii) L-inversion of elements in $R(L)$; and (iii) Moore–Penrose pseudoinverse of elements in a larger Hilbert space $\mathcal{G} : \mathcal{H} \subset \mathcal{G}$, where by larger, we mean the imbedding (identity) operator $I : \mathcal{H} \to \mathcal{G}$ is bounded. The present paper is devoted to the just stated question (i) for the concrete contexts induced by the SDBVPs and SCIVPs given by (1.1)–(1.4). They, in particular, fit into the notation system where the potential kernels $\{h_p\}_{p \in D}, D = \mathcal{E}$, D being the unit ball or the upper-half space, and the associated RKHSs coincide with the mentioned classical Hardy spaces. The following lemma addresses several easy facts in the contexts.

Lemma 1. *With the Poisson kernel and the heat kernel contexts we have* (i) $\{h_p\}$ *is a dictionary, that is*

$$\|h_p\|_{L^2(\partial D)} = 1 \quad \text{and} \quad \overline{\text{span}\{h_p\}_p} = L^2(\partial D); \tag{2.15}$$

(ii) $\|K_p\|_{H_K} = 1$ *and hence* $K_p = E_p = h_p$; (iii) *the lifting up operator* L *in the setting is an isometry with a bounded inverse*; *and* (iv) $\langle h_p, K_q(p) \rangle_{H_K} = h_q$ *on* ∂D.

Proof. (i) is approximation to identity property of the Poisson and the heat kernels.[52] (ii) is due to the relations

$$1 = \langle h_q, h_q \rangle_{L^2(\mathbb{R}^n)} = K(q,q) = \langle K_q, K_q \rangle_{H_K} = \|K_q\|^2_{H_K}.$$

(iii) holds because in the case $N(L) = \{0\}$ and thus $f = f^+, F = L(f) = L(f^+), \|F\|_{H_K} = \|f\|_{\mathcal{H}}$. To prove (iv) we perform change of order of taking limits based on the Lebesgue Dominated Convergence Theorem:

$$\begin{aligned}\langle h_p(\tilde{p}), K_q(p)\rangle_{H_K} &= \langle \lim_{p'\to\tilde{p}} K_p(p'), K_q(p)\rangle_{H_K} \\ &= \lim_{p'\to\tilde{p}} \langle K_p(p'), K_q(p)\rangle_{H_K} \\ &= \lim_{p'\to\tilde{p}} K_q(p') \\ &= h_q(\tilde{p}). \end{aligned}$$

□

Next, we turn to stochastic boundary and stochastic IVPs. Consider the Dirichlet problem with a random boundary data, i.e.

$$\begin{cases} \Delta u(\underline{x},\omega) = 0, & \underline{x} \in D, \ \text{a.s. } \omega \in \Omega, \\ u(\underline{x}',\omega) = f(\underline{x}',\omega), & \text{a.e. } \underline{x}' \in \partial D, \ \text{a.s. } \omega \in \Omega, \end{cases} \tag{2.16}$$

where $f \in L^2(\Omega, L^2(\partial D)),\ D = B_1$. We also write $f_\omega(\cdot)$ as $f(\cdot,\omega)$. For initial studies of this problem we refer to Refs. 6,8,9 and 48. Equation (2.16) can be viewed as a family of equations labeled with $\omega \in \Omega$. Thus for a.s. $\omega \in \Omega$ we can consider the stochastic process formed by the classical *Poisson integral*

$$u_{f_\omega}(\underline{x}) \triangleq \int_{\partial B_1} P_{\underline{x}}(\underline{t}') f_\omega(\underline{t}') d\sigma(\underline{t}'), \quad \underline{x} \in B_1, \tag{2.17}$$

where $P_{\underline{x}}(\underline{t}')$ is the Poisson potential kernel in B_1 defined by (2.3). For a.s. $\omega \in \Omega$, it follows that $u_{f_\omega}(\underline{x})$ solves the (2.16) with

$$\lim_{\rho\to 1} u_{f_\omega}(\rho\underline{x}') = f(\underline{x}',\omega),$$

in both the L^2-norm and the a.e. pointwise sense.

Similarly, the Cauchy problem with a stochastic initial value is formulated in $\mathbb{R}^{n+1}_+$,

$$\begin{cases} (\partial_t - \Delta_{\underline{x}}) u(t+\underline{x},\omega) = 0, & (t,\underline{x}) \in \mathbb{R}^1_+ \times \mathbb{R}^n, \quad \text{a.s. } \omega \in \Omega, \\ u(0+\underline{x},\omega) = f(\underline{x},\omega), & \text{a.e. } \underline{x} \in \mathbb{R}^n, \quad \text{a.s. } \omega \in \Omega, \end{cases} \tag{2.18}$$

with $f \in L^2(\Omega, L^2(\mathbb{R}^n))$. For a.s. ω, $f_\omega \in L^2(\mathbb{R}^n)$, $p = t+\underline{x}$, the convolution operator with the heat kernel given has the role as the lifting up operator of the context:

$$u_{f_\omega}(t+\underline{x}) \triangleq \langle f_\omega, h_p\rangle = \int_{\mathbb{R}^n} f_\omega(\underline{y}) H_t(\underline{x}-\underline{y}) d\underline{y} = (H_t * f_\omega)(\underline{x}). \tag{2.19}$$

If f_ω is in the Bochner space, then for a.s. ω, the classical solution u_{f_ω} belongs to the heat-Hardy space $H^2_{\text{heat}}(\mathbb{R}^{1+n}_+)$.

3. The SPOAFD Methodology to Solve Random Boundary and Initial Value Problems

We start with recalling POAFD for deterministic functions.[11,35]

3.1. *POAFD*

We are based on the $\mathcal{H}$-H_K formulation cited in Remark 2.1 adapted to the BVP and IVP. We are working with a Hilbert space $L^2(\partial D)$ and its dictionary $\mathcal{D}$. Under the convention we adopt that the dictionary elements are of norm-one, denoted as $\{E_q\}_{q\in\mathcal{E}}$, whose span is dense in $L^2(\partial D)$. Note that the dictionary elements as functions in $L^2(\partial D)$ are parameterized by elements in $\mathcal{E} = D$. For the RKHS H_K, the naturally associated dictionary consists of the elements of the form $E_q = \frac{K_q}{\|K_q\|}$, where K_q are the reproducing kernels. In each of our contexts the boundary values of the normalized reproducing kernels constitute a dictionary of $L^2(\partial D)$. We sometimes call a family $\{K_q\}$ a *pre-dictionary* if the corresponding norm-one normalizations $\{K_q/\|K_q\|\}$ forms a dictionary. The following BVC is sufficient for implementing the POAFD process: For any function $G \in L^2(\partial D)$ there holds

$$\lim_{q\to\partial D} |\langle G, E_q\rangle| = 0. \tag{3.1}$$

With the case $D = \mathbb{R}^{1+n}_+$, ∂D is defined as $\mathbb{R}^n \cup \{\infty\}$, and the limit $q \to \partial D$ is in the topology of the one-point-compactification of $\mathbb{R}^{1+n}$, where this "one point" is ∞.[43] BVC implies the following maximal selection principle (MSP): There exists $p \in D$ such that

$$p = \arg\sup\{|\langle G, E_q\rangle| \ : \ q \in D\}.$$

To ensure attainability of the supreme at each matching pursuit it is necessary to define *multiple dictionary elements*, also called *multiple kernels*. Let $(q_1, \ldots, q_n)$ be an n-tuple of parameters in D. Denote by $l(k)$ the multiplicity of q_k in the k-tuple $(q_1, \ldots, q_k), 1 \leq k \leq n$. With a little abuse of notation we define *the* $(q_1, \ldots, q_k)$-*related multiple kernel* as

$$\tilde{K}_{q_k} = \left[\left(\frac{\partial}{\partial \overline{q}}\right)^{(l(k)-1)} K_q\right]_{q=q_k}, \quad k = 1, 2, \ldots, n,$$

where $\frac{\partial}{\partial \overline{q}}$ is a prescribed directional derivative with the variable $\overline{q}$ in the parameter space. In most cases, including those in this study, multiple kernels belong to the underlying Hilbert space, and even to the underlying RKHS. Under these assumptions we have the following proposition.

