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Homogeneous even kernels on surfaces

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0 Introduction

Let K be a kernel of the form

$$K(x)=\frac{P_k(x)}{|x|^{k+n}}\,,$$

where P_k is a homogeneous polynomial of degree k defined on $\mathbb{R}^{n+1}$. In [Se], [D1] and [D2] it was proved that if P_k is odd, then for an n-dimensional surface Σ in their classes, the operator T_Σ, defined on a nice class of functions on the surface by

$$T_\Sigma f(x)=\text{p.v.}\int_\Sigma K(x-y)f(y)\,d\sigma(y),\quad x\in\Sigma\,,\tag{2}$$

where $d\sigma(y)$ denotes the area measure on the surface, extends to a bounded operator on $L^2(\Sigma)$. It follows that, if $L(x)$ is odd, homogeneous of degree $-n$, and real-analytic away from 0 with a good enough radius of convergence on $\mathbb{R}^{n+1}\cup\{\infty\}$, then the associated operator defined as in (2) by using the kernel L gives rise to an L^2-bounded operator on the surface (see [Se]). Similar results on Lipschitz surfaces were established earlier in [CMcM] and [CDM], by using the rotation method, based on L^2-boundedness of the Cauchy integral of Calderón along Lipshitz curves. Notice that the rotation method can only produce odd kernels. It is natural to ask what happens when the kernels of the form (1) are even. When $\Sigma=\mathbb{R}^n$, it was proved that (see [St, SW], for example) a sufficient condition for $T_{\mathbb{R}^n}$ to be $L^2(\mathbb{R}^n)$ bounded is that:

$$\int_{S^{n-1}} P_k(x)\,dx=0\,,\tag{3}$$

where dx is the area measure on the $(n-1)$-unit sphere S^{n-1}. It is known that (3) is also a necessary condition for boundedness [J]. In this paper we will prove that if an even kernel gives rise to an L^2-bounded operator on every member of a set of hyperplanes of a certain variety of directions; or if it gives rise to an L^2-bounded

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operator on a surface with a set of tangent hyperplanes of a certain variety of directions, then the kernel must be identically zero. In our theorem, however, we will make a weaker assumption than L^2-boundedness of T_Σ in terms of a pointwise property of the associated maximal singular integral operator.

1 Statement of results

Let K be a kernel of the form (1). We assume throughout this paper that P_k is a non-trivial even function, which means that k is a positive even integer. Without loss of generality, we assume that the coefficients of P_k are real numbers and the test functions under consideration are real-valued. We also assume that P_k is not divisible by $|x|^2$.

To simplify our argument we assume that all the surfaces under consideration are n-dimensional orientable connected smooth surfaces each having a nowhere vanishing normal vector field. We will be talking about a set of surfaces together with the above-mentioned normals. For a given even kernel of the form (1) and a surface Σ as specified, it might not be possible to define a principal value singular integral as in (2). Nevertheless, we can always define the following maximal operator:

$$T^*_\Sigma f(x) = \sup_{\varepsilon>0} \Big| \int_{y\in\Sigma, |x-y|>\varepsilon} K(x-y) f(y)\, d\sigma(y) \Big|, \quad x\in\Sigma . \tag{4}$$

We will call an open ball of $\mathbb{R}^{n+1}$ a Σ-ball if the centre of the ball is in Σ. For a fixed kernel K of the form (1) and a set of surfaces Σ_i, $i\in I$, we define

$$\mathscr{A}(I) = \{x\in\mathbb{R}^{n+1} : \exists\, i=i(x)\in I,\ \exists\, \Sigma_i-\text{ball } B \text{ such that } x\in\Sigma_i\cap B,$$
$$\text{and } T^*_{\Sigma_i}\chi_B(x)<\infty\}$$

and

$$\mathscr{N}(I) = \{n\in\mathbf{S}^n : \exists\, x\in\mathscr{A}(I) \text{ such that } n=n_x, \text{ the normal to } \Sigma_{i(x)} \text{ at } x\} .$$

In this way, every x in $\mathscr{A}(I)$ corresponds to some points on the n-dimensional unit sphere $\mathbf{S}^n$. If I contains only a single point, as we will consider in Corollary 2, we denote the corresponding objects by $\mathscr{A}$ and $\mathscr{N}$, respectively.

