A kind of multilinear operator and the Schatten—von Neumann classes

Peng Lizhong and Qian Tao

1. Introduction

Let $H^l(\mathbf{R}^d)$ denote the collection of all distributions m satisfying

(i) $m\in C^\infty(\mathbf{R}^d\setminus\{0\})$,

(ii) m is homogeneous of degree l, $l\geqq 0$.

Let R^N denote the operator which maps a function m to its Taylor remainder of order N, i.e.

$$R^N m(\eta, \Delta\eta) = m(\eta+\Delta\eta)-\sum_{|\alpha|\leqq N-1}\frac{1}{\alpha!}D^\alpha m(\eta)(\Delta\eta)^\alpha. \tag{1.1}$$

In general we consider

$$R^{N_1,\dots,N_n}m(\eta, \Delta\eta_1, \dots, \Delta\eta_n) = R^{N_n}R^{N_1,\dots,N_{n-1}}m(\eta, \Delta\eta_1, \dots, \Delta\eta_{n-1}, \Delta\eta_n).$$

In this paper we study the operator $T_{b_1,\dots,b_n}(R^{N_1,\dots,N_n}m)$ defined by

$$\begin{aligned}&[T_{b_1,\dots,b_n}(R^{N_1,\dots,N_n}m)f]\hat{}\,(\xi)\\ &= (2\pi)^{-nd}\int_{\mathbf{R}^{nd}}\prod_{j=1}^n \hat{b}_j(\eta_{j-1}-\eta_j)R^{N_1,\dots,N_n}m(\eta_n, \eta_{n-1}-\eta_n, \dots, \eta_0-\eta_1)\hat{f}(\eta_n)\,d\eta\end{aligned} \tag{1.2}$$

where $d\eta=d\eta_1, \dots, d\eta_n$, $\eta_0=\xi$.

In fact many multilinear singular integrals have the form (1.2). Let $d=1$, $m(\xi)=|\xi|$, then $[b, |D|]=[b, HD]=T_b(R^1 m)$, where H is the Hilbert transform. According to Janson and Peetre [5], $[b, |D|]$ is a paracommutator of the Toeplitz type, it is bounded on $L^2(\mathbf{R})$ if and only if $b'\in L^\infty$, and it is never compact unless $b'=0$. But $D[b, H]=T_b(R^2 m)$ is a paracommutator of the Hankel type; it is bounded on $L^2(\mathbf{R})$ if and only if $b'\in\mathrm{BMO}$, and $D[b, H]\in S_p$ (the Schatten—von Neumann class) if and only if $b\in B_p^{1+(1/p)}$ $(1\leqq p<\infty$, the Besov space). This is the motivation for studying the multilinear operator (1.2) using the Taylor remainder $R^N m$ instead of the difference $m(\xi)-m(\eta)$. Several authors have studied the boundedness of the multilinear operator (1.2) and obtained the BMO-results (direct results), e.g. Cohen [1, 2], Coifman and Meyer [3], Hu [4], Qian [9, 10], Qian and Li [11]. In this paper we study, in the framework of paracommutators (Janson and Peetre [5], Peng [6], [7]) and multi-fold paracommutators (Peng [8]), the boundedness, compactness, and the Schatten—von Neumann properties of the multilinear operator (1.2).

We adopt the notation for the Schatten—von Neumann class S_p, the Besov space B_p^s, the assumptions $A0$, $A1$, $A2$, $A3(\alpha)$, $A4$, $A4\frac{1}{2}$, $A5$, $A10(\alpha)$, A^* of the Fourier kernel $A(\xi, \eta)$, fractional integration or differentiation $I^l, \ldots$, in [5, 6, 7, 8].

In § 2, we study the direct results. In § 3, we study the converse results and the Janson—Wolff phenomena. In § 4, we discuss some examples.

2. Direct results

First of all, we study the case $n=1$, i.e. the bilinear operator.