Proposition 3.1. *If K_p is a reproducing kernel and $\tilde{K}_{q_k}$ the $(q_1, \ldots, q_k)$-related multiple kernel, then*

$$\langle K_p, \tilde{K}_{q_k}\rangle = \overline{\tilde{K}_{q_k}(p)} \triangleq (\tilde{K}_p)(q_k). \tag{3.2}$$

Moreover,

$$\langle f, \tilde{K}_{q_k}\rangle = \left[\left(\frac{\partial}{\partial p}\right)^{(l(k)-1)} f\right](q_k).$$

Thanks to density of the span of the kernels, the second assertion of the proposition is a conclusion of the first.

In solving a stochastic PDE problems we use the "lifting up" mechanism if a potential kernel is available. Asserted by the proposition, the mechanism even extends to the multiple kernels. If, in addition, the potential solution kernel possesses the semigroup property, then the lifted kernels and the multiple kernels preserve the same form as the potential kernel. In the upper half space Poisson kernel case, for instance, the semigroup property reads as $P_t * P_{q_k}(x) = P_{t+q_k}(x)$. In the case the lifting up process projects the Poisson kernel on $L^2(\mathbb{R}^n)$ to the Poisson kernels on $L^2(\mathbb{R}^{n+1}_+)$ preserving the same form. On multiple Poisson kernels it reads as $P_t * \tilde{P}_{q_k} = \tilde{P}_{t+q_k}$, called *generalized semigroup property*, where $\tilde{P}_{q_k}$ is the multiple kernel corresponding to $(q_1, \ldots, q_k)$.

Necessity of multiple kernels is clearly seen from pre-orthogonal maximal selection principle (POMSP) defined in (3.3). Suppose we already have a k-tuple of parameters $(q_1, \ldots, q_k)$ in which multiplicity may occur, and, accordingly, we have an k-tuple of multiple kernels, $(\tilde{K}_{q_1}, \ldots, \tilde{K}_{q_k})$ (some may have multiple 1). Applying the Gram–Schmidt (GS) orthonormalization process consecutively, we have, in the linear span sense, an equivalent k-orthonormal system, $(E_1, \ldots, E_k)$. Note that the notation E_k used here is of different meaning with E_{q_k} : the former being the kth function in the orthonormal system and the latter being the dictionary element parameterized by q_k. Under this convention we have $E_1 = E_{q_1}$. Now for any given G in the Hilbert space $N^\perp(L)$ or H_K in the $\mathcal{H}$-H_K formulation, see Remark 2.1, we investigate whether there exists a $q_k \in D$ corresponding to a function E_k giving rise to

$$|\langle G, E_k\rangle| = \sup\{|\langle G, E_k^q\rangle| \ : \ q \in D, \quad q \neq q_1, \ldots, q_{k-1}\}. \tag{3.3}$$

For q different from the preceding $q_l, l = 1, \ldots, k-1$, the function E_k^q on the right-hand side of (3.3) is the GS orthonormalization

$$E_k^q = \frac{K_q - \sum_{l=1}^{k-1} \langle K_q, E_l\rangle E_l}{\left\| K_q - \sum_{l=1}^{k-1} \langle K_q, E_l\rangle E_l \right\|}. \tag{3.4}$$

We remark that in (3.4) K_q may be replaced by E_q as we will do in the proof of Theorem 3.3. Detailed analysis shows that BVC implies that there exists $q_k \in D$ such that the supreme of (3.3) is attainable at

$$E_k = \frac{\tilde{K}_{q_k} - \sum_{l=1}^{k-1} \langle \tilde{K}_{q_k}, E_l\rangle E_l}{\sqrt{\|\tilde{K}_{q_k}\|^2 - \sum_{l=1}^{k-1} |\langle \tilde{K}_{q_k}, E_l\rangle|^2}}, \tag{3.5}$$

being the GS orthonormalization of the multiple kernel $\tilde{K}_{q_k}$ with the proceeding $E_l, l = 1, \ldots, k-1$ (see Refs. 35 or 11,44). Existence of $q_k \in D$ and a related E_k that make (3.3) hold is phrased as POMSP.

It is proved that BVC of a dictionary in the underlying space $L^2(\partial D)$ implies its MSP and POMSP. Iteratively applying POMSP to G, we obtain a sequence of such optimally selected $\{q_k\}_{k=1}^{\infty}$ with the property

$$G = \sum_{k=1}^{\infty} \langle G, E_k \rangle E_k. \tag{3.6}$$

Working with projected remainders and projected dictionaries at each step, POAFD is, in fact, the greediest matching pursuit method among all types of matching pursuit methods, including the *general* GA and the OGA.[11,34,54] Moreover, at each step the greatest matching pursuit is attainable. The obtained orthonormal system $\{E_k\}$ is not necessarily a basis. It, however, can well express the given signal. Without assuming smoothness of a signal of finite energy, the resulted POAFD series has the convergence rate $O(\frac{1}{\sqrt{n}})$. Not with coincidence, this convergence rate as the best estimation for none smooth signals in the function space is the same as that of the Shannon expansion and the KL expansion, and cannot in general be improved.[11,17,25,35] We note that quite like the Fourier series case when the data function is smooth or analytic on the circle, or the Shannon sampling case when the sampling frequency is higher than two times of the Nyquist frequency, then much faster convergence rates are gained. In the POAFD case if the boundary data is analytic across the unit circle, then the energy of the remainder is exponentially decay.[41]

As the next stage, we incorporate probability and formulate stochastic POAFD 1 and 2 (SPOAFD1 and 2, Ref. 38).

3.2. *SPOAFD1*

The SPOAFD1 algorithm is designed for random signals $f(t, \omega)$, also written as $f_\omega(t)$. Assuming the existence of mathematical expectation (or mean) for a.e. t, a random signal may be written as $f_\omega(t) \triangleq \tilde{f}(t) + r_\omega(t)$, where $\tilde{f}(t) \triangleq \mathbb{E}(f_\omega(t))$, the mathematical expectation, and $r_\omega(t) \triangleq f_\omega(t) - \tilde{f}(t)$, with the property $\mathbb{E}(r_\omega(t)) = 0$ for a.e. $t \in D$. In this section the error term $r_\omega(t)$ is assumed to have a small $\mathcal{N}$-norm.

The SPOAFD1 method to study the stochastic BVPs and IVPs is divided into three steps. Below, we only take as examples the SDBVP on the unit ball and the SCIVP on the upper half space. The SDBVP on the upper half space is similar.

On the ball the first step is to compute the average $\tilde{f}$ as a deterministic signal. The second step is to apply POAFD to $\tilde{f} \in L^2(\partial B_1)$ and obtain the L^2-expansion on ∂B_1,

$$\tilde{f}(\underline{x}') = \sum_{k=1}^{\infty} \langle \tilde{f}, E_k \rangle E_k(\underline{x}'), \tag{3.7}$$

where E_k are, consecutively, the terms of the GS orthogonalizations of the multiple dictionary elements (kernels) under the maximal selections q_k. The BVC is proved in Ref. 43. The third step is to solve the classical boundary value problem (BVP) with each term of the series (3.7), regarded as "lifting up" process. Under the notation set in (2.2) the semigroup property $P[E_k](\underline{x}) = \tilde{P}_{|\underline{x}|q_k}(\underline{x}'), \underline{x} \in B_1$, is used to obtain the lifting up functions. They then be added together, invoking the isometry relation between $h^2(B_1)$ and $L^2(\partial B_1)$, to result in a sparse series representation of the solution $u_{\tilde{f}}$ of the BVP $\tilde{f}$. To summarize, we have

$$u_{\tilde{f}}(\underline{x}) \overset{H_K}{=} \sum_{k=1}^{\infty} \langle \tilde{f}, E_k \rangle u_{E_k}(\underline{x}) = \sum_{k=1}^{\infty} \langle \tilde{f}, E_k \rangle P[E_k](\underline{x})$$

$$= \sum_{k=1}^{\infty} c_k \tilde{P}_{|\underline{x}|q_k}(\underline{x}'), \quad \underline{x} \in B_1, \tag{3.8}$$

where $\underline{x} = |\underline{x}|\underline{x}'$, $q_k = |q_k|q_k'$, $\tilde{P}_{q_k}$ the multiple Poisson kernels, and $\{c_k\}$ are the coefficients referred to the latter half of the proof of Theorem 3.5 below.