We call N a **k-lattice array** on $\mathbf{S}^n$, if there are $n+1$ sets $N_1, \ldots, N_{n+1}$, each consisting of $k+1$ distinct real numbers, and N consists of the projection points onto S^n of the Cartesian product $N_1\times\cdots\times N_{n+1}\subset\mathbb{R}^{n+1}$.

If P is a polynomial, the notation $P=0$ means that all the coefficients of P are zero. There are different conditions on the zero set of a polynomial that guarantee the polynomial to be zero (see Lemma 3 below, for example).

Now we are ready to state our theorem.

Theorem 1 *Let K be a kernel of the form* (1) *and Σ_i, $i\in I$, be a set of surfaces. If the set $\mathscr{N}(I)$ contains a k-lattice array or a measurable set of positive measure on $\mathbf{S}^n$, then* $K=0$.

The theorem is stated in terms of the maximal singular integral operator. To apply it to the associated singular integral operators, in case they exist, we need the following notion.

We will call an n-dimensional smooth surface Σ an **admissible surface with respect to the kernel** K, if there is an operator T_Σ satisfying the following conditions:

(i) T_Σ maps $L_c^\infty(\Sigma)$, the space of L^∞ functions with compact supports, into $L^1{}_{\text{loc}}(\Sigma)$;
(ii) T_Σ extends to a bounded linear operator on $L^2(\Sigma)$;
(iii) For every $f \in L_c^\infty(\Sigma)$,

$$T_\Sigma f(x) = \int_\Sigma K(x-y) f(y)\, d\sigma(y), \quad \text{for } x \in \{\operatorname{supp} f\}^c,$$

where $L^p(\Sigma)$ are the L^p spaces of functions defined on Σ with respect to the area measure $d\sigma(y)$. If an admissible surface happens to be a graph, then in terms of the parametrization by $\mathbb{R}^n$, T_Σ is just a so called Calderón–Zygmund operator with standard kernel K (see [J]).

References [Se], [D1] and [D2] gave two kinds of admissible surfaces with respect to odd kernels, each consisting of a wide class of surfaces and neither containing the other. In both cases it was assumed that the area of the surface inside an n-sphere centred at a point on the surface is equivalent to the area of the sphere. A surface Σ of Semmes' type is smooth and $\mathbb{R}^{n+1} \backslash \Sigma$ satisfies a property similar to Whitney's decomposition in the standard case (see [Se]); and a surface of David's type is assumed to have a parametrization satisfying certain estimates (see [D2]).

Remark. If Σ is admissible, then the associated singular integral operator is locally L^2-bounded. From the standard Calderón–Zygmund operator theory, the maximal operator $T_\Sigma^*$is locally L^2-bounded, and so for any Σ-ball B,

$$T_\Sigma^* \chi_B(x) < \infty \quad \text{a.e.} \quad x \in \Sigma \cap B\,.$$

It follows that $x \in \mathscr{A}(I)$ for a.e. $x \in \Sigma$.

We will sometimes denote, for $x \in \mathbb{R}^{n+1}$, $x = (\mathbf{x}, x_{n+1})$, where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

The theorem has the following easy corollaries.

Corollary 1 *Let $\mathbb{H}_i$, $i \in I$, be a set of admissible hyperplanes with respect to an even kernel of the form* (1). *If the set of the normals of these hyperplanes contains a k-lattice array or a measurable set of positive measure on $\mathbf{S}^n$, then $K = 0$. In particular, let ε be an arbitrary positive number. If every hyperplane in the sector $\mathbb{S}_\varepsilon = \{x = (\mathbf{x}, x_{n+1}) \in \mathbb{R}^{n+1} : |x_{n+1}| \leqq |\mathbf{x}| \tan \varepsilon\}$ is admissible with respect to the kernel K, then $K = 0$.*

The following corollary is related to only one surface.