Let $\varphi \in C_0^\infty(0, \infty)$ with $\varphi(t)=1$ on $[\delta^2, \delta^{-2}]$ for some small δ and define

(2.1) $$A_1(\xi, \eta) = \left(1-\varphi\left(\frac{|\eta|}{|\xi|}\right)\right)\frac{R^N m(\eta, \xi-\eta)}{|\xi-\eta|^l},$$

(2.2) $$A_2(\xi, \eta) = \varphi\left(\frac{|\eta|}{|\xi|}\right)\frac{R^N m(\eta, \xi-\eta)}{|\xi|^l}.$$

Thus

(2.3) $$T_b(R^N m) = T_{I^{-l}b}(A_1)+T_b^{l,0}(A_2).$$

By Lemma 3.1, 3.2 and 3.4 of Janson and Peetre [5],

$$T_b(R^N m)\in S_p \quad \text{if and only if both} \quad T_{I^{-l}b}(A_1) \quad \text{and} \quad T_b^{l,0}(A_2)\in S_p,$$

for $1 \leqq p \leqq \infty$,

$$T_b(R^N m) \text{ is compact if and only if both } T_{I^{-l}b}(A_1) \text{ and } T_b^{l,0}(A_2)$$

are compact.

So we can treat the two pieces separately.

Lemma 2.1. *Suppose that* $m\in H^l(\mathbf{R}^d)$, $l\geqq 0$, $N=[l]+1$. *Then* A_1 *satisfies* $A0$, $A1$, $A2$, $A3(\infty)$ *and* A_2 *satisfies* $A0$, $A1$, $A2$, $A3(\infty)$ *of* [5]. *Also* A_2 *satisfies* $A0$, $A1$, $A2$, $A3(N)$ *of* [5] *and vanishes on* $\Delta_j\times\Delta_k$ *when* $|j-k|$ *is large.*

Proof. It is obvious that A_1 and A_2 satisfy $A0$. If $|j-k|$ is small, $A_1=0$; if $|j-k|$ is large, e.g. $j\gg k$, $\eta\in\Delta_k$, $\zeta\in\Delta_j$, then $|\eta|<\delta|\xi|$. By Lemma 3.6 of [5], we have

$$\|A_1(\zeta,\eta)\|_{M(\Delta_j\times\Delta_k)}$$

$$\leqq \left\|1-\varphi\left(\frac{|\eta|}{|\zeta|}\right)\right\|_{M(\mathbf{R}^d\times\mathbf{R}^d)}\left\|\frac{|\zeta|^l}{|\zeta-\eta|^l}\right\|_{M(\Delta_j\times\Delta_k)}\left\|\frac{R^N m(\eta,\xi-\eta)}{|\zeta|^l}\right\|_{M(\Delta_j\times\Delta_k)}$$

$$\leqq c\left(\left\|\frac{m(\xi)}{|\xi|^l}\right\|_{M(\Delta_j\times\Delta_k)}+\sum_{|\alpha|\leqq N-1}\frac{1}{\alpha!}\left\|\frac{D^\alpha m(\eta)(\xi-\eta)^\alpha}{|\xi|^l}\right\|_{M(\Delta_j\times\Delta_k)}\right)$$

$$\leqq c\left(\left\|\frac{m(\xi)}{|\xi|^l}\right\|_{L^\infty(\Delta_j)}+\sum_{|\alpha|\leqq N-1}C_\alpha\sup_{\alpha_1+\alpha_2=\alpha}\||\xi|^{\alpha_1-l}\|_{L^\infty(\Delta_j)}\|D^\alpha m(\eta)|\eta|^{\alpha_2}\|_{L^\infty(\Delta_k)}\right)$$

$$\leqq c(1+2^{(k-j)(l+1-N)})\leqq c.$$

So A_1 satisfies $A1$.

It is similar to show that A_1 satisfies $A2$ and A_2 satisfies $A1$. Notice that A_1 vanishes on a neighbourhood of $\{\xi=\eta\}$, it follows that A_1 satisfies $A3(\infty)$.

Let us show that A_2 satisfies $A3(N)$. For any $B=B(\xi_0,r)$ with $r<\delta|\xi_0|$, by Lemma 3.10 of [5], we have

$$\|A_2(\xi,\eta)\|_{M(B\times B)}\leqq c\left(\frac{r}{|\xi_0|}\right)^N\sup_{|\alpha|\leqq m}\ \sup_{\xi,\eta\in B(\xi_0,2r)}|\xi_0|^{|\alpha|}|D^\alpha A_2(\xi,\eta)|\leqq c\left(\frac{r}{|\xi_0|}\right)^N.$$

It is obvious that A_2 vanishes on $\Delta_j\times\Delta_k$ when $|k-j|$ is large. □

Remark. By the definitions of A_p1, A_p3 of Peng [7], we can also show that A_1 satisfies A_p1, $A_p3(\infty)$ and that A_2 satisfies A_p1, $A_p3(N)$, for $0<p\leqq 1$.

