Revista Matemática Iberoamericana
Vol. 10, N.° 3, 1994

Clifford algebras, Fourier transforms and singular convolution operators on Lipschitz surfaces

Chun Li, Alan McIntosh and Tao Qian

1. Introduction.

In the Fourier theory of functions of one variable, it is common to extend a function and its Fourier transform holomorphically to domains in the complex plane $\mathbb{C}$, and to use the power of complex function theory. This depends on first extending the exponential function $e^{ix\xi}$ of the real variables x and ξ to a function $e^{iz\zeta}$ which depends holomorphically on both the complex variables z and ζ.

Our thesis is this. The natural analog in higher dimensions is to extend a function of m real variables monogenically to a function of $m+1$ real variables (with values in a complex Clifford algebra), and to extend its Fourier transform holomorphically to a function of m complex variables. This depends on first extending the exponential function $e^{i\langle \mathbf{x},\xi\rangle}$ of the real variables $\mathbf{x} \in \mathbb{R}^m$ and $\xi \in \mathbb{R}^m$ to a function $e(x,\zeta)$ which depends monogenically on $x = \mathbf{x} + x_L e_L \in \mathbb{R}^{m+1}$ and holomorphically on $\zeta = \xi + i\eta \in \mathbb{C}^m$.

We explore this thesis for functions Φ whose monogenic extensions are bounded by a constant multiple of $|x|^{-m}$ on a cone

$$C^{\circ}_{\mu+} = \{x = \mathbf{x} + x_L e_L \in \mathbb{R}^{m+1} : \; x_L > -|\mathbf{x}| \tan \mu\} \; ,$$

on $C^{\circ}_{\mu-} = -C^{\circ}_{\mu+}$, or on $S^{\circ}_{\mu} = C^{\circ}_{\mu+} \cap C^{\circ}_{\mu-}$. The Fourier transforms b of these functions extend holomorphically to bounded functions on certain cones $S^{\circ}_{\nu}(\mathbb{C}^m)$ in $\mathbb{C}^m$ (for all $\nu < \mu$). Conversely, every bounded holomorphic function b on $S^{\circ}_{\mu}(\mathbb{C}^m)$ can be decomposed as $b = b_+ + b_-$, where $b_{\pm}$ are the Fourier transforms of functions f whose monogenic extensions are bounded by $c\,|x|^{-m}$ on $C^{\circ}_{\nu\pm}$ (for all $\nu < \mu$).

Such functions were studied in [LMcS], where it was shown that if Φ is a right-monogenic function which is bounded by $c\,|x|^{-m}$ on $C^{\circ}_{\mu+}$, then the singular convolution operator T_{Φ} defined by

$$(T_{\Phi} u)(x) = \lim_{\delta \to 0+} \int_{\Sigma} \Phi(x + \delta e_L - y)\, n(y)\, u(y)\, dS_y$$

is a bounded linear operator on $L_p(\Sigma)$ for $1 < p < \infty$. Here Σ is a Lipschitz surface consisting of all the points $x = \mathbf{x} + g(\mathbf{x})\, e_L \in \mathbb{R}^{m+1}$, where $\mathbf{x} \in \mathbb{R}^m$, and g is a real-valued Lipschitz function which satisfies $\|\nabla g\|_{\infty} \le \tan \omega$ for some $\omega < \mu$. We have embedded $\mathbb{R}^m$ in a complex Clifford algebra with at least m generators in the usual way, and identified the extra basis element e_L of $\mathbb{R}^{m+1}$ with either another generator such as e_{m+1}, or with the identity e_0. We use Clifford multiplication in the above integrand, in which $n(y)$ denotes the unit normal (which is defined at almost all $y \in \Sigma$).

So the Fourier transform b of Φ can be thought of as the Fourier multiplier corresponding to T_{Φ}. But we can also think of the mapping from b to $T_{\Phi} \in \mathcal{L}(L_p(\Sigma))$ as giving us a bounded H_{∞} functional calculus of a differential operator

$$-i\, \mathbf{D}_{\Sigma} = \sum_{k=1}^{m} -i\, e_k\, D_{k,\Sigma} \; ,$$

and write

$$T_{\Phi} = b(-i\, \mathbf{D}_{\Sigma}) = b(-i\, D_{1,\Sigma}, -i\, D_{2,\Sigma}, \dots, -i\, D_{m,\Sigma}) \, .$$

Such functional calculi are studied at length later in this paper. The operators $D_{k,\Sigma}$ are given by

$$D_{k,\Sigma}\, u = \frac{\partial U}{\partial x_k}\Big|_{\Sigma}$$

when u is the restriction to Σ of a function U which is left-monogenic on a neighbourhood of Σ.

Not surprisingly, $\mathbf{D}_\Sigma$ is the operator considered previously by Murray [M] and McIntosh [McI], when using Clifford analysis to prove the L_p-boundedness of the Cauchy singular integral C_Σ on Σ. For, if Σ is parametrized by $x = \mathbf{s} + g(\mathbf{s})\, e_L$, $\mathbf{s} \in \mathbb{R}^m$, then

$$(\mathbf{D}_\Sigma\, e_L u)(\mathbf{s} + g(\mathbf{s})\, e_L) = (e_L - \mathbf{D}g)^{-1}\mathbf{D}_s\, u(\mathbf{s} + g(\mathbf{s})\, e_L)\,, \quad u \in W_p^1(\Sigma)\,.$$

It is not easy to extend a given function monogenically from a domain in $\mathbb{R}^m$ to a domain in $\mathbb{R}^{m+1}$. In particular, it is not easy to tell whether a given function Φ defined on $\mathbb{R}^m \setminus \{0\}$ extends to a right-monogenic function which is bounded by $c|x|^{-m}$ on S_μ°, and hence whether the results of [LMcS] can be used to conclude that the singular convolution operator T_Φ is a bounded linear operator on $L_p(\Sigma)$. The use of Fourier theory helps. For example, consider the functions defined on $\mathbb{R}^m \setminus \{0\}$ by

$$\Phi_k(\mathbf{x}) = -\frac{2\,x_k}{\sigma_m\,|\mathbf{x}|^{m+1}}\,.$$

Their Fourier transforms are $r_k(\xi) = i\,\xi_k/|\xi|$, which extend holomorphically to bounded functions on the subsets $S_\mu^\circ(\mathbb{C}^m)$ of $\mathbb{C}^m$ (as we shall see). Therefore the functions Φ_k extend monogenically to functions satisfying the appropriate bounds on S_μ°, and the corresponding singular convolution operators $R_{k,\Sigma}$ are bounded on $L_p(\Sigma)$ for $1 < p < \infty$. These operators can be thought of as *Riesz transforms* on Σ. They satisfy $R_{j,\Sigma}R_{k,\Sigma} = R_{k,\Sigma}R_{j,\Sigma}$, $\sum e_k R_{k,\Sigma} = C_\Sigma$ and $\sum (R_{k,\Sigma})^2 = -I$.

As other examples, take $b_s(\xi) = |\xi|^{2is}$ for s real. Such b_s extend holomorphically to bounded functions on $S_\mu^\circ(\mathbb{C}^m)$, so their inverse Fourier transforms extend monogenically to functions satisfying the appropriate bounds, and the corresponding singular convolution operators are bounded on $L_p(\Sigma)$ for $1 < p < \infty$. These operators are purely imaginary powers of $\mathbf{D}_\Sigma^2$, which can be thought of as the negative of the Laplacian on Σ. See Sections 6 to 8.

In Section 7, we indicate the kind of application of these results that we have in mind by considering a boundary value problem for harmonic functions.

Let us briefly recall the main results from [McQ] concerning Fourier transforms of holomorphic functions defined on sectors in the complex plane. This material is generalized to higher dimensions in Sections 3 and 4.

For $0 < \mu \le \pi/2$, define the sectors

$$S^{\circ}_{\mu^+}(\mathbb{C}) = \{\lambda \in \mathbb{C} : \lambda \neq 0,\ |\arg(\lambda)| < \mu\}, \qquad S^{\circ}_{\mu^-}(\mathbb{C}) = -S^{\circ}_{\mu^+}(\mathbb{C}),$$

and the cones

$$C^{\circ}_{\mu^+}(\mathbb{C}) = \{Z = X + iY \in \mathbb{C} :\ Z \neq 0,\ Y > |X| \tan\mu\},$$
$$C^{\circ}_{\mu^-}(\mathbb{C}) = -C^{\circ}_{\mu^+}(\mathbb{C}),$$

and also the double sector

$$S^{\circ}_{\mu}(\mathbb{C}) = S^{\circ}_{\mu^+}(\mathbb{C}) \cup S^{\circ}_{\mu^-}(\mathbb{C}) = C^{\circ}_{\mu^+}(\mathbb{C}) \cap C^{\circ}_{\mu^-}(\mathbb{C}).$$

Let $H_\infty(S^{\circ}_{\mu^+}(\mathbb{C}))$ be the Banach space of bounded complex-valued holomorphic functions defined on $S^{\circ}_{\mu^+}(\mathbb{C})$, and let $K(C^{\circ}_{\mu^+}(\mathbb{C}))$ be the Banach space of complex-valued holomorphic functions Φ for which $\sup\{|Z\,\Phi(Z)| :\ Z \in C^{\circ}_{\mu^+}(\mathbb{C})\} < +\infty$.

For every function $\Phi \in K(C^{\circ}_{\mu^+}(\mathbb{C}))$, there is a unique holomorphic function B defined on $S^{\circ}_{\mu^+}(\mathbb{C})$ which satisfies Parseval's identity

$$\begin{aligned}\frac{1}{2\pi}\int_0^{+\infty} B(\lambda)\,\hat{u}(-\lambda)\,d\lambda &= \lim_{\alpha\to 0^+}\int_{\mathbb{R}} \Phi(X + i\alpha)\,u(X)\,dX \\ &= \lim_{\varepsilon\to 0}\Big(\int_{|X|\ge\varepsilon} \Phi(X)\,u(X)\,dX + \Phi_1(\varepsilon)\,u(0)\Big),\end{aligned}$$

for all u in the Schwartz space $\mathcal{S}(\mathbb{R})$, where $\Phi_1(\varepsilon) = \int_{\delta(\varepsilon)} \Phi(Z)\,dZ$, the integral being along a contour $\delta(\varepsilon)$ from $-\varepsilon$ to ε in $C^{\circ}_{\mu^+}$. Moreover, if $0 < \nu < \mu$, then $B \in H_\infty(S^{\circ}_{\nu^+}(\mathbb{C}))$, and

$$\|B\|_\infty \le c_{\nu,\mu}\,\sup\{|Z\,\Phi(Z)| :\ Z \in C^{\circ}_{\mu^+}(\mathbb{C})\}.$$

Conversely, for every function $B \in H_\infty(S^{\circ}_{\mu^+}(\mathbb{C}))$, there is a unique holomorphic function Φ defined on $C^{\circ}_{\mu^+}(\mathbb{C})$ which satisfies Parseval's identity. Moreover, if $0 < \nu < \mu$, then $\Phi \in K(C^{\circ}_{\nu^+}(\mathbb{C}))$, and

$$\sup\{|Z\,\Phi(Z)| :\ Z \in C^{\circ}_{\nu^+}(\mathbb{C})\} \le c_{\nu,\mu}\,\|B\|_\infty\,.$$

We write $B = \mathcal{F}(\Phi)$, and call B the *Fourier transform* of Φ, and we write $\Phi = \mathcal{G}(B)$, and call Φ the *inverse Fourier transform* of B. Also let $\Phi_1 = \mathcal{G}_1(B)$.

Similar results hold when $C^\circ_{\mu+}(\mathbb{C})$ is replaced by $C^\circ_{\mu-}(\mathbb{C})$ and $S^\circ_{\mu+}(\mathbb{C})$ is replaced by $S^\circ_{\mu-}(\mathbb{C})$, provided the limit in α is taken over negative α, and the contour $\delta(\varepsilon)$ from $-\varepsilon$ to ε is in $C^\circ_{\mu-}$.

Characterization of the inverse Fourier transform of functions in $H_\infty(S^\circ_\mu(\mathbb{C}))$ is slightly more complicated.

Define $\chi_{\mathrm{Re}>0} \in H_\infty(S^\circ_\mu(\mathbb{C}))$ by $\chi_{\mathrm{Re}>0}(\lambda) = 1$ when $\mathrm{Re}\,\lambda > 0$, and $\chi_{\mathrm{Re}>0}(\lambda) = 0$ when $\mathrm{Re}\,\lambda < 0$. Similarly define $\chi_{\mathrm{Re}<0}$.

Now consider $B \in H_\infty(S^\circ_\mu(\mathbb{C}))$. Write $B = B_+ + B_-$, where

$$B_+ = B\,\chi_{\mathrm{Re}>0} \in H_\infty(S^\circ_{\mu+}),$$

and

$$B_- = B\,\chi_{\mathrm{Re}<0} \in H_\infty(S^\circ_{\mu-}),$$

and let $\Phi = \mathcal{G}(B) = \mathcal{G}(B_+) + \mathcal{G}(B_-)$ and $\Phi_1(B) = \mathcal{G}_1(B) = \mathcal{G}_1(B_+) + \mathcal{G}_1(B_-)$. Then

i) Φ is a holomorphic function on $S^\circ_\mu(\mathbb{C})$,

ii) Φ_1 is a holomorphic function on $S^\circ_{\mu+}(\mathbb{C})$ which satisfies $\Phi_1'(Z) = \Phi(Z) + \Phi(-Z)$, and

iii) $$\frac{1}{2\pi}\int_0^{+\infty} B(\lambda)\,\hat{u}(-\lambda)\,d\lambda = \lim_{\varepsilon\to 0}\Big(\int_{|X|\geq\varepsilon} \Phi(X)\,u(X)\,dX + \Phi_1(\varepsilon)\,u(0)\Big),$$

for all $u \in \mathcal{S}(\mathbb{R})$. Moreover, if $0 < \nu < \mu$, then

$$\sup\{|Z\,\Phi(Z)| : \; Z \in S^\circ_\nu(\mathbb{C})\} \leq c_{\nu,\mu}\,\|B\|_\infty$$

and

$$\sup\{|\Phi_1(Z)| : \; Z \in S^\circ_\nu(\mathbb{C})\} \leq c_{\nu,\mu}\,\|B\|_\infty\,.$$

Conversely, given functions Φ and Φ_1 which satisfy i) and ii), such that $Z\,\Phi(Z)$ and $\Phi_1(Z)$ are bounded in Z, then there exists a unique holomorphic function B on $S^\circ_\mu(\mathbb{C})$ which satisfies Parseval's identity iii). Moreover, if $0 < \nu < \mu$, then, on $S^\circ_\nu(\mathbb{C})$,

$$\|B\|_\infty \leq c_{\nu,\mu} \sup\{|Z\,\Phi(Z)| : Z \in S^\circ_\mu(\mathbb{C})\} + \sup\{|\Phi_1(Z)| : Z \in S^\circ_{\mu^+}(\mathbb{C})\}\,.$$

In [McQ], this material is used to show that singular convolution operators $T_{(\Phi,\Phi_1)}$ are bounded linear operators on $L_p(\gamma)$ when $1 < p < \infty$, where γ is a Lipschitz curve in the complex plane $\mathbb{C}$. This is done by representing $T_{(\Phi,\Phi_1)}$ as $B(-iD_\gamma)$, where

$$D_\gamma = \frac{d}{dz}\Big|_\gamma\,,$$

and proving that $-iD_\gamma$ has a bounded H_∞ functional calculus in $L_p(\gamma)$. See also [McQ1].

We would like to take this opportunity to thank all those with whom we have discussed this material, in particular Raphy Coifman and John Ryan. The work was mainly done at Macquarie University, but parts were achieved at Yale University, the Mittag-Leffler Institute and Flinders University, and we thank them for their support. It may be worth noting that some of the early ideas concerning the role of the exponential functions $e(x,\zeta)$ in relating functional calculi of $\mathbf{D}_\Sigma\, e_L$ to singular convolution operators were developed while the second author was visiting Coifman at Yale in 1987.

2. Clifford analysis.

Throughout this paper m and M denote positive integers, L is equal to either 0 or $m+1$, and $M \geq \max\{m, L\}$.

The real 2^M-dimensional Clifford algebra $\mathbb{R}_{(M)}$ or the complex 2^M-dimensional Clifford algebra $\mathbb{C}_{(M)}$ have basis vectors e_S, where S is any subset of $\{1, 2, \ldots, M\}$. Under the identifications $e_0 = e_\phi$ and $e_j = e_{\{j\}}$ for $1 \leq j \leq M$, the associative multiplication of basis vectors satisfies

$$\begin{gathered} e_0 = 1\,, \qquad e_j^2 = -e_0 = -1\,, \quad \text{for } 1 \leq j \leq M\,, \\ e_j\, e_k = -e_k\, e_j = e_{\{j,k\}}\,, \quad \text{for } 1 \leq j < k \leq M\,, \quad \text{and} \\ e_{j_1}\, e_{j_2} \cdots e_{j_s} = e_S\,, \quad \text{if } 1 \leq j_1 < j_2 < \cdots < j_s \leq M\,, \quad \text{and} \\ S = \{j_1, j_2, \ldots, j_s\}\,. \end{gathered}$$

The product of two elements $u = \sum_S u_S e_S$ and $v = \sum_T v_T e_T$ in $\mathbb{R}_{(M)}$ (or in $\mathbb{C}_{(M)}$) is $uv = \sum_{S,T} u_S v_T e_S e_T$, where u_S, $v_T \in \mathbb{R}$ (or $\mathbb{C}$). The term $u_\phi e_\phi$ is usually written as $u_0 e_0$ or just u_0, and is called the *scalar part* of u.

We embed the vector space $\mathbb{R}^m$ in the Clifford algebras $\mathbb{R}_{(M)}$ and $\mathbb{C}_{(M)}$ by identifying the standard basis vectors $e_1, e_2, \ldots, e_m$ of $\mathbb{R}^m$ with their counterparts in $\mathbb{R}_{(M)}$ or $\mathbb{C}_{(M)}$.

There are two common ways of embedding $\mathbb{R}^{m+1}$ in the Clifford algebras. Both ways are useful. We treat them together by denoting standard basis vectors of $\mathbb{R}^{m+1}$ by $e_1, e_2, \ldots, e_m, e_L$ and identifying e_L with either e_0 or e_{m+1}.

We use the euclidean norms $|u| = (\sum |u_S|^2)^{1/2}$ on $\mathbb{R}_{(M)}$ and on $\mathbb{C}_{(M)}$, and remark that $|uv| \leq C\,|u|\,|v|$ for some constant C depending only on M. This constant can be taken as 1 if $u \in \mathbb{R}^{m+1}$, and as $\sqrt{2}$ if $u \in \mathbb{C}^{m+1}$.

We write an element $x \in \mathbb{R}^{m+1}$ as $x = \mathbf{x} + x_L e_L$ where $x \in \mathbb{R}^m$ and $x_L \in \mathbb{R}$, and its Clifford conjugate as $\bar{x} = -\mathbf{x} + x_L \bar{e}_L$ where $\bar{e}_L e_L = 1$. Then $\bar{x}\,x = x\,\bar{x} = \sum_{j=1}^m x_j^2 + x_L^2 = |x|^2$.

The Clifford algebras $\mathbb{R}_{(0)}, \mathbb{R}_{(1)}$ and $\mathbb{R}_{(2)}$ are the real numbers, complex numbers, and quaternions, respectively. An important property of these three algebras is that every non-zero element has an inverse. Although this is not true in general it is an important fact that every element $x = \mathbf{x} + x_L e_L$ of $\mathbb{R}^{m+1}$ does have an inverse x^{-1} in $\mathbb{R}_{(M)}$. Indeed $x^{-1} = |x|^{-2}\bar{x} \in \mathbb{R}^{m+1} \subset \mathbb{R}_{(M)}$.

For $\xi \in \mathbb{R}^m$, $\xi \neq \mathbf{0}$, define $\chi_\pm(x) = (1 \pm i\,\xi e_L |\xi|^{-1})/2$, so that $\chi_+(x) + \chi_-(x) = 1$. Using $(i\,\xi e_L)^2 = |\xi|^2$, we obtain

$$\chi_+(\xi)^2 = \chi_+(\xi)\,, \qquad \chi_-(\xi)^2 = \chi_-(\xi)\,,$$
$$\chi_+(\xi)\,\chi_-(\xi) = 0 = \chi_-(\xi)\,\chi_+(\xi)\,.$$

Further, $i\,\xi e_L = |\xi|\,\chi_+(\xi) - |\xi|\,\chi_-(\xi)$, and indeed, for any polynomial $P(\lambda) = \sum a_k \lambda_k$ in one variable with scalar coefficients, we have $P(i\xi e_L) = \sum a_k (i\xi e_L)^k = P(|\xi|)\,\chi_+(\xi) + P(-|\xi|)\,\chi_-(\xi)$. Therefore the polynomial p in m variables defined by $p(\xi) = P(i\xi e_L)$ satisfies $p(\mathbf{0}) = P(0)$ and

$$p(\xi) = P(i\xi e_L) = P(|\xi|)\,\chi_+(\xi) + P(-|\xi|)\,\chi_-(\xi)\,, \qquad \xi \neq \mathbf{0}\,.$$

It is natural to associate with every function B of one real variable a function b of m real variables defined at x by

$$b(\xi) = B(i\xi e_L) = B(|\xi|)\,\chi_+(\xi) + B(-|\xi|)\,\chi_-(\xi)$$

if $|\xi|$ and $-|\xi|$ are in the domain of B, and by $b(\mathbf{0}) = B(0)$ if 0 is in the domain of B.

Let us repeat this procedure for holomorphic functions of complex variables. First extend $|\xi|^2$ holomorphically to $\mathbb{C}^m$ by defining

$$|\zeta|_{\mathbb{C}}^2 = \sum_{j=1} \zeta_j^2 = |\xi|^2 - |\eta|^2 + 2i\langle \xi, \eta\rangle\,,$$

for $\zeta = \xi + i\eta \in \mathbb{C}^m$ (where $\xi, \eta \in \mathbb{R}^m$) and note that $(i\zeta e_L)^2 = |\zeta|_{\mathbb{C}}^2$. When $|\zeta|_{\mathbb{C}}^2 \neq 0$, take $\pm|\zeta|_{\mathbb{C}}$ as its two square roots, and define $\chi_\pm(\zeta) = (1 \pm i\,\zeta e_L |\zeta|_{\mathbb{C}}^{-1})/2$, so that, as before,

$$\chi_+(\zeta) + \chi_-(\zeta) = 1\,, \qquad \chi_+(\zeta)^2 = \chi_+(\zeta)\,, \qquad \chi_-(\zeta)^2 = \chi_-(\zeta)\,,$$
$$\chi_+(\zeta)\,\chi_-(\zeta) = 0 = \chi_-(\zeta)\,\chi_+(\zeta)\,,$$

and $i\,\zeta e_L = |\zeta|_{\mathbb{C}}\,\chi_+(\zeta) - |\zeta|_{\mathbb{C}}\,\chi_-(\zeta)$.