Corollary 2 *Let K be a kernel of the form* (1) *and Σ be an admissible surface with respect to the kernel K. If the set of the normals of Σ contains a k-lattice array or a measurable set of positive measure on $\mathbf{S}^n$, then $K = 0$.*

2 Proofs of the theorem and the corollaries

Proof of the theorem. First, let us fix a normal $n \in \mathscr{N}(I)$. Let $x \in \mathscr{A}(I)$ be such that $n = n_x$. From the definition of $\mathscr{A}(I)$, there is an index $i \in I$, and a Σ_i-ball B such that

$x \in \Sigma_i \cap B$ and

$$T^*_{\Sigma_i} \chi_B(x) = \sup_{\varepsilon > 0} \left| \int_{y \in \Sigma_i \cap B, |x-y| > \varepsilon} K(x-y)\, d\sigma(y) \right| < \infty .$$

To simplify our notation we temporarily suppress the subscript i.

We start with the following lemma.

Lemma 1 *If* $\mathbf{T}_x$ *is the tangent hyperplane of* Σ *at* x, *then*

$$\int_{\mathbf{S}^n \cap (\mathbf{T}_x - x)} P_k(x)\, dx = 0 , \tag{5}$$

where $\mathbf{T}_x - x = \{ y - x : y \in \mathbf{T}_x \}$, *the hyperplane that passes through the origin and has the same direction as* T_x *does, and* dx *denotes the normalized* $(n-1)$*-dimensional spherical area measure on* $\mathbf{S}^n \cap (\mathbf{T}_x - x)$.

Proof. Since there is a local parametrization of Σ at x, there is a Σ-ball B_1 centred at x and contained in B such that inside B_1 both Σ and $\mathbf{T}_x$ can be parametrized by, say, the first n variables. Write a typical element x of $\mathbb{R}^{n+1}$ as $x = (\mathbf{x}, x_{n+1})$. Denote by σ and t the parametrizations of Σ and T, respectively. So, for example, the piece of Σ inside the ball B_1 consists of the points $\{(\mathbf{y}, \sigma(\mathbf{y}))\}$, where $\mathbf{y}$ varies in the projection ball $\mathbf{B}_1$ of B_1 onto $\mathbb{R}^n$.

For every $\varepsilon > 0$,

$$T^\varepsilon_\Sigma \chi_B(x) = \int_{y \in \Sigma \cap B_1, |x-y| > \varepsilon} K(x-y)\, d\sigma(y) + \int_{y \in \Sigma \cap (B \setminus B_1), |x-y| > \varepsilon} K(x-y)\, d\sigma(y) .$$

Notice that the second integral of the above sum is uniformly bounded with respect to ε, and, therefore, so is the first integral.

Using the parametrization, the last mentioned fact can be written as

$$\left| \int_{\{\mathbf{y} : \delta > |\mathbf{y} - \mathbf{x}| > \varepsilon\}} \frac{P_k((\mathbf{x}, \sigma(\mathbf{x})) - (\mathbf{y}, \sigma(\mathbf{y})))}{|(\mathbf{x}, \sigma(\mathbf{x})) - (\mathbf{y}, \sigma(\mathbf{y}))|^{k+n}} \sqrt{1 + |D\sigma(\mathbf{y})|^2}\, d\mathbf{y} \right| < C ,$$

where C is a constant independent of ε, δ is the radius of $\mathbf{B}_1$, $D = (D_1, \ldots, D_n)$ and

$$D_j = \frac{d}{dx_j}, \quad j = 1, 2, \ldots, n+1 .$$

Consider the corresponding integral on the tangent hyperplane:

$$\int_{\{\mathbf{y} : \delta > |\mathbf{y} - \mathbf{x}| > \varepsilon\}} \frac{P_k((\mathbf{x}, t(\mathbf{x})) - (\mathbf{y}, t(\mathbf{y})))}{|(\mathbf{x}, t(\mathbf{x})) - (\mathbf{y}, t(\mathbf{y}))|^{k+n}} \sqrt{1 + |Dt(\mathbf{y})|^2}\, d\mathbf{y} . \tag{6}$$