Combining Lemma (2.1), Theorems 7.3, 8.1, 13.1, 13.3 (and its extension) of [5], and Theorem 1 of [7], we get the following.

Theorem 2.1. *Suppose that* $m\in H^l(\mathbf{R}^d)$, $l\geqq 0$, $N=[l]+1$, $s,t>\max[-d/2,-d/p]$, $s+t+l+d/p<N$, $1<p\leqq\infty$. *Then*

(i) $b\in I^l(\mathrm{BMO})$ *implies that* $T_b(R^Nm)\in S_\infty$,

(ii) $b\in I^l(\mathrm{CMO}$ *implies that* $T_b(R^Nm)$ *is compact,*

(iii) $b\in B_p^{s+t+l+(d/p)}$ *implies that* $T_b^{s,t}(R^Nm)\in S_p$,

(iv) $b\in b_\infty^{s+t+l}$ *implies that* $T_b^{s,t}(R^Nm)$ *is compact.* □

Now we study the case $n\geqq 2$. Let X_p denote the space $B_p^{1/p}$ (if $p<\infty$) or the space BMO (if $p=\infty$).

Theorem 2.2. *Suppose that* $m\in H^l(\mathbf{R}^d)$, $l\geqq 0$, $0<\alpha_i\leqq 1$, $N_i\in\mathbf{N}$, $d/\alpha_i<p_i\leqq\infty$, *for* $i=1,\dots,n$, *and that* $\sum_{i=1}^n(N_i-\alpha_i)=l$, $1/p=\sum_{i=1}^n 1/p_i$, $1\leqq p\leqq\infty$. *Then*

(2.4) $$\|T_{b_1,\dots,b_n}(R^{N_1,\dots,N_n}m)\|_{S_p}\leqq C\prod_{i=1}^n\|b_i\|_{I^{N_i-\alpha_i}(X_{p_i})}.$$

Proof. If $l=0$, $N_i=\alpha_i=1$, for $i=1,\dots,n$, then Theorem 2.2 implies Theorem 3 of [8]. We prove this theorem using the procedure of the proof of Theorem 3 in [8].

Let $\varphi\in C^\infty(0,\infty)$ be such that $\varphi\equiv 1$ on $(0,n+1)$ and $\varphi\equiv 0$ on $(n+2,\infty)$, $\psi=1-\varphi$. Then we have

$$R^{N_1,\dots,N_n}m(\eta_n,\eta_{n-1}-\eta_n,\dots,\eta_0-\eta_1)$$

$$=R^{N_1,\dots,N_n}m(\eta_n,\eta_{n-1}-\eta_n,\dots,\eta_0-\eta_1)\,\Pi_{i=1}^n\left[\psi\left(\frac{|\eta_0|}{|\eta_i-\eta_{i-1}|}\right)+\varphi\left(\frac{|\eta_0|}{|\eta_i-\eta_{i-1}|}\right)\right]$$

$$=\sum_{J\in G_n}A_J(\eta_0,\eta_1,\dots,\eta_n)$$

where G_n is the set of subsets J of $\{1,\dots,n\}$,

$$A_J(\eta_0,\eta_1,\dots,\eta_n)=R^{N_1,\dots,N_n}m(\eta_n,\eta_{n-1}-\eta_n,\dots,\eta_0-\eta_1)$$

$$\cdot\,\Pi_{j\in J}\psi\left(\frac{|\eta_0|}{|\eta_j-\eta_{j-1}|}\right)\Pi_{j'\in J'}\varphi\left(\frac{|\eta_0|}{|\eta'_j-\eta'_{j-1}|}\right),$$

J' is the complement of J in $\{1,\dots,n\}$.

It suffices to show (2.4) for each A_J.