Similarly, for the SCIVP, the SPOAFD1 expansion based on a POAFD expansion of $\tilde{f}$ on $\mathbb{R}^n$, the lifting up to $\mathbb{R}^{1+n}_+$, and the semigroup property of the heat kernel gives

$$u_{\tilde{f}}(t + \underline{x}) \overset{H_K}{=} \sum_{k=1}^{\infty} \langle \tilde{f}, E_k \rangle u_{E_k}(t + \underline{x}) = \sum_{k=1}^{\infty} \langle \tilde{f}, E_k \rangle (H_t * E_k)(\underline{x})$$

$$= \sum_{k=1}^{\infty} d_k \tilde{H}_{t+q_k}(\underline{x}), \tag{3.9}$$

where $q_k = s_k + \underline{y}_k, x \in \mathbb{R}^n$, $\tilde{H}_{q_k}$ are the related multiple heat kernels, and $\{d_k\}$ are the corresponding coefficients in the equivalent non-orthogonal system.

Let, temporarily, ω be a fixed sample point. Like what we have done to $\tilde{f}$ on the ball, we will use the notations u_{f_ω} and u_{r_ω} for the solutions of the Dirichlet problem on the ball for, respectively, the boundary data f_ω and r_ω, as two deterministic functions. The theory for the heat IVP in $\mathbb{R}^{1+n}_+$, is parallel to the spherical Dirichlet problem we will use the same set of notations. Hence the following arguments and results are valid for both contexts. They are valid for the SDBVP context, too.

Define

$$d_{u_{f_\omega}}(x) \triangleq u_{f_\omega}(x) - \sum_{k=1}^{\infty} \langle f_\omega, E_k \rangle u_{E_k}(x). \tag{3.10}$$

Since the system $\{E_k\}$ is obtained via the average $\tilde{f}$, for a particular $\omega \in \Omega$, one cannot expect

$$\|d_{u_{f_\omega}}\|_{H_K} = 0, \tag{3.11}$$

and let alone in the $\mathcal{N}$-norm sense

$$\|d_{u_f}\|_{\mathcal{N}(H_K)} = 0. \tag{3.12}$$

The series expansion on the right-hand side of (3.10) in terms of $\{E_k\}$, therefore, may not be u_{f_ω} for a.s. $\omega \in \Omega$. Due to the relation $\|r\|^2_{\mathcal{N}} = \|\text{var} f\|_{L^1(\partial D)}$, the estimations of Ref. 38 generate the following theorem.

Theorem 3.2.

$$\|u_{f_\omega} - u_{\tilde{f}}\|_{\mathcal{N}(H_K)} \leq \|r\|_{\mathcal{N}(H_K)},$$

and

$$\|d_{u_{f_\omega}}\|^2_{\mathcal{N}(H_K)} \leq \|r\|^2_{\mathcal{N}(H_K)} - \sum_{j=1}^{\infty} \mathbb{E}|\langle r_\omega, E_j\rangle|^2,$$

and, in particular,

$$\|d_{u_{f_\omega}}\|_{\mathcal{N}(H_K)} \leq \|r\|_{\mathcal{N}(H_K)}.$$

In summary SPOAFD1 amounts to solving the stochastic BVP and IVP up to an error dominated by the $L^1(\partial D)$-norm of the variation of f_ω.

3.3. *SPOAFD2*

In this section, SPOAFD2, or SPOAFD in brief, as the main method of this paper is introduced. By using SPOAFD we will get a sparse series expansion, converging in the stochastic Hilbert space $\mathcal{N}$ sense (4.5). There holds (3.12), and, as a consequence, there holds (3.11) almost surely. Due to the strong convergence SPOAFD is an improvement of SPOAFD1 (Theorem 3.2). The strong convergence is at the cost of computational complexity. Being supplementary to Ref. 38 the following theorem proves more results on stochastic pre-orthogonal maximal selection principle (SPOMSP) upon which SPOAFD2 is built.

Theorem 3.3. *If a dictionary $\mathcal{D} = \{E_q\}$ of $L^2(\partial D)$ satisfies BVC* (3.1), *then in $L^2(\Omega, L^2(\partial D))$ it satisfies stochastic BVC, that is, for any $f \in L^2(\Omega, L^2(\partial D))$,*

$$\lim_{q \to \partial D} \mathbb{E}|\langle f_\omega, E_q\rangle|^2 = 0. \tag{3.13}$$

As a consequence, the stochastic maximal selection principle (SMSP) is valid, that is, there exists $q_1 \in D$ such that

$$q_1 = \arg\max_{q \in D} \mathbb{E}|\langle f_\omega, E_q\rangle|^2. \tag{3.14}$$

Moreover, under the BVC assumption, SPOBVC for the terms obtained in the G-S process holds: For every k,

$$\lim_{q \to \partial D} \mathbb{E}|\langle f_\omega, E_k^q\rangle|^2 = 0. \tag{3.15}$$

(Here, we remind again the difference between E_q and E_k^q.) As a consequence, the SPOMSP is implementable: There exists $q_k \in D$ such that

$$q_k = \arg\max_{q \in D} \mathbb{E}|\langle f_\omega, E_k\rangle|^2. \tag{3.16}$$

Proof. Since $f \in \mathcal{N}(L^2)$, there exists a zero-probability event Q such that $\omega \in \Omega \backslash Q$ implies $f_\omega \in L^2(\partial D)$. The assumption that $\{E_q\}$ satisfies BVC then implies

$$\lim_{q \to \partial D} |\langle f_\omega, E_q \rangle|^2 = 0, \quad \omega \in \Omega \setminus Q. \tag{3.17}$$

$f \in \mathcal{N}(L^2)$ also implies $\|f_\omega\|^2_{H^2}$ in $L^1(\Omega, d\mathbb{P})$, the latter being a dominating function of $|\langle f_\omega, E_q\rangle|^2$ for all q :

$$\begin{aligned} |\langle f_\omega, E_q \rangle|^2 &\leq \|f_\omega\|^2_{L^2(\partial D)} \|E_q\|^2_{L^2(\partial D)} \\ &= \|f_\omega\|^2_{L^2(\partial D)} \in L^1(\Omega, d\mathbb{P}). \end{aligned} \tag{3.18}$$

By the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{q \to \partial D} \mathbb{E}|\langle f_\omega, E_q \rangle|^2 = 0. \tag{3.19}$$

By employing a Bolzano–Weierstrass-type compact argument we conclude that a maximal selection of $q \in D$ is available and (3.14) holds.