The difference between the last two integrals is dominated by

$$\left| \int_{\{\mathbf{y} : \delta > |\mathbf{y} - \mathbf{x}| > \varepsilon\}} \frac{P_k((\mathbf{x}, \sigma(\mathbf{x})) - (\mathbf{y}, \sigma(\mathbf{y})))}{|(\mathbf{x}, \sigma(\mathbf{x})) - (\mathbf{y}, \sigma(\mathbf{y}))|^{k+n}} (\sqrt{1 + |D\sigma(\mathbf{y})|^2} - \sqrt{1 + |D\sigma(\mathbf{x})|^2}\, d\mathbf{y} \right|$$

$$+ \left| \int_{\{\mathbf{y} : \delta > |\mathbf{y} - \mathbf{x}| > \varepsilon\}} \left(\frac{P_k((\mathbf{x}, \sigma(\mathbf{x})) - (\mathbf{y}, \sigma(\mathbf{y})))}{|(\mathbf{x}, \sigma(\mathbf{x})) - (\mathbf{y}, \sigma(\mathbf{y}))|^{k+n}} - \frac{P_k((\mathbf{x}, t(\mathbf{x})) - (\mathbf{y}, t(\mathbf{y})))}{|(\mathbf{x}, t(\mathbf{x})) - (\mathbf{y}, t(\mathbf{y}))|^{k+n}} \right) \cdot \sqrt{1 + |Dt(\mathbf{y})|^2}\, d\mathbf{y} \right| .$$

By using the mean value theorem, the second term of the above sum is dominated by

$$C\left|\int_{\{\mathbf{y}:\delta>|\mathbf{y}-\mathbf{x}|>\varepsilon\}}\left(\frac{P_k((\mathbf{x},\sigma(\mathbf{x}))-(\mathbf{y},\sigma(\mathbf{y})))}{|(\mathbf{x},\sigma(\mathbf{x}))-(\mathbf{y},\sigma(\mathbf{y}))|^{k+n}}-\frac{P_k((\mathbf{x},t(\mathbf{x}))-(\mathbf{y},t(\mathbf{y})))}{|(\mathbf{x},t(\mathbf{x}))-(\mathbf{y},t(\mathbf{y}))|^{k+n}}\right)d\mathbf{y}\right|$$

$$\leqq C\int_{\{\mathbf{y}:\delta>|\mathbf{y}-\mathbf{x}|>\varepsilon\}}\frac{1}{|\mathbf{x}-\mathbf{y}|^{n+1}}|(\sigma(\mathbf{x})-\sigma(\mathbf{y}))-t(\mathbf{x})-t(\mathbf{y}))|\,d\mathbf{y}$$

$$\leqq C\int_{\{\mathbf{y}:\delta>|\mathbf{y}-\mathbf{x}|>\varepsilon\}}\frac{1}{|\mathbf{x}-\mathbf{y}|^{n-1}}\,d\mathbf{y}$$

$$\leqq C\,,$$

where the last constant C depends on δ, and we used the fact that the surface is smooth.

The first term can be estimated similarly, and is also dominated by a constant.

Therefore, the integral in (6) is uniformly bounded with respect to ε. Since the mapping t is linear, we have

$$\sup_{\varepsilon}\left|\int_{\{\mathbf{y}:\delta>|\mathbf{y}|>\varepsilon,\,\mathbf{y}\in(\mathbf{T}_x-x)\}}K(x)\,dx\right|<C\,.$$

On letting $\varepsilon\to 0$, since K is homogeneous, we have that

$$\int_{\mathbf{S}^n\cap(\mathbf{T}_x-x)}P_k(x)\,dx=0\,.$$

This concludes the proof of Lemma 1.

