The Schwarzian Derivative in $I\!R^n$ [†]

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Abstract. In this note Schwarzian derivative in $I\!R^n$ is defined and its properties are investigated. Relationship between Möbius transformation and linear fractional transformation is established in terms of Schwarzian derivative. An injective criterion is obtained.

Keywords: Schwarzian derivative, linear fractional transformation, Möbius transformation, immersion, embedding

MSC 2000: 32Axx, 32Hxx

1. Introduction

In analysis of one complex variable, the Schwarzian derivative of holomorphic functions is

$$Sf(z) = \frac{f^{'''}(z)}{f^{'}(z)} - \frac{3}{2}\left(\frac{f^{''}(z)}{f^{'}(z)}\right)^2.$$

The most important result in relation to Schwarzian derivative is that if a function f is of the form $f(z) = \dfrac{az+b}{cz+d}$, then $Sf(z) = 0$, and vice versa (see [5], for instance). If, in particular, $ad - bc = 1$, then the above linear fractional transformation becomes a Möbius Transformation. All Möbius transformations constitute a group, namely the Möbius group. The generators of the Möbius group are translations, reflexions with respect to the unit circle, dilations and rotations. Based on this point of view Ahlfors obtained the Möbius transformations in $I\!R^n$. Möbius transformation, therefore, is a natural starting point for generalizing Schwarzian derivative to different contexts ([3,7,8]).

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On the other hand, in $\mathbb{R}^2 = \mathbb{C}$, every holomorphic function is a conformal mapping. One therefore can also depart from conformal mapping by studying the order 3 approximation to conformal mappings by Möbius transformations, and hence obtain Schwarzian derivatives of conformal mappings between Riemannian manifolds ([3,6]).
In higher dimensional spaces holomorphic mappings are no longer conformal. Study of Schwarzian derivatives of holomorphic mappings in this case hence becomes a new direction ([5]). In the study of Gong, by treating Schwarzian derivatives as the infinite small form of cross ratio of four points, many significant results were obtained ([5]).

Let $\{e_1, e_2, \cdots\cdots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$. The real *Clifford algebra* generated by $e_1, e_2, \cdots\cdots, e_n$ is the 2^n-dimensional associative algebra, $\mathcal{A}_n$, whose multiplication satisfies

$$e_i e_j + e_j e_i = -2\delta_{ij}.$$

Denote by $e_{i_1 i_2 \cdots i_r}$ the product $e_{i_1} e_{i_2} \cdots e_{i_r}$, $1 \le i_1 < i_2 < \cdots i_r \le n$. If $x \in \mathcal{A}_n$ then

$$x = \sum_{A \subseteq \{1,2,\cdots,n\}} x^A e_A,$$

where A runs over all subsets of $\{1, 2, \cdots, n\}$. According to different cardinality of subsets A, $\mathcal{A}_n$ can be expressed as direct sum of $n+1$ subspaces:

$$\mathcal{A}_n = \mathcal{A}_n^0 \oplus \mathcal{A}_n^1 \oplus \cdots \oplus \mathcal{A}_n^n,$$

where $\mathcal{A}_n^0$ is isomorphic to $\mathbb{R}$, whose elements are scalars, $\mathcal{A}_n^1$ is isomorphic to $\mathbb{R}^n$ whose elements are vectors. The elements of $\mathcal{A}_n^k$, $k \ge 2$, are called k-vectors. If $x \in \mathcal{A}_n$, then x can be decomposed as

$$x = [x]_0 + [x]_1 + \cdots + [x]_n.$$

The elements of $\mathcal{A}_n$ are called *Clifford numbers*. The *Clifford group* of $\mathcal{A}_n$ is the Lie group, denoted by Γ_n, consisting of all finite products of vectors in $\mathbb{R}^n$. The subset of Γ_n consisting of products of an even number of vectors forms a subgroup of Γ_n, denoted by Γ_n^+. The complement set of Γ_n^+ in Γ_n is denoted by Γ_n^-. We introduce the following operations:

$$(e_{i_1} e_{i_2} \cdots e_{i_r})^* = e_{i_r} e_{i_{r-1}} \cdots e_{i_2} e_{i_1}$$

and

$$(e_{i_1} e_{i_2} \cdots e_{i_r})' = (e_{i_1})' (e_{i_2})' \cdots (e_{i_r})',$$

where $(e_0)' = 1' = 1$, $(e_i)' = -e_i$, $i \ne 0$. The operations $*$ and $'$ are extended to $\mathcal{A}_n$ by linearity and we define the third operation$^-$ by

$$\overline{x} = (x')^* = (x^*)'.$$

Note that for $v \in I\!R^n$, we have $v^* = v$ and $\bar{v} = -v$. For general $a, b \in \mathcal{A}_n$, we have $(ab)^* = b^* a^*$ and $\overline{ab} = \bar{b}\bar{a}$. If $x \in \mathcal{A}_n$, $x = \sum\limits_{A \subset \{1,2,\cdots,n\}} x^A e_A$, then the *norm* of x is defined to be