Let $\bar A_J=R^{N_1,\dots,N_n}m(\eta_n,\eta_{n-1}-\eta_n,\dots,\eta_0-\eta_1)$

$$\cdot\,\Pi_{j\in J}\psi\left(\frac{|\eta_0|}{|\eta_j-\eta_{j-1}|}\right)\frac{1}{|\eta_j|^{N_j-\alpha_j}}\,\Pi_{j'\in J'}\varphi\left(\frac{|\eta_0|}{|\eta'_j-\eta'_{j-1}|}\right)\frac{1}{|\eta'_j-\eta'_{j-1}|^{N'_j-\alpha'_j}},$$

then

$$T_{b_1,\dots,b_n}(R^{N_1,\dots,N_n}m)=T^{s_0,s_1,\dots,s_n}_{I^{\beta_1}b_1,\dots,I^{\beta_n}b_n}(\bar A_J),$$

where $\beta_j=0$ if $j\in J$, $\beta_{j'}=N_{j'}-\alpha_{j'}$ if $j'\in J'$,

$$s_j=N_{j+1}-\alpha_{j+1}\quad\text{if}\quad j+1\in J,\quad s_{j'}=0\quad\text{if}\quad j'+1\in J',\quad s_n=0.$$

It is not too hard to check $\bar A_J$ satisfies the assumption $A^*(N_1-\alpha_1,\dots,N_n-\alpha_n)$ in Theorem 2 of [8]. So Theorem 2 of [8] shows that

$$\|T_{b_1,\dots,b_n}(A_J)\|_{S_p}\le C\,\Pi_{i=1}^n\|b_i\|_{I^{N_i-\alpha_i}(X_{p_i})}.\quad\square$$

3. Converse results and the Janson—Wolff phenomena

We need some non-degeneracy assumptions on m.

ND1. If l is an integer, $m\in H^l(\mathbf{R}^d)$, for any $\xi_0\in S_{d-1}$, there exists $0\ne\eta_0\in\mathbf{R}^d$ such that

$$m(\xi_0)-\sum_{|\alpha|=l}\frac{1}{\alpha!}D^\alpha m(\eta_0)\xi_0^\alpha\ne 0.$$

ND2. If l is a non-integer, $m\in H^l(\mathbf{R}^d)$, for any $\xi_0\in S_{d-1}$,

$$m(\xi_0)\ne 0.$$

ND3. If $m\in H^l(\mathbf{R}^d)$, $l\geqq 0$, $N=[l]+1$, for any $\xi_0\in S_{d-1}$. There exists $0\neq\eta_0\in\mathbf{R}^d$ such that

$$D^N_{\xi_0}m(\eta_0)\neq 0,$$

where $D^N_{\xi_0}m(\eta_0)$ denote the direction derivative of order N along $\xi_0\in S_{d-1}$.

We consider the converse results and the Janson—Wolff phenomena only for the case $n=1$.

Lemma 3.1. *If* $m\in H^l(\mathbf{R}^d)$, $l\geqq 0$, $N=[l]+1$, *m satisfies* ND1 (when l is an integer) *or* ND2 (when l is a non-integer), *then* A_1 *in* (2.1) *satisfies* $A4\frac{1}{2}$ *and* $A5$. (For $A4\frac{1}{2}$, see Peng [6].)

Proof. When l is an integer, $N=l+1$. For any $\xi_0\in S_{d-1}$, by ND1, we can take $0\neq\eta_0'\in\mathbf{R}^d$ such that

$$k=\left|m(\xi_0)-\sum_{|\alpha|=N-1}\frac{1}{\alpha!}D^\alpha m(\eta_0')\xi_0^\alpha\right|>0.$$

By the homogeneity of degree 0 of $D^\alpha m(\eta)$, for any $t\in(0,\infty)$,

$$\left|m(\xi_0)-\sum_{|\alpha|=N-1}\frac{1}{\alpha!}D^\alpha m(t\eta_0')\xi_0^\alpha\right|=k.$$