Now let n vary in $\mathcal{N}(I)$. According to our assumption, the set $\mathcal{N}(I)$ contains a k-lattice array or a measurable set of positive measure on $\mathbf{S}^n$. Denote the k-lattice array or the measurable set of positive measure on $\mathbf{S}^n$ by $\{n_\lambda:\lambda\in\Lambda\}$, the set of the associated tangent hyperplanes by $\{\mathbf{T}_\lambda:\lambda\in\Lambda\}$, and the set of the tangent points by $\{x_\lambda:\lambda\in\Lambda\}$. We have that

$$\int_{\mathbf{S}^n\cap(\mathbf{T}_\lambda-x_\lambda)}P_k(x)\,dx=0,\quad\lambda\in\Lambda\,,\tag{7}$$

and we have to prove that (7) implies that $P_k=0$.

By changing variables, the above can be written as

$$\int_{\mathbf{S}^{n-1}}P_k(A_\lambda\mathbf{x})\,d\mathbf{x}=0,\quad\lambda\in\Lambda\,,\tag{8}$$

where A_λ are rotations such that $A_\lambda e_{n+1}=n_\lambda$ and $d\mathbf{x}$ denotes the normalized spherical area measure on $\mathbf{S}^{n-1}$. To simplify the notation, we suppress the subscript λ for a moment and write $A=A_\lambda=(a_{jl})$, an $(n+1)\times(n+1)$ orthonormal matrix. We therefore have $Ae_{n+1}=n_\lambda=n=(a_{j(n+1)})^t_{j=1,\dots,n+1}$, and $A\mathbf{x}=(\sum_{l=1}^n a_{jl}x_l)^t_{j=1,\dots,n+1}$.

Lemma 2 *If Q is a homogeneous function of degree k on $\mathbb{R}^n$ which satisfies*

$$\int_{\mathbf{S}^{n-1}}Q(\mathbf{x})\,d\mathbf{x}=0\,,$$

then

$$\Delta_n^{k/2}Q=0\,,$$

where $\Delta_n=\sum_{j=1}^n D_j^2$, the Laplacian in the first n variables.

Proof. If ρ is the outer unit normal on $\mathbf{S}^{n-1}$, then

$$\frac{\partial Q}{\partial \rho} = kQ .$$

Using Green's formula, we have

$$\begin{aligned} 0 &= k \int_{\mathbf{S}^{n-1}} Q(\mathbf{x})\, d\mathbf{x} \\ &= \int_{\mathbf{S}^{n-1}} \frac{\partial Q}{\partial \rho}(\mathbf{x})\, d\mathbf{x} \\ &= \int_{x \in \mathbb{R}^n,\, |\mathbf{x}| \leqq 1} \Delta_n Q(\mathbf{x})\, d\mathbf{x} . \end{aligned}$$

Since $\Delta_n Q$ is homogeneous of degree $k-2$, the above reduces to

$$0 = \int_0^1 r^{n+k-3}\, dr \cdot \int_{\mathbf{S}^{n-1}} \Delta_n Q(\mathbf{x})\, d\mathbf{x} ,$$

and, so,

$$\int_{\mathbf{S}^{n-1}} \Delta_n Q(\mathbf{x})\, d\mathbf{x} = 0 .$$

Repeat the above argument $k/2$ times to deduce that

$$\int_{\mathbf{S}^{n-1}} \Delta_n^{k/2} Q(\mathbf{x})\, d\mathbf{x} = 0 .$$

Since Q is a polynomial of degree k, the above integrand is a constant. We therefore conclude the desired result (also see Lemma 4 below).

Now let $Q(\mathbf{x}) = P_k(A\mathbf{x})$. Using (8) and Lemma 2, we obtain

$$\Delta_n^{k/2}(P_k(A\mathbf{x})) = 0 . \tag{9}$$

So our task has reduced to proving that if (9) holds for all $A = A_\lambda$, $\lambda \in \Lambda$, then $P_k = 0$.