Given any polynomial $P(\lambda) = \sum a_k \lambda_k$ in one variable with complex coefficients, the associated polynomial p in m variables defined by $p(\zeta) = P(i\zeta e_L) = \sum a_k (i\zeta e_L)^k$ satisfies

$$\begin{aligned} p(\zeta) &= P(i\zeta e_L) = P(|\zeta|_{\mathbb{C}})\,\chi_+(\zeta) + P(-|\zeta|_{\mathbb{C}})\,\chi_-(\zeta) \\ &= \frac{1}{2}(P(|\zeta|_{\mathbb{C}}) + P(-|\zeta|_{\mathbb{C}})) + \frac{1}{2}\,\frac{(P(|\zeta|_{\mathbb{C}}) - P(-|\zeta|_{\mathbb{C}}))i\,\zeta e_L}{|\zeta|_{\mathbb{C}}}\,, \end{aligned}$$

if $|\zeta|_{\mathbb{C}}^2 \neq 0$, and

$$p(\zeta) = P(0) + P'(0)\,i\,\zeta e_L\,, \qquad \text{if } |\zeta|_{\mathbb{C}}^2 = 0\,.$$

It is natural to associate with every complex-valued holomorphic function B of one variable, defined on an open subset S of $\mathbb{C}$, a Clifford-valued holomorphic function b of m complex variables, defined, for all $\zeta \in \mathbb{C}^m$ such that $\{\pm|\zeta|_{\mathbb{C}}\} \subset S$, by

$$\begin{aligned} b(\zeta) &= B(i\zeta e_L) = B(|\zeta|_{\mathbb{C}})\,\chi_+(\zeta) + B(-|\zeta|_{\mathbb{C}})\,\chi_-(\zeta) \\ &= \frac{1}{2}(B(|\zeta|_{\mathbb{C}}) + B(-|\zeta|_{\mathbb{C}})) + \frac{1}{2}\,\frac{(B(|\zeta|_{\mathbb{C}}) - B(-|\zeta|_{\mathbb{C}}))i\,\zeta e_L}{|\zeta|_{\mathbb{C}}}\,, \end{aligned}$$

if $|\zeta|_{\mathbb{C}}^2 \neq 0$, and

$$b(\zeta) = B(0) + B'(0)\, i\,\zeta e_L\,, \qquad \text{if } |\zeta|_{\mathbb{C}}^2 = 0\,.$$

The reason we say that this is natural, is not only because b is the required polynomial when B is a polynomial, but also because the mapping from B to b is an algebra homomorphism. That is, if F is another holomorphic function defined on S, and $c_1, c_2 \in \mathbb{C}$, then $(c_1F + c_2B)(i\zeta e_L) = c_1F(i\zeta e_L) + c_2B(i\zeta e_L)$ and $(FB)(i\zeta e_L) = F(i\zeta e_L)B(i\zeta e_L)$.

Important examples, defined for each real t, are the holomorphic functions of $\lambda \in \mathbb{C}$ given by $E_t(\lambda) = e^{-t\lambda}$. The associated functions of m variables are given by

$$\begin{aligned} e(te_L, \zeta) = E_t(i\zeta e_L) &= e^{-t|\zeta|_{\mathbb{C}}}\chi_+(\zeta) + e^{t|\zeta|_{\mathbb{C}}}\chi_-(\zeta) \\ &= \cosh(t\,|\zeta|_{\mathbb{C}}) - \sinh(t\,|\zeta|_{\mathbb{C}})\,|\zeta|_{\mathbb{C}}^{-1} i\,\zeta e_L\,, \end{aligned}$$

if $|\zeta|_{\mathbb{C}}^2 \neq 0$, and

$$e(te_L, \zeta) = 1 - t\,i\,\zeta e_L\,, \qquad \text{if } |\zeta|_{\mathbb{C}}^2 = 0\,.$$

Then $e(te_L, \zeta)\, e(se_L, \zeta) = e((t+s)e_L, \zeta)$ and $e(te_L, -\zeta) = e(-te_L, \zeta)$. Also,

$$\frac{d}{dt} e(te_L, \zeta) = -i\,\zeta e_L\, e(te_L, \zeta) = -e(te_L, \zeta)\, i\,\zeta e_L\,.$$

Other important examples, defined for each complex α, are the functions $R_\alpha(\lambda) = (\lambda - \alpha)^{-1}$, $\lambda \neq \alpha$. Then

$$R_\alpha(i\,\zeta e_L) = (i\zeta e_L - \alpha)^{-1} = (i\zeta e_L + \alpha)\,(|\zeta|_{\mathbb{C}}^2 - \alpha^2)^{-1}\,, \qquad |\zeta|_{\mathbb{C}}^2 \neq \alpha^2\,.$$

(What we are really doing, is studying the spectral theory of the elements $i\zeta e_L$ of the complex algebra $\mathbb{C}_{(M)}$. The spectrum of $i\zeta e_L$ is $\{\pm|\zeta|_{\mathbb{C}}\}$, and the spectral decomposition of $i\zeta e_L$ is $i\zeta e_L = |\zeta|_{\mathbb{C}}\,\chi_+(\zeta) - |\zeta|_{\mathbb{C}}\,\chi_-(\zeta)$ when $|\zeta|_{\mathbb{C}}^2 \neq 0$, while $i\zeta e_L$ is nilpotent when $|\zeta|_{\mathbb{C}}^2 = 0$).

So far it has been unimportant which sign we assign to each square-root of $|\zeta|_{\mathbb{C}}^2$, though from now on we shall assume that $|\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0]$, and take $\operatorname{Re}|\zeta|_{\mathbb{C}} > 0$.

It is time to prove some estimates.

Theorem 2.1. *Let $\zeta = \xi + i\eta \in \mathbb{C}^m$ (where $\xi, \eta \in \mathbb{R}^m$), and assume that $|\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0]$. Let*

$$\theta = \tan^{-1}\left(\frac{|\eta|}{\operatorname{Re}|\zeta|_{\mathbb{C}}}\right) \in [0, \pi/2)\,.$$

Then

a) $0 < \operatorname{Re}|\zeta|_{\mathbb{C}} \le |\xi| \le \sec\theta \operatorname{Re}|\zeta|_{\mathbb{C}}$,

b) $\operatorname{Re}|\zeta|_{\mathbb{C}} \le ||\zeta|_{\mathbb{C}}| \le \sec\theta \operatorname{Re}|\zeta|_{\mathbb{C}} \le |\zeta| \le (1 + 2\tan^2\theta)^{1/2} \operatorname{Re}|\zeta|_{\mathbb{C}}$,

c) $-\theta \le \arg|\zeta|_{\mathbb{C}} \le \theta$, *and*

d) $|\chi_{\pm}(\zeta)| \le \dfrac{\sec\theta}{\sqrt{2}}$.

PROOF. The simplest to prove is

$$||\zeta|_{\mathbb{C}}|^2 = ||\zeta|_{\mathbb{C}}^2| = \left((|\xi|^2 - |\eta|^2)^2 + 4\langle\xi,\eta\rangle^2\right)^{1/2} \le |\xi|^2 + |\eta|^2 = |\zeta|^2\,,$$

so that

$$\text{e)} \qquad \operatorname{Re}|\zeta|_{\mathbb{C}} \le ||\zeta|_{\mathbb{C}}| \le |\zeta|\,.$$

On taking real parts in the identity

$$-(\xi + i\eta)^2 = -\zeta^2 = |\zeta|_{\mathbb{C}}^2 = (\operatorname{Re}|\zeta|_{\mathbb{C}} + i\operatorname{Im}|\zeta|_{\mathbb{C}})^2$$

we obtain

$$(\#) \qquad |\xi|^2 - |\eta|^2 = (\operatorname{Re}|\zeta|_{\mathbb{C}})^2 - (\operatorname{Im}|\zeta|_{\mathbb{C}})^2$$

or

$$2\,|\xi|^2 - |\zeta|^2 = 2\,(\operatorname{Re}|\zeta|_{\mathbb{C}})^2 - ||\zeta|_{\mathbb{C}}|^2\,,$$

so that, by *e*), we obtain $\operatorname{Re}|\zeta|_{\mathbb{C}} \le |\xi|$. Also, from (#), we have $|\xi|^2 \le |\eta|^2 + (\operatorname{Re}|\zeta|_{\mathbb{C}})^2 = (\tan^2\theta + 1)(\operatorname{Re}|\zeta|_{\mathbb{C}})^2$, leading to $|\xi| \le \sec\theta \operatorname{Re}|\zeta|_{\mathbb{C}}$. Thus we have proved *a*).

Another consequence of (#) is

$$2\sec^2\theta(\operatorname{Re}|\zeta|_{\mathbb{C}})^2 = 2\big((\operatorname{Re}|\zeta|_{\mathbb{C}})^2 + |\eta|^2\big) = |\zeta|^2 + ||\zeta|_{\mathbb{C}}|^2\,,$$

which, by *e*), gives $||\zeta|_{\mathbb{C}}| \le \sec\theta \operatorname{Re}|\zeta|_{\mathbb{C}} \le |\zeta|$. A further application of (#) gives

$$|\zeta|^2 = 2\,|\eta|^2 + (\operatorname{Re}|\zeta|_{\mathbb{C}})^2 - (\operatorname{Im}|\zeta|_{\mathbb{C}})^2 \le (1 + 2\tan^2\theta)\,(\operatorname{Re}|\zeta|_{\mathbb{C}})^2\,,$$

so that we have proved *b*).

Part *c*) is an immediate consequence of the inequality $||\zeta|_{\mathbb{C}}| \le \sec\theta \operatorname{Re}|\zeta|_{\mathbb{C}}$, while *d*) follows from $|\zeta| \le (1 + 2\tan^2\theta)^{1/2}||\zeta|_{\mathbb{C}}|$.

On defining

$$S^\circ_\mu(\mathbb{C}^m) = \{\zeta = \xi + i\eta \in \mathbb{C}^m : |\zeta|^2_{\mathbb{C}} \notin (-\infty, 0] \text{ and } |\eta| < \mathrm{Re}(|\zeta|_{\mathbb{C}}) \tan\mu\},$$

we see, from part *c*) of the above theorem, that whenever $\zeta \in S^\circ_\mu(\mathbb{C}^m)$, then $|\zeta|_{\mathbb{C}} \in S^\circ_{\mu+}(\mathbb{C})$ and $-|\zeta|_{\mathbb{C}} \in S^\circ_{\mu-}(\mathbb{C})$ (these sectors are defined in Section 1). So, for every holomorphic function B defined on $S^\circ_\mu(\mathbb{C}) = S^\circ_{\mu+}(\mathbb{C}) \cup S^\circ_{\mu-}(\mathbb{C})$, the associated holomorphic function b of m variables given by

$$b(\zeta) = B(i\zeta e_L) = B(|\zeta|_{\mathbb{C}})\,\chi_+(\zeta) + B(-|\zeta|_{\mathbb{C}})\,\chi_-(\zeta)$$

is defined on $S^\circ_\mu(\mathbb{C}^m)$. Moreover, by part *d*), if B is bounded, then

$$\|b\|_\infty \leq \sqrt{2}\sec\mu\,\|B\|_\infty\,.$$

On letting $H_\infty(S^\circ_\mu(\mathbb{C}^m)) = H_\infty(S^\circ_\mu(\mathbb{C}^m), \mathbb{C}_{(M)})$, the Banach space of bounded Clifford-valued holomorphic functions defined on $S^\circ_\mu(\mathbb{C}^m)$, we obtain the following result

Theorem 2.2. *The mapping $B \mapsto b$ defined above is a one-one bounded algebra homomorphism from $H_\infty(S^\circ_\mu(\mathbb{C}))$ to $H_\infty(S^\circ_\mu(\mathbb{C}^m))$.*

PROOF. All that remains to be proved is that the mapping is one-one. Actually we can do better and recover B from b by means of the formula

$$B(\lambda) = \frac{2}{\sigma_{m-1}} \int_{|\xi|=1} b(\lambda\xi)\,\chi_\pm(\xi)\,d\xi\,, \qquad \lambda \in S^\circ_{\mu\pm}(\mathbb{C})$$

where σ_{m-1} is the volume of the unit $(m-1)$-sphere in $\mathbb{R}^m$. (Beware that, when $m = 1$, functions in $H_\infty(S^\circ_\mu(\mathbb{C}))$ are complex-valued, while functions in $H_\infty(S^\circ_\mu(\mathbb{C}^1))$ take their values in $\mathbb{C}_{(M)}$.)

It may seem strange to use the inequality $|\eta| < \mathrm{Re}(|\zeta|_{\mathbb{C}})\tan\mu$ in the definition of $S^\circ_\mu(\mathbb{C}^m)$, rather than a simpler one such as $|\eta| < |\xi|\tan\mu$. (These inequalities are the same when $m = 1$.) However, when it comes to characterizing the Fourier transforms of monogenic functions on cones $C^\circ_{\mu\pm}$ and S°_μ in $\mathbb{C}^{m+1}$, then, as we shall see in Section 4, there is no choice but to use the sets $S^\circ_\mu(\mathbb{C}^m)$ as defined above.

So far we have been considering Clifford-valued holomorphic functions of m complex variables. What is usually called *Clifford analysis* is the study of monogenic functions of $m+1$ real variables. In Section 4, we shall relate these two concepts via the Fourier transform. This involves extensive use of the exponential function

$$\begin{aligned} e(x,\zeta) &= e(\mathbf{x}+x_L e_L,\zeta) \\ &= e^{i\langle \mathbf{x},\zeta\rangle} e(x_L e_L,\zeta) \\ &= e^{i\langle \mathbf{x},\zeta\rangle}(e^{-x_L|\zeta|_{\mathbb{C}}}\chi_+(\zeta)+e^{x_L|\zeta|_{\mathbb{C}}}\chi_-(\zeta)), \end{aligned}$$

which is a holomorphic function of $\zeta \in \mathbb{C}^m$ for each $x = \mathbf{x}+x_L e_L \in \mathbb{R}^{m+1}$, and is a left-monogenic function of $x \in \mathbb{R}^{m+1}$ for each $\zeta \in \mathbb{C}^m$. It satisfies $e(x,\zeta)\,e(y,\zeta) = e(x+y,\zeta)$ and $e(x,-\zeta) = e(-x,\zeta)$. Of course, when $\mathbf{x} \in \mathbb{R}^m$ and $\xi \in \mathbb{R}^m$, then $e(\mathbf{x},\xi) = e^{i\langle \mathbf{x},\xi\rangle}$, the usual exponential function in Fourier theory.

Also, $e(x,\zeta)\,\overline{e_L}$ is a right-monogenic function of $x \in \mathbb{R}^{m+1}$ for each $\zeta \in \mathbb{C}^m$.

We remark that

$$e(x,\zeta) = \exp i(\langle \mathbf{x},\zeta\rangle - x_L\zeta e_L) = \sum_{k=0} \frac{1}{k!}\big(i(\langle \mathbf{x},\zeta\rangle - x_L\zeta e_L)\big)^k .$$

Let us briefly review some facts about Clifford analysis. The differential operator

$$D = \mathbf{D} + \frac{\partial}{\partial x_L}\, e_L\,, \qquad \text{where } \mathbf{D} = \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\, e_k\,,$$

acts on C^1-functions $f = \sum f_S e_S$ of $m+1$ real variables to give

$$Df = \sum_{k=1}^{m} \frac{\partial f_S}{\partial x_k}\, e_k e_S + \frac{\partial f_S}{\partial x_L}\, e_L e_S$$

and also

$$fD = \sum_{k=1}^{m} \frac{\partial f_S}{\partial x_k}\, e_S e_k + \frac{\partial f_S}{\partial x_L}\, e_S e_L\ .$$

A C^1-function defined on an open subset of $\mathbb{R}^{m+1}$ with values in $\mathbb{R}_{(M)}$ or $\mathbb{C}_{(M)}$ is called *left-monogenic* if $Df = 0$ and *right-monogenic* if $fD = 0$.

We remark that each component of every left-monogenic function is harmonic, as is each component of every right-monogenic function.

The function $e(x,\zeta)$ is a left-monogenic function of x (for fixed ζ) because

$$\begin{aligned}\frac{\partial}{\partial x_L} e_L e(x,\zeta) &= -e_L i\,\zeta e_L e(x,\zeta)\\ &= -e_L i\,\overline{e_L}\zeta e(x,\zeta)\\ &= -i\,\zeta e(x,\zeta) = -\mathbf{D}\, e(x,\zeta).\end{aligned}$$

Similar reasoning shows that $e(x,\zeta)\overline{e_L}$ is right-monogenic in x.

Define the function k on $\mathbb{R}^{m+1}\setminus\{0\}$ by

$$k(x) = \frac{1}{\sigma_m}\frac{\bar{x}}{|x|^{m+1}}, \qquad \text{for } x \neq 0,$$

(where σ_m is the volume of the unit m-sphere in $\mathbb{R}^{m+1}$).

The Cauchy kernels $k(y-x)$ are left- and right-monogenic functions of both x and y (when $x \neq y$).

Let us state Cauchy's theorem and the Cauchy integral formula.

Theorem 2.3. *Let Ω be a bounded open subset of $\mathbb{R}^{m+1}$ with Lipschitz boundary $\partial\Omega$ and exterior unit normal $n(y)$ defined for almost all $y \in \partial\Omega$. Suppose f is left-monogenic and g is right-monogenic on a neighbourhood of $\Omega^{\mathrm{cl}} = \Omega \cup \partial\Omega$. Then*

i) $$\int_{\Sigma} g(y)\,n(y)\,f(y)\,dS_y = 0,$$

ii) $$\int_{\partial\Omega} g(y)\,n(y)\,k(y-x)\,dS_y = \begin{cases} g(x), & x\in\Omega,\\ 0, & x\notin\Omega^{\mathrm{cl}},\end{cases}$$

iii) $$\int_{\partial\Omega} k(y-x)\,n(y)\,f(y)\,dS_y = \begin{cases} f(x), & x\in\Omega,\\ 0, & x\notin\Omega^{\mathrm{cl}}.\end{cases}$$

Part i) is a direct consequence of Gauss' divergence theorem, while parts ii) and iii) follow from i) in the usual way, together with the easily verified identity

$$\int_{|y-x|=r} n(y)\,k(y-x)\,dS_y = \int_{|y-x|=r} k(y-x)\,n(y)\,dS_y = 1, \quad r>0.$$

These reduce to theorems known by Cauchy when $m = 1$, in which case $\mathbb{R}^{1+1} = \mathbb{R}_{(1)} \cong \mathbb{C}$. It appears that, for $m = M = 2$, A. C. Dixon [D] published the first such results in 1904.

Further information about monogenic functions can be found in the books [BDS], [GM], and in the papers of F. Sommen, J. Ryan and others. See [S1] in particular, where Sommen introduces the exponential function $e(x, \xi)$. The paper [PS] contains some related material.

We remark that parts i) and iii) of Theorem 2.1 remain valid when f is a left-monogenic function taking its values in a finite-dimensional left Clifford module $\mathcal{X}$. That is, $\mathcal{X}$ is a finite-dimensional real (or complex) linear space together with a representation of $\mathbb{R}_{(M)}$ (or $\mathbb{C}_{(M)}$) as linear operators on $\mathcal{X}$. If $u \in \mathbb{R}_{(M)}$ (or $\mathbb{C}_{(M)}$) and $v \in \mathcal{X}$, we denote the action of u on v by uv.

We consider $\mathcal{X}$ together with a norm $\|\cdot\|$ and note that there exists a constant C such that $\|uv\| \leq C\,|u|\,\|v\|$ for all $u \in \mathbb{R}_{(M)}$ (or $\mathbb{C}_{(M)}$) and $v \in \mathcal{X}$. (We do not equip $\mathcal{X}$ with an inner-product, and in particular, do not require that the basis vectors e_j are represented by skew-adjoint operators).

Part I: Monogenic extensions of functions and holomorphic extensions of their Fourier transforms.

3. Monogenic functions on cones.

We present a mildly generalized version of results presented in [LMcS]. Here we consider monogenic functions defined on cones in $\mathbb{R}^{m+1}$ which are unions of half-spaces, whereas in [LMcS] we only considered cones which are rotationally symmetric about the x_0 axis. Allowing e_L to be e_{m+1} rather than e_0 causes no problems. (One reason for generalizing in this way, is to incorporate the application to boundary value problems which is presented in Section 7.)

We start by specifying some sets of unit vectors in $\mathbb{R}^{m+1}_+ = \{x = \mathbf{x} + x_L e_L \in \mathbb{R}^{m+1} : x_L > 0\}$. The metric $\angle(n, y) = \cos^{-1}\langle n, y\rangle$ is used on these unit vectors.

Let N be a compact set of unit vectors in $\mathbb{R}^{m+1}_+$ which contains e_L, and let $\mu_N = \sup\{\angle(n, e_L) : n \in N\}$. Then $0 \leq \mu_N < \pi/2$. For

$0 < \mu \le \pi/2 - \mu_N$, define the open neighbourhoods N_μ of N in the unit sphere by $N_\mu = \{y \in \mathbb{R}^{m+1}_+ : |y| = 1, \angle(y,n) < \mu \text{ for some } n \in N\}$. (In Section 6, N is the set of unit vectors normal to a surface).

For each unit vector n, let C_n^+ be the open half-space $C_n^+ = \{x \in \mathbb{R}^{m+1} : \langle x, n\rangle > 0\}$, and define open cones in $\mathbb{R}^{m+1}$ as follows. Let $C_{N_\mu}^+ = \cup\{C_n^+ : n \in N_\mu\}$, $C_{N_\mu}^- = -C_{N_\mu}^+$ and $S_{N_\mu} = C_{N_\mu}^+ \cap C_{N_\mu}^-$.

We remark that in [LMcS] we considered the case when N is rotationally symmetric, namely $N = \{n = \mathbf{n} + n_L e_L \in \mathbb{R}^{m+1}_+ : |n| = 1,\ n_L \ge |n| \operatorname{cotan} w\}$ for some $\omega \in [0, \pi/2)$. Then $\mu_N = \omega$. In this case we use the symbols

$$C^\circ_{\mu^+} = C^+_{N_{\mu-\omega}} = \{x = \mathbf{x} + x_L e_L \in \mathbb{R}^{m+1} : x_L > -|x| \tan\mu\},$$
$$C^\circ_{\mu^-} = -C^\circ_{\mu^+}, \qquad S^\circ_\mu = C^\circ_{\mu^+} \cap C^\circ_{\mu^-},$$

consistent with [LMcS].

Define the Banach space $K(C^+_{N_\mu})$ to be the space of right-monogenic functions Φ from $C^+_{N_\mu}$ to $\mathbb{C}_{(M)}$ for which

$$\|\Phi\|_{K(C^+_{N_\mu})} = \frac{1}{2}\sigma_m \sup\{|x|^m |\Phi(x)| : x \in C^+_{N_\mu}\} < +\infty.$$

Similarly define $K(C^-_{N_\mu})$.