Thus, if δ is small enough, we have

$$\begin{aligned}
&\left\|m(\xi_0)-\sum_{|\alpha|=N-1}\frac{1}{\alpha!}D^\alpha m(t\eta_0')\xi_0^\alpha-\frac{R^N m(\eta,\xi-\eta)}{|\xi|^l}\right\|_{M(U\times V)}\\
&=\left\|\frac{m(\xi)}{|\xi|^l}-m(\xi_0)\sum_{|\alpha|\leqq N-2}\frac{1}{\alpha!}D^\alpha m(\eta)(\xi-\eta)^\alpha/|\xi|^l\right.\\
&\quad-\sum_{|\alpha|=N-1}\frac{1}{\alpha!}D^\alpha m(\eta)\sum_{\substack{\alpha_1+\alpha_2=\alpha\\|\alpha_2|<0}}C_\alpha\xi^{\alpha_1}\eta^{\alpha_2}/|\xi|^l\\
&\left.-\sum_{|\alpha|=N-1}\frac{1}{\alpha!}D^\alpha m(\eta)\frac{\xi^\alpha}{|\xi|^l}-D^\alpha m(t\eta_0')\xi_0^\alpha\right\|_{M(U\times V)}\leqq\left\|\frac{m(\xi)}{|\xi|^l}-m(\xi_0)\right\|_{L^\infty(U)}\\
&\quad+\sum_{|\alpha|\leqq N-2}\frac{1}{\alpha!}\|D^\alpha m(\eta)\|_{L^\infty(V)}\sum_{\alpha_1+\alpha_2=\alpha}|C_{\alpha_1}|\left\|\frac{\xi^{\alpha_1}}{|\xi|^l}\right\|_{L^\infty(U)}\|\eta^{\alpha_2}\|_{L^\infty(V)}\\
&\quad+\sum_{|\alpha|=N-1}\frac{1}{\alpha!}\sum_{\substack{\alpha_1+\alpha_2=\alpha\\|\alpha_2|>0}}|C_{\alpha_1}|\left\|\frac{\xi^{\alpha_1}}{|\xi|^l}\right\|_{L^\infty(U)}\|D^\alpha m(\eta)\eta^{\alpha_2}\|_{L^\infty(V)}\\
&\quad+\sum_{|\alpha|=N-1}\frac{1}{\alpha!}\|D^\alpha m(\eta)-D^\alpha m(t\eta_0')\|_{L^\infty(V)}\left\|\frac{\xi^\alpha}{|\xi|^l}\right\|_{L^\infty(U)}\\
&\quad+\sum_{|\alpha|=N-1}\frac{1}{\alpha!}|D^\alpha m(t\eta_0')|\left\|\frac{\xi^\alpha}{|\xi|^l}-\xi_0^\alpha\right\|_{L^\infty(U)}\\
&\leqq c\delta^{\frac{1}{2}}\ (\text{choose } t \text{ so that } |t\eta_0'|=|\eta_0|=\delta^{\frac{1}{2}})<k
\end{aligned}$$

which implies that $R^N m(\eta, \xi-\eta)/|\xi|^l$ is invertible in $M(U\times V)$, moreover by Lemma 3.6 of [5], A_1 is invertible in $M(U\times V)$.

When l is a non-integer, $l>0$, $N=[l]+1$, ND2 implies that, for any $\xi_0\in S_{d-1}$, $|m(\xi_0)|=k>0$. If δ is small enough, we have

$$\left\| m(\xi_0)-\frac{R^N m(\eta,\xi-\eta)}{|\xi|^l}\right\|_{M(U\times V)}$$

$$=\left\| m(\xi_0)-\frac{m(\xi)}{|\xi|^l}+\sum_{|\alpha|\leqq N-1}\frac{1}{\alpha!}D^\alpha m(\eta)\sum_{\alpha_1+\alpha_2=\alpha}C_{\alpha_1}\xi^{\alpha_1}\eta^{\alpha_2}/|\xi|^l\right\|_{M(U\times V)}$$

$$\leqq \left\|\frac{m(\xi)}{|\xi|^l}-m(\xi_0)\right\|_{L^\infty(U)}+\sum_{|\alpha|\leqq N+1}\frac{1}{\alpha!}\sum_{\alpha_1+\alpha_2=\alpha}|C_{\alpha_1}|\left\|\frac{\xi^\alpha}{|\xi|^l}\right\|_{L^\infty(U)}\|D^\alpha(\eta)\eta^{\alpha_2}\|_{L^\infty(V)}$$

$$\leqq c\delta^{l+1-N}\ (\text{choose } |\eta_0|=2\delta)<k.$$

This implies that $R^N m(\eta,\xi-\eta)/|\xi|^l$ is invertible in $M(U\times V)$, again by Lemma 3.6 of [5], A_1 is invertible in $M(U\times V)$.

Because A_1 satisfies $A0$, that A_1 satisfies $A5$ implies that A_1 satisfies $A4\frac{1}{2}$.

Lemma 3.2. *If* $m\in H^l(\mathbf{R}^d)$, $l\geqq 0$, $N=[l]+1$, *m satisfies* ND3, *then* A_2 *satisfies* $A10(N)$. (For $A10(N)$, see Peng [7].)

Remark 3.1. It is easy to see from the proof that A_1 satisfies also $A_p4\frac{1}{2}$ of [7] for any $0<p<1$.