$$\|x\| = (\sum_{A \subset \{1,2,\cdots,n\}} |x^A|^2)^{1/2}.$$

When $v \in I\!R^n$, we have $\|v\|^2 = v\bar{v} = \bar{v}v$. For $a \in \mathcal{A}_n$, $\|a\|^2 = [a\bar{a}]_0 = [\bar{a}a]_0$. If $v, w \in I\!R^n$, then the inner product

$$< v, w >= \frac{1}{2}(vw + wv) \in I\!R = \mathcal{A}_n^0.$$

The outer product of v, w is defined to be

$$v \wedge w = \frac{1}{2}(vw - wv) \in \mathcal{A}_n^2.$$

Inner product is symmetric, and $< v, w >= 0$ if and only if v and w are orthogonal. Outer product is anti-symmetric, and $v \wedge w = 0$ if and only if v and w are parallel.

2. Criterion of Linear Fractional Transformation

Let $\Omega \subseteq I\!R^n, f : \Omega \to \mathcal{A}_n, \lambda \in I\!R^n, \lambda = \sum_i \lambda^i e_i$.

Definition. The *directional derivative* of f in the direction of λ is

$$D_\lambda f(x) = \lim_{t \to 0} \frac{f(x + t\lambda) - f(x)}{t} = \sum_i \lambda^i \frac{\partial f}{\partial x^i}.$$

We shall prove the following theorem.

Theorem 1. Let $0 \in \Omega \subseteq I\!R^n$, $n \geq 2$, where Ω is a starlike domain around the origin, and $f : \Omega \to I\!R^n$ be an immersion. Then a sufficient and necessary condition for f being a linear fractional transformation is

$$D_\lambda f \wedge D_\lambda^2 f \equiv 0, \qquad \text{for any } \lambda \in I\!R^n.$$

Proof. i) Necessity. We can get the proof by direct computation.
ii) Sufficiency. Without loss of generality, we assume that $f(0) = 0$, $\frac{\partial f^\alpha}{\partial x^i}(0) = \delta_i^\alpha$. Let $x_0 \in \Omega$ be fixed. Set $y : [0,1] \to I\!R^n, y(t) = f(tx_0)$, then

$$\dot{y} = \frac{dy}{dt} = D_{x_0} f(tx_0), \qquad \ddot{y} = \frac{d^2 y}{dt^2} = D_{x_0}^2 f(tx_0).$$

Owing to $D_{x_0}f(tx_0) \wedge D^2_{x_0}f(tx_0) = 0$, we have

$$D^2_{x_0}f(tx_0) = aD_{x_0}f(tx_0), \qquad a \in I\!R,$$

or

$$\ddot{y}(t) = a\dot{y}(t) \quad \text{and} \quad \ddot{y}^\alpha(t) = a\dot{y}^\alpha(t).$$

Therefore, we have

$$a = \frac{\ddot{y}^\alpha}{\dot{y}^\alpha} = \frac{\ddot{y}^\beta}{\dot{y}^\beta}, \qquad \alpha, \beta = 1, 2, \cdots\cdots, n.$$

Integrating from 0 to t, we have

$$\frac{\dot{y}^\alpha(t)}{\dot{y}^\alpha(0)} = \frac{\dot{y}^\beta(t)}{\dot{y}^\beta(0)}. \tag{3}$$

Since $\dot{y}^\alpha(0) = x_0^i \dfrac{\partial f^\alpha}{\partial x^i}(0) = x_0^i \delta_i^\alpha = x_0^\alpha$, (3) may be written as

$$\frac{\dot{y}^\alpha(t)}{x_0^\alpha} = \frac{\dot{y}^\beta(t)}{x_0^\beta}.$$

Integrating above equality, we obtain $\dfrac{y^\alpha(t)}{x_0^\alpha} = \dfrac{y^\beta(t)}{x_0^\beta}$. For $t = 1$, we have

$$\frac{f^\alpha(x_0)}{x_0^\alpha} = \frac{f^\beta(x_0)}{x_0^\beta}. \tag{4}$$

(4) holds for all $x \in \Omega$. Hence there exists a function $z : I\!R^n \to I\!R$ such that

$$\frac{f^\alpha(x)}{x^\alpha} = z(x), \qquad \text{for any} \quad \alpha = 1, 2, \cdots, n. \tag{5}$$

Hence,

$$\frac{\partial f^\alpha}{\partial x^i} = \delta_i^\alpha z + x^\alpha z_i, \tag{6}$$

where $z_i = \dfrac{\partial z}{\partial x^i}$. For $x = 0$, $\delta_i^\alpha = \dfrac{\partial f^\alpha}{\partial x^i}(0) = \delta_i^\alpha z(0)$, so

$$z(0) = 1. \tag{7}$$

We have, from (6),

$$\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} = \delta_i^\alpha z_j + \delta_j^\alpha z_i + x^\alpha z_{ij}. \tag{8}$$

where $z_{ij} = \dfrac{\partial^2 z}{\partial x^i \partial x^j}$. Based on the assumption of the theorem, we have

$$\sum_{i,j} \lambda^i \lambda^j (\delta_i^\alpha z_j + \delta_j^\alpha z_i + x^\alpha z_{ij}) = a \sum_i \lambda^i (\delta_i^\alpha z + x^\alpha z_i).$$