Also define the Banach space $K(S_{N_\mu})$ to be the space of pairs $(\Phi, \underline{\Phi})$ of functions with Φ right-monogenic from S_{N_μ} to $\mathbb{C}_{(M)}$ and with $\underline{\Phi}$ continuous on $(0, +\infty)\, e_L$, such that

$$\underline{\Phi}(R\, e_L) - \underline{\Phi}(r\, e_L) = \int_{r \le |\mathbf{x}| \le R} \Phi(\mathbf{x})\, d\mathbf{x}\, e_L,$$

and

$$\|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})} = \frac{1}{2}\sigma_m \sup\{|x|^m |\Phi(x)| : x \in S_{N_\mu}\} + \sup\{|\underline{\Phi}(r\, e_L)| : r > 0\} < +\infty.$$

Note that $\underline{\Phi}$ is determined by Φ up to an additive constant, and that

$$\underline{\Phi}'(r\, e_L) = \int_{|\mathbf{x}| = r} \Phi(\mathbf{x})\, d\mathbf{x}\, e_L.$$

Moreover, $\underline{\Phi}$ has a unique continuous extension to the cone

$$T_{N_\mu} = \{y = \mathbf{y} + y_L e_L \in \mathbb{R}_+^{m+1} : \; y^\perp \subset S_{N_\mu}\},$$

which satisfies

$$\underline{\Phi}(y) - \underline{\Phi}(z) = \int_{A(y,z)} f(x)\, n(x)\, dS_x ,$$

where $A(y,z)$ is a smooth oriented m-manifold in S_{N_μ} joining the $(m-1)$-sphere $S_y = \{x \in \mathbb{R}^{m+1} : \; \langle x, y\rangle = 0 \text{ and } |x| = |y|\}$ to the $(m-1)$-sphere S_z, in which case $|\underline{\Phi}(y)| \leq \|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})}$ for all $y \in T_{N_\mu}$.

When N is rotationally symmetric, namely

$$N = \{n = \mathbf{n} + n_L e_L \in \mathbb{R}_+^{m+1} : \; |n| = 1 \,, \; n_L \geq |\mathbf{n}| \operatorname{cotan} \omega\},$$

we use the symbol

$$T_\mu^\circ = T_{N_{\mu-\omega}} = \{y = \mathbf{y} + y_L e_L \in \mathbb{R}^{m+1} : \; y_L > |x| \operatorname{cotan} \mu\},$$

consistent with [LMcS].

Let us state the relationship between these spaces. Here $H_{y^\pm}$ denote the hemispheres $H_{y^\pm} = \{x \in \mathbb{R}^{m+1} : \; \pm\langle x, y\rangle \geq 0 \text{ and } |x| = |y|\}$ with boundaries S_y.

Theorem 3.1. i) *Given* $\Phi_\pm \in K(C_{N_\mu}^\pm)$, *define the functions* $\underline{\Phi_\pm}$ *on* T_{N_μ} *by*

$$\underline{\Phi_\pm}(y) = \pm \int_{H_{y^\pm}} \Phi_\pm(x)\, n(x)\, dS_x , \qquad y \in T_{N_\mu},$$

where $n(x) = x/|x|$ *is normal to the hemisphere* $H_{y^\pm}$. *Then*

$$(\Phi, \underline{\Phi}) = (\Phi_+ + \Phi_-, \underline{\Phi_+} + \underline{\Phi_-}) \in K(S_{N_\mu})$$

and

$$\|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})} \leq \|\Phi_+\|_{K(C_{N_\mu}^+)} + \|\Phi_-\|_{K(C_{N_\mu}^-)} .$$

ii) *Conversely, given* $(\Phi, \underline{\Phi}) \in K(S_{N_\mu})$, *there exist unique functions* $\Phi_\pm \in K(C^\pm_{N_\mu})$ *which satisfy* $\Phi = \Phi_+ + \Phi_-$ *and* $\underline{\Phi} = \underline{\Phi_+} + \underline{\Phi_-}$. *These functions are given by the formulae*

$$\Phi_\pm(x) = \pm \lim_{\varepsilon \to 0} \Big(\int\limits_{\substack{\langle y,n\rangle=0 \\ |y|\geq\varepsilon}} \Phi(y)\, n\; k(x-y)\, dS_y + \underline{\Phi}(\varepsilon\, e_L)\, k(x) \Big) ,$$

with $x \in C^\pm_n \subset C^\pm_{N_\mu}$, *for all* $n \in N_\mu$, *where*

$$k(x) = \frac{1}{\sigma_m} \frac{\bar{x}}{|x|^{m+1}} ,$$

and they satisfy the estimates

$$\|\Phi_\pm\|_{K(C^\pm_{N_\mu})} \leq c\, \|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})} ,$$

where c *depends only on* μ_N, μ *(and the dimension* m*).*

PROOF. i) To see that

$$\underline{\Phi_\pm}(y) - \underline{\Phi_\pm}(z) = \int\limits_{A(y,z)} \Phi_\pm(x)\, n(x)\, dS_x ,$$

apply Cauchy's theorem to the right-monogenic functions $\Phi_\pm$. The bound is straightforward.

ii) This is a slight generalization of results proved in [LMcS].

In other words, there is a natural isomorphism

$$K(S_{N_\mu}) \simeq K(C^+_{N_\mu}) \oplus K(C^-_{N_\mu}) .$$

We also need the closed linear subspaces $M(C^\pm_{N_\mu})$ of $K(C^\pm_{N_\mu})$ which consist of those functions $\Phi \in K(C^\pm_{N_\mu})$ which are left-monogenic as well as right-monogenic. The subspace $M(S_{N_\mu})$ of $K(S_{N_\mu})$ for which

$$M(S_{N_\mu}) \simeq M(C^+_{N_\mu}) \oplus M(C^-_{N_\mu})$$

is then

$$M(S_{N_\mu}) = \{(\Phi, \underline{\Phi}) \in K(S_{N_\mu}) :\ \Phi \text{ is left-monogenic and } (*) \text{ holds}\} ,$$

where

$$(*)\qquad \int_{|\mathbf{y}|=r} \langle \mathbf{y},\mathbf{x}\rangle\, r^{-1}\,(e_L\Phi(\mathbf{y})\mathbf{y} - \mathbf{y}\Phi(\mathbf{y})e_L)\,dS_y + \mathbf{x}\,\underline{\Phi}(re_L) - e_L\,\underline{\Phi}(re_L)\,\mathbf{x}\,e_L = 0\,,$$

when $r > 0$. It is not difficult to see i) that the value of the integral is independent of r, and ii) that it equals 0 when $\Phi \in M(C^{\pm}_{N_\mu})$. The difficult part is to show that when $(\Phi,\underline{\Phi}) \in M(S_{N_\mu})$, then the functions $\Phi_\pm$ defined in Theorem 3.1.ii are left-monogenic. See [LMcS]. Section 7 of [LMcS] contains further information about $(*)$.

Let us now consider convolutions. Given $\Phi \in K(C^+_{N_\mu})$, $\Psi \in M(C^+_{N_\mu})$ and $x \in C^+_n \subset C^+_{N_\mu}$, define $(\Phi * \Psi)(x)$ by

$$\begin{aligned}(\Phi * \Psi)(x) &= \int_{\langle y,n\rangle=\delta} \Phi(x-y)\,n\,\Psi(y)\,dS_y \\ &= \lim_{\varepsilon\to 0}\Big(\int_{\substack{\langle y,n\rangle=0\\ |y|\geq\varepsilon}} \Phi(x-y)\,n\,\Psi(y)\,dS_y + \underline{\Phi}(\varepsilon e_L)\,\Psi(x)\Big)\,,\end{aligned}$$

where $0 < \delta < \langle x,n\rangle$. It follows from Cauchy's theorem and the hypotheses of Φ being right-monogenic and Ψ being left-monogenic, that the integral is independent of the precise surface chosen. On the other hand, it is a consequence of Ψ being right-monogenic, that $\Phi * \Psi$ is right-monogenic, and indeed that

$$\|\Phi * \Psi\|_{K(C^+_{N_\nu})} \leq c_{\nu,\mu}\,\|\Phi\|_{K(C^+_{N_\mu})}\,\|\Psi\|_{K(C^+_{N_\mu})}\,,$$

for all $\nu < \mu$, as is shown in [LMcS].

If moreover $\Psi_1 \in M(C^+_{N_\mu})$, then $\Psi * \Psi_1$ is left- as well as right-monogenic, and $\Phi * (\Psi * \Psi_1) = (\Phi * \Psi) * \Psi_1$.

Corresponding results hold for functions defined on $C^-_{N_\mu}$.

If $(\Phi,\underline{\Phi}) \in K(S_{N_\mu})$ and $(\Psi,\underline{\Psi}) \in M(S_{N_\mu})$, define

$$(\Phi,\underline{\Phi}) * (\Psi,\underline{\Psi}) = (\Phi_+ * \Psi_+ + \Phi_- * \Psi_-, \underline{\Phi_+ * \Psi_+} + \underline{\Phi_- * \Psi_-})\,.$$

It then follows from the above material that

$$\|(\Phi,\underline{\Phi}) * (\Psi,\underline{\Psi})\|_{K(S^\circ_{\nu+})} \le C_{\nu,\mu}\, \|(\Phi,\underline{\Phi})\|_{K(S^\circ_{\mu+})}\, \|(\Psi,\underline{\Psi})\|_{K(S^\circ_{\mu+})}\,,$$

for all $\nu < \mu$.

Let K_N^+ be the linear space of functions Φ on $\mathbb{R}^m \setminus \{0\}$ which extend monogenically to $\Phi \in K(C^+_{N_\mu})$ for some $\mu > 0$. Similarly define K_N^-, K_N, M_N^+, M_N^- and M_N, so that $K_N \simeq K_N^+ \oplus K_N^-$ and $M_N \simeq M_N^+ \oplus M_N^-$, while M_N^+, M_N^- and M_N are all convolution algebras. (We do not introduce topologies on these spaces, so that $\oplus$ is merely the direct sum of linear spaces.) We remark that the only functions Φ which belong to both K_N^+ and K_N^- are those of the form $\Phi(\mathbf{x}) = c\,k(\mathbf{x})$ for some $c \in \mathbb{C}_{(M)}$, where

$$k(\mathbf{x}) = \frac{1}{\sigma_m} \frac{-\mathbf{x}}{|\mathbf{x}|^{m+1}}\,, \qquad \text{for } \mathbf{x} \in \mathbb{R}^m \setminus \{0\}\,,$$

with monogenic extension

$$k(x) = \frac{1}{\sigma_m} \frac{\bar{x}}{|x|^{m+1}}\,.$$

(See Section 12 of [BDS].) The embedding of K_N^+ into K_N defined above takes $c\,k \in K_N^+$ to $(c\,k, c/2) \in K_N$, while the embedding of K_N^- into K_N takes $c\,k \in K_N^-$ to $(c\,k, -c/2) \in K_N$.

4. Fourier transforms.

Our aim in this section is to identify the Fourier transforms $\mathcal{F}_\pm(\Phi)$ of the functions $\Phi \in K_N^\pm$, and to define the Fourier transforms $\mathcal{F}(\Phi,\underline{\Phi})$ of $(\Phi,\underline{\Phi})$. The transforms turn out to be bounded holomorphic functions defined on cones in $\mathbb{C}^m$. We also show that $\mathcal{F}_+$, $\mathcal{F}_-$ and $\mathcal{F}$ are algebra homomorphisms from the convolution algebras M_N^+, N_N^- and M_N to algebras of holomorphic functions.

We first associate with every unit vector $n = \mathbf{n} + n_L e_L \in \mathbb{R}^{m+1}$ satisfying $n_L > 0$, a real m-dimensional surface $n(\mathbb{C}^m)$ in $\mathbb{C}^m$, defined as follows.

$$\begin{aligned} n(\mathbb{C}^m) &= \{\zeta = \xi + i\eta \in \mathbb{C}^m : \ \xi \neq 0 \text{ and } n_L\eta = (n_L^2|\xi|^2 + \langle x,\mathbf{n}\rangle^2)^{1/2}\mathbf{n}\} \\ &= \{\zeta = \xi + i\eta \in \mathbb{C}^m : \ |\zeta|^2_{\mathbb{C}} \notin (-\infty,0] \text{ and } n_L\eta = \operatorname{Re}(|\zeta|_{\mathbb{C}})\mathbf{n}\} \\ &= \{\zeta = \xi + i\eta \in \mathbb{C}^m : \ |\zeta|^2_{\mathbb{C}} \notin (-\infty,0] \text{ and} \\ &\qquad\qquad \eta + \operatorname{Re}(|\zeta|_{\mathbb{C}})e_L = \kappa\, n \text{ for some } \kappa > 0\}\,, \end{aligned}$$

where

$$|\zeta|_{\mathbb{C}}^2 = \sum_{j=1}^{m} \zeta_j^2 = |\xi|^2 - |\eta|^2 + 2i\langle x, \eta\rangle .$$

The surfaces associated with distinct unit vectors are disjoint, with, in particular, $e_L(\mathbb{C}^m) = \mathbb{R}^m \setminus \{0\}$.

On these surfaces $|\xi|$, $|\zeta|$, $\mathrm{Re}(|\zeta|_{\mathbb{C}})$ and $||\zeta|_{\mathbb{C}}|$ are all equivalent. Indeed, by Theorem 2.1,

$$\mathrm{Re}\,|\zeta|_{\mathbb{C}} \le |\xi| \le (n_L)^{-1}\,\mathrm{Re}\,|\zeta|_{\mathbb{C}} ,$$

and

$$\mathrm{Re}\,|\zeta|_{\mathbb{C}} \le ||\zeta|_{\mathbb{C}}| \le (n_L)^{-1}\,\mathrm{Re}\,|\zeta|_{\mathbb{C}} \le |\zeta| \le (n_L)^{-1}\,(1+|\mathbf{n}|^2)^{1/2}\mathrm{Re}\,|\zeta|_{\mathbb{C}} ,$$

for all $\zeta \in n(\mathbb{C}^m)$. Further, the parametrization $\xi \mapsto \zeta = \xi + i\eta$ used in the first definition of $n(\mathbb{C}^m)$ is smooth, with

$$\left|\det\left(\frac{\partial \zeta_j}{\partial \xi_k}\right)\right| \le \frac{1}{n_L} , \qquad \xi \ne 0 .$$

In proving this, we can assume, without loss of generality, that $n = n_1 e_1 + n_L e_L$, so that

$$\zeta = \xi + i\frac{n_1}{n_L}\left(|\xi|^2 n_L^2 + \xi_1^2 n_1^2\right)^{1/2} e_1 .$$

Then, if $j \ge 2$, $\partial\zeta_j/\partial\xi_k = \delta_{jk}$, and

$$\frac{\partial \zeta_1}{\partial \xi_k} = \delta_{1k} + \frac{i\,n_1\xi_k(n_L^2 + \delta_{1k}n_1^2)}{n_L(|\xi|^2 n_L^2 + \xi_1^2 n_1^2)^{1/2}} .$$

Therefore

$$\left|\frac{\partial \zeta_1}{\partial \xi_1}\right| \le \frac{1}{n_L} \qquad \text{and} \qquad \left|\frac{\partial \zeta_1}{\partial \xi_k}\right| \le n_1 , \qquad \text{when } k \ge 2 .$$

The estimate for the Jacobian follows.

For the open sets N_μ of unit vectors defined in Section 3, we associate the open cones $N_\mu(\mathbb{C}^m)$ in $\mathbb{C}^m$ given by

$$\begin{aligned} N_\mu(\mathbb{C}^m) &= \bigcup\{n(\mathbb{C}^m) :\ n \in N_\mu\} \\ &= \{\zeta = \xi + i\eta \in \mathbb{C}^m :\ |\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0] \text{ and } \\ &\qquad \eta + \mathrm{Re}\,(|\zeta|_{\mathbb{C}})\, e_L = \kappa\, n \\ &\qquad \text{for some } \kappa > 0 \text{ and } n \in N_\mu\} . \end{aligned}$$

Since $N_\mu(\mathbb{C}^m) \subset S^\circ_{\mu_N+\mu}(\mathbb{C}^m)$, the estimates in Theorem 2.1 all hold with $\theta = \mu_N + \mu$.

When N is rotationally symmetric, namely

$$N = \{n = \mathbf{n} + n_L e_L \in \mathbb{R}^{m+1}_+ : \ |n| = 1,\ n_L \geq |\mathbf{n}| \cot w\},$$

we have $S^\circ_\mu(\mathbb{C}^m) = N_{\mu-w}(\mathbb{C}^m)$. Again we allow functions to take their values in the complex Clifford algebra $\mathbb{C}_{(M)}$, so for example $H_\infty(N_\mu(\mathbb{C}^m))$ denotes the Banach space of all bounded holomorphic functions from $N_\mu(\mathbb{C}^m)$ to $\mathbb{C}_{(M)}$ under the norm $\|b\|_\infty = \sup\{|b(\zeta)| : \ \zeta \in N_\mu(\mathbb{C}^m)\}$.

Crucial to this paper are the exponential functions

$$e(x,\zeta) = e_+(x,\zeta) + e_-(x,\zeta),$$

where

$$e_+(x,z) = e^{i\langle \mathbf{x},\zeta\rangle}\, e^{-x_L|\zeta|_{\mathbb{C}}}\, \chi_+(\zeta)$$

and

$$e_-(x,\zeta) = e^{i\langle \mathbf{x},\zeta\rangle}\, e^{x_L|\zeta|_{\mathbb{C}}}\, \chi_-(\zeta).$$

They are entire left-monogenic functions of $x \in \mathbb{R}^{m+1}$ (for fixed ζ), and holomorphic functions of $\zeta \in N_\mu(\mathbb{C}^m)$ (for fixed x), which satisfy the bounds

$$\begin{aligned}|e_+(x,\zeta)| &= e^{-\langle \mathbf{x},\eta\rangle - x_L \operatorname{Re}|\zeta|_{\mathbb{C}}}\, |\chi_+(\zeta)| \\ &\leq \frac{\sec(\mu_N+\mu)}{\sqrt{2}}\, e^{-\langle x,n\rangle \operatorname{Re}|\zeta|_{\mathbb{C}}/n_L}, \qquad \zeta \in n(\mathbb{C}^m),\end{aligned}$$

and

$$\begin{aligned}|e_-(x,\zeta)| &= e^{-\langle \mathbf{x},\eta\rangle + x_L \operatorname{Re}|\zeta|_{\mathbb{C}}}\, |\chi_-(\zeta)| \\ &\leq \frac{\sec(\mu_N+\mu)}{\sqrt{2}}\, e^{\langle x,n\rangle \operatorname{Re}|\zeta|_{\mathbb{C}}/n_L}, \qquad \zeta \in \tilde{n}(\mathbb{C}^m).\end{aligned}$$

Let

$$H^\pm_\infty(N_\mu(\mathbb{C}^m)) = \{b \in H_\infty(N_\mu(\mathbb{C}^m)) : \ b\chi_\pm = b\}.$$

Then every function $b \in H_\infty(N_\mu(\mathbb{C}^m))$ can be uniquely decomposed as $b = b_+ + b_-$ where $b_\pm = b\chi_\pm \in H^\pm_\infty(N_\mu(\mathbb{C}^m))$ These are closed linear subspaces of $H_\infty(N_\mu(\mathbb{C}^m))$, and indeed

$$H_\infty(N_\mu(\mathbb{C}^m)) = H^+_\infty(N_\mu(\mathbb{C}^m)) \oplus H^-_\infty(N_\mu(\mathbb{C}^m))$$

because

$$\|b\chi_\pm\|_\infty \le \sqrt{2}\,\|b\|_\infty\,\|\chi_\pm\|_\infty \le \sec(\mu_N+\mu)\,\|b\|_\infty\,,$$

for all $b \in H_\infty(N_\mu(\mathbb{C}^m))$.

We also introduce the subalgebras

$$\mathcal{A}(N_\mu(\mathbb{C}^m)) = \{b \in H_\infty(N_\mu(\mathbb{C}^m)) :\ \zeta e_L b(\zeta) = b(\zeta)\zeta e_L \text{ for all } \zeta\}\,.$$

Define $\mathcal{A}^\pm(N_\mu(\mathbb{C}^m))$ similarly, and note that if $b \in \mathcal{A}(N_\mu(\mathbb{C}^m))$ then $b_\pm = b\chi_\pm \in \mathcal{A}^\pm(N_\mu(\mathbb{C}^m))$, so that

$$\mathcal{A}(N_\mu(\mathbb{C}^m)) = \mathcal{A}^+(N_\mu(\mathbb{C}^m)) \oplus \mathcal{A}^-(N_\mu(\mathbb{C}^m))\,.$$

Particular functions b belonging to $\mathcal{A}(N_\mu(\mathbb{C}^m))$ are those of the form

$$b(\zeta) = B(i\,\zeta e_L) = B(|\zeta|_\mathbb{C})\,\chi_+(\zeta) + B(-|\zeta|_\mathbb{C})\,\chi_-(\zeta)$$

defined in Section 2, where $B \in H_\infty(S^\circ_{\mu_N+\mu}(\mathbb{C}))$. Also in $\mathcal{A}(N_\mu(\mathbb{C}^m))$ are all scalar-valued holomorphic functions in $H_\infty(N_\mu(\mathbb{C}^m))$, the simplest being $r_k(\zeta) = i\,\zeta_k/|\zeta|_\mathbb{C}\,,\ k = 1, 2, \ldots, m\,.$

Let H_N^+ be the algebra of all those functions b on $\mathbb{R}^m \setminus \{0\}$ which extend holomorphically to $b \in H_\infty^+(N_\mu(\mathbb{C}^m))$ for some $\mu > 0$. The algebra H_N^- however is defined to be the algebra of all those functions b on $\mathbb{R}^m \setminus \{0\}$ which extend holomorphically to $b \in H_\infty^-(\overline{N}_\mu(\mathbb{C}^m))$ for some $\mu > 0$, where $\overline{N} = \{\overline{n} \in \mathbb{R}^{m+1} :\ n \in N\}$. Then $H_N^+ \cap H_N^- = \{0\}$.

Define H_N by $H_N = H_N^+ + H_N^-$. Then $H_N = H_N^+ \oplus H_N^-$. Let $\mathcal{A}_N^+$, $\mathcal{A}_N^-$ and $\mathcal{A}_N$ be the subspaces of H_N^+, H_N^- and H_N consisting of all those functions which satisfy $\xi e_L b(\xi) = b(\xi)\xi e_L$ for all $\xi \neq \mathbf{0}$. Then $\mathcal{A}_N = \mathcal{A}_N^+ \oplus \mathcal{A}_N^-\,.$

We need to ensure that these holomorphic extensions are unique, which we can do by assuming that N is connected. In fact we shall make the stronger assumption that the compact sets N of unit vectors in $\mathbb{R}_+^{m+1}$ are starlike about e_L (in the sense that, whenever $n \in N$ and $0 \le \tau \le 1$, then $(\tau n + (1-\tau)e_L)/|\tau n + (1-t)e_L| \in N$). In this case, the open sets N_μ are also starlike about e_L in the same sense, and $N_\mu(\mathbb{C}^m)$ are connected open subsets of $\mathbb{C}^m$.