Proof. Recall the assumption $A10(N)$: for any $0\neq\theta\in\mathbf{R}^d$, there exist a positive number $\delta<\frac{1}{2}$ and a subset V_θ of $\mathbf{R}^d$ such that if N_r denote the number of integer points contained in $V_\theta\cap B_r$, where $B_r=B(0,r)$, then $\overline{\lim}_{r\to\infty}N_r/r^d>0$, and for every $\underline{n}\in V_\theta$,

$$\left\|\frac{1}{A(\cdot+\underline{n}+\theta,\ \cdot+\underline{n})}\right\|_{M(B\times B)}\leqq c|\underline{n}|^N,\quad\text{where}\quad B=B(0,\delta).$$

For any $0\neq\theta\in S_{d-1}$, by ND3, there exists $0\neq\eta_0\in\mathbf{R}^d$ such that

$$D_\theta^N m(\eta_0)=\sum_{|\alpha|=N}D^\alpha m(\eta_0)\theta^\alpha\neq 0.$$

We can assume that $|\eta_0|=1$, $k=|D_\theta^N m(\eta_0)|>0$. By the continuity, there exists δ such that if $|\xi-\theta|<\delta$, $|\eta-\eta_0|<\delta$, then

$$\left|\sum_{|\alpha|=N}D^\alpha m(\eta)\xi^\alpha\right|\geqq k/2.$$

Let $V_\theta=\left\{\eta\in\mathbf{R}^d\colon \left|\frac{\eta}{|\eta|}-\eta_0\right|<\delta,\ |\eta|>23/\delta\right\}$, then V_θ satisfies the condition of $A10(N)$.

Let $\underline{n}\in V_\theta$, if $u\in B$, $v\in B$, $B=B(0,\delta)$, then

$$\left|R^N m(v+\underline{n}, (u+\underline{n}+\theta)-(v+\underline{n}))\right| = \left|\sum_{|\alpha|=N}\frac{1}{\alpha!}D^\alpha m(\bar{\eta})(u+\theta-v)^\alpha\right| \geqq ck|\underline{n}|^{l-N}.$$

Note that $R^N m(v+\underline{n}, (u+\underline{n}+\theta)-(v+\underline{n}))\in C^\infty(2B\times 2B)$, so

$$1/R^N m(v+\underline{n}, (u+\underline{n}+\theta)-(v+\underline{n}))$$

can be expressed as the absolutely convergent Fourier series:

$$\frac{1}{R^N m(v+\underline{n}, (u+\underline{n}+\theta)-(v+\underline{n}))}\sum_{j,k\in Z^d} a_{\underline{j},\underline{k}}\beta_{\underline{j},\underline{k}}(u)\gamma_{\underline{j},\underline{k}}(v),$$

where

$$\sum |a_{\underline{j},\underline{k}}| \leqq c\sum_{|\alpha|\leqq M}\left\|D^\alpha\frac{1}{R^N m(\cdot+\underline{n}, (\cdot+\underline{n}+\theta)-(\cdot+\underline{n}))}\right\|_{L^\infty(2B\times 2B)} \leqq c|\underline{n}|^{N-l}.$$

Therefore

$$\left\|\frac{1}{A_2(\cdot+\underline{n}+\theta, \cdot+\underline{n})}\right\|_{M(B\times B)} \leqq c\,|n|^N,$$

i.e. $A10(N)$ holds. □

Lemma 3.1, Theorem 10.1 of [5] and Theorem 2 of [6] and its extension give the following converse results.

Theorem 3.1. *Suppose that* $m\in H^l(\mathbf{R}^d)$, $l\geqq 0$, $N=[l]+1$, *and m satisfies* ND1 *(when l is an integer) or* ND2 *(when l is a non-integer). Then* $T_b(R^N m)$ *is bounded on* $L^2(\mathbf{R}^d)$ *implies that* $I^{-l}b\in\mathrm{BMO}$, *and* $T_b(R^N m)$ *is compact implies that* $I^{-l}b\in\mathrm{CMO}$.

Lemma 3.1, Theorem 9.1 of [5] and Theorem 2 of [7] give the following converse results.