A direct calculation gives

$$\Delta_n(P_k(A\mathbf{x})) = \left(\sum_{l=1}^{n} \left(\sum_{j=1}^{n+1} a_{jl} D_j \right)^2 P_k \right)(A\mathbf{x}) . \tag{10}$$

Using the relations

$$1 - a_{j(n+1)}^2 = \sum_{l=1}^{n} a_{jl}^2 ,$$

$$-a_{j(n+1)} a_{h(n+1)} = \sum_{l=1}^{n} a_{jl} a_{hl} ,$$

where $j, h = 1, \ldots, n+1$, and $j \neq h$, (10) can be written as

$$\Delta_n(P_k(A\mathbf{x})) = \left(\Delta_{n+1} - \left(\sum_{j=1}^{n+1} a_{j(n+1)} D_j \right)^2 P_k \right)(A\mathbf{x}) ,$$

where $\Delta_{n+1} = \sum_{j=1}^{n+1} D_j^2$, the Laplacian in $n+1$ variables.

It follows that (9) can be rewritten as

$$0=\Delta_n^{k/2}\left(P_k(A\mathbf{x})\right)$$
$$=\left(\Delta_{n+1}-\left(\sum_{j=1}^{n+1} a_{j(n+1)}D_j\right)^2\right)^{\frac{k}{2}}P_k$$
$$=\left(-\sum_{j=1}^{n+1} a_{j(n+1)}D_j\right)^k P_k+\sum_{i=1}^{\frac{k}{2}}\binom{\frac{k}{2}}{i}\Delta_{n+1}^i\left(-\sum_{j=1}^{n+1} a_{j(n+1)}D_j\right)^{k-2i} P_k(A\mathbf{x})\,.$$

So,

$$\left(\sum_{j=1}^{n+1} a_{j(n+1)}D_j\right)^k P_k=(-1)^{k+1}\left(\sum_{i=1}^{\frac{k}{2}}\binom{\frac{k}{2}}{i}\Delta_{n+1}^i\left(-\sum_{j=1}^{n+1} a_{j(n+1)}D_j\right)^{k-2i} P_k\right),$$

which is an equality between P_k's coefficients.

Taylor's expansion and the above equality give

$$P_k(Ae_{n+1})=P_k((a_{j(n+1)})^t_{j=1,\ldots,n+1})$$
$$=\frac{1}{k!}\left(\sum_{j=1}^{n+1} a_{j(n+1)}D_j\right)^k P_k(0)$$
$$=\frac{1}{k!}(-)^{k+1}\left(\sum_{i=1}^{\frac{k}{2}}\binom{\frac{k}{2}}{i}\Delta_{n+1}^i\left(-\sum_{j=1}^{n+1} a_{j(n+1)}D_j\right)^{k-2i} P_k\right)(0)\,.$$

Now consider the polynomial

$$Q_1(x)=P_k(x)-|x|^2\frac{1}{k!}(-1)^{k+1}$$
$$\times\left(\sum_{i=1}^{\frac{k}{2}}\binom{\frac{k}{2}}{i}|x|^{k-2}\Delta_{n+1}^i\left(-\sum_{j=1}^{n+1}\frac{x_j}{|x|}D_j\right)^{k-2j} P_k(0)\right),$$

where we treated x_j as constants when we applied the differential operators D_j. Q_1 is therefore a homogeneous polynomial of degree k on $\mathbb{R}^{n+1}$ and satisfies

$$Q_1(Ae_{n+1})=0\,. \tag{11}$$

Equation (11) holds in fact for all $A_\lambda e_{n+1}=n_{x_\lambda}$, $\lambda\in\Lambda$. By using the following lemma we can conclude that $Q_1=0$, and therefore P_k is divisible by $|x|^2$. This is a contradiction if $P_k\neq 0$.