Theorem 4.1. *Let N be a compact set of unit vectors in $\mathbb{R}^{m+1}_+$ which is starlike about e_L. For every $\Phi \in K^+_N$, there exists a unique function $b \in H^+_N$ which satisfies Parseval's identity*

$$
\text{(P)}\qquad
\begin{aligned}
(2\pi)^{-m} \int_{\mathbb{R}^m} b(\xi)\, \hat{u}(-\xi)\, d\xi
&= \lim_{\alpha \to 0+} \int_{\mathbb{R}^m} \Phi(\mathbf{x} + \alpha\, e_L)\, e_L\, u(\mathbf{x})\, d\mathbf{x} \\
&= \lim_{\varepsilon \to 0} \Big(\int_{|\mathbf{x}| \geq \varepsilon} \Phi(\mathbf{x})\, e_L\, u(\mathbf{x})\, d\mathbf{x} + \underline{\Phi}(\varepsilon\, e_L)\, u(\mathbf{0}) \Big),
\end{aligned}
$$

for all u in the Schwartz space $\mathcal{S}(\mathbb{R}^m)$, where $\underline{\Phi}$ is defined in Theorem 3.1. *So b is the distribution Fourier transform of Φe_L, and we write $b = \mathcal{F}_+(\Phi)e_L$. We also call Φ the inverse Fourier transform of $b\overline{e_L}$, and write $\Phi = \mathcal{G}_+(b\overline{e_L})$.*

The Fourier transform $\mathcal{F}_+$ is a linear transformation with the following properties.

i) *$\mathcal{F}_+$ is a one-one map of K^+_N onto H^+_N. That is, for every $b \in H^+_N$ there exists a unique function $\Phi \in K^+_N$ such that $b = \mathcal{F}_+(\Phi)e_L$.*

ii) *If $0 < \nu < \mu \leq \pi/2 - \mu_N$ and $\Phi \in K(C^+_{N_\mu})$ then $b \in H^\pm_\infty(N_\nu(\mathbb{C}^m))$ and $\|b\|_\infty \leq c_\nu \|\Phi\|_{K(C^+_{N_\mu})}$ for some constant c_ν which depends on ν (as well as on μ_N and μ).*

iii) *If $0 < \nu < \mu \leq \pi/2 - \mu_N$ and $b \in H^\pm_\infty(N_\mu(\mathbb{C}^m))$ then $\Phi \in K(C^+_{N_\nu})$ and $\|\Phi\|_{K(C^+_{N_\nu})} \leq c_\nu \|b\|_\infty$ for some constant c_ν which depends on ν (as well as on μ_N and μ).*

iv) *$\Phi \in M^+_N$ if and only if $b \in \mathcal{A}^+_N$.*

v) *If $\Phi \in K^+_N$, $\Psi \in M^+_N$, $b = \mathcal{F}_+(\Phi)e_L$ and $f = \mathcal{F}_+(\Psi)e_L$, then*

$$bf = \mathcal{F}_+(\Phi * \Psi)e_L\ .$$

vi) *The mapping $\Phi \mapsto b$ is an algebra homomorphism from the convolution algebra M^+_N onto the function algebra $\mathcal{A}^+_N$.*

vii) *Let p be a polynomial in m variables with values in $\mathbb{C}_{(M)}$, and let*

$$\Psi = p\Big(-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_m}\Big)\, \Phi\,.$$

Then $\Psi \in K_N^+$ *if and only if* $p\,b \in H_N^+$, *in which case* $\mathcal{F}_+(\Psi)e_L = p\,b$.

viii) *If* $0 < \nu < \mu \le \pi/2 - \mu_N$, $s > -m$, *and* b *extends holomorphically to a bounded function which satisfies* $|b(\zeta)| \le c\,|\zeta|^s$ *for some* c_s *and all* $\zeta \in N_\mu(\mathbb{C}^m)$, *then there exists* $c_{s,\nu}$ *such that* $|\Phi(x)| \le c_{s,\nu}\,|x|^{-m-s}$ *for all* $x \in C_{N_\nu}^+$.

Hence $|\underline{\Phi}(y)| \le c_{s,\nu}\,|y|^{-s}$ *for all* $y \in T_{N_\mu}$, *so that, in particular, when* $-m < s < 0$, *we have* $\lim_{y\to 0} \underline{\Phi}(y) = 0$.

PROOF. In the estimates which follow, constants c may depend on μ_N, μ and the dimension m, and may vary from line to line. Dependence on ν will be specified by using c_ν.

Let $\Phi \in K(C_{N_\mu}^+)$. It is easy to see that either form of Parseval's identity uniquely determines b on $\mathbb{R}^m$ and therefore on $N_\mu(\mathbb{C}^m)$ (because it is a connected open set).

To construct b, we proceed as follows. For $\alpha > 0$, define $\Phi_\alpha(x) = \Phi(x + \alpha e_L)$, $x + \alpha e_L \in C_{N_\mu}^+$, in which case

$$\begin{aligned}\|\Phi_\alpha\|_{K(C_{N_\mu}^+)} &= \frac{1}{2}\sigma_m \sup\{|x|^m|\Phi(x+\alpha e_L)| :\ x \in C_{N_\mu}^+\}\\ &\le \sup\{|y|^m|\Phi(y)| :\ y \in C_{N_\mu}^+ + \alpha e_L\} \le \|\Phi\|_{K(C_{N_\mu}^+)}\,.\end{aligned}$$

For $\zeta \in n(\mathbb{C}^m) \subset N_\nu(\mathbb{C}^m) \subset N_\mu(\mathbb{C}^m)$, $\nu < \mu$, define

$$b_\alpha(\zeta) = \int_\sigma \Phi_\alpha(x)\,n(x)\,e_+(-x,\zeta)\,dS_x\,,$$

where σ is the surface defined by

$$\sigma = \{x \in \mathbb{R}^{m+1} :\ \langle x, n\rangle = -|x|\,\sin(\mu-\nu)\}\,.$$

Note that the integrand is continuous and exponentially decreasing at infinity. (As usual, $n(x)$ denotes the normal to σ with $n_L(x) > 0$). Indeed, for $x \in \sigma$,

$$\begin{aligned}|e_+(-x,\zeta)| &\le \frac{\sec(\mu_N+\mu)}{\sqrt{2}}\,e^{\langle x,n\rangle \mathrm{Re}\,|\zeta|_{\mathbb{C}}/n_L}\\ &\le \frac{\sec(\mu_N+\mu)}{\sqrt{2}}\,e^{-|x|\,|\xi|\sin\theta}\,,\end{aligned}$$

where $\theta = \mu - \nu$.

Because of this fact, and Cauchy's theorem for monogenic functions (noting that Φ_α is right-monogenic and $e_+(-x,\zeta)$ is left-monogenic in x), we see that the definition of $b_\alpha(\zeta)$ does not depend on the precise surface σ chosen. So $b_\alpha(\zeta)$ depends holomorphically on $\zeta \in N_\mu(\mathbb{C}^m)$. Moreover

$$\begin{aligned} b_\alpha(\zeta)\, e^{\alpha|\zeta|_{\mathbb{C}}} &= \int_\sigma \Phi(x+\alpha e_L)\, n(x)\, e_+(-(x+\alpha e_L),\zeta)\, dS_x \\ &= \int_\sigma \Phi(x+\beta e_L)\, n(x)\, e_+(-(x+\beta e_L),\zeta)\, dS_x \\ &= b_\beta(\zeta)\, e^{\beta|\zeta|_{\mathbb{C}}}\,, \end{aligned}$$

for all $\alpha, \beta > 0$, so it makes sense to define b as the holomorphic function on $N_\mu(\mathbb{C}^m)$ which satisfies

$$b(z) = b_\alpha(\zeta)\, e^{\alpha|\zeta|_{\mathbb{C}}}\,, \qquad \text{for all } \alpha > 0\,.$$

We shall prove in a moment that

$$(\#) \qquad |b_\alpha(\zeta)| \le c_\nu\, \|\Phi\|_{K(C^+_{N_\mu})}\,, \qquad \text{for all } z \in N_\mu(\mathbb{C}^m)\,,$$

(where c_ν is independent of α) and

$$(\#\#) \qquad (2\pi)^{-m} \int_{\mathbb{R}^m} b_\alpha(\xi)\, \hat{u}(-\xi)\, d\xi = \int_{\mathbb{R}^m} \Phi(\mathbf{x}+\alpha e_L)\, e_L\, u(\mathbf{x})\, d\mathbf{x}\,.$$

The first version of Parseval's identity (P) follows as a consequence, as does the estimate in ii).

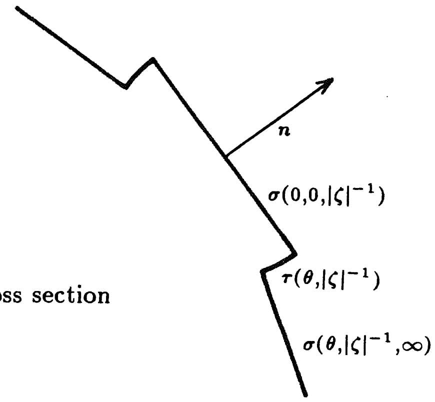
Let us prove (#). With $\zeta \in n(\mathbb{C}^m) \subset N_\nu(\mathbb{C}^m) \subset N_\mu(\mathbb{C}^m)$ and $\theta = \mu - \nu$ as before, apply Cauchy's theorem to change the surface of integration, so that

$$b_\alpha(\zeta) = \Big(\int_{\sigma(0,0,|\zeta|^{-1})} + \int_{\tau(\theta,|\zeta|^{-1})} + \int_{\sigma(\theta,|\zeta|^{-1},\infty)} \Big)\, \Phi_\alpha(x)\, n(x)\, e_+(-x,\zeta)\, dS_x\,,$$

where

$$\sigma(\theta,r,R) = \{x \in \mathbb{R}^{m+1} :\ \langle x,n\rangle = |x|\,\sin\theta,\ r \le |x| \le R\}\,,$$

$$\tau(\theta,R) = \{x \in \mathbb{R}^{m+1} :\ |x| = R,\ 0 \ge \langle x,n\rangle \ge -R\sin\theta\}\,.$$



Cross section

We need some estimates.

a)

$$
\begin{aligned}
\Big| \int_{\sigma(0,0,R)} \Phi_\alpha(x)\, n\, e_+(-x,\zeta)\, dS_x \Big|
&\le c \Big| \int_{\sigma(0,0,R)} \Phi_\alpha(x)\, n\, e(-x,\zeta)\, dS_x \Big| \\
&\le c \Big| \int_{\sigma(0,0,R)} \Phi_\alpha(x)\, n\, \big(e(-x,\zeta)-1\big)\, dS_x \Big| \\
&\quad + c \Big| \int_{\substack{\langle x,n\rangle \ge 0 \\ |x|=R}} \Phi_\alpha(x)\, n\, dS_x \Big| \\
&\le c\, \|\Phi_\alpha\|_{K(C^+_{N_\mu})} \Big(\sup\{ |\nabla_y e(-y,\zeta)| :\ y \in \sigma(0,0,R)\} \\
&\qquad \cdot \int_{\sigma(0,0,R)} |x|^{-m} |x|\, dS_x + 1 \Big) \\
&\le c\, \|\Phi_\alpha\|_{K(C^+_{N_\mu})} (R\,|\zeta| + 1) \\
&\le c\, \|\Phi_\alpha\|_{K(C^+_{N_\mu})}, \qquad \text{provided } R \le |\zeta|^{-1},
\end{aligned}
$$

$$\Big| \int\limits_{\tau(\theta,R)} \Phi_\alpha(x)\, n(x)\, e_+(-x,\zeta)\, dS_x \Big|$$

b)

$$\begin{aligned}
&\le c\,\|\Phi_\alpha\|_{K(C^+_{N_\mu})}\, R^{-m} \int\limits_{\tau(\theta,R)} e^{\langle x,n\rangle \operatorname{Re}|\zeta|_{\mathbb{C}}/n_L}\, dS_x \\
&= c\,\|\Phi_\alpha\|_{K(C^+_{N_\mu})} \int\limits_{\tau(\theta,1)} e^{\langle x,n\rangle R \operatorname{Re}|\zeta|_{\mathbb{C}}/n_L}\, dS_x \\
&= c\,\|\Phi_\alpha\|_{K(C^+_{N_\mu})} \int_{-\theta}^{0} e^{R \operatorname{Re}|\zeta|_{\mathbb{C}} \sin\Phi/n_L}\, d\Phi \\
&\le \frac{c}{R\,|\zeta|}\, \|\Phi_\alpha\|_{K(C^+_{N_\mu})} \\
&\le c\,\|\Phi_\alpha\|_{K(C^+_{N_\mu})}\,, \qquad \text{provided } R \ge |\zeta|^{-1}\,,
\end{aligned}$$

$$\Big| \int\limits_{\sigma(\theta,R,\infty)} \Phi_\alpha(x)\, n(x)\, e_+(-x,\zeta)\, dS_x \Big|$$

c)

$$\begin{aligned}
&\le c\,\|\Phi_\alpha\|_{K(C^+_{N_\mu})} \int\limits_{\sigma(\theta,R,\infty)} |x|^{-m} e^{\langle x,n\rangle \operatorname{Re}|\zeta|_{\mathbb{C}}/n_L}\, dS_x \\
&= c\,\|\Phi_\alpha\|_{K(C^+_{N_\mu})} \int_R^\infty s^{-1} e^{-s\sin\theta \operatorname{Re}|\zeta|_{\mathbb{C}}/n_L}\, ds \\
&\le \frac{c_\nu}{R\,|\zeta|}\, \|\Phi_\alpha\|_{K(C^+_{N_\mu})} \\
&\le c_\nu\, \|\Phi_\alpha\|_{K(C^+_{N_\mu})}\,, \qquad \text{provided } R \ge |\zeta|^{-1}\,.
\end{aligned}$$

On using the above three estimates with $R = |\zeta|^{-1}$, together with the preceding representation of b_α, we find that we have proved (#).

Now we shall prove (##). If we define $b_{\alpha,N}(\xi)$ for $\xi \in \mathbb{R}^m$ by

$$b_{\alpha,N}(\xi) = \int\limits_{|\mathbf{x}|\le N} \Phi_\alpha(\mathbf{x})\, e_L\, e^{i\langle \mathbf{x},\xi\rangle}\, d\mathbf{x}\,,$$

then the usual Parseval's identity gives

$$(2\pi)^{-m} \int_{\mathbb{R}^m} b_{\alpha,N}(\xi)\, \hat{u}(-\xi)\, d\xi = \int\limits_{|\mathbf{x}|\le N} \Phi(\mathbf{x} + \alpha e_L)\, e_L\, u(\mathbf{x})\, d\mathbf{x}\,,$$

for all $u \in \mathcal{S}(\mathbb{R}^m)$. We shall prove that

(*) $|b_{\alpha,N}(\xi)| \le c\,\|\Phi\|_{K(C^+_{N_\mu})}$ for all $\xi \in \mathbb{R}^m$ and $N > 0$,

(**) for each $\xi \in \mathbb{R}^m$, $b_{\alpha,N}(\xi)\,\chi_+(\xi) \to b_\alpha(\xi)$, as $N \to \infty$, and

(* * *) for each $\xi \in \mathbb{R}^m$, $b_{\alpha,N}(\xi)\,\chi_-(\xi) \to 0$, as $N \to \infty$.

Then (##) follows from these results and the Lebesgue dominated convergence theorem.

In proving (*) and (**), we use the estimates *a*), *b*) and *c*) above with $n = e_L$ in the definitions of σ, $\sigma(\theta, r, R)$ and $\tau(\theta, R)$.

First we prove (*) when $|\xi|^{-1} \le N$. Choose $0 < \theta < \mu$, and apply Cauchy's theorem to write

$$b_{\alpha,N}(\xi)\,\chi_+(\xi) = \Big(\int_{\sigma(0,0,|\xi|^{-1})} + \int_{\tau(\theta,|\xi|^{-1})} + \int_{\sigma(\theta,|\xi|^{-1},N)} - \int_{\tau(\theta,N)} \Big)\, \Phi_\alpha(x)\, n(x)\, e_+(-x,\xi)\, dS_x \ ,$$

so the uniform boundedness of $b_{\alpha,N}(\xi)\,\chi_+(\xi)$ in ξ and N follows from *a*), *b*) and *c*). On the other hand,

$$b_{\alpha,N}(\xi)\,\chi_-(\xi) = \int_{H_{N+}} \Phi_\alpha(x)\, n(x)\, e_-(-x,\xi)\, dS_x \ ,$$

so, on using similar reasoning to the proof of *b*),

$$|b_{\alpha,N}(\xi)\,\chi_-(\xi)| \le \frac{c}{N\,|\xi|}\, \|\Phi\|_{K(C^+_{N_\mu})} \le c\,\|\Phi\|_{K(C^+_{N_\mu})} \ .$$

To prove (*) when $|\xi|^{-1} \ge N$, only *a*) is needed.

To prove (**), fix $\xi \in \mathbb{R}^m$, $\xi \ne \mathbf{0}$, and apply Cauchy's theorem to write

$$b_\alpha(\xi) - b_{\alpha,N}(\xi)\,\chi_+(\xi) = \Big(\int_{\tau(\theta,N)} + \int_{\sigma(\theta,N,\infty)} \Big)\, \Phi_\alpha(x)\, n(x)\, e_+(-x,\xi)\, dS_x \ ,$$

so, by b) and c),

$$|b_\alpha(\xi) - b_{\alpha,N}(\xi)\,\chi_+(\xi)| \le \frac{c}{N\,|\xi|}\,\|\Phi_\alpha\|_{K(C^+_{N_\mu})} \to 0\,, \qquad \text{as } N \to \infty\,.$$

Moreover, $(***)$ follows from the estimate a few lines above.

As noted previously, the first version of Parseval's identity (P) follows. Our next task is to prove the second version of (P). Let $\varepsilon > 0$. Then

$$\begin{aligned}
(2\pi)^{-m} \int_{\mathbb{R}^m} & b(\xi)\,\hat{u}(-\xi)\,d\xi \\
&= \lim_{\alpha\to 0+} \Big(\int\limits_{|\mathbf{x}|\ge\varepsilon} \Phi(\mathbf{x}+\alpha\,e_L)\,e_L\,u(\mathbf{x})\,d\mathbf{x} \\
&\qquad + \int\limits_{|\mathbf{x}|\le\varepsilon} \Phi(\mathbf{x}+\alpha\,e_L)\,e_L\,u(\mathbf{0})\,d\mathbf{x} \\
&\qquad + \int\limits_{|\mathbf{x}|\le\varepsilon} \Phi(\mathbf{x}+\alpha\,e_L)\,e_L\,(u(\mathbf{x})-u(\mathbf{0}))\,d\mathbf{x}\Big) \\
&= \int\limits_{|\mathbf{x}|\ge\varepsilon} \Phi(\mathbf{x})\,e_L\,u(\mathbf{x})\,d\mathbf{x} + \underline{\Phi}(\varepsilon)\,u(\mathbf{0}) \\
&\quad + \lim_{\alpha\to 0+} \Big(\int\limits_{|\mathbf{x}|\le\varepsilon} \Phi(\mathbf{x}+\alpha\,e_L)\,e_L\,(u(\mathbf{x})-u(\mathbf{0}))\,d\mathbf{x}\Big)\,,
\end{aligned}$$

(with Cauchy's theorem being used to evaluate the second integral). Now

$$\begin{aligned}
\overline{\lim_{\varepsilon\to 0}}\, \lim_{\alpha\to 0+} & \Big(\int\limits_{|\mathbf{x}|\le\varepsilon} |\Phi(\mathbf{x}+\alpha\,e_L)\,e_L(u(\mathbf{x})-u(\mathbf{0}))|\,d\mathbf{x}\Big) \\
&\le \overline{\lim_{\varepsilon\to 0}}\, \lim_{\alpha\to 0+} \Big(C \int\limits_{|\mathbf{x}|\le\varepsilon} |\mathbf{x}+\alpha\,e_L|^{-m}\,|u(\mathbf{x})-u(\mathbf{0})|\,d\mathbf{x}\Big) \\
&\le \overline{\lim_{\varepsilon\to 0}}\, \lim_{\alpha\to 0+} \Big(C \int\limits_{|\mathbf{x}|\le\varepsilon} |\mathbf{x}|^{-m}|u(\mathbf{x})-u(\mathbf{0})|\,d\mathbf{x}\Big) = 0
\end{aligned}$$

(as $u \in \mathcal{S}(\mathbb{R}^m)$), so

$$(2\pi)^{-m} \int_{\mathbb{R}^m} b(\xi)\,\hat{u}(-\xi)\,d\xi = \lim_{\varepsilon \to 0} \Big(\int\limits_{|\mathbf{x}| \geq \varepsilon} \Phi(\mathbf{x})\,e_L\,u(\mathbf{x})\,d\mathbf{x} + \underline{\Phi}(\varepsilon)\,u(\mathbf{0}) \Big)\,,$$

as required.

This completes the proof of the introductory statement in the theorem, together with the estimates in ii).

PROOF OF i) AND iii). It is easily verified that $\mathcal{F}_+$ is one-one. We prove it maps onto H_N^+ by constructing the inverse Fourier transform $\mathcal{G}_+$.

Consider functions $b \in H_\infty^+(N_\mu(\mathbb{C}^m))$. For $n \in N_\mu$, and $x = \mathbf{x} + x_L e_L \in C_n^+ \subset C_{N_\mu}^+$, define

$$\begin{aligned}\Phi_n(x) &= (2\pi)^{-m} \int\limits_{n(\mathbb{C}^m)} b(\zeta)\,e(x,\zeta)\,d\zeta_1 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_m\,\overline{e_L} \\ &= (2\pi)^{-m} \int\limits_{n(\mathbb{C}^m)} b(\zeta)\,e_+(x,\zeta)\,d\zeta_1 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_m\,\overline{e_L}\,.\end{aligned}$$

On the surface $n(\mathbb{C}^m)$, the integrand is exponentially decreasing at infinity. Indeed, when $\zeta \in n(\mathbb{C}^m)$, then

$$|e^{i\langle \mathbf{x},\zeta\rangle}\,e^{-x_L|\zeta|_{\mathbb{C}}}| \leq c\,e^{-\langle x,n\rangle\,\mathrm{Re}\,|\zeta|_{\mathbb{C}}/n_L}$$

and $\langle x,n\rangle > 0$. Moreover, $e(x,\zeta)\overline{e_L}$ is right-monogenic, so Φ_n is a right-monogenic function on C_n^+ which satisfies

$$|\Phi_n(x)| \leq \frac{c}{\langle x,n\rangle^m}\,\|b\|_\infty\,, \qquad x \in C_n^+\,,$$

where c depends only on μ_N and μ.