Theorem 3.2. *Suppose that* $m\in H^l(\mathbf{R}^d)$, $l\geqq 0$, $N=[l]+1$, *and m satisfies* ND1 *(when l is an integer) or* ND2 *(when l is a non-integer). Then for* $1\leqq p\leqq\infty$, *any* s, t, $T_b^{s,t}(R^N m)\in S_p$ *implies that* $b\in B_p^{s+t+l+d/p}$. *For* $0<p<1$, $s, t>-d/2$, *and* $T_b^{s,t}(R^N m)\in S_p$ *implies that the following a priori inequality holds*

$$\|b\|_{B_p^{s+t+l+d/p}} \leqq c\,\|T_b^{s,t}(R^{N}m)\|_{S_p}.$$

Lemma 3.2 and Theorem 4 of [7] give the following results about the Janson—Wolff phenomena.

Theorem 3.3. *Suppose that* $m\in H^l(\mathbf{R}^d)$, $l\geqq 0$, $N=[l]+1$, *and m satisfies* ND3. *Then for* $1\leqq p\leqq d/N-l-s-t$, $T_b^{s,t}(R^N m)\in S_p$ *implies that b is a polynomial. For* $0<p\leqq\min(d/N-l-s-t, 1)$, $b\in S'(\mathbf{R}^d)$ *with* $\hat{b}$ *with compact support such that* $T_b^{s,t}(R^N m)\in S_p$ *implies that b is a polynomial.*

Applications.

1. Combining Theorem 2.1, 3.1, 3.2 and 3.3, we get the following

Theorem Σ. *Suppose that* $m \in H^l(\mathbf{R}^d)$, $l \geqq 0$, $N=[l]+1$, *and* m *satisfies* ND1 *(when* l *is an integer) or* ND2 *(when* l *is a non-integer) and* ND3. *Then*

(i) $T_b(R^N m)$ *is bounded on* $L^2(\mathbf{R}^d)$ *if and only if* $I^{-l} b \in \mathrm{BMO}$,

(ii) $T_b(R^N m)$ *is compact if and only if* $I^{-l} b \in \mathrm{CMO}$,

(iii) *for* $d/N - l < p < \infty$ *and* $p \geqq 1$, $T_b(R^N m) \in S_p$ *if and only if* $b \in B_p^{l+d/p}$; *for* $0 < p < 1$, *directly,* $b \in B_p^{l+d/p}$ *implies* $T_b(R^N m) \in S_p$ *and, conversely, an a priori inequality holds.*

(iv) *for* $1 \leqq p \leqq d/N - l$, $T_b(R^N m) \in S_p$ *if and only if* b *is a polynomial; for* $0 \leqq p \leqq \min(d/N - l, 1)$, $b \in S'(\mathbf{R}^d)$ *with* $\hat{b}$ *with compact support implies that* b *is a polynomial.*

2. Higher commutators of fractional integration.

In particular, if $m(\xi) = |\xi|^l$, $l > 0$, then $m \in H^l(R^d)$, and m satisfies ND1 (or ND2) and ND3. So Theorem Σ gives a generalization of Example 8 in [5] from the commutators of fractional integration to the higher commutators.

3. Multilinear singular integrals.

Lemma (Qian [10]). *Suppose that* $\Omega \in H^0(\mathbf{R}^d)$, *and* $\int_{S^{d-1}} \Omega(x) x^\beta \, d\sigma(x) = 0$, *for* $|\beta| \leqq l$ *and* $l > 0$. *Denote, for* $N_1 + \ldots + N_n \leqq l + n$,

$$T_{b_1, \ldots, b_n}^{N_1, \ldots, N_n}(\Omega) f(x) = \text{p.v.} \int \prod_{j=1}^n P^{N_j} b_j(x, y-x) \frac{\Omega(x-y)}{|x-y|^{d+l}} f(y)\, dy.$$

Then

$$T_{b_1, \ldots, b_n}^{N_1, \ldots, N_n}(\Omega) f = T_{b_1, \ldots, b_n}(R^{N_1, \ldots, N_n} m) f \quad \textit{for every} \quad f \in C_0^\infty(\mathbf{R}^d),$$

where

$$m(\xi) = c|\xi|^l \int_{S_{d-1}} \Omega(y) L(\xi' y)\, d\sigma(y), \quad \xi' = \xi/|\xi|, \quad L = L_1 + L_2,$$

$$L_1(t) = \int_0^\infty \frac{e^{itr}}{r^{l+1}}\, dr, \quad L_2(t) = \frac{(it)^{l+1}}{l!} \int_0^1 \int_0^1 u^l e^{irt(1-u)}\, du\, dr.$$

(See Qian [10], Theorem 1.) □

Many authors have studied the boundedness (direct results) of $T_{b_1, \ldots, b_n}^{N_1, \ldots, N_n}(\Omega)$. Cohen [2] obtained the result for the case $n=1$, $N_1=1$, Hu [4] obtained the result for the case $N_1 = \ldots = N_n = 1$. Qian [9] obtained the result for the general case.