In order to show (3.16), it suffices to show (3.15). For the fixed $q_1, \ldots, q_{k-1}$, denote by $Q_{q_1,\ldots,q_{k-1}}$ the projection operator from $L^2(\partial D)$ to the orthogonal complement of span$\{E_1, \ldots, E_{k-1}\}$. Taking into account the remark on (3.4) we can replace K_q by E_q in the expression of E_k^q. Since $Q_{q_1,\ldots,q_{k-1}}$ is self-adjoint, we have

$$\langle f_\omega, E_k^q \rangle = \left\langle f_\omega, \frac{Q_{q_1,\ldots,q_{k-1}} E_q}{\|Q_{q_1,\ldots,q_{k-1}} E_q\|} \right\rangle = \left\langle Q_{q_1,\ldots,q_{k-1}} f_\omega, \frac{E_q}{\|Q_{q_1,\ldots,q_{k-1}} E_q\|} \right\rangle.$$

Owing to (3.4), for $q \to \partial D$, we have

$$\mathbb{E}|\langle f_\omega, E_k^q \rangle|^2 = \frac{1}{1 - \sum_{l=1}^{k-1} |\langle E_q, E_l \rangle|^2} \mathbb{E}|\langle Q_{q_1,\ldots,q_{k-1}} f_\omega, E_q \rangle|^2 \to 0.$$

The proof is complete. □

Owing to Theorem 3.3, if $\mathcal{D}$ is a dictionary in $L^2(\partial D)$ enjoying BVC, then in $L^2(\Omega, L^2(\partial D))$ the dictionary $\mathcal{D}$ can be used to produce a SPOAFD algorithm. If, moreover, $H_K(D)$ is a RKHS with reproducing kernel K generating a pre-dictionary $\mathcal{D}$, then in $L^2(\Omega, H_K(D))$, $\mathcal{D}$ also enjoys SPOBVC and enables a SPOAFD algorithm. In all the contexts considered in the paper the space $L^2(\Omega, H_K(D))$ is isometric with the space $L^2(\Omega, L^2(\partial D))$. We state the result regarding sparse expansion of the random boundary data and initial values:

Theorem 3.4 (Ref. 38). *Let $f \in \mathcal{N} = L^2(\Omega, L^2(\partial D))$. Assume that $\{q_k\}$ are selected under SPOMSP, where, through* (3.5), *q_k gives rise to the supreme*

$$\sup_{q \in D, q \neq q_1, \ldots, q_{k-1}} \mathbb{E}|\langle f_\omega, E_k^q \rangle|^2. \tag{3.20}$$

Let $\{E_k\}$ *be the consecutive GS orthonormalization of the correspondingly selected multiple kernels* $\{\tilde{K}_{q_k}\}$*. Then there holds*

$$f_\omega \overset{\mathcal{N}(L^2)}{=} \sum_{k=1}^{\infty} \langle f_\omega, E_k \rangle E_k. \tag{3.21}$$

In the rest of the paper we will call $\{q_k\}, \{\tilde{K}_{q_k}\}$ and $\{E_k\}$, respectively, the *SPOAFD parameter sequence*, the *SPOAFD multiple kernel sequence*, and the *SPOAFD orthonormal system of the stochastic signal* (random field) $f(t, \omega)$ in $\mathcal{N}(L^2)$ (or $\mathcal{N}(H_K)$, see Theorem 4.2).

BVC of the Poisson potential kernel in the unit ball and BVC of the heat potential kernel in $\mathbb{R}^{1+n}_+$ are proved in literature.[43] Under the convention of the notation u_{f_ω} (3.10), by invoking Theorem 3.4, we have the following theorem.

Theorem 3.5. *Let* $f \in \mathcal{N} = L^2(\Omega, L^2(\partial D))$*. Under the SPOAFD2 expansion* (3.21) *on* $L^2(\partial D)$ *there further holds*

$$u_{f_\omega} \overset{\mathcal{N}(H_K)}{=} \sum_{k=1}^{\infty} \langle f_\omega, E_k \rangle u_{E_k} = \sum_{k=1}^{\infty} c_k(\omega) u_{\tilde{K}_{q_k}}, \tag{3.22}$$

where $\{q_k\}, \{\tilde{K}_{q_k}\}$ *and* $\{E_k\}$ *are, respectively, the SPOAFD parameter sequence, the SPOAFD multiple kernel sequence, and the SPOAFD orthonormal system of* $f(t, \omega)$*, and* $c_k(\omega)$ *are accordingly determined random coefficients. In* (2.16) *and* (2.18) *cases, we have, respectively,*

$$u_{\tilde{K}_{q_k}}(x) = \tilde{K}_{|x|q_k}(x') \quad \text{and} \quad u_{\tilde{K}_{q_k}}(t + x) = \tilde{K}_{t+q_k}(x). \tag{3.23}$$

Moreover, in both cases,

$$\|d_{u_{f_\omega}}\|_{\mathcal{N}(H_K)} = 0.$$

Proof. By using SPOAFD we expand the random boundary data f_ω on the boundary ∂D into an orthonormal system converging in the $\mathcal{N}$ norm, where the system is made from the maximally selected, possibly, multiple dictionary elements. That is

$$f_\omega \overset{\mathcal{N}(L^2)}{=} \sum_{k=1}^{\infty} \langle f_\omega, E_k \rangle E_k. \tag{3.24}$$

Using this decomposition we solve the problem term by term as

$$u(\langle f_\omega, E_n \rangle E_n)(x) = \langle f_\omega, E_n \rangle u(E_n)(x), \quad x \in D,$$

and then add them together. The validity of the adding up is due to the linearity and boundedness of the lifting up operator $u : f_\omega \to u_{f_\omega}$ from $L^2(\Omega, L^2(\partial D))$ to

$L^2(\Omega, H_K(D))$. We thus also have, in the norm sense,

$$u_{f_\omega} \overset{\mathcal{N}(H_K)}{=} \sum_{k=1}^{\infty} \langle f_\omega, E_k \rangle u_{E_k}.$$

Next, we deduce the coefficients $c_k(\omega)$. Denote, for every positive integer n, $\mathcal{E}_n = (E_1, \ldots, E_n)$, $\mathcal{K}_n = (\tilde{K}_{q_1}, \ldots, \tilde{K}_{q_n})$, $\mathcal{C}_n = (c_1(\omega), \ldots, c_n(\omega))$, and $\mathcal{F}_n = (\langle f_\omega, E_1 \rangle, \ldots, \langle f_\omega, E_n \rangle)$. Then there follows

$$\mathcal{K}_n{}^t = \mathcal{A}_n \mathcal{E}_n{}^t,$$

where the superscript "t" stands for transpose of matrix, and $\mathcal{A}_n = (a_{ij})_{1 \le i,j \le n}$ is an $n \times n$ lower triangle matrix, where

$$a_{ij} = \begin{cases} E_{q_i}(q_j) = \dfrac{K(q_j, q_i)}{\|K_{q_i}\|}, & i \ge j, \\ 0, \quad i < j. \end{cases}$$

Hence,

$$\mathcal{E}_n{}^t = \mathcal{A}_n^{-1} \mathcal{K}_n{}^t.$$

Since $\mathcal{F}_n \mathcal{E}_n{}^t = \mathcal{C}_n \mathcal{K}_n{}^t$, we have $\mathcal{C}_n = \mathcal{F}_n \mathcal{A}_n^{-1}$. The proof is complete. □

Theorem 3.5 shows that with a stochastic signal or a random field f in the Bochner-type space $L^2(\Omega, L^2(\partial D))$ the SDBVP and the SCIVP are solvable in their respective Bochner-type stochastic Hardy spaces $L^2(\Omega, H_K(D))$ in the form of sparse series as linear combinations of the respective potential kernels with random coefficients (also see Refs. 11 or 37).

Remark 3.6. There is comparatively a small computation cost for performing SPOAFD1 through $\tilde{f}$ and a large cost for SPOAFD2 due to more complicated MSP and the GS orthogonalization (except in the SAFD case when TM system is available). To solve a stochastic BVP or a stochastic IVP the third step "lifting up" does not actually raise a cost due to semi-group property of the potential kernel in context.