Moreover the integrand depends holomorphically on the single complex variable $z = \langle \zeta, \mathbf{n}\rangle$ (on writing $\zeta = z\mathbf{n} + \zeta'$ where $\langle \zeta', \mathbf{n}\rangle = 0$, and holding ζ' constant). So, by the starlike nature of N_μ, and Cauchy's theorem in the z-plane, we find that $\Phi_n(x) = \Phi_{e_L}(x)$ for all $x \in C_n^+$ with $x_L > 0$. Hence there is a unique right-monogenic function Φ on $C_{N_\mu}^+$ which coincides with each of the functions $\Phi_n(x)$ on C_n^+. We call

Φ the *inverse Fourier transform* of $b\overline{e_L}$, and write $\Phi = \mathcal{G}_+(b\overline{e_L})$. The above estimates for Φ_n imply that $\Phi \in K(C^+_{N_\nu})$ for all $\nu < \mu$, and

$$\|\Phi\|_{K(C^+_{N_\nu})} \leq c_\nu \, \|b\|_\infty \, .$$

We remark that, in the particular case when $x_L = 0$ and $|b(\zeta)| \leq c(1 + |\zeta|^{m+1})^{-1}$ for all $\zeta \in N_\mu(\mathbb{C}^m)$, then, by Cauchy's theorem, we can change the surface of integration to conclude that

$$\mathcal{G}_+(b\overline{e_L})(\mathbf{x}) = \Phi(\mathbf{x}) = (2\pi)^{-m} \int_{\mathbb{R}^m} b(\xi)\, e^{i\langle \mathbf{x},\xi\rangle}\, d\xi\, \overline{e_L} = \check{b}(\mathbf{x})\,\overline{e_L}\, ,$$

which is the usual inverse Fourier transform of $b\overline{e_L}$.

Let us show that b and $\Phi = \mathcal{G}_+(b\overline{e_L})$ satisfy Parseval's identity (P), from which we conclude that $\mathcal{G}_+$ really is the inverse of the Fourier transform $\mathcal{F}_+$, and complete our proof of i) and iii).

Let $b_\alpha(\zeta) = b(\zeta)\, e^{-\alpha|\zeta|_{\mathbb{C}}}$ for $\alpha > 0$. Then, for $\mathbf{x} \in \mathbb{R}^m$,

$$\Phi(\mathbf{x} + \alpha e_L) = \mathcal{G}_+(b\overline{e_L})(\mathbf{x} + \alpha e_L) = \mathcal{G}_+(b_\alpha \overline{e_L})(\mathbf{x}) = (b_\alpha)^\vee(\mathbf{x})\,\overline{e_L}$$

(by the above remark). Apply the usual Parseval's identity to obtain

$$(2\pi)^{-m} \int_{\mathbb{R}^m} b_\alpha(\xi)\, \hat{u}(-\xi)\, d\xi = \int_{\mathbb{R}^m} \Phi(\mathbf{x} + \alpha e_L)\, e_L\, u(\mathbf{x})\, d\mathbf{x}\, ,$$

and hence

$$(2\pi)^{-m} \int_{\mathbb{R}^m} b(\xi)\, \hat{u}(-\xi)\, d\xi = \lim_{\alpha \to 0+} \int_{\mathbb{R}^m} \Phi(\mathbf{x} + \alpha e_L)\, e_L\, u(\mathbf{x})\, d\mathbf{x}\, ,$$

for all $u \in \mathcal{S}(\mathbb{R}^m)$, as required.

PROOF OF iv). Choose $\Phi \in K(C^+_{N_\mu})$. Then Φ is left- (as well as right-monogenic) if and only if

$$\mathbf{D}\, e_L \Phi(x) = (\Phi e_L)\, \mathbf{D}(x)\, , \qquad \text{for all } x \in C^+_{N_\mu}\, .$$

(Both sides equal $-\dfrac{\partial \Phi}{\partial x_L}(x)$).

Let $b\overline{e_L} = \mathcal{F}_+(\Phi)$, define b_α as above, and use twice the version of Parseval's identity involving b_α, to see that, for all $u \in \mathcal{S}(\mathbb{R}^m)$,

$$(2\pi)^{-m} \int_{\mathbb{R}^m} \xi\, e_L\, b_\alpha(\xi)\, \hat{u}(-\xi)\, d\xi = -i \int_{\mathbb{R}^m} (\mathbf{D}\, e_L \Phi)(\mathbf{x} + \alpha e_L)\, e_L\, u(\mathbf{x})\, d\mathbf{x}$$

and

$$(2\pi)^{-m}\int_{\mathbb{R}^m} b_\alpha(\xi)\,\xi\, e_L\,\hat{u}(-\xi)\,d\xi = -i\int_{\mathbb{R}^m}(\Phi e_L\mathbf{D})(\mathbf{x}+\alpha e_L)\,e_L\,u(\mathbf{x})\,d\mathbf{x}\,.$$

So $\Phi \in M(C^+_{N_\mu})$ if and only if $\mathbf{D}\,e_L\Phi(x) = (\Phi e_L)\mathbf{D}(x)$ for all $x \in C^+_{N_\mu}$, which holds if and only if

$$\mathbf{D}\,e_L\Phi(\mathbf{x}+\alpha e_L) = (\Phi e_L)\mathbf{D}(\mathbf{x}+\alpha e_L)\,, \qquad \text{for all } x \in \mathbb{R}^m \setminus \{0\}$$

(by the right-monogenicity of the functions on both sides of this equation). By the above identities, this is true if and only if $\xi e_L b_\alpha(\xi) = b_\alpha(\xi)\xi e_L$. And this equation is satisfied if and only if $\zeta e_L b(\zeta) = b(\zeta)\zeta e_L$ for all $z \in N_\mu(\mathbb{C}^m)$. This proves iv).

The remaining parts can be proved in a similar way, with the estimates in viii) requiring a modification of the proof of iii).

Theorem 4.2. *The statement of Theorem 4.1 remains valid when the following changes are made.*

Replace $C^+_{N_\mu}$, $N_\mu(\mathbb{C}^m)$, K^+_N, M^+_N, $H^+_\infty(N_\mu(\mathbb{C}^m))$, H^+_N, $\mathcal{A}^+_N$ *and* $\mathcal{F}_+$ *by* $C^-_{N_\mu}$, $\overline{N}_\mu(\mathbb{C}^m)$, K^-_N, M^-_N, $H^-_\infty(\overline{N}_\mu(\mathbb{C}^m))$, H^-_N, $\mathcal{A}^-_N$ *and* $\mathcal{F}_-$ *respectively, and take the limit in* α *over negative* α.

Denote the inverse of $\mathcal{F}_-$ by $\mathcal{G}_- : H^-_N \longrightarrow K^-_N$, and call $\mathcal{F}_-$ the *Fourier transform* and $\mathcal{G}_-$ the *inverse Fourier transform.*

On combining Theorems 4.1 and 4.2, and using Theorem 3.1, the following result is obtained.

Theorem 4.3. *Let* N *be a compact set of unit vectors in* $\mathbb{R}^{m+1}_+$ *which is starlike about* e_L. *For every* $(\Phi,\underline{\Phi}) \in K_N$, *there exists a unique function* $b \in H_N$ *which satisfies Parseval's identity*

$$\text{(P)}\qquad \begin{aligned}&(2\pi)^{-m}\int_{\mathbb{R}^m} b(\xi)\,\hat{u}(-\xi)\,d\xi\\ &\qquad = \lim_{\varepsilon\to 0}\Big(\int_{|\mathbf{x}|\geq\varepsilon}\Phi(\mathbf{x})\,e_L\,u(\mathbf{x})\,d\mathbf{x} + \underline{\Phi}(\varepsilon\, e_L)\,u(\mathbf{0})\Big)\,,\end{aligned}$$

for all u *in the Schwartz space* $\mathcal{S}(\mathbb{R}^m)$. *So* $b\,\overline{e_L}$ *is the distribution Fourier transform of* $(\Phi,\underline{\Phi})$, *and we write* $b = \mathcal{F}(\Phi,\underline{\Phi})\,e_L$.

The Fourier transform $\mathcal{F}$ *is a linear transformation with the following properties.*

i) *$\mathcal{F}$ is a one-one map of K_N onto H_N. That is, for every $b \in H_N$ there exists a unique $(\Phi, \underline{\Phi}) \in K_N$ such that $b = \mathcal{F}(\Phi, \underline{\Phi})\, e_L$. Actually, if $b = b_+ + b_-$ with $b_\pm = b\,\chi_\pm \in H_N^\pm$, then $(\Phi, \underline{\Phi}) = (\Phi_+, \underline{\Phi_+}) + (\Phi_-, \underline{\Phi_-})$ where $\Phi_\pm = \mathcal{G}_\pm(b_\pm \overline{e_L}) \in K_N^\pm$.*

We write $(\Phi, \underline{\Phi}) = \mathcal{G}(b\overline{e_L})$, and call $\mathcal{G}$ the inverse Fourier transform.

ii) *If $0 < \nu < \mu \leq \pi/2 - \mu_N$ and $(\Phi, \underline{\Phi}) \in K(S_{N_\mu})$ then $b_+ \in H_\infty^+(N_\nu(\mathbb{C}^m))$, $b_- \in H_\infty^-(\overline{N}_\nu(\mathbb{C}^m))$ and $\|b_\pm\|_\infty \leq c_\nu\, \|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})}$ for some constant c_ν which depends on ν (as well as on μ_N and μ).*

iii) *If $0 < \nu < \mu \leq \pi/2 - \mu_N$ and $b_+ \in H_\infty^+(N_\mu(\mathbb{C}^m))$ and $b_- \in H_\infty^-(\overline{N}_\mu(\mathbb{C}^m))$, then $(\Phi, \underline{\Phi}) \in K(S_{N_\nu})$ and*

$$\|(\Phi, \underline{\Phi})\|_{K(S_{N_\nu})} \leq c_\nu\, (\|b_+\|_\infty + \|b_-\|_\infty)$$

for some constant c_ν which depends on ν (as well as on μ_N and μ).

iv) *$(\Phi, \underline{\Phi}) \in M_N$ if and only if $b \in \mathcal{A}_N$.*

v) *If $(\Phi, \underline{\Phi}) \in K_N$ and $(\Psi, \underline{\Psi}) \in M_N$ and $b = \mathcal{F}(\Phi, \underline{\Phi})\, e_L$ and $f = \mathcal{F}(\Psi, \underline{\Psi})\, e_L$, then*

$$bf = \mathcal{F}((\Phi, \underline{\Phi}) * (\Psi, \underline{\Psi}))\, e_L\,.$$

vi) *The mapping $(\Phi, \underline{\Phi}) \mapsto b$ is an algebra homomorphism from the convolution algebra M_N onto the function algebra $\mathcal{A}_N$.*

vii) *If $(\Phi, \underline{\Phi}), (\Psi, \underline{\Psi}) \in K_N$, $b = \mathcal{F}(\Phi, \underline{\Phi})\, e_L$, $f = \mathcal{F}(\Psi, \underline{\Psi})\, e_L$, and if $f = p\, b$ where p is a polynomial in m variables with values in $\mathbb{C}_{(M)}$, then*

$$\Psi = p\Big(-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_m}\Big)\Phi\,.$$

viii) *If $0 < \nu < \mu \leq \pi/2 - \mu_N$, $s > -m$, and b_+ (and b_-) extend holomorphically to bounded functions which satisfy $|b_\pm(\zeta)| \leq c_s\, |\zeta|^s$ for some c_s and all $\zeta \in N_\mu(\mathbb{C}^m)$ (respectively, $\zeta \in \overline{N}_\mu(\mathbb{C}^m)$), then there exists $c_{s,\nu}$ such that $|\Phi(x)| \leq c_{s,\nu}\, |x|^{-m-s}$ for all $x \in C_{N_\nu}^+$ and $|\underline{\Phi}(y)| \leq c_{s,\nu}\, |y|^{-s}$ for all $y \in T_{N_\mu}$.*

Hence, in particular, when $-m < s < 0$, we have $\lim_{y\to 0} \underline{\Phi}(y) = 0$.

This result is a little nicer when $N = \overline{N}$. For then,

$$b_+ \in H_\infty^+(N_\mu(\mathbb{C}^m)) \qquad \text{and} \qquad b_- \in H_\infty^-(\overline{N}_\mu(\mathbb{C}^m))$$

if and only if

$$b \in H_\infty(N_\mu(\mathbb{C}^m)).$$

One application of the results of this section is to the investigation of monogenic extensions of functions defined on $\mathbb{R}^m \setminus \{0\}$. For example, consider

$$k_j(\mathbf{x}) = \frac{-x_j}{\sigma_m |\mathbf{x}|^{m+1}}, \qquad j = 1, 2, \ldots, m.$$

Our knowledge that the monogenic extension of $k(\mathbf{x}) = \sum_{j=1}^m k_j(\mathbf{x})\, e_j$ is

$$k(x) = \frac{\bar{x}}{|x|^{m+1}}, \qquad x \in \mathbb{R}^{m+1} \setminus \{0\},$$

does not help, because the individual components of k are not monogenic.

But we do know that the Fourier transform of $(2k_j, 0)$ is $r_j(\xi) = i\,\xi_j |\xi|^{-1}$, in the sense that $\Phi = 2k_j$, $\underline{\Phi} = 0$ and $b(\xi) = r_j(\xi)\, e_L$ satisfy Parseval's identity (P) in Theorem 4.3. We also know that r_j has a holomorphic extension which belongs to $H_\infty(S^\circ_\mu(\mathbb{C}^m))$ for all $\mu < \pi/2$, namely $r_j(\zeta) = i\,\zeta_j |\zeta|_{\mathbb{C}}^{-1}$. Hence $(2k_{(j)}, 0) = \mathcal{G}(r_j) \in K(S^\circ_\mu)$.

Therefore k_j has the right-monogenic extension $k_{(j)}$ to S°_μ for all $\mu < \pi/2$, and this extension satisfies $|k_{(j)}(x)| \leq c_\mu / |x|^m$ for all $x \in S^\circ_\mu$.

We remark that, although the results in Section 3 were derived in [LMcS] without recourse to the Fourier theory just developed, many of these results were discovered, at least informally, by using Fourier transforms. In particular, the decomposition $K(S_{N_\mu}) \simeq K(C^+_{N_\mu}) \oplus K(C^-_{N_\mu})$ given in Theorem 3.1 can be obtained in this way, at least when $N = \overline{N}$, since Theorem 4.3 can be proved without making use of Theorem 3.1.

5. Connection with holomorphic functions of one variable.

Let $B \in H_\infty(S^\circ_\mu(\mathbb{C}))$, where $0 < \mu < \pi/2$. In Section 2 we saw that it is natural to associate with B the function $b \in H_\infty(S^\circ_\mu(\mathbb{C}^m))$, defined by $b(\zeta) = B(i\zeta e_L) = B(|\zeta|_{\mathbb{C}})\,\chi_+(\zeta) + B(-|\zeta|_{\mathbb{C}})\,\chi_-(\zeta)$. Actually $b \in \mathcal{A}(S^\circ_\mu(\mathbb{C}^m)) = \{b \in H_\infty(S^\circ_\mu(\mathbb{C}^m)) : \zeta e_L b(\zeta) = b(\zeta)\zeta e_L \text{ for all } \zeta\}$, and the mapping $B \mapsto b$ is a one-one bounded algebra homomorphism from $H_\infty(S^\circ_\mu(\mathbb{C}))$ to $\mathcal{A}(S^\circ_\mu(\mathbb{C}^m))$.

Let us recall the symbols we are using for subsets of $\mathbb{C}$, $\mathbb{R}^{m+1}$ and $\mathbb{C}^m$:

$$C^\circ_{\mu^+}(\mathbb{C}) = \{Z = X + iY \in \mathbb{C} : \ Z \neq 0,\ Y > -|X| \tan\mu\}\,,$$

$$C^\circ_{\mu^-}(\mathbb{C}) = -C^\circ_{\mu^+}(\mathbb{C})\,,$$

$$S^\circ_{\mu^+}(\mathbb{C}) = \{\lambda \in \mathbb{C} : \ \lambda \neq 0,\ |\arg(\lambda)| < \mu\}\,, \qquad S^\circ_{\mu^-}(\mathbb{C}) = -S^\circ_{\mu^+}(\mathbb{C})\,,$$

$$S^\circ_\mu(\mathbb{C}) = S^\circ_{\mu^+}(\mathbb{C}) \cup S^\circ_{\mu^-}(\mathbb{C}) = C^\circ_{\mu^+}(\mathbb{C}) \cap C^\circ_{\mu^-}(\mathbb{C})\,,$$

$$C^\circ_{\mu^+} = \{x = \mathbf{x} + x_L e_L \in \mathbb{R}^{m+1} : \ x_L > -|\mathbf{x}| \tan\mu\}\,,$$

$$C^\circ_{\mu^-} = -C^\circ_{\mu^+}\,, \qquad S^\circ_\mu = C^\circ_{\mu^+} \cap C^\circ_{\mu^-}\,,$$

$$T^\circ_\mu = \{y = \mathbf{y} + y_L e_L \in \mathbb{R}^{m+1} : \ y_L > |\mathbf{y}| \cot\mu\}\,,$$

$$S^\circ_\mu(\mathbb{C}^m) = \{\zeta = \xi + i\eta \in \mathbb{C}^m : |\zeta|^2_{\mathbb{C}} \notin (-\infty, 0] \text{ and } |\eta| < \operatorname{Re}(|\zeta|_{\mathbb{C}}) \tan\mu\}\,.$$

Let us find the inverse Fourier transform of b in terms of the inverse Fourier transform of B.

We do this first for $B \in H_\infty(S^\circ_{\mu^+}(\mathbb{C}))$. In this case the inverse Fourier transform $\Phi = \mathcal{G}(B)$ of B is a complex-valued holomorphic function defined on $C^\circ_{\mu^+}(\mathbb{C})$. See Section 1. In particular

$$\Phi(Z) = \frac{1}{2\pi} \int_0^{+\infty} B(r)\, e^{irZ}\, dr\,, \qquad \text{when } \operatorname{Im}(Z) > 0\,.$$

The associated function b satisfies $b(\zeta) = B(i\zeta e_L) = B(|\zeta|_{\mathbb{C}})\, \chi_+(\zeta)$, and therefore $b \in H^+_\infty(S^\circ_\mu(\mathbb{C}^m))$. Let $\Phi = \mathcal{G}_+(b\overline{e_L})$. Thus, when $x_L > 0$,

$$\begin{aligned}
\Phi(x) &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} B(|\xi|)\, e_+(x,\xi)\, d\xi\, \overline{e_L} \\
&= \frac{1}{2(2\pi)^m} \int_{\mathbb{R}^m} \left(\overline{e_L} + \frac{i\xi}{|\xi|}\right) B(|\xi|)\, e^{-x_L|\xi|}\, e^{i\langle \mathbf{x},\xi\rangle}\, d\xi \\
&= \frac{1}{2(2\pi)^m} \int_{S^{m-1}} (\overline{e_L} + i\tau) \\
&\qquad \cdot \int_0^{+\infty} B(r)\, e^{-x_L r}\, e^{i\langle \mathbf{x},\tau\rangle r}\, r^{m-1}\, dr\, dS_\tau
\end{aligned}$$

$$
\begin{aligned}
(+) \qquad &= \frac{1}{2(2\pi i)^{m-1}} \int_{S^{m-1}} (\overline{e_L} + i\tau)\, \mathbf{\Phi}^{(m-1)}(\langle \mathbf{x}, \tau\rangle + ix_L)\, dS_\tau \\
&= \frac{1}{2(2\pi i)^{m-1}} \int_{S^{m-1}} (\overline{e_L} + i\langle \mathbf{x}, \tau\rangle \mathbf{x}|\mathbf{x}|^{-2}) \\
&\qquad\qquad \cdot \mathbf{\Phi}^{(m-1)}(\langle \mathbf{x}, \tau\rangle + ix_L)\, dS_\tau \\
&= \frac{\sigma_{m-2}}{2(2\pi i)^{m-1}} \int_{-1}^{1} (1-t^2)^{(m-3)/2} \Big(\overline{e_L} + \frac{it\,\mathbf{x}}{|\mathbf{x}|}\Big) \\
&\qquad\qquad \cdot \mathbf{\Phi}^{(m-1)}(|\mathbf{x}|t + ix_L)\, dt\,,
\end{aligned}
$$

where $\mathbf{\Phi}^{(m-1)}$ is the $(m-1)$-st derivative of $\mathbf{\Phi}$. As we know, Φ extends to a right- and left-monogenic function on $C^{\circ}_{\mu+}$ which belongs to $M(C^{\circ}_{\nu+})$ for all $\nu < \mu$.

When $B \in H_\infty(S^{\circ}_{\mu^-}(\mathbb{C}))$, $\mathbf{\Phi} = \mathcal{G}(B)$ and $b(\zeta) = B(i\zeta e_L) = B(-|\zeta|_{\mathbb{C}})\,\chi_-(\zeta)$, then $b \in H^-_\infty(S^{\circ}_\mu(\mathbb{C}^m))$, so we can form $\Phi = \mathcal{G}_-(b\,\overline{e_L})$. We see that when $x_L < 0$,

$$
\begin{aligned}
\Phi(x) &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} B(-|\xi|)\, e_-(x,\xi)\, d\xi\, \overline{e_L} \\
&= \frac{1}{2(2\pi)^m} \int_{\mathbb{R}^m} \Big(\overline{e_L} - \frac{i\xi}{|\xi|}\Big) B(-|\xi|)\, e^{-x_L|\xi|}\, e^{i\langle \mathbf{x},\xi\rangle}\, d\xi \\
&= \frac{1}{2(2\pi)^m} \int_{S^{m-1}} (\overline{e_L} - i\tau) \int_0^{+\infty} B(-r)\, e^{x_L r}\, e^{i\langle \mathbf{x},\tau\rangle r}\, r^{m-1}\, dr\, dS_\tau \\
&= \frac{(-1)^{m-1}}{2(2\pi)^m} \int_{S^{m-1}} (\overline{e_L} + i\tau) \int_{-\infty}^{0} B(r)\, e^{-x_L r}\, e^{i\langle \mathbf{x},\tau\rangle r}\, r^{m-1}\, dr\, dS_\tau \\
&= \frac{1}{2(-2\pi i)^{m-1}} \int_{S^{m-1}} (\overline{e_L} + i\tau)\, \mathbf{\Phi}^{(m-1)}(\langle \mathbf{x}, \tau\rangle + ix_L)\, dS_\tau \\
&= \frac{\sigma_{m-2}}{2(-2\pi i)^{m-1}} \int_{-1}^{1} (1-t^2)^{(m-3)/2} \Big(\overline{e_L} + \frac{it\mathbf{x}}{|\mathbf{x}|}\Big) \\
&\qquad\qquad \cdot \mathbf{\Phi}^{(m-1)}(|\mathbf{x}|t + ix_L)\, dt\ .
\end{aligned}
$$

When $B \in H_\infty(S^{\circ}_\mu(\mathbb{C}))$, write $B = B_+ + B_-$, where $B_+ = B\,\chi_{\mathrm{Re}>0} \in H_\infty(S^{\circ}_{\mu^+}(\mathbb{C}))$, and $B_- = B\,\chi_{\mathrm{Re}<0} \in H_\infty(S^{\circ}_{\mu^-})(\mathbb{C}))$. Then $b = b_+ + b_-$, where $b_\pm$ is associated with $B_\pm$. We can use this decomposition to relate the inverse Fourier transform $\mathcal{G}(b\,\overline{e_L}) = (\Phi, \underline{\Phi})$ of $b\,\overline{e_L}$ to the inverse Fourier transform $\mathcal{G}(B) = (\mathbf{\Phi}, \mathbf{\Phi}_1)$ of B.