Lemma 3 *Assume that Q is a homogeneous polynomial of degree k defined on $\mathbb{R}^{n+1}$ with real coefficients. If it vanishes on a k-lattice array, or on a measurable set of positive measure on $\mathbf{S}^n$, then $Q=0$.*

Proof. Let us first deal with the case in which Q vanishes on a k-lattice array. Owing to the homogeneity of Q, the assertion is a consequence of the following algebraic proposition stated in [Y]:

Proposition *Let $R(u_1, u_2, \ldots, u_m)$ be a polynomial in the variables $u_1, u_2, \ldots, u_m$ with real coefficients. Suppose that the degree of R in u_j is not more than n_j,*

$j=1, 2, \ldots, m$. If there is an m-Cartesian product:

$$S=S_1 \times S_2 \times \cdots \times S_m ,$$

where S_j contains n_j+1 elements, and R vanishes on S, then $R=0$.

The proposition is a consequence of the fundamental theorem of algebra and induction.

Now we deal with the second case in which Q vanishes on a measurable set of positive measure on $\mathbf{S}^n$. If $Q \neq 0$, consider the zero set Z of Q: $Z=\{x \in \mathbb{R}^{n+1} : Q(x)=0\}$. We will prove that $m(Z)=0$, where m denotes the $(n+1)$-dimensional Lebesgue measure. It will then follow that the zero set on the unit sphere $\mathbf{S}^n$ is a null set with respect to the n-dimensional spherical area measure, which is a contradiction; and therefore $Q=0$. That $m(Z)=0$ is a consequence of the following more general

Lemma 4 *Let R be a non-trivial polynomial defined on $\mathbb{R}^{n+1}$ with real coefficients. If $Z=\{x \in \mathbb{R}^{n+1} : R(x)=0\}$, the zero set of R, then $m(Z)=0$.*

Proof. We use induction. When $n=0$, the assertion follows from the fundamental theorem of algebra, and the fact that finite sets are null. For general n, we have

$$m(Z)=\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} \chi_Z(\mathbf{x}, x_{n+1})\, d\mathbf{x} \right) dx_{n+1} .$$

For every fixed x_{n+1}, $Z_{x_{n+1}}=\{\mathbf{x} \in \mathbb{R}^n : (\mathbf{x}, x_{n+1}) \in Z\}$ is the zero set of the polynomial $Q(\cdot, x_{n+1})$ in n variables. By the induction hypothesis, the n-dimensional measure of $Z_{x_{n+1}}$ equals 0, i.e.

$$\int_{\mathbb{R}^n} \chi_Z(\mathbf{x}, x_{n+1})\, d\mathbf{x}=0 ,$$

and so by using the Fubini–Tonelli theorem, $m(Z)=0$. This concludes the proof of the lemma, and so that of Theorem 1.

Corollary 1 is straightforward; and we will prove only Corollary 2. From the remark about admissible surfaces in Sect. 1, it follows that $\mathscr{A}$ is dense in Σ with respect to the induced topology on Σ.

Since

$$\int_{\mathbf{S}^n \cap (\mathbf{T}_x - x)} P_k(y)\, dy \tag{12}$$

is continuous in $x \in \Sigma$ and vanishes on $\mathscr{A}$ (see Lemma 1), (12) too vanishes on Σ. Now we are in the position as in (8), and the same reasoning gives $P_k=0$.

Note added in proof. We are grateful to S. Semmes for pointing out that, if the even nonzero kernel K is not homogeneous, it may still give rise to a bounded operator on L^2, provided it satisfies supplementary conditions, for instance boundedness of truncated integrals on all k-hyperplanes. For a comprehensive account, see Part III of G. David, Wavelet and singular integrals on curves and surfaces. Lecture Notes Math.. Springer-Verlag, vol. 1465 (1991); also, the article by R. R. Coifman and S. Semmes, L^2 estimates in nonlinear Fourier analysis. Proceedings of Harmonic Analysis, Sendai. Lecture Notes Math.. Springer-Verlag, 79–95 (1990).

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