Qian and Li [11] obtained the boundedness (direct results) of $T_{b_1, \ldots, b_n}(R^{N_1, \ldots, N_n} m)$.

Theorem 2.2 of this paper gives the characterization of the boundedness and the Schatten—von Neumann properties for $T_{b_1, \ldots, b_n}(R^{N_1, \ldots, N_n} m)$. It includes the result of Qian and Li [11].

Theorem 2.2 and Lemma 4.1 give the characterization of the boundedness and the Schatten—von Neumann properties for $T^{N_1,\dots,N_n}_{b_1,\dots,b_n}(\Omega)$. It includes the results of Cohen [2], Hu [4] and Qian [9].

For the case $n=1$, Theorem Σ and Lemma 4.1 give a perfect characterization of the boundedness, the compactness, the Schatten—von Neumann properties and the Janson—Wolff phenomena for both $T^N_b(\Omega)$ and $T_b(R^N m)$.

Remark. Finally, we say a few words why we deal only with the case $N=[l]+1$. In this case, the operator $T_b(R^N m)$ behaves as a Hankel operator, so we can study its compactness and Schatten—von Neumann properties. For the case $N=[l]$ some results on boundedness are obtained in [4], [10], [11]. But then $T_b(R^N m)$ behaves as a Toeplitz operator and, therefore, cannot be compact in general. We will study this case elsewhere.

Notice also that in the proof of Lemma 3.1, the choice $|\eta_0|=\delta^{1/2}$ guarantees that the fourth term is small; the choice $|\eta_0|=2\delta$ can not do this job.

Acknowledgement

We would like to thank The Centre for Mathematical Analysis, the Australian National University, for its financial support.

References

1. Cohen, J., Multilinear singular integrals, *Studia Math.* **68** (1980), 261—280.
2. Cohen, J., A sharp estimate for a multilinear singular integral in $\mathbf{R}^n$, *Indiana Univ. Math. J.* **30** (1981), 693—702.
3. Coifman, R. and Meyer, U., Commutateurs d'intégrales singulières et opérateurs multilinéares, *Ann. Inst. Fourier (Grenoble)* **28** (1978), 177—202.
4. Hu, Y., An estimate for multilinear singular integrals in $\mathbf{R}^n$, *Acta Sci. Natur. Univ. Pekinensis* 3 (1985), 19—26. (In Chinese.)
5. Janson, S. and Peetre, J., Paracommutators — boundedness and Schatten—von Neumann properties, *Trans. Am. Math. Soc.* **305** (1988), 467—504.
6. Peng, L. Zh., On the compactness of paracommutators, *Ark. Mat.* **26** (1988), 315—325.
7. Peng, L. Zh., Paracommutators of Schatten—von Neumann class S_p, $0<p<1$, *Math. Scand.* **61** (1987), 68—92.
8. Peng, L. Zh., *Multilinear singular integrals of Schatten—von Neumann class* S_p, Institut Mittag-Leffler, report 7:1986.
9. Qian, T., On estimate for a multilinear singular integrals, *Sci. Sinica (Series A)* **27**:11 (1984), 1143—1154.
10. Qian, T., Commutators of homogeneous multiplier operators, *Sci. Sinica (Series A)* **28**:3 (1985), 225—234.

11. QIAN, T. and LI, CH., Pointwise estimates for a class of singular integrals and higher commutators, *Acta Math. Sinica* **2** (1986), 249—259.
12. ROCHBERG, R., Decomposition theorems for Bergman spaces and their applications, in: S. C. Power (ed.) *Operators and function theory*, pp. 225—278. Reidel, Dordrecht–Boston–Lancaster, 1984.
13. SEMMES, S., *The Cauchy integral and related operators on smooth curves*, dissertation, Washington Univ., St. Louis, 1983.

Received Dec. 15, 1987

Peng Lizhong
Department of Mathematics
Peking University
Beijing
China

Qian Tao
School of Mathematics, Physics
Computing and Electronics
Macquarie University
NSW 2109
Australia