In the following examples,

$$k(x) = \frac{1}{\sigma_m} \frac{\bar{x}}{|x|^{m+1}}$$

as usual.

$(\Phi(Z), \Phi_1(Z))$	$B(\lambda)$	$b(\zeta)$
$(0,1)$	1	1
$\left(\frac{i}{2\pi Z}, \frac{1}{2}\right)$	$\chi_{\mathrm{Re}>0}(\lambda)$	$\chi_+(\zeta)$
$\left(\frac{-i}{2\pi Z}, \frac{1}{2}\right)$	$\chi_{\mathrm{Re}<0}(\lambda)$	$\chi_-(\zeta)$
$\left(\frac{i}{\pi Z}, 0\right)$	$\mathrm{sgn}\,(\lambda)$	$\frac{i\zeta e_L}{\|\zeta\|_{\mathbb{C}}}$
$\frac{i}{2\pi}\Big(\frac{1}{Z+it},$ $-i\pi + \log\left(\frac{Z+it}{Z-it}\right)\Big)$	$\chi_{\mathrm{Re}>0}(\lambda)\, e^{-t\lambda}$ $(t>0)$	$\chi_+(\zeta)\, e^{-t\|\zeta\|_{\mathbb{C}}}$ $= \chi_+(\zeta)\, e^{-t(i\zeta e_L)}$
$\frac{t}{2\pi}\left(\frac{-1}{(Z+it)^2}, \frac{2Z}{Z^2+t^2}\right)$	$\chi_{\mathrm{Re}>0}(\lambda) t\lambda e^{-t\lambda}$ $(t>0)$	$\chi_+(\zeta)\, t\, \|\zeta\|_{\mathbb{C}} e^{-t\|\zeta\|_{\mathbb{C}}}$ $= i\chi_+(\zeta) t e_L \zeta\, e^{-t(i\zeta e_L)}$
$\Gamma(1+is)\big(\frac{i}{2\pi} e^{-\pi s/2} Z^{-1-is},$ $(\pi s)^{-1}\sinh(\pi s/2) Z^{-is}\big)$	$\chi_{\mathrm{Re}>0}(\lambda)\, \lambda^{is}$ $(s \in \mathbb{R})$	$\chi_+(\zeta)\, \|\zeta\|_{\mathbb{C}}^{is}$ $= \chi_+(\zeta)\, (i\zeta e_L)^{is}$
$(i\chi_{\mathrm{Re}>0}(Z)\, e^{i\alpha Z},$ $\alpha^{-1}(e^{i\alpha Z} - 1))$	$(\lambda-\alpha)^{-1}$ $(\mathrm{Im}\,\alpha > 0)$	$(i\zeta e_L - \alpha)^{-1}$
$(-i\chi_{\mathrm{Re}<0}(Z)\, e^{i\alpha Z},$ $\alpha^{-1}(e^{i\alpha Z} - 1))$	$(\lambda-\alpha)^{-1}$ $(\mathrm{Im}\,\alpha < 0)$	$(i\zeta e_L - \alpha)^{-1}$

$(\Phi(Z), \Phi_1(Z))$	$(\Phi(x), \underline{\Phi}(y))$
$(0,1)$	$(0,1)$
$\left(\frac{i}{2\pi Z}, \frac{1}{2}\right)$	$\left(k(x), \frac{1}{2}\right)$
$\left(\frac{-i}{2\pi Z}, \frac{1}{2}\right)$	$\left(-k(x), \frac{1}{2}\right)$
$\left(\frac{i}{\pi Z}, 0\right)$	$(2\,k(x), 0)$
$\frac{i}{2\pi}\left(\frac{1}{Z+it}, -i\pi + \log\left(\frac{Z+it}{Z-it}\right)\right)$ $(t>0)$	$(k(x+te_L), \underline{\Phi}(y))$ $\lim_{y\to 0} \underline{\Phi}(y) = 0$
$\frac{t}{2\pi}\left(\frac{-1}{(Z+it)^2}, \frac{2Z}{Z^2+t^2}\right)$ $(t>0)$	$\left(-t\frac{\partial k}{\partial t}(x+te_L), \underline{\Phi}(y)\right)$ $\lim_{y\to 0} \underline{\Phi}(y) = 0$
$\Gamma(1+is)\left(\frac{i}{2\pi} e^{-\pi s/2} Z^{-1-is}\right.$ $\left.(\pi s)^{-1}\sinh(\pi s/2) Z^{-is}\right)$ $(s \in \mathbb{R})$	$\left(\frac{-1}{\Gamma(1-is)}\right.$ $\left.\int_0^{+\infty} t^{-is}\frac{\partial k}{\partial t}(x+te_L)\,dt, \underline{\Phi}_s(y)\right)$ (see below)

The function $\underline{\Phi}_s$ (in the last row) has the form

$$\underline{\Phi}_s(r\,n) = \frac{r^{-is}}{\Gamma(1-is)} \int_0^{+\infty} t^{is-1} F(m, n_L, \tau)\, d\tau\, \overline{e_L}\, n\,,$$

where $r > 0$, $|n| = 1$, and F is a real-valued function which satisfies

$$|F(m, n_L, t)| \leq c(m, n_L) \frac{t^m}{(1+t)^{m+1}}\,.$$

In particular, when $n = e_L$, then

$$\underline{\Phi}_s(r\,e_L) = \frac{\sigma_{m-1} r^{-is}}{\Gamma(1-is)} \int_0^{+\infty} \frac{t^{m+is-1}}{(1+t^2)^{(m+1)/2}}\, dt\,, \qquad r > 0\,.$$

(To prove this, first show that the function $\underline{\Phi}$ in the preceding row has the form $\underline{\Phi}(r\,n) = F(m, n_L, r/t)\,\overline{e_L}\,n\,$.)

The functions $\mathbf{\Phi}_1$ and $\underline{\Phi}$ are really only of interest near zero, and indeed do not enter into Parseval's identity or the convolution formulae when they tend to nought at zero. It is shown in [McQ] that if $|B(\lambda)| \le c_s\,|\lambda|^s$ for all $\lambda \in S^\circ_\mu(\mathbb{C})$ and some $s < 0$, then $\mathbf{\Phi}_1(Z) \to 0$ as $Z \to 0$ $(Z \in S_{\nu+}(\mathbb{C}),\ \nu < \mu)$. It also follows that $|b(\zeta)| \le c_s\,|\zeta|^s$ for all $\zeta \in S^\circ_\mu(\mathbb{C})$, and hence from Theorem 4.3.viii) that $\underline{\Phi}(y) \to 0$ as $y \to 0$ $(y \in T^\circ_\nu,\ \nu < \mu)$. So there is really no need to find $\mathbf{\Phi}_1$ and $\underline{\Phi}$ when $|B(\lambda)| \le c_s\,|\lambda|^s$, $s < 0$.

It is important to realize however, that $\mathbf{\Phi}_1$ and $\underline{\Phi}$ do not always have limits at zero, in which case they are needed in Parseval's identity and in the definitions of the convolution operators presented in the next section. Principal-value integrals do not suffice. For example, Parseval's identity (P) connecting the function $\chi_+(\xi)\,|\xi|^{is}_{\mathbb{C}}$ with its inverse Fourier transform involves the function $\underline{\Phi}_s$ given above.

Let us turn our attention to the function $B = B_+ = B\,\chi_{\mathrm{Re}>0}$, and substitute the corresponding values of $\mathbf{\Phi}$ and $\underline{\Phi}$ into the formula (+). We obtain (on using the fact that $(\overline{e_L}+i\tau)(a+ib)^k = (\overline{e_L}+i\tau)(a-be_L\tau)^k$ whenever $\tau \in S^{m-1}$ and $a, b \in \mathbb{R}$)

$$\frac{\bar{x}}{\sigma_m |x|^{m+1}} = \frac{1}{2(2\pi i)^{m-1}} \int\limits_{S^{m-1}} (\overline{e_L} + i\tau)\frac{i}{2\pi}\,\frac{(-1)^{m-1}(m-1)!}{(\langle \mathbf{x}, \tau\rangle + i x_L)^m}\, dS_\tau$$

$$= \frac{(m-1)!}{2}\left(\frac{i}{2\pi}\right)^m \int\limits_{S^{m-1}} (\overline{e_L} + i\tau)\,(\langle \mathbf{x}, \tau\rangle - x_L e_L \tau)^{-m}\, dS_\tau\,,$$

where $x_L > 0$, which, on taking the real part of the right hand side, is the plane wave decomposition of the Cauchy kernel presented by Sommen in [S] (at least in the case $L = 0$).

For the function $B = \chi_{\mathrm{Re}<0}$ we obtain

$$\frac{\bar{x}}{\ldots} = \frac{-(m-1)!}{\ldots}\left(\frac{-i}{\ldots}\right)^m \int\limits_{S^{m-1}} \ldots$$

Part II: Convolution singular integrals on surfaces, and functional calculi.*

6. Convolution singular integrals on Lipschitz surfaces.

Let Σ denote the Lipschitz surface consisting of points $x = \mathbf{x} +$ [illegible]

$g(\mathbf{x})\, e_L \in \mathbb{R}^{m+1}$, where $\mathbf{x} \in \mathbb{R}^m$, and g is a real-valued Lipschitz function which satisfies

$$\|\nabla g\|_\infty = \sup \Big(\sum^{m} \Big| \frac{\partial g}{\partial x_i} \Big|^2 \Big)^{1/2} \leq \tan \omega < +\infty, \quad \text{where } 0 < \omega < \frac{\pi}{2}$$ [illegible]

[illegible] $(x) \in \mathbb{R}^{m+1}$ [illegible]

Moreover, if $\Phi \in K(C^{\pm}_{N_\mu})$ *for* $0 < \mu \leq \pi/2 - \omega$, *then*

$$\|T_\Phi u\|_p \leq C_{\omega,\mu,p} \, \|\Phi\|_{K(C^{+}_{N_\mu})} \, \|u\|_p \, ,$$

for some constants $C_{\omega,\mu,p}$ *which depend only on* ω, μ *and* p.

ii) *Given* $(\Phi, \underline{\Phi}) \in K_N$, *there exists* $T_{(\Phi,\underline{\Phi})} \in \mathcal{L}(L_p(\Sigma))$, *defined for all* $u \in L_p(\Sigma)$ *and almost all* $x \in \Sigma$, *by*

$$(T_{(\Phi,\underline{\Phi})}u)(x) = \lim_{\varepsilon \to 0} \Big(\int\limits_{\substack{|x-y|\geq\varepsilon \\ y\in\Sigma}} \Phi(x-y)\, n(y)\, u(y)\, dS_y + \underline{\Phi}(\varepsilon\, n(x))\, u(x) \Big) .$$

Moreover, if $(\Phi, \underline{\Phi}) \in K(S_{N_\mu})$ *for* $0 < \mu \leq \pi/2 - \omega$, *then*

$$\|T_{(\Phi,\underline{\Phi})}u\|_p \leq C_{\omega,\mu,p} \, \|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})} \, \|u\|_p \, ,$$

for some constants $C_{\omega,\mu,p}$ *which depend only on* ω, μ *and* p.

We remark that part ii) follows directly from part i), together with Theorem 3.1, and that $T_{(\Phi,\underline{\Phi})} = T_{\Phi_+} + T_{\Phi_-}$ with Φ_+ and Φ_- the functions specified there.

Recall that the spaces K_N^+, K_N^- and K_N are not convolution algebras, but that the subspaces M_N^+, M_N^- and M_N are.

Theorem 6.2. *The mappings from* $\Phi \in M_N^{\pm}$ *to* $T_\Phi \in \mathcal{L}(L_p(\Sigma))$ *and from* $(\Phi, \underline{\Phi}) \in M_N$ *to* $T_{(\Phi,\underline{\Phi})} \in \mathcal{L}(L_p(\Sigma))$ *are algebra homomorphisms.*

Let $k(x) = \bar{x}/(\sigma_m |x|^{m+1})$, $x \neq 0$. Then k belongs to both M_N^+ and M_N^-, so let us write it as k_+ when considered in M_N^+ (with $\underline{k_+} = 1/2$), and as k_- when considered in M_N^- (with $\underline{k_-} = -1/2$). Also $(2k, 0) = (k_+, 1/2) + (k_-, -1/2) \in M_N$. The corresponding bounded linear operators on $L_p(\Sigma)$ are the Cauchy singular integral operators $C_\Sigma = T_{(2k,0)}$, $P_+ = T_{k_+}$ and $P_- = -T_{k_-}$. By Theorem 6.1 we know that they are defined for all $u \in L_p(\Sigma)$ and almost all $x \in \Sigma$ by

$$(P_\pm u)(x) = \pm \lim_{\delta \to 0+} \int_\Sigma k(x \pm \delta\, e_L - y)\, n(y)\, u(y)\, dS_y$$

and

$$(C_\Sigma u)(x) = 2 \lim_{\varepsilon \to 0} \int\limits_{\substack{|x-y|\geq\varepsilon \\ y\in\Sigma}} k(x-y)\, n(y)\, u(y)\, dS_y \, .$$

It is the inquiry into the boundedness of C_Σ that really started this study.

As noted in [LMcS], the following properties are immediate consequences of Theorems 6.1 and 6.2.

Theorem 6.3. *Let $\Phi_\pm \in M_N^\pm$. The Cauchy singular integral operators P_+, P_- and C_Σ are bounded linear operators on $L_p(\Sigma)$ which satisfy the following identities.*

(0) $P_+ + P_- = I\,, \qquad P_+ - P_- = C_\Sigma \qquad$ (*the Plemelj formulae*);

(1) $$P_+ T_{\Phi_+} = T_{\Phi_+} P_+ = T_{\Phi_+}\,, \qquad P_- T_{\Phi_+} = T_{\Phi_+} P_- = 0\,,$$
$$P_- T_{\Phi_-} = T_{\Phi_-} P_- = T_{\Phi_-}\,, \qquad P_+ T_{\Phi_-} = T_{\Phi_-} P_+ = 0\,;$$

(2) $$P_+^2 = P_+\,, \quad P_-^2 = P_-\,, \quad P_+ P_- = P_- P_+ = 0\,, \quad C_\Sigma^2 = I\,;$$

(3) $$T_{\Phi_+} T_{\Phi_-} = T_{\Phi_-} T_{\Phi_+} = 0\,.$$

Define the *Hardy spaces* $L_p^\pm(\Sigma)$ to be the images of the projections $P_\pm$, so that $L_p(\Sigma) = L_p^+(\Sigma) \oplus L_p^-(\Sigma)$. The operators T_{Φ_+} map $L_p(\Sigma)$ into $L_p^+(\Sigma)$ and are zero on $L_p^-(\Sigma)$, while the operators T_{Φ_-} map $L_p(\Sigma)$ into $L_p^-(\Sigma)$ and are zero on $L_p^+(\Sigma)$. So alternatively we could define $T_{\Phi_\pm} \in \mathcal{L}(L_p^\pm(\Sigma))$, in which case $T_{(\Phi,\underline{\Phi})} = T_{\Phi_+} \oplus T_{\Phi_-}$, where $(\Phi, \underline{\Phi})$ is related to Φ_+ and Φ_- as in Theorem 3.1.

Let us make one observation which depends on Section 4. We used Fourier theory at the end of that section to show that $(2k_j, 0) \in K_N$, where $k_j(x) = -x_j/(\sigma_m |x|^{m+1})$, $x \in \mathbb{R}^m \setminus \{0\}$, $j = 1, 2, \ldots, m$. So the operators $R_{j,\Sigma} = T_{2k_j}$ are bounded linear operators on $L_p(\Sigma)$. The question as to whether these operators, which can be thought of as Riesz transforms on Σ, are L_p-bounded, was actually one of the motivations for developing the Fourier theory of this paper. (The boundedness of these operators is not a direct consequence of the boundedness of the Cauchy operator $C_\Sigma = \sum e_j R_{j,\Sigma}$, because $R_{j,\Sigma}$ is not merely the j-th component of C_Σ).

Theorem 6.4. *The Riesz transforms $R_{j,\Sigma}$ are bounded linear operators on $L_p(\Sigma)$ which satisfy $R_{j,\Sigma} R_{k,\Sigma} = R_{k,\Sigma} R_{j,\Sigma}$, $\sum e_j R_{j,\Sigma} = C_\Sigma$ and $\sum (R_{j,\Sigma})^2 = -I$.*

Here are some further consequences of Theorems 6.1 and 6.2. When $\Phi \in K_N^+$, and $\delta > 0$, then $\Phi_\delta \in K_N^+$ is defined by $\Phi_\delta(x) = \Phi(x + \delta e_L)$. In particular, $k_\delta \in M_N^+$, where $k_\delta(x) = k_{+\delta}(x) = k_+(x + \delta e_L)$.

If p is a polynomial in m variables with values in $C_{(M)}$, then $p(-i\,\mathbf{D})\,k_\delta \in K_N^+$, where

$$p(-i\,\mathbf{D})k_\delta(x) = p\Big(-i\frac{\partial}{\partial x_1}, -i\frac{\partial}{\partial x_2}, -i\frac{\partial}{\partial x_m}\Big)\, k_+(x + \delta\, e_L)\,.$$

Theorem 6.5. *Let $\alpha > 0$ and $\delta > 0$.*

i) *If $\Phi \in K_N^+$, then $\Phi * k_\delta = \Phi_\delta \in K_N^+$, and $T_\Phi\, T_{k_\delta} = T_{\Phi_\delta}$.*

ii) *If $\Phi \in M_N^+$, then $k_\delta * \Phi = \Phi_\delta \in M_N^+$, and $T_{k_\delta}\, T_\Phi = T_{\Phi_\delta}$.*

iii) *$k_\alpha * k_\delta = k_{\alpha+\delta} \in M_N^+$, and $T_{k_\alpha}\, T_{k_\delta} = T_{k_{\alpha+\delta}}$.*

Suppose that p and q are two polynomials, with p satisfying $p(\xi)\xi e_L = \xi e_L p(\xi)$. Then $p(-i\,\mathbf{D})k_\delta \in M_N^+$, and

iv) *$k_\alpha * p(-i\,\mathbf{D})k_\delta = p(-i\,\mathbf{D})k_{\alpha+\delta} \in M_N^+$, and*

$$T_{k_\alpha}\, T_{p(-i\,\mathbf{D})k_\delta} = T_{p(-i\,\mathbf{D})k_{\alpha+\delta}}\,, \qquad \textit{and}$$

v) *$q(-i\,\mathbf{D})k_\alpha * p(-i\,\mathbf{D})k_\delta = (qp)(-i\,\mathbf{D})k_{\alpha+\delta} \in K_N^+$, and*

$$T_{q(-i\,\mathbf{D})k_\alpha}\, T_{p(-i\,\mathbf{D})k_\delta} = T_{(qp)(-i\,\mathbf{D})k_{\alpha+\delta}}\,.$$

Let Ω_+ be the open subset of $\mathbb{R}^{m+1}$ above Σ. That is, $\Omega_+ = \{X \in \mathbb{R}^{m+1} : X = x + \delta e_L,\ x \in \Sigma,\ \delta > 0\}$. For $u \in L_p(\Sigma)$, define $\mathcal{C}_\Sigma^+ u$ to be the left-monogenic function on Ω_+ given by

$$(\mathcal{C}_\Sigma^+ u)(X) = \int_\Sigma k(X - y)\, n(y)\, u(y)\, dS_y\,, \qquad X \in \Omega_+\,.$$

Then $(\mathcal{C}_\Sigma^+ u)(x + \delta e_L) = T_{k_\delta} u(x) \to P_+ u(x)$ as $\delta \to 0^+$ for almost all $x \in \Sigma$. This limit also exists in the L_p sense [LMcS]. That is, $\|T_{k_\delta} u - P_+ u\|_p \to 0$ as $\delta \to 0^+$.

Let us consider functions $u \in L_p^+(\Sigma)$, in which case $\|T_{k_\delta} u - u\|_p \to 0$ as $\delta \to 0^+$.

We could differentiate $(\mathcal{C}_\Sigma^+ u)(X)$ before taking the limit as X approaches Σ, though the limit need not always exist. More generally, given any polynomial p in m variables with values in $\mathbb{C}_{(M)}$, we could form

$$p(-i\,\mathbf{D})(\mathcal{C}_\Sigma^+ u)(X) = p\Big(-i\frac{\partial}{\partial X_1}, -i\frac{\partial}{\partial X_2}, \dots, -i\frac{\partial}{\partial X_m}\Big)(\mathcal{C}_\Sigma^+ u)(X),$$

though again the limit as X approaches Σ need not always exist. But let us define $p(-i\,\mathbf{D}_\Sigma)u(x)$ to be the limit of $p(-i\,\mathbf{D})(\mathcal{C}_\Sigma^+ u)(x+\delta e_L) = T_{p(-i\,\mathbf{D})k_\delta} u(x)$ as $\delta \to 0^+$, when it does exist in $L_p(\Sigma)$.

To be precise, define $p(-i\,\mathbf{D}_\Sigma)$ to be the linear transformation, from its domain $\mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma)) \subset L_p^+(\Sigma)$, into $L_p(\Sigma)$, given by

$$\mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma)) = \{u \in L_p^+(\Sigma) :\ T_{p(-i\,\mathbf{D})k_\delta} u \to w \in L_p(\Sigma)\}$$

and $p(-i\,\mathbf{D}_\Sigma)u = w$.

If u is itself of the form $u = T_{k_\alpha} v$ for some $v \in L_p^+(\Sigma)$, then u is the restriction of the left monogenic function U to Σ, where

$$U(X) = (\mathcal{C}_\Sigma^+ v)(X + \alpha e_L), \qquad X + \alpha e_L \in \Omega_+ .$$

Such functions u belong to $\mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))$ and

$$p(-i\,\mathbf{D}_\Sigma)u = (p(-i\,\mathbf{D})U)|_\Sigma .$$

Consider, in particular, the functions $q_k(x) = i\xi_k$, $k = 1, 2, \dots, m$, and $q(\xi) = i\,\xi e_L = \sum_{k=1}^m i\xi_k e_k e_L$. Using them, we define the operators $D_{k,\Sigma} = q_k(-i\,\mathbf{D}_\Sigma)$, and $\mathbf{D}_\Sigma e_L = q(-i\,\mathbf{D}_\Sigma)$, so that, for functions u of the type specified in the preceding paragraph,

$$\mathbf{D}_\Sigma e_L u = (\mathbf{D} e_L U)|_\Sigma$$

and

$$D_{k,\Sigma} u = \frac{\partial U}{\partial X_k}, \qquad k = 1, 2, \dots, m .$$

It may be interesting to write these functions out in terms of the parameter $\mathbf{s}$, when Σ is parametrized by $x = \mathbf{s} + g(\mathbf{s})e_L$. We obtain

$$D_{k,\Sigma} u(\mathbf{s} + g(\mathbf{s})e_L) = \Big(\frac{\partial}{\partial s_k} + \frac{\partial g}{\partial s_k}(e_L - \mathbf{D}g)^{-1}\mathbf{D}_\mathbf{s}\Big)\, u(\mathbf{s} + g(\mathbf{s})e_L)$$

and

$$\begin{aligned}\mathbf{D}_\Sigma e_L u(\mathbf{s} + g(\mathbf{s})e_L) &= \sum_{k=1}^m e_k e_L D_{k,\Sigma} u(\mathbf{s} + g(\mathbf{s})e_L)\\ &= (e_L - \mathbf{D}g)^{-1}\mathbf{D}_\mathbf{s} u(\mathbf{s} + g(\mathbf{s})e_L),\end{aligned}$$

for all functions u such that $u = T_{k_\alpha} v$ for some $v \in L_p^+(\Sigma)$. In Theorem 8.2 we shall see that this expression for $\mathbf{D}_\Sigma$ is valid for every function u in its domain.