4. Stochastic Fractional Heat and Fractional Poisson Equations

Likewise, the SPOAFD algorithm can be used to solve the stochastic fractional heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t + \underline{x}, \omega) = (-\Delta)^\alpha u(t + \underline{x}, \omega), & (t, \underline{x}) \in \mathbb{R}_+^{1+n}; \\ u(0 + \underline{x}, \omega) = f(\underline{x}, \omega), & \underline{x} \in \mathbb{R}^n, \end{cases} \tag{4.1}$$

where $\alpha \in (0, 1]$.

We first recall some necessary knowledge on these equations.[46] The fractional Laplacian operator $(-\Delta)^\alpha$ can be defined via Fourier transform: For $\underline{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$,

$$\widehat{(-\Delta)^\alpha} f(\underline{\xi}) = |\underline{\xi}|^{2\alpha} \widehat{f}(\underline{\xi}).$$

The fractional heat semigroup is referred to

$$(\widehat{e^{-t(-\Delta)^\alpha}}) f(\underline{\xi}) = e^{-t|\underline{\xi}|^{2\alpha}} \widehat{f}(\underline{\xi}), \quad 0 < \alpha \leq 1.$$

The cases $\alpha = 1/2$ and $\alpha = 1$ correspond to, respectively, the classical Laplace and the classical heat equation problems. Denote by $k_{\alpha,t}(\cdot)$ the potential kernel, or precisely the fractional heat kernel associated to $e^{-t(-\Delta)^\alpha}$, i.e.

$$e^{-t(-\Delta)^\alpha} f(\underline{x}) = \int_{\mathbb{R}^n} k_{\alpha,t}(\underline{x} - \underline{y}) f(\underline{y}) d\underline{y}.$$

By the inverse Fourier transform, the fractional heat kernel, as a kernel of approximation to the identity when $t \to 0+$, can be represented as

$$k_{\alpha,t}(\underline{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-t|\underline{\xi}|^{2\alpha}} e^{i\underline{x}\underline{\xi}} d\underline{\xi}. \tag{4.2}$$

It is shown that the unique solution $u(t + \underline{x})$ of the IVP (1.3) can be represented through the fractional integral transform, the lifting up,

$$u(t + \underline{x}) = \int_{\mathbb{R}^n} k_{\alpha,t}(\underline{x} - \underline{y}) f(\underline{y}) d\underline{y}.$$

In Ref. 46, it is shown that the solutions of (1.3) constitute a RKHS with the reproducing kernel $K(q, p)$ given by: For $q = t + \underline{x}$ and $p = s + \underline{y}$,

$$\begin{aligned} K(q,p) &= \langle k_{\alpha,t+\underline{x}}(\cdot),\ k_{\alpha,s+\underline{y}}(\cdot)\rangle_{L^2} \\ &= \int_{\mathbb{R}^n} e^{-(t+s)|\underline{\xi}|^{2\alpha}} e^{i(\underline{x}-\underline{y})\underline{\xi}} d\underline{\xi} \\ &= k_{\alpha,t+s}(\underline{x} - \underline{y}). \end{aligned}$$

The kernel is of the same form as the potential kernel, and thus possesses the semigroup property. For $\alpha = 1/2, 1$, the above reduces to, respectively, the semigroup properties of the Poisson and the heat potential kernels:

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n} \frac{t}{(t^2 + |\underline{x} - \underline{z}|)^{(n+1)/2}} \frac{s}{(s^2 + |\underline{y} - \underline{z}|)^{(n+1)/2}} d\underline{z} = c_n \frac{t+s}{((t+s)^2 + |\underline{x} - \underline{y}|)^{(n+1)/2}}; \\ \displaystyle\frac{1}{(4\pi)^n (ts)^{n/2}} \int_{\mathbb{R}^n} e^{-|\underline{x}-\underline{z}|^2/4t} e^{-|\underline{y}-\underline{z}|^2/4s} d\underline{z} = \frac{1}{(4\pi(t+s))^{n/2}} e^{-|\underline{x}-\underline{y}|^2/4(t+s)}, \end{cases} \tag{4.3}$$

The pre-dictionary $\{K_p\}$ satisfies BVC: For any fixed $p = s + \underline{y}$,[46]

$$\lim_{q\to\partial^*\mathbb{R}_+^{1+n}} |\langle K_p,\ E_q\rangle_{H_K}| = 0 \tag{4.4}$$

By invoking Theorem 3.3 the BVC implies the corresponding SPOBVC, the latter allowing a sparse series representation of the random data on $\mathbb{R}^n$ via the SPOAFD method, using K_p as a pre-dictionary of $L^2(\mathbb{R}^n)$. We apply the lifting up mechanism to each E_n in the orthonormal system generalized by the maximally selected multiple kernels $\tilde{K}_{\alpha,t_k+\underline{x}_k}$ and then add the results with the corresponding coefficients in the series to get the solution of the stochastic IVP (4.1).

Theorem 4.1. *The unique solution of the fractional heat equations* (4.1) *is given by*

$$u_{f_\omega}(t+\underline{x}) \overset{\mathcal{N}(H_K)}{=} \sum_{k=1}^{\infty}\langle f_\omega, E_k\rangle u_{E_k}(t+\underline{x}) = \sum_{k=1}^{\infty} c_k(\omega)\tilde{K}_{t+q_k}(\underline{x}), \tag{4.5}$$

where $\{q_k\}, \{\tilde{K}_{q_k}\}$ *and* $\{E_k\}$ *are, respectively, the SPOAFD parameter sequence, the SPOAFD multiple kernel sequence, and the SPOAFD orthonormal system of* $f(\underline{x},\omega)$ *on the boundary, and* $c_k(\omega)$ *are the corresponding random coefficients,* u_{E_k} *are the solutions of Eq.* (4.1) *with the initial data* E_k.

Next, we turn to the stochastic fractional Poisson equation

$$\begin{cases} \operatorname{div}(t^\sigma\nabla u)(t+\underline{x},\omega) = 0, & t+\underline{x}\in\mathbb{R}_+^{1+n},\ \text{a.s. } \omega\in\Omega; \\ u(0+\underline{x},\omega) = f(\underline{x},\omega), & \text{a.s. } \omega\in\Omega, \end{cases} \tag{4.6}$$

where σ is restricted to be in $(0, 2]$.[10] For a smooth deterministic boundary function f the Caffarelli–Silvestre extension of f, or the solution of (1.3) with the initial data f, is explicitly given by the convolution of f with the fractional Poisson potential kernel

$$p_t^\sigma(\underline{x}) := \frac{c(n,\sigma)t^\alpha}{(|\underline{x}|^2+t^2)^{(n+\sigma)/2}},$$

where $c(n,\sigma)$ is a constant such that $\int_{\mathbb{R}^n} p_t^\sigma(\underline{x})d\underline{x} = 1$ (see Ref. 46). The potential kernel gives rise to an approximation to identity. It, however, does not possess the semigroup property. In such case one still uses the lifting up method, but there will be no relations such as those in (3.23), and no convenience as $u_{\tilde{k}_{q_k}}(t+\underline{x}) = \tilde{k}_{t+q_k}(\underline{x})$ as used in (4.5). One, in the case, in the underlying stochastic RKHS consisting of the solution functions. Next theorem addresses and compares the two types of methods. We first recall the RKHS structure associated with this equation.[36]

Let $\mathcal{H} = L^2(\mathbb{R}^n)$ and $D = \mathbb{R}^{1+n}_+$. The potential kernel, as the kernel of the lifting up operator is, for $p = s + \underline{y}$ in D,

$$h_p(\underline{x}) = \frac{c(n,\sigma)s^\sigma}{(s^2 + |\underline{x} - \underline{y}|^2)^{(n+\sigma)/2}}. \tag{4.7}$$

The space H_K is defined as

$$H_K := \{u : \mathbb{R}^{1+n}_+ \to \mathbb{R},\ u(p) = \langle f,\ h_p\rangle_{L^2(\mathbb{R}^n)}\},$$

with the induced Hilbert space norm on the boundary for the associated NBL functions. For $q = t + \underline{x}$ the reproducing kernel is computed

$$\begin{aligned} K_q(p) &= K(q,p) \\ &= \langle h_q,\ h_p\rangle_{L^2(\mathbb{R}^n)} \\ &= c^2(n,\sigma)\int_{\mathbb{R}^n} \frac{t^\sigma}{(t^2 + |\underline{x} - \underline{z}|^2)^{(n+\sigma)/2}} \frac{s^\sigma}{(s^2 + |\underline{y} - \underline{z}|^2)^{(n+\sigma)/2}} dz. \end{aligned} \tag{4.8}$$

In particular, as functions defined on $\mathbb{R}^n$, $h_q = \frac{K_q}{\|K_q\|}$, concluded from the fact that h_p is an approximation to identity. In the proof of Theorem 4.2 we further show $\|K_q\|_{H_K} = 1$, and so $h_q = K_q$, restricted to the boundary.