It follows from the next theorem that these are closed linear operators in $L_p^+(\Sigma)$. In the following two sections, we shall explore ways in which the convolution operators of Theorem 6.1 can be represented as bounded holomorphic functions of $(D_{k,\Sigma})$ and $\mathbf{D}_\Sigma$. We are still supposing that $1 < p < \infty$.

Theorem 6.6. *Let p be a polynomial in m variables with values in $\mathbb{C}_{(M)}$. Then $p(-i\,\mathbf{D}_\Sigma)$ is a linear transformation from $L_p^+(\Sigma)$ to $L_p(\Sigma)$ with domain $\mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))$ dense in $L_p^+(\Sigma)$.*

If $p(\xi)\xi e_L = \xi e_L p(\xi)$, then

$$p(-i\,\mathbf{D}_\Sigma)u \in L_p^+(\Sigma)\,, \qquad \text{for all } u \in \mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))\,,$$

and indeed $p(-i\,\mathbf{D}_\Sigma)$ is a closed linear operator in $L_p^+(\Sigma)$.

Suppose that p and q are two polynomials, with p satisfying $p(\xi)\xi e_L = \xi e_L p(\xi)$. Let $u \in \mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))$. Then $p(-i\,\mathbf{D}_\Sigma)u \in \mathcal{D}^+(q(-i\,\mathbf{D}_\Sigma))$ if and only if $u \in \mathcal{D}^+((qp)(-i\,\mathbf{D}_\Sigma))$, in which case $q(-i\,\mathbf{D}_\Sigma)p(-i\,\mathbf{D}_\Sigma)u = (qp)(-i\,\mathbf{D}_\Sigma)u$.

PROOF. The domain $\mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))$ is dense in $L_p^+(\Sigma)$, because every function $u \in L_p^+(\Sigma)$ is the limit of $T_{k_\alpha}u \in \mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))$ as $\alpha \to 0$.

For the remainder of this proof, suppose that $p(\xi)\xi e_L = \xi e_L p(\xi)$. Let $u \in \mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))$. We saw in Theorem 6.5 that $p(-i\mathbf{D})k_\delta \in M_N^+$ when $\delta > 0$, and that $T_{k\alpha}T_{p(-i\,\mathbf{D})k_\delta}u = T_{p(-i\,\mathbf{D})k_{\alpha+\delta}}u$ when $\alpha > 0$ also. On letting δ tend to 0, we obtain $T_{k_\alpha}p(-i\,\mathbf{D}_\Sigma)u = T_{p(-i\,\mathbf{D})k_\alpha}u$. On letting α to 0, we conclude that $p(-i\,\mathbf{D}_\Sigma)u = P_+p(-i\,\mathbf{D}_\Sigma)u \in L_p^+(\Sigma)$.

To prove that $p(-i\,\mathbf{D}_\Sigma)$ is a closed operator in $L_p^+(\Sigma)$, choose a sequence (v_n) in $\mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))$ such that $v_n \to v \in L_p^+(\Sigma)$ and $p(-i\,\mathbf{D}_\Sigma)v_n \to w \in L_p^+(\Sigma)$. We need to show that $v \in \mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))$ and $p(-i\,\mathbf{D}_\Sigma)v = w$. For each $\alpha > 0$, $T_{k_\alpha}p(-i\,\mathbf{D}_\Sigma)v_n \to T_{k_\alpha}w$, and $T_{k_\alpha}p(-i\,\mathbf{D}_\Sigma)v_n = T_{p(-i\,\mathbf{D})k_\alpha}v_n \to T_{p(-i\,\mathbf{D})k_\alpha}v$, so that $T_{p(-i\,\mathbf{D})k_\alpha}v = T_{k_\alpha}w$. Therefore $T_{p(-i\,\mathbf{D})k_\alpha}v = T_{k_\alpha}w \to w$ as $\alpha \to 0$. We conclude that $v \in \mathcal{D}(p(-i\,\mathbf{D}_\Sigma))$ and that $p(-i\,\mathbf{D}_\Sigma)v = w$ as required.

From Theorem 6.5, we also have that

$$T_{q(-i\,\mathbf{D})k_\alpha}T_{p(-i\,\mathbf{D})k_\delta}u = T_{(qp)(-i\,\mathbf{D})k_{\alpha+\delta}}u\,,$$

and so, letting δ tend to 0, we obtain

$$T_{q(-i\,\mathbf{D})k_\alpha}p(-i\,\mathbf{D}_\Sigma)u = T_{(qp)(-i\,\mathbf{D})k_\alpha}u\,.$$

On letting α tend to 0, we conclude that $p(-i\,\mathbf{D}_\Sigma)u \in \mathcal{D}^+(q(-i\,\mathbf{D}_\Sigma))$ if and only if $u \in \mathcal{D}^+((qp)(-i\,\mathbf{D}_\Sigma))$, in which case $q(-i\,\mathbf{D}_\Sigma)p(-i\,\mathbf{D}_\Sigma)u = (qp)(-i\,\mathbf{D}_\Sigma)u$.

In a similar way, we can define the linear transformation $p(-i\,\mathbf{D}_\Sigma)$ from its domain $\mathcal{D}^-(p(-i\,\mathbf{D}_\Sigma)) \subset L_p^-(\Sigma)$ into $L_p(\Sigma)$.

Finally, we define the linear operator $p(-i\,\mathbf{D}_\Sigma)$ in $L_p(\Sigma)$ with dense domain

$$\begin{aligned}\mathcal{D}(p(-i\,\mathbf{D}_\Sigma)) &= \mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma)) \oplus \mathcal{D}^-(p(-i\,\mathbf{D}_\Sigma))\\ &\subset L_p^+(\Sigma) \oplus L_p^-(\Sigma) = L_p(\Sigma)\end{aligned}$$

by $p(-i\,\mathbf{D}_\Sigma)u = p(-i\,\mathbf{D}_\Sigma)P_+u + p(-i\,\mathbf{D}_\Sigma)P_-u$.

Theorem 6.7. *The statement of Theorem 6.6 remains valid when $L_p^+(\Sigma)$ is replaced by $L_p(\Sigma)$ and $\mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))$ is replaced by $\mathcal{D}(p(-i\,\mathbf{D}_\Sigma))$.*

Suppose that U is a left-monogenic function on the strip $\Sigma + (-t,t)e_L$, and that the functions u_α defined by $u_\alpha(x) = U(x+\alpha e_L)$, $x \in \Sigma$, are uniformly bounded in $L_p(\Sigma)$, $\alpha \in (-t,t)$. Let $u = u_0 = U|_\Sigma$. Then, for each polynomial p,

$$p(-i\,\mathbf{D}_\Sigma)u = (p(-i\,\mathbf{D})U)\big|_\Sigma\,.$$

This follows from the remark following the definition of $p(-i\,\mathbf{D}_\Sigma)$ in $L_p^+(\Sigma)$, and the fact that $P_+u = T_{k_\alpha}P_+u_{-\alpha}$, together with a similar result for P_-u.

In particular, for such left-monogenic functions U,

$$\mathbf{D}_\Sigma\, e_L u = (\mathbf{D}e_L U)\big|_\Sigma \qquad \text{and} \qquad D_{k,\Sigma}u = \frac{\partial U}{\partial X_k}\Big|_\Sigma\,,$$

$k = 1, 2, \ldots, m$, when $u = U|_\Sigma$.

7. H_∞ functional calculi for functions of m variables.

Let $(\Phi, \underline{\Phi}) \in K(S_{N_\mu})$. We can think of $b = \mathcal{F}(\Phi, \underline{\Phi})e_L$ as the Fourier multiplier corresponding to the bounded linear operator $T_{(\Phi,\underline{\Phi})}$. But we can also think of the mapping from $b \in H_N$ to $T_{(\Phi,\underline{\Phi})} \in \mathcal{L}(L_p(\Sigma))$ as giving us a bounded H_∞ functional calculus of $-i\,\mathbf{D}_\Sigma = \sum_{k=1}^m -i\,e_k D_{k,\Sigma}$, and write

$$T_{(\Phi,\underline{\Phi})} = b(-i\,\mathbf{D}_\Sigma) = b(-iD_{1,\Sigma}, -iD_{2,\Sigma}, \dots, -iD_{m,\Sigma}).$$

In order to see that this is a natural thing to do, let us introduce a larger algebra than H_N, namely $\mathcal{P}_N$, consisting of all functions b from $\mathbb{R}^m \setminus \{0\}$ to $\mathbb{C}_{(M)}$ such that $b_+ = b\chi_+$ extends holomorphically to $N_\mu(\mathbb{C}^m)$ for some $\mu > 0$, and $b_- = b\chi_-$ extends holomorphically to $\overline{N_\mu}(\mathbb{C}^m)$, and the extensions satisfy $|b_\pm(\zeta)| \le c(1+|\zeta|^s)$ for some s and $c \ge 0$.

For such $b \in \mathcal{P}_N$, the functions $b_{+\delta}$ and $b_{-\delta}$ belong to H_N^+ and H_N^- respectively, where $b_{+\delta}(\zeta) = b_+(\zeta)\,e^{-\delta|\zeta|_{\mathbb{C}}}$ and $b_{-\delta}(\zeta) = b_-(\zeta)\,e^{-\delta|\zeta|_{\mathbb{C}}}$ for $\delta > 0$. Therefore $\Phi_{\pm\delta} = \mathcal{G}_\pm(b_{\pm\delta}\overline{e_L}) \in K_N^+$. Define $b(-i\,\mathbf{D}_\Sigma)$ to be the linear operator in $L_p(\Sigma)$ with domain

$$\mathcal{D}(b(-i\,\mathbf{D}_\Sigma)) = \{u \in L_p(\Sigma) : T_{\Phi_{\pm\delta}}u \to w_\pm \in L_p(\Sigma) \text{ as } \delta \to 0\}$$

by

$$b(-i\,\mathbf{D}_\Sigma)u = w_+ + w_- \,.$$

The fact that this terminology is reasonable follows from the following facts.

Theorem 7.1. *Suppose that $1 < p < \infty$. Let $b \in \mathcal{P}_N$.*

i) *If $b \in H_N$, then $b(-i\,\mathbf{D}_\Sigma) = T_{(\Phi,\underline{\Phi})} \in \mathcal{L}(L_p(\Sigma))$, where $(\Phi, \underline{\Phi})e_L = \mathcal{G}(b)$. In particular, $1(-i\,\mathbf{D}_\Sigma) = I$, $\chi_\pm(-i\,\mathbf{D}_\Sigma) = P_\pm$, $(r_j e_L)(-i\,\mathbf{D}_\Sigma) = R_{j,\Sigma}$, and $r(-i\,\mathbf{D}_\Sigma) = C_\Sigma = \sum e_j R_{j,\Sigma}$ where $r(\xi) = i\,\xi|\xi|^{-1}e_L$.*

ii) *If $b_+ = b\chi_+ \in H_\infty^+(N_\mu(\mathbb{C}^m))$ and $b_- = b\chi_- \in H_\infty^-(\overline{N_\mu}(\mathbb{C}^m))$ with $0 < \mu \le \pi/2 - \omega$, then*

$$\|b(-i\,\mathbf{D}_\Sigma)u\|_p \le C_{\omega,\mu,p}\left(\|b_+\|_\infty + \|b_-\|_\infty\right)\|u\|_p \,,$$

for some constants $C_{\omega,\mu,p}$ which depend only on ω, μ, p (and the dimension m).

iii) *If b is a polynomial in m variables, then the definition of $b(-i\,\mathbf{D}_\Sigma)$ coincides with that given in Section 6.*

iv) *The domain $\mathcal{D}(b(-i\,\mathbf{D}_\Sigma))$ of $b(-i\,\mathbf{D}_\Sigma)$ is dense in $L_p(\Sigma)$.*

v) *If $b(\xi)\xi e_L = \xi e_L b(\xi)$ for all $\xi \in \mathbb{R}^m \setminus \{0\}$, then $b(-i\,\mathbf{D}_\Sigma)$ is a closed linear operator in $L_p(\Sigma)$.*

vi) *If $u \in \mathcal{D}(b(-i\,\mathbf{D}_\Sigma))$, $f \in \mathcal{P}_N$ and $c \in \mathbb{C}_{(M)}$, then*

$$u \in \mathcal{D}(f(-i\,\mathbf{D}_\Sigma)) \quad \textit{if and only if} \quad u \in \mathcal{D}((c\,b + f)(-i\,\mathbf{D}_\Sigma)),$$

in which case $c\,b(-i\,\mathbf{D}_\Sigma)u + f(-i\,\mathbf{D}_\Sigma)u = (c\,b + f)(-i\,\mathbf{D}_\Sigma)u$.

vii) *If $b(\xi)\xi e_L = \xi e_L b(\xi)$ for all $\xi \in \mathbb{R}^m \setminus \{0\}$, $u \in \mathcal{D}(b(-i\,\mathbf{D}_\Sigma))$ and $f \in \mathcal{P}_N$, then*

$$b(-i\,\mathbf{D}_\Sigma)u \in \mathcal{D}(f(-i\,\mathbf{D}_\Sigma)) \quad \textit{if and only if} \quad u \in \mathcal{D}((fb)(-i\,\mathbf{D}_\Sigma)),$$

in which case $f(-i\,\mathbf{D}_\Sigma)b(-i\,\mathbf{D}_\Sigma)u = (fb)(-i\,\mathbf{D}_\Sigma)u$.

PROOF. When $b \in H_N$, let $b_+ = b\chi_+$ and $\Phi_+ = \mathcal{G}_+(b_+\overline{e_L})$, so that $\Phi_{+\delta}(x) = \mathcal{G}_+(b_{+\delta}\overline{e_L})(x) = \Phi_+(x + \delta e_L)$. Therefore, for all $u \in L_p(\Sigma)$, $T_{\Phi_{+\delta}}u \to T_{\Phi_+}u$ in $L_p(\Sigma)$ as $\delta \to 0$. Similarly $T_{\Phi_{-\delta}}u \to T_{\Phi_-}u$. So $u \in \mathcal{D}(b(-i\,\mathbf{D}_\Sigma))$, and $b(-i\,\mathbf{D}_\Sigma)u = T_{\Phi_+}u + T_{\Phi_-}u = T_{(\Phi,\underline{\Phi})}$.

The estimate in ii) is a consequence of Theorems 4.3.iii) and 6.1.ii).

To prove iii), use the identities $\mathcal{F}_\pm(p(-i\,\mathbf{D}_\Sigma)k_{\pm\delta})e_L = p_{\pm\delta}$, which are consequences of Theorem 4.1.vii) and Theorem 4.2.

The proof of the remaining parts mimics that of Theorem 6.6.

Let us indicate the kind of application of these results that we have in mind. Details will appear elsewhere.

Consider the following boundary value problem for harmonic functions.

$$\begin{cases} \Delta U(X) = \displaystyle\sum_{k=1}^m \frac{\partial^2 U}{\partial X_k^2}(X) + \frac{\partial^2 U}{\partial x_L^2}(X) = 0\,, \qquad X \in \Omega_+\,, \\ \Big(\displaystyle\sum_{k=1}^m \beta_k \frac{\partial U}{\partial X_k} + \beta_L \frac{\partial U}{\partial X_L}\Big)\Big|_\Sigma = w \in L_p(\Sigma, \mathbb{C})\,, \end{cases}$$

where $\beta_k, \beta_L \in \mathbb{C}$ and $2 \le p < \infty$.

In the special case when $\beta_L = 1$ and $\beta_k = 0$, $k = 1, 2, \ldots, m$, the solution to this problem is given by

$$U(X) = U(\mathbf{X} + X_L\, e_L) = -\int_{X_L}^{+\infty} (\mathcal{C}^+_{\Sigma_0} v)(\mathbf{X} + t\, e_L)\, dt\,,$$

where $v = (P_{+0})^{-1} w \in L_p(\Sigma)$. Here $\mathcal{C}^+_{\Sigma_0}$ denotes the scalar part of the Cauchy integral $\mathcal{C}^+_\Sigma$, namely

$$(\mathcal{C}^+_{\Sigma_0} v)(X) = \int_\Sigma \langle \overline{k(X-y)}, n(y)\rangle\, v(y)\, dS_y\ , \qquad X \in \Omega_+\ ,$$

which is the double-layer potential operator on Σ, and $P_{+0} = \frac{1}{2}(I + C_{\Sigma_0})$, where C_{Σ_0} is the singular double-layer potential operator on Σ. The invertibility of P_{+0} in $L_p(\Sigma, \mathbb{C})$ was proved by Verchota [V].

In the general case of complex β_k and β_L, we assume that, for some $\kappa > 0$,

$$(\#) \qquad |\langle \beta, n + it\rangle| \geq \kappa\,, \quad \text{for all } n \in N \text{ and } t \in \mathbb{R}^{m+1} \text{ such that } |t| = 1 \text{ and } \langle n, t\rangle = 0\,,$$

where $\beta = \sum \beta_k e_k + \beta_L e_L$. (This is the weakest condition on β under which we can expect to solve the boundary value problem, because, if Σ is smooth in a neighbourhood of a point $x \in \Sigma$, then the covering condition of Agmon, Douglis, Nirenberg for this problem is, that there does not exist a unit tangent vector t to Σ at x satisfying $\langle \beta, n(x) + it\rangle = 0$, where $n(x)$ is the unit normal to Σ at x.)

It can be shown that $(\#)$ implies

$$(\#\#) \qquad |\langle \beta, |\zeta|_{\mathbb{C}}\, e_L - i\zeta\rangle| \geq \kappa |\, |\zeta|_{\mathbb{C}}|\,, \qquad \text{for all } \zeta \in N(\mathbb{C}^m)\,,$$

and hence that the holomorphic function b defined by

$$b(\zeta) = \frac{|\zeta|_{\mathbb{C}}}{\langle \beta, |\zeta|_{\mathbb{C}}\, e_L - i\zeta\rangle}$$

is bounded by κ^{-1} on $N(\mathbb{C}^m)$, and indeed is bounded by $2\kappa^{-1}$ on $N_\mu(\mathbb{C}^m)$ for μ small enough (to derive $(\#\#)$ from $(\#)$, choose $\zeta \in N(\mathbb{C}^m)$, meaning that there exists $n \in N$ and $c > 0$ such that $\eta +$

$\mathrm{Re}(|\zeta|_{\mathbb{C}})e_L = cn$. Apply (#) with this choice of n, and with $t = c^{-1}(-\xi + \mathrm{Im}(|\zeta|_{\mathbb{C}})e_L))$.

Therefore $b(-i\,\mathbf{D}_\Sigma)$ is a bounded linear operator on $L_p(\Sigma, \mathbb{C}_{(M)})$.

On noting the identity,

$$\Big(\sum_{k=1}^{m} \beta_k\zeta_k - \beta_L\zeta e_L\Big)\, b(\zeta)\,\chi_+(\zeta) = -\zeta e_L \chi_+(\zeta)\,,$$

it is straightforward to verify that the solution to our boundary value problem is

$$U(X) = U(\mathbf{X} + X_L\, e_L) = -\int_{X_L}^{+\infty} (\mathcal{C}_\Sigma^+ b(-i\,\mathbf{D}_\Sigma)v)_0(\mathbf{X} + t\, e_L)\, dt\,,$$

where $X \in \Omega_+$ and $v = (P_{+0})^{-1}w \in L_p(\Sigma, \mathbb{C})$.

Further, $(\mathcal{C}_\Sigma^+ b(-i\,\mathbf{D}_\Sigma)v)_0(x + \delta e_L) = (T_{\Phi_\delta}v)_0(x)$ when $x \in \Sigma$ and $\delta > 0$, where $\Phi = \mathcal{G}_+(b\,\chi_+\,\overline{e_L}) \in M_N^+$. So the integrand can be expressed as

$$(\mathcal{C}_\Sigma^+ b(-i\,\mathbf{D}_\Sigma)v)_0(X) = \int_\Sigma \langle\overline{\Phi(X-y)}, n(y)\rangle\, v(y)\, dS_y\,,$$

where $X \in \Omega_+$.

We stress that the Fourier theory developed in Section 4 has been used to show that the assumption (#) implies that $\Phi \in M_N^+$, and hence that $T_\Phi \in \mathcal{L}(L_p(\Sigma, \mathbb{C}_{(M)}))$.

In our other papers involving H_∞ functional calculi, frequent use is made of a Convergence Lemma of the following nature. In particular it can be used to show that other reasonable definitions of $b(-i\,\mathbf{D}_\Sigma)$ lead to the same operator as ours. We are still supposing that $1 < p < \infty$.

Convergence Lemma. *Suppose* $0 < \mu \le \pi/2 - \omega$. *Let*

$$b_{(\alpha)} = b_{(\alpha)+} + b_{(\alpha)-}\,,$$

where $b_{(\alpha)+}$ *is a uniformly bounded net of functions in* $H_\infty^+(N_\mu(\mathbb{C}^m))$ *which converges to a function* $b_+ \in H_\infty^+(N_\mu(\mathbb{C}^m))$ *uniformly on every set of the form* $\{\zeta \in N_\mu(\mathbb{C}^m) : 0 < \delta \le |\zeta| \le \Delta < \infty\}$, *and where* $b_{(\alpha)-}$ *is a uniformly bounded net of functions in* $H_\infty^-(\overline{N}_\mu(\mathbb{C}^m))$ *which converges in a similar way to a function* $b_- \in H_\infty^-(\overline{N}_\mu(\mathbb{C}^m))$. *Let* $b = b_+ + b_-$. *Then* $b_{(\alpha)}(-i\,\mathbf{D}_\Sigma)u$ *converges to* $b(-i\,\mathbf{D}_\Sigma)u$ *for every* $u \in L_p(\Sigma)$, *and consequently* $\|b(-i\,\mathbf{D}_\Sigma)\| \le \sup_\alpha \|b_{(\alpha)}(-i\,\mathbf{D}_\Sigma)\|$.

PROOF. It is actually quite straightforward to use the definitions to show that $\Phi_{(\alpha)\pm} = \mathcal{G}_\pm(b_{(\alpha)\pm}\overline{e_L})$ converge in a similar way to $\Phi_\pm = \mathcal{G}_\pm(b_\pm\overline{e_L})$ and then that, for each $u \in L_p(\Sigma)$, $b_{(\alpha)}(-i\,\mathbf{D}_\Sigma)u = T_{\Phi_{(\alpha)+}}u + T_{\Phi_{(\alpha)-}}u$ converges to $T_{\Phi_+}u + T_{\Phi_-}u = b(-i\,\mathbf{D}_\Sigma)u$.

Here is a small corollary. Let us state it for functions defined on sets of the form $S^\circ_\mu(\mathbb{C}^m)$, rather than on the more general sets $N_\mu(\mathbb{C}^m)$ and $\overline{N}_\mu(\mathbb{C}^m)$).

Theorem 7.2. *Let b be a holomorphic function which satisfies $|b(\zeta)| \le c(1+|\zeta|^d)$ on $S^\circ_\mu(\mathbb{C}^m)$ for some $\mu \in (\omega, \pi/2)$, and d and $c \ge 0$. Assume that $b(\xi)\xi e_L = \xi e_L b(\xi)$ for all $\xi \in \mathbb{R}^m$, and suppose that $b(\zeta)$ has an inverse $b(\zeta)^{-1} \in \mathbb{C}_{(M)}$ for all $\zeta \in S^\circ_\mu(\mathbb{C}^m)$, and that there exists $s \ge 0$ such that*

$$|b(\zeta)^{-1}| \le c(|\zeta|^s + |\zeta|^{-s}), \qquad \zeta \in S^\circ_\mu(\mathbb{C}^m).$$

Then, the operator $b(-i\,\mathbf{D}_\Sigma)$ is one-one, and has dense range $\mathcal{R}(b(-i\,\mathbf{D}_\Sigma))$ in $L_p(\Sigma)$.

PROOF. Let $F_{(n)}(\lambda) = (n\lambda)^s(i+n\lambda)^{-s}(\chi_{\mathrm{Re}>0}(\lambda)e^{-\lambda/n} + \chi_{\mathrm{Re}<0}(\lambda)e^{\lambda/n})$, $\lambda \in S^\circ_\mu(\mathbb{C})$, $n = 1, 2, 3, \ldots$ Then the sequence $(F_{(n)})$ is uniformly bounded and converges uniformly to 1 on every set of the form $\{\lambda \in S^\circ_\mu(\mathbb{C}) : 0 < \delta \le |\lambda| \le \Delta < +\infty\}$. For each n, define $f_{(n)} \in H_\infty(S^\circ_\mu(\mathbb{C}^m))$ by

$$f_{(n)}(\zeta) = F_{(n)}(|\zeta|_{\mathbb{C}})\,\chi_+(\zeta) + F_{(n)}(-|\zeta|_{\mathbb{C}})\,\chi_-(\zeta)$$

as in Section 5. Then the sequence $(f_{(n)})$ is uniformly bounded and converges uniformly to 1 on every set of the form $\{z \in S^\circ_\mu(\mathbb{C}^m) : 0 < \delta \le |\zeta| \le \Delta < +\infty\}$. Let

$$g_{(n)} = f_{(n)}b^{-1} \in H_\infty(S^\circ_\mu(\mathbb{C}^m)) \quad \text{and} \quad h_{(n)} = b^{-1}f_{(n)} \in H_\infty(S^\circ_\mu(\mathbb{C}^m)),$$

so that $f_{(n)} = g_{(n)}b = b\,h_{(n)}$.

Suppose that $u \in \mathcal{D}(b(-i\,\mathbf{D}_\Sigma))$, and that $b(-i\,\mathbf{D}_\Sigma)u = 0$. By Theorem 7.1.vii), $f_{(n)}(-i\,\mathbf{D}_\Sigma)u = g_{(n)}(-i\,\mathbf{D}_\Sigma)b(-i\,\mathbf{D}_\Sigma)u = 0$, and, by the Convergence Lemma, $f_{(n)}(-i\,\mathbf{D}_\Sigma)u$ tends to u. So $u = 0$. We conclude that $b(-i\,\mathbf{D}_\Sigma)$ is a one-one operator.

Let $w \in L_p(\Sigma)$. Then $f_{(n)}(-i\,\mathbf{D}_\Sigma)w = b(-i\,\mathbf{D}_\Sigma)h_{(n)}(-i\,\mathbf{D}_\Sigma)w \in \mathcal{R}(b(-i\,\mathbf{D}_\Sigma))$, and

$$\lim_{n\to\infty} f_{(n)}(-i\,\mathbf{D}_\Sigma)w = w.$$

We conclude that $\mathcal{R}(b(-i\,\mathbf{D}_\Sigma))$ is dense in $L_p(\Sigma)$.

8. H_∞ functional calculi for functions of one variable.

Let us turn our attention to functions b which are associated with holomorphic functions of one variable. Recall that, for every holomorphic function B defined on $S^\circ_\mu(\mathbb{C})$, where $\omega < \mu \le \pi/2$, there is an associated function b defined on $S^\circ_\mu(\mathbb{C}^m)$ by

$$b(\zeta) = B(i\,\zeta e_L) = B(|\zeta|_{\mathbb{C}})\,\chi_+(\zeta) + B(-|\zeta|_{\mathbb{C}})\,\chi_-(\zeta)\,.$$

So it is natural to define the operator $B(\mathbf{D}_\Sigma e_L)$ by $B(\mathbf{D}_\Sigma e_L) = b(-i\,\mathbf{D}_\Sigma)$ whenever $b(-i\,\mathbf{D}_\Sigma)$ is itself defined.

It is a consequence of Theorems 2.2 and 7.1 that the mapping from $H_\infty(S^\circ_\mu(\mathbb{C}))$ to $\mathcal{L}(L_p(\Sigma))$ given by $B \mapsto B(\mathbf{D}_\Sigma e_L)$ is a bounded algebra homomorphism.

We remark that the condition $b(\zeta)\zeta e_L = \zeta e_L b(\zeta)$ which we often use, is automatically satisfied by functions b of the type $b(\zeta) = B(i\zeta e_L)$.

Let H_ω be the linear space of functions B on $\mathbb{R}\setminus\{0\}$ which have holomorphic extensions $B \in H_\infty(S^\circ_\mu(\mathbb{C}))$ for some $\mu > \omega$, and let $\mathcal{P}_\omega$ be the linear space of functions B on $\mathbb{R}\setminus\{0\}$ which extend holomorphically to $S^\circ_\mu(\mathbb{C})$ for some $\mu > \omega$ and satisfy $|B(\zeta)| \le c(1+|\zeta|^s)$ on $S^\circ_\mu(\mathbb{C})$ for some s and $c \ge 0$.

Theorem 8.1. *Suppose that* $1 < p < \infty$. *Let* $B \in \mathcal{P}_\omega$.

i) *The operator* $B(\mathbf{D}_\Sigma e_L)$ *is a closed linear operator in* $L_p(\Sigma)$ *with domain* $\mathcal{D}(B(\mathbf{D}_\Sigma e_L))$ *dense in* $L_p(\Sigma)$.

ii) *If* $B \in H_\omega$, *then*

$$B(\mathbf{D}_\Sigma e_L) = T_{(\Phi,\underline{\Phi})} \in \mathcal{L}(L_p(\Sigma))\,,$$

where $\mathcal{F}(\Phi,\underline{\Phi})e_L = b$ *and* $b(\xi) = B(i\,\xi e_L)$. *In particular,* $1(\mathbf{D}_\Sigma e_L) = I$, $\chi_{\mathrm{Re}>0}(\mathbf{D}_\Sigma e_L) = P_+$, $\chi_{\mathrm{Re}<0}(\mathbf{D}_\Sigma e_L) = P_-$ *and* $\mathrm{sgn}\,(\mathbf{D}_\Sigma e_L) = C_\Sigma$.

iii) *If* $B \in H_\infty(S^\circ_\mu(\mathbb{C}))$ *with* $\omega < \mu \le \pi/2$, *then*

$$\|B(\mathbf{D}_\Sigma e_L)u\|_p \le C_{\omega,\mu,p}\,\|B\|_\infty\,\|u\|_p\,, \qquad u \in L_p(\Sigma)\,,$$

for some constants $C_{\omega,\mu,p}$ *which depend only on* ω, μ, p *(and the dimension* m*).*

iv) *If* $u \in \mathcal{D}(B(\mathbf{D}_\Sigma e_L))$, $F \in \mathcal{P}_\omega$ *and* $c \in \mathbb{C}$, *then* $u \in \mathcal{D}(F(\mathbf{D}_\Sigma e_L))$ *if and only if* $u \in \mathcal{D}((cB+F)(\mathbf{D}_\Sigma e_L))$, *in which case*

$$c\,B(\mathbf{D}_\Sigma e_L)u + F(\mathbf{D}_\Sigma e_L)u = (c\,B + F)(\mathbf{D}_\Sigma e_L)u\,.$$

v) *If* $u \in \mathcal{D}(B(\mathbf{D}_\Sigma e_L))$ *and* $F \in \mathcal{P}_\omega$, *then* $B(\mathbf{D}_\Sigma e_L)u \in \mathcal{D}(F(\mathbf{D}_\Sigma e_L))$ *if and only if* $u \in \mathcal{D}((FB)(\mathbf{D}_\Sigma e_L))$, *in which case*

$$F(\mathbf{D}_\Sigma e_L)B(\mathbf{D}_\Sigma e_L)u = (FB)(\mathbf{D}_\Sigma e_L)u\,.$$

vi) *The complex spectrum* $\sigma(B(\mathbf{D}_\Sigma e_L))$ *is a subset of*

$$\bigcap\{(B(S^\circ_\mu(\mathbb{C}))^{\mathrm{cl}}\,:\ \mu > \omega\}\,.$$

Indeed

$$\|(B(\mathbf{D}_\Sigma e_L) - \alpha I)^{-1}u\|_p \le C_{\omega,\mu,p}\,\frac{\|u\|_p}{\operatorname{dist}\{\alpha, B(S^\circ_\mu(\mathbb{C}))\}}\,,$$

for all $u \in L_p(\Sigma)$.

vii) *Suppose that there exists* $\mu \in (\omega, \pi/2)$, $s \ge 0$ *and* $c > 0$ *such that*

$$|B(\lambda)| \ge c\,|\lambda|^s(1 + |\lambda|^{2s})^{-1}\,, \qquad \lambda \in S^\circ_\mu(\mathbb{C})\,.$$

Then, the operator $B(\mathbf{D}_\Sigma e_L)$ *is one-one, and has dense range* $\mathcal{R}(B(\mathbf{D}_\Sigma e_L))$ *in* $L_p(\Sigma)$.

PROOF. The first five parts are immediate corollaries of Theorem 7.1.

To prove vi), let α be a complex number such that

$$d = \operatorname{dist}\{\alpha, B(S^\circ_\mu(\mathbb{C}))\} > 0\,, \qquad \text{for some } \mu > \omega\,.$$

Then $F = (B - \alpha)^{-1} \in H_\infty(S^\circ_\mu(\mathbb{C}))$ and $\|F\|_\infty \le d^{-1}$, so, by ii) and iii), $F(\mathbf{D}_\Sigma e_L) \in \mathcal{L}(L_p(\Sigma))$ and $\|F(\mathbf{D}_\Sigma e_L)u\|_p \le C_{\omega,\mu,p}\, d^{-1}\|u\|_p$ for all $u \in L_p(\Sigma)$.

Therefore, by iv) and v), $(B(\mathbf{D}_\Sigma e_L) - \alpha I)F(\mathbf{D}_\Sigma e_L)u = u$ for all $u \in L_p(\Sigma)$, and $F(\mathbf{D}_\Sigma e_L)(B(\mathbf{D}_\Sigma e_L) - \alpha I)u = u$ for all $u \in D(B(\mathbf{D}_\Sigma e_L))$. Hence $(B(\mathbf{D}_\Sigma e_L) - \alpha I)^{-1} = F(\mathbf{D}_\Sigma e_L)$. The result follows.

Part vii) is a consequence of Theorem 7.2.

The closed linear operator $\mathbf{D}_\Sigma e_L$ is defined on $L_p(\Sigma)$ by $\mathbf{D}_\Sigma e_L = B(\mathbf{D}_\Sigma e_L)$ when $B(\lambda) = \lambda$. It is a consequence of part vi) above, that its spectrum $\sigma(\mathbf{D}_\Sigma e_L)$ is a subset of $S_\omega(\mathbb{C}) = S_{\omega+}(\mathbb{C}) \cup S_{\omega-}(\mathbb{C})$ where $S_{\omega\pm}(\mathbb{C}) = \{\lambda \in \mathbb{C} : \lambda = 0 \text{ or } |\arg(\pm\lambda)| \le \omega\}$. Further, for all $\mu > \omega$, there exists $c_{\omega,\mu,p}$ such that

$$\|(\mathbf{D}_\Sigma e_L - \alpha I)^{-1}u\|_p \le c_{\omega,\mu,p}\,|\alpha|^{-1}\|u\|_p\,,$$

for all $\alpha \notin S_\mu(\mathbb{C})$ and all $u \in L_p(\Sigma)$. That is, $\mathbf{D}_\Sigma e_L$ is of type ω in $L_p(\Sigma)$ (provided "type ω" is defined using the double sector $S_\omega(\mathbb{C})$ as in [McQ]). Indeed, on applying part vii) as well, we find that $\mathbf{D}_\Sigma e_L$ is a one-one operator of type ω in $L_p(\Sigma)$ with dense domain $\mathcal{D}(\mathbf{D}_\Sigma e_L)$ and dense range $\mathcal{R}(\mathbf{D}_\Sigma e_L)$ in $L_p(\Sigma)$.

We see that the restrictions of $\mathbf{D}_\Sigma e_L$ to $L_p^\pm(\Sigma)$ are closed linear operators in $L_p^\pm(\Sigma)$ with spectra in $S_{\omega\pm}(\mathbb{C})$, and indeed that $\mp\mathbf{D}_\Sigma e_L$ are the infinitesimal generators of the holomorphic C_0-semigroups $u \mapsto T_{k_{\pm\alpha}} u$, $\alpha > 0$, in $L_p^\pm(\Sigma)$.

The next theorem states that resolvents and polynomials of $\mathbf{D}_\Sigma e_L$ are equal to their counterparts $B(\mathbf{D}_\Sigma e_L)$. Thus it is reasonable to say that the mapping $B \mapsto B(\mathbf{D}_\Sigma e_L)$ is a functional calculus of the single operator $\mathbf{D}_\Sigma e_L$, as well as to say that the mapping $b \mapsto b(-i\mathbf{D}_\Sigma) = b(-iD_{1,\Sigma}, -iD_{2,\Sigma}, \dots, -iD_{m,\Sigma})$ defined in Section 7 is a functional calculus of the m commuting operators $-iD_{k,\Sigma}$, $k = 1, 2, \dots, m$. It also states that $\mathbf{D}_\Sigma e_L$ is, not surprisingly, precisely the operator considered previously by Murray [M] and McIntosh [McI] (when $L = 0$). See also [GM].

Theorem 8.2. *Suppose that* $1 < p < \infty$.

i) *If* $\alpha \notin S_\omega(\mathbb{C})$, *define* $R_\alpha(\lambda) = (\lambda - \alpha)^{-1}$, *in which case* $R_\alpha(i\zeta e_L) = (i\zeta e_L - \alpha)^{-1}$. *Then* $R_\alpha(\mathbf{D}_\Sigma e_L) = (\mathbf{D}_\Sigma e_L - \alpha I)^{-1} \in \mathcal{L}(L_p(\Sigma))$.

ii) *For* k *a positive integer, define* $S_k(\lambda) = \lambda^k$, *in which case* $S_k(i\zeta e_L) = (i\zeta e_L)^k$. *Then* $\mathcal{D}(S_k(\mathbf{D}_\Sigma e_L)) = \mathcal{D}((\mathbf{D}_\Sigma e_L)^k)$ *and* $S_k(\mathbf{D}_\Sigma e_L)u = (\mathbf{D}_\Sigma e_L)^k u$ *for all* $u \in \mathcal{D}((\mathbf{D}_\Sigma e_L)^k)$.

iii) *Given a complex valued polynomial* $P(\lambda) = \sum_{k=0}^{k=d} a_k \lambda^k$ *of one variable with* $a_d \neq 0$, *define*

$$P(\mathbf{D}_\Sigma e_L)u = \sum a_k (\mathbf{D}_\Sigma e_L)^k u\,, \quad u \in \mathcal{D}(P(\mathbf{D}_\Sigma e_L)) = \mathcal{D}((\mathbf{D}_\Sigma e_L)^d)\,.$$

Then $\mathcal{D}(P(\mathbf{D}_\Sigma e_L)) = \mathcal{D}((\mathbf{D}_\Sigma e_L)^d)$, *and* $P(\mathbf{D}_\Sigma e_L)u = P(\mathbf{D}_\Sigma e_L)u$ *for all* $u \in \mathcal{D}(\mathbf{D}_\Sigma e_L)$.

iv) *If* Σ *is parametrized by* $x = \mathbf{s} + g(\mathbf{s})\,e_L$, $s \in \mathbb{R}^m$, *then*

$$\mathcal{D}(\mathbf{D}_\Sigma e_L) = W_p^1(\Sigma) = \Big\{ u \in L_p(\Sigma) : \frac{\partial}{\partial s_j} u(\mathbf{s} + g(\mathbf{s})\,e_L) \in L_p(\mathbb{R}^m, ds), \\ j = 1, 2, \dots, m \Big\}$$

and

$$(\mathbf{D}_\Sigma e_L u)(\mathbf{s}+g(\mathbf{s})\,e_L) = (e_L - \mathbf{D}g)^{-1}\mathbf{D}_s u(\mathbf{s}+g(\mathbf{s})\,e_L)\,, \quad u \in W_p^1(\Sigma)\,.$$

PROOF. The proofs of parts i) to iii) require repeated use of parts iv) and v) of Theorem 8.1. (*cf.* the proof of Theorem 8.1.vi).

To prove iv), let, for the moment, $\mathbf{A}_\Sigma$ be the closed linear operator with domain $W_p^1(\Sigma)$ in $L_p(\Sigma)$ defined by

$$(\mathbf{A}_\Sigma u)(\mathbf{s}+g(\mathbf{s})\,e_L) = (e_L - \mathbf{D}g)^{-1}\mathbf{D}_s u(\mathbf{s}+g(\mathbf{s})\,e_L)\,,$$

for all $u \in W_p^1(\Sigma)$, and note that $\mathbf{A}_\Sigma - iI$ is one-one [McI] (actually, one can see directly that $\mathbf{A}_\Sigma$ is of type ω).

Given $u \in \mathcal{D}(\mathbf{D}_\Sigma e_L)$, write $u = u_+ + u_-$ where $u_\pm = P_\pm u$, and, for $\delta > 0$, let $u_{+\delta} = T_{k+\delta}u_+$. We saw in Section 6 that $u_{+\delta} \in \mathcal{D}(\mathbf{D}_\Sigma e_L)$, $u_{+\delta} \to u_+$ and $\mathbf{D}_\Sigma e_L u_{+\delta} \to \mathbf{D}_\Sigma e_L u_+$ as $\delta \to 0$. Also $u_{+\delta} \in W_p^1(\Sigma)$, and we saw in Section 6 that $\mathbf{D}_\Sigma e_L u_{+\delta} = \mathbf{A}_\Sigma u_{+\delta}$. The fact that the operator $\mathbf{A}_\Sigma$ is closed implies that $u_+ \in \mathcal{D}(\mathbf{A}_\Sigma)$ and that $\mathbf{D}_\Sigma e_L u_+ = \mathbf{A}_\Sigma u_+$. On treating u_- in a similar way, we find that $u \in \mathcal{D}(\mathbf{A}_\Sigma)$ and that $\mathbf{D}_\Sigma e_L u = \mathbf{A}_\Sigma u$. Using the facts that $(\mathbf{A}_\Sigma - iI)$ is one-one and that $(\mathbf{D}_\Sigma e_L - iI)$ maps onto $L_p(\Sigma)$, we see that $\mathcal{D}(\mathbf{A}_\Sigma)$ can be no larger than $\mathcal{D}(\mathbf{D}_\Sigma e_L)$, and thus conclude the proof.

It is also true that, for $B \in H_\omega$, and indeed for $B \in \mathcal{P}_\omega$, the operator $B(\mathbf{D}_\Sigma e_L)$ coincides with that obtained using the definitions of holomorphic functional calculi in [Mc], [McQ], [McY], [CDMcY]. This is derived from Theorem 8.2 by using the Convergence Lemma of Section 7, and the Convergence Lemmas of those papers. We shall not go into details, but just wish to draw attention to the fact that the boundedness of the algebra homomorphism $B \mapsto B(\mathbf{D}_\Sigma e_L)$ is equivalent to the fact that $\mathbf{D}_\Sigma$ satisfies square function estimates in $L_p(\Sigma)$.

One particular consequence in the case $p = 2$ is the square function estimate,

$$\|u\|_2 \le C\left(\int_0^{+\infty} \|\Psi_+(t\mathbf{D}_\Sigma e_L)u\|_2 \,\frac{dt}{t}\right)^{1/2}, \qquad u \in L_2^+(\Sigma)\,,$$

where $\Psi_+(\lambda) = \chi_{\mathrm{Re}>0}(\lambda)\lambda e^{\lambda}$, or in other words, letting $U = \mathcal{C}_\Sigma^+ u$ denote the left-monogenic extension of u to Ω_+,

$$\|u\|_2 \le C\Big(\iint_{\Omega_+} |(\mathbf{D}U)(X)|^2 \operatorname{dist}\{X,\Sigma\}\, dX\Big)^{1/2}$$

$$\le C\Big(\iint_{\Omega_+} \Big(\sum_{k=1}^m \Big|\frac{\partial U}{\partial X_k}(X)\Big|^2 + \Big|\frac{\partial U}{\partial X_L}(X)\Big|^2\Big) \operatorname{dist}\{X,\Sigma\}\, dX\Big)^{1/2},$$

where $u \in L_2^+(\Sigma)$. But this is the estimate proved in Theorem 4.1 of [LMcS], on which all our bounds are based. Now that we have traversed a full circle in [LMcS] and the current paper, it is time to stop.

ADDED IN PROOF. The reader may be interested in the lectures of Alan McIntosh on Clifford algebras, singular integrals, and harmonic functions on Lipschitz domains. These will appear as part of the Proceedings of the Conference on Clifford Algebras in Analysis held in Fayetteville, Arkansas, 1993, to be published by CRC Press.

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Recibido: 16 de abril de 1.992
Revisado: 18 de octubre de 1.993

Chun Li* and Alan McIntosh*
School of Mathematics, Physics
Computing and Electronics
Macquarie University
NSW 2109, AUSTRALIA

Tao Qian*
Department of Mathematics
University of New England
Armidale
NSW 2351, AUSTRALIA

* All three authors were supported by the Australian Government through the Australian Research Council.