It is shown in Ref. 46 that $\{K_q\}$ as a pre-dictionary in $L^2(\mathbb{R}^n)$ satisfies BVC; and, equivalently, in H_K also satisfies BVC. Hence, SPOAFD may be performed in both the spaces $L^2(\mathbb{R}^n)$ and H_K. We have the following theorem.

Theorem 4.2. (1) *The initial random data f_ω in $L^2(\mathbb{R}^n)$ has a SPOAFD expansion*

$$f_\omega(\underline{x}) \overset{\mathcal{N}(L^2)}{=} \sum_{k=1}^{\infty} \langle f_\omega, E_k\rangle E_k(\underline{x}) = \sum_{k=1}^{\infty} c_k(\omega)\tilde{K}_{q_k}(\underline{x}), \tag{4.9}$$

where $\{q_k\}, \{\tilde{K}_{q_k}\}$ and $\{E_k\}$ are, respectively, the SPOAFD parameter sequence, the SPOAFD multiple kernel sequence, and the SPOAFD orthonormal system of $f(\underline{x},\omega)$ on the boundary, and $c_k(\omega)$ are the corresponding random coefficients. Subsequently, the solution of the fractional Poisson equations (4.6) *is*

$$u_{f_\omega} \overset{\mathcal{N}(H_K)}{=} \sum_{k=1}^{\infty} \langle f_\omega, E_k\rangle u_{E_k} = \sum_{k=1}^{\infty} c_k(\omega) u_{\tilde{K}_{q_k}}, \tag{4.10}$$

where u_{f_ω}, u_{E_k} and $u_{\tilde{K}_{q_k}}$ are, respectively, the lifting up's of f_ω, E_k and $\tilde{K}_{q_k}$ through the potential kernel h_p in the pattern of (2.19), *where, in particular,*

$$u_{\tilde{K}_{q_k}}(p) = \langle \tilde{K}_{q_k}, h_p\rangle = \tilde{K}_{q_k}(p).$$

(2) *Directly using SPOAFD in* H_K *to the potential solution of* (4.6) *given by*

$$u_{f_\omega}(p) = \langle f_\omega, h_p \rangle,$$

we have a sparse expansion

$$u_{f_\omega} \overset{\mathcal{N}(H_K)}{=} \sum_{k=1}^{\infty} \langle u_{f_\omega}, E_k^* \rangle E_k^* = \sum_{k=1}^{\infty} c_k(\omega) \tilde{K}_{q_k^*}, \tag{4.11}$$

where $\{q_k^*\}, \{\tilde{K}_{q_k^*}\}$ *and* $\{E^*{}_k\}$ *are, respectively, the SPOAFD parameter sequence, the SPOAFD multiple kernel sequence, and the SPOAFD orthonormal system of* u_{f_ω} *in* $\mathcal{N}(H_K)$. *The last series* (4.11), *term by term, equals with the expansion* (4.10) *obtained in* (1).

Proof. Since the SPOAFD method is valid in both cases, only the term-by-term equality relation needs to be checked. As shown in Lemma 1, $E_q = K_q$. On the one hand, the MSP of the formulation (2) to generate (4.11) is based on the quantity

$$\langle u_{f_\omega}, E_q \rangle_{H_K} = \langle u_{f_\omega}, K_q \rangle_{H_K} = u_{f_\omega}(q).$$

On the other hand, by invoking (iv) of Lemma 1, the same quantity is expressed as

$$\begin{aligned} \langle u_{f_\omega}, K_q \rangle_{H_K} &= \langle \langle f_\omega, h_p \rangle_{L^2(\mathbb{R}^n)}, K_q(p) \rangle_{H_K} \\ &= \langle f_\omega, \langle h_p, K_q(p) \rangle_{H_K} \rangle_{L^2(\mathbb{R}^n)} \\ &= \langle f_\omega, h_q \rangle_{L^2(\mathbb{R}^n)} \\ &= \langle f_\omega, E_q \rangle_{L^2(\mathbb{R}^n)}. \end{aligned}$$

The last quantity is what is based on to have the expansion (4.9), and so the selection of each q_k can be the same, that is $q_k^* = q_k$. This also concludes

$$u_{\tilde{K}_{q_k}}(p) = \tilde{K}_{q_k^*}(p),$$

for each $p \in D$. □

Remark 4.3. Computing the quantity $\mathbb{E}|\langle f_\omega, E_l^q \rangle|^2$ for the energy matching pursuit is an important step for the SPOAFD algorithm. In practice, we have three methods to compute the expectation. The first is that the stochastic signal f_ω itself is a random variable when the space variable $\underline{x}$ or the time variable t is fixed. In the case when the probability distribution or density function of X is known as a function of the space or time variable, then the expectation is computable. Two examples of such type with concrete computation are presented in Sec. 5. The second method is to use the covariance function as it does for the KL decomposition. Not like in the KL case, in performing SPOAFD, one does not need to compute out the eigensystem of the integral operator defined by the covariance function. Therefore, compared with KL, SPOAFD saves a great amount of computation complexity. We provide an example of this method in Sec. 5. The third method to compute the expectation would be based on sampling and use the Large Number Law.

5. Examples of Random Boundary Value Problem

In this section we provide details of algorithm of SPOAFD2 (SPOAFD in short), which is applied to numerical simulations of the two basic types of random PDEs studied in Sec. 3. Denote by $\{E_1, \ldots, E_j\}$ the GS orthonormalization of the maximally selected parameterized multiple kernels under the SPOAFD program.

Example 1. Consider the Laplace equation in the unit disk $D \triangleq \{z \in \mathbb{C} : \ |z| < 1\}$ with the random Dirichlet boundary condition:

$$\begin{cases} \Delta u(z, \omega) = 0, \quad z \in D, \ \text{a.s. } \omega \in \Omega, \\ u(e^{it}, \omega) = \dfrac{1}{\sqrt{5 + (\sin t - X(\omega))^2}}, \quad t \in [0, 2\pi], \ \text{a.s. } \omega \in \Omega, \end{cases} \tag{5.1}$$

where $t \in [0, 2\pi]$ and X is a random variable with the density function $p(s) = \frac{1}{m}\alpha(s)$, where

$$\alpha(s) = \begin{cases} e^{-\frac{1}{(s-\pi)^2+1}}, & 0 \le s \le \pi, \\ e^{-\frac{1}{(s+\pi)^2+1}}, & -\pi \le s < 0, \end{cases}$$

and $m = \int_{-\pi}^{\pi} \alpha(s) ds$.

Since the boundary data is a bi-variate function of t and random variable X, we can make use the distribution of the random variable X. An optimal $q_n \in D$ is

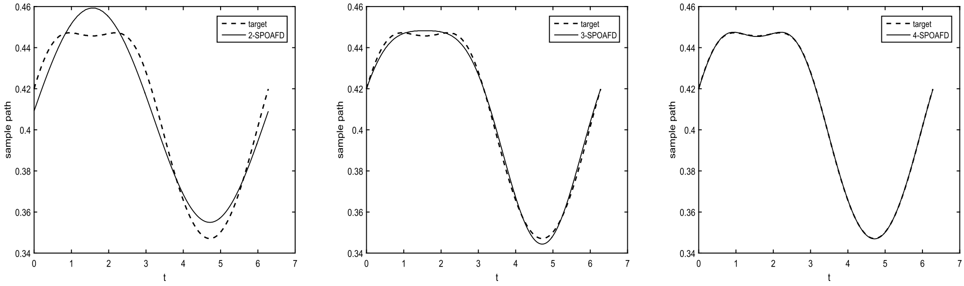
Algorithm 1. SPOAFD.

Input: boundary random function $f(t, \omega)$
Output: solution u_n of random equation
Initialize $u_n = 0, \ j = 0$, error=10, energy$_\omega = 0$.
While error $\ge 10^{-4}$ **do**
  $j \leftarrow j + 1$
  $q_j \leftarrow \arg\max_{q \in D} \mathbb{E}_\omega |\langle f_\omega, E_j^q \rangle|^2$
  $u_n \leftarrow u_n + \langle f_\omega, E_j^{q_j} \rangle E_j^{q_j}$
  energy$_\omega \leftarrow$ energy$_\omega + |\langle f_\omega, B_j^{q_j} \rangle|^2$
  error $\leftarrow \frac{\|f\|^2_{\mathcal{N}(L^2)} - \mathbb{E}(\text{energy}_\omega)}{\|f\|^2_{\mathcal{N}(L^2)}}$
End while

selected according to

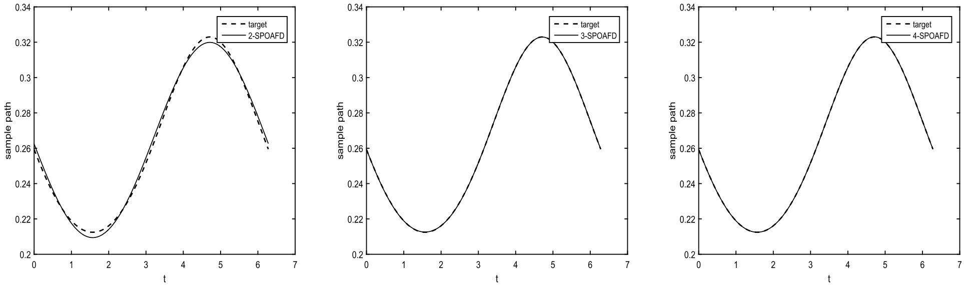
$$\mathbb{E}|\langle f_\omega, E_n^q\rangle|^2 = \int_{-\pi}^{\pi} \left| \int_0^{2\pi} \frac{1}{\sqrt{5 + (\sin t - s)^2}} E_n^q(e^{it})dt \right|^2 p(s)ds.$$

We have three groups of figures, respectively, for visual effects of the experiments at those ω such that $X(\omega) = 0, -\pi$, and 2.4504, as shown in Figs. 1–3. The relative



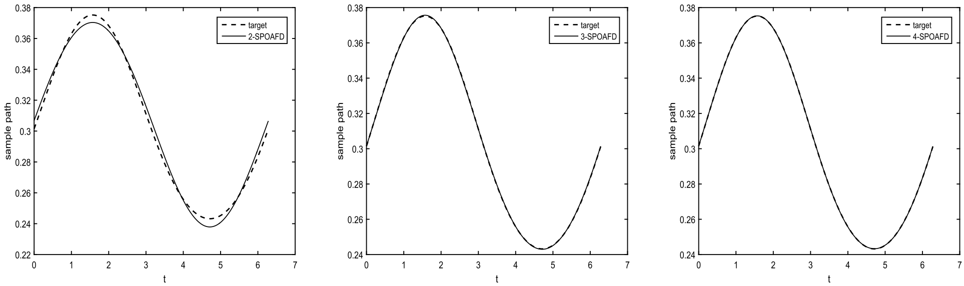
2 partial sum 3 partial sum 4 partial sum

Fig. 1. $X(\omega) = 0$.



2 partial sum 3 partial sum 4 partial sum

Fig. 2. $X(\omega) = -\pi$.



2 partial sum 3 partial sum 4 partial sum

Fig. 3. $X(\omega) = 2.4504$.

Table 1. Relative error.

SPOAFD	3.6902×10^{-4}	2.2177×10^{-5}	1.8945×10^{-7}

Table 2. Relative error.

SPOAFD	6.7418×10^{-5}	6.2401×10^{-8}	6.7669×10^{-9}

Table 3. Relative error.

SPOAFD	1.4236×10^{-4}	6.5114×10^{-7}	6.0066×10^{-8}

errors in Tables 1–3 are computed according to the formulas given in the relevant algorithms.

Example 2. The other case is the heat equation with the random initial condition:

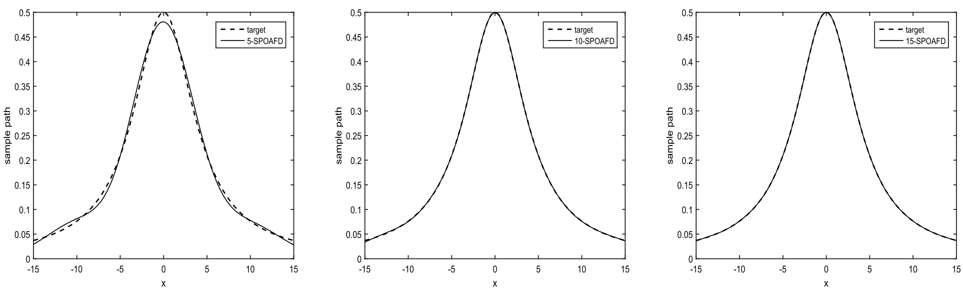
$$\begin{cases} (\partial_t - \Delta_x)u(t+\underline{x},\omega) = 0, \quad (t,\underline{x}) \in \mathbb{R}^1_+ \times \mathbb{R}^1, \ \text{a.s. } \omega \in \Omega, \\ u(0+\underline{x},\omega) = \dfrac{1}{2+\left(\dfrac{x}{3} - X(\omega)\right)^2}, \quad x \in \mathbb{R}^1, \ \text{a.s. } \omega \in \Omega, \end{cases} \tag{5.2}$$

where X is the random variable with the density function $p(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}}$.

Again for this example we can make direct use of the distribution of X. An optimal $q_n \in \mathbb{R}^1_+ \times \mathbb{R}^1$ is selected according to

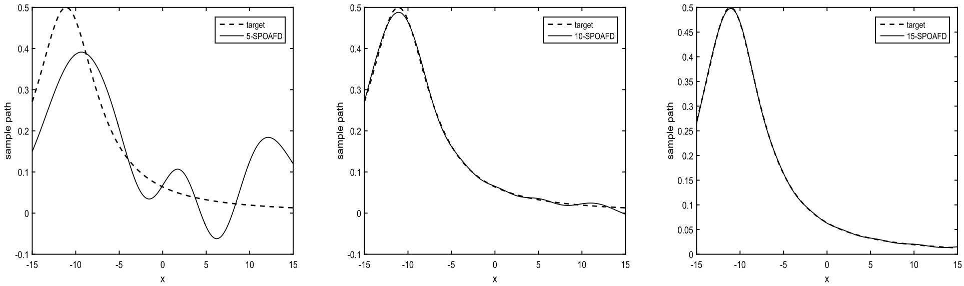
$$\mathbb{E}|\langle f_\omega, E^q_n \rangle|^2 = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \frac{1}{\sqrt{2+(\frac{x}{3}-s)^2}} E^q_n(x)dx \right|^2 p(s)ds.$$

We have three groups of figures, respectively, for visual effects of the experiments at those ω for which $X(\omega) = 0, -3.7$, and 3.1, as shown in Figs. 4–6. The relative



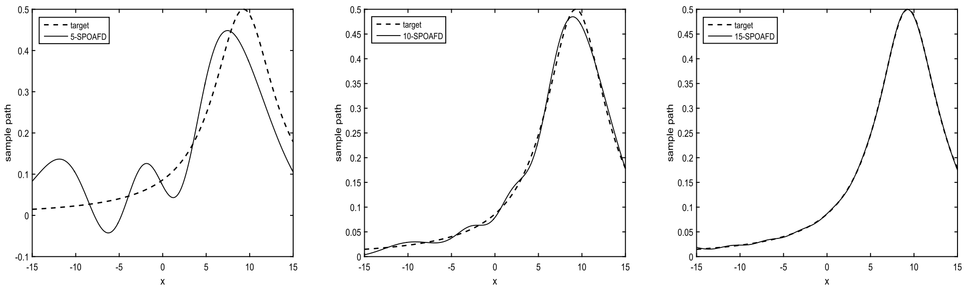
5 partial sum | 10 partial sum | 15 partial sum

Fig. 4. $X(\omega) = 0$.



5 partial sum 10 partial sum 15 partial sum

Fig. 5. $X(\omega) = -3.7$.



5 partial sum 10 partial sum 15 partial sum

Fig. 6. $X(\omega) = 3.1$.

Table 4. Relative error.

SPOAFD	0.0014	9.0613×10^{-6}	7.1549×10^{-7}

Table 5. Relative error.

SPOAFD	0.1759	4.8514×10^{-4}	2.7590×10^{-5}

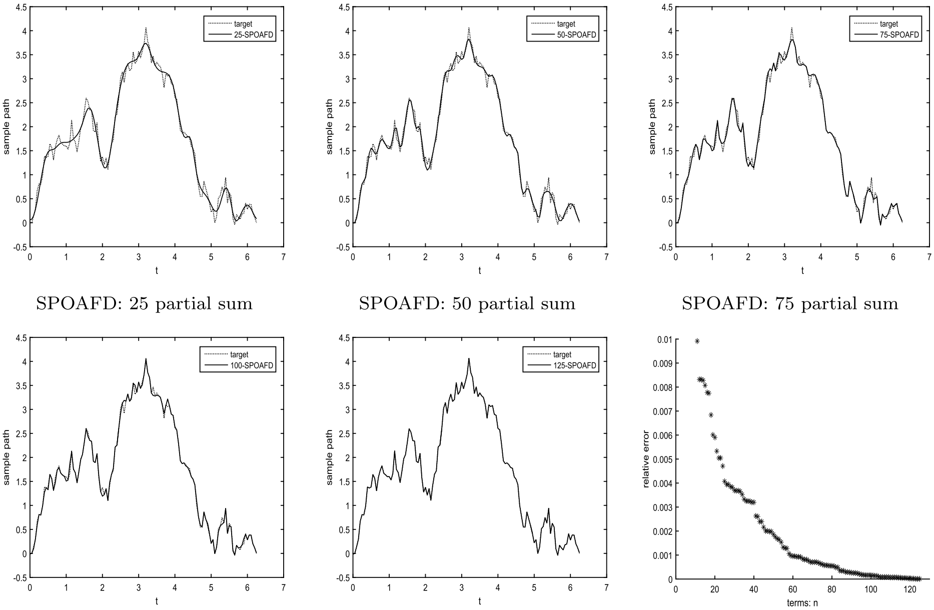
Table 6. Relative error.

SPOAFD	0.1064	0.0016	2.5211×10^{-5}

errors in Tables 4–6 are computed according to the formulas given in the relevant algorithms.

Example 3. In this example, we decompose Brownian bridge over $[0, 2\pi]$. We are considering the stochastic Dirichlet problem

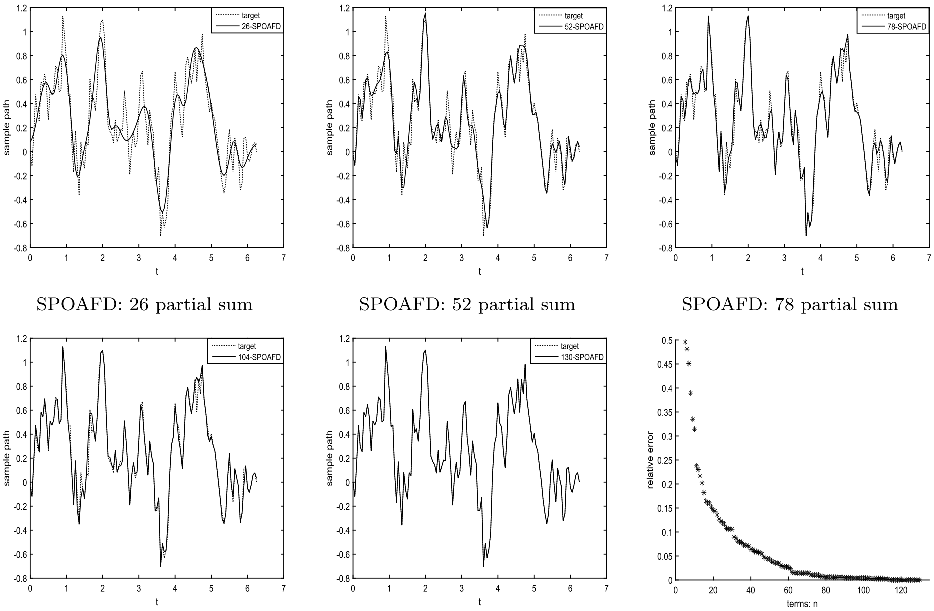
$$\begin{cases} \Delta u(z,\omega) = 0, \ z \in D, \\ u(e^{it},\omega) = W(t,\omega), \ t \in [0, 2\pi], \end{cases} \tag{5.3}$$



SPOAFD: 25 partial sum | SPOAFD: 50 partial sum | SPOAFD: 75 partial sum

SPOAFD: 100 partial sum | SPOAFD2: 125 partial sum | relative error

Fig. 7. Sample path I of Brownian bridge.



SPOAFD: 26 partial sum | SPOAFD: 52 partial sum | SPOAFD: 78 partial sum

SPOAFD: 104 partial sum | SPOAFD: 130 partial sum | relative error

Fig. 8. Sample path II of Brownian bridge.

where D is the unit disc in the complex plane, and $(t,\omega) \mapsto W(t,\omega)$ denotes a sample path or a Brownian bridge of the process. By using SPOAFD for the Poisson kernel in the unit disc we obtain an orthonormal system uniformly applicable for all Brownian bridges. For $n \geq 1$, an optimal q_n in the unit disc is selected according to

$$\mathbb{E}|\langle f_\omega, E_n^q\rangle|^2 = \int_0^{2\pi}\int_0^{2\pi} C(s,t)E_n^q(e^{is})E_n^q(e^{it})dsdt,$$

where $C(s,t)$ is the covariance function of the Brownian bridge.[37] We arbitrarily chose two sample paths to test effectiveness of the obtained SPOAFD system. As shown in Figs. 7 and 8, the results are promising.

6. Conclusion

The study demonstrates the successful application of the state-of-the-art SPOAFD method for solving Dirichlet BVPs and Cauchy IVPs involving random data, including stochastic processes. This method utilizes dictionaries of the Poisson and heat type potential kernels and produces sparse series representations of solutions that converge quickly. Furthermore, the researchers extended these methods to address BVPs and IVPs associated with the fractional heat equation and the fractional Poisson equation, which are currently of significant interest in the field. The numerical experimental results indicate promising convergence speeds, further highlighting the effectiveness of the approach.

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