Hence, we obtain

$$a = \frac{2\lambda^\alpha \sum_j \lambda^j z_j + x^\alpha \sum_{i,j} \lambda^i \lambda^j z_{ij}}{\lambda^\alpha z + x^\alpha \sum_i \lambda^i z_i}, \qquad \alpha = 1, 2, \cdots\cdots, n. \tag{9}$$

(9) implies that

$$\frac{2\lambda^\alpha D_\lambda z + x^\alpha D_\lambda^2 z}{\lambda^\alpha z + x^\alpha D_\lambda z} = \frac{2\lambda^\beta D_\lambda z + x^\beta D_\lambda^2 z}{\lambda^\beta z + x^\beta D_\lambda z}.$$

The above is equivalent to

$$(x^\alpha \lambda^\beta - \lambda^\alpha x^\beta)(z D_\lambda^2 z - 2(D_\lambda z)^2) = 0, \qquad \alpha = 1, 2, \cdots\cdots, n.$$

So,

$$z D_\lambda^2 z - 2(D_\lambda z)^2 = 0. \tag{10}$$

For any $\lambda \in I\!\!R^n$ (10) is valid. Setting $x = tx_0, \lambda = x_0$, we have

$$z\ddot{z} = 2\dot{z}^2,$$

where $\dot{z} = \dfrac{dz}{dt}, \ddot{z} = \dfrac{d^2 z}{dt^2}$. The above equality is just

$$\frac{\ddot{z}}{\dot{z}} = 2\frac{\dot{z}}{z}. \tag{11}$$

Integrating (11), we obtain

$$\frac{\dot{z}(tx_0)}{\dot{z}(0)} = z^2(tx_0).$$

Integrating the above equality and noting $z(0) = 1$, we have

$$1 - \frac{1}{z(tx_0)} = t\dot{z}(0),$$

or

$$z(tx_0) = \frac{1}{1 - t\dot{z}(0)}.$$

For $t = 1$, we have

$$z(x_0) = \frac{1}{1 - \dot{z}(0)}. \tag{12}$$

Now we compute $\dot{z}(0)$:

$$\dot{z}(0) = \left.\frac{dz}{dt}\right|_{t=0} = x_0^i \frac{\partial z}{\partial x^i}(0) =< x_0, d >, \qquad \text{where } d = \sum_i \frac{\partial z}{\partial x_i} e_i.$$

Hence, we have

$$z(x_0) = \frac{1}{1- < x_0, d >},$$

and

$$f = \frac{x}{1- < x, d >}.$$

The proof of Theorem 1 is now complete. □

3. Schwarzian Derivative

Schwarzian derivative may be viewed as the infinite small form of the cross ratio of four points([5]). It can be defined (also see [7]) as follows.

Definition. Let $f : \mathbb{R}^n \supseteq \Omega \to \mathbb{R}^n$ be an immersion, $\lambda \in \mathbb{R}^n$, then the *Schwarzian derivative* of f in the direction λ is defined to be

$$Sf(\lambda, x) = (D_\lambda f)^{-1} D_\lambda^3 f - \frac{3}{2} \left[(D_\lambda f)^{-1} D_\lambda^2 f \right]^2.$$

In [7], Ryan introduces

$$S_r f(\lambda, x) = D_\lambda^3 f (D_\lambda f)^{-1} - \frac{3}{2} [D_\lambda^2 f (D_\lambda f)^{-1}]^2.$$

Since $\overline{D_\lambda f} = D_\lambda \overline{f}$, we have the relation

$$\overline{Sf(\lambda, x)} = S_r \overline{f}(\lambda, x).$$

A direct computation gives

Theorem 2. If $f : \Omega \to \mathbb{R}^n$ is a linear fractional transformation, then for any $\lambda \in \mathbb{R}^n$ and $x \in \Omega$ we have

$$Sf(\lambda, x) \equiv 0.$$

The following result also holds.

Theorem 3. Let Ω be a starlike domain about the origin in $I\!R^n$ and $f : \Omega \to I\!R^n$ be a immersion satisfying

i) $Sf(\lambda, x) = 0, \quad$ for any $\lambda \in I\!R^n, x \in \Omega; \quad$ and

ii) $D_\lambda f(0) \wedge D_\lambda^2 f(0) = 0, \quad$ for any $\lambda \in I\!R^n$.

Then f is a linear fractional transformation.

Proof. Let $x_0 \in \Omega$ be fixed. Set $g : [0,1] \to I\!R^n, g(t) = f(tx_0) - f(0)$. Obviously,

$$\frac{dg}{dt} = x_0^i \frac{\partial f}{\partial x^i}(tx_0) = D_{x_0} f(tx_0), \qquad \frac{d^2 g}{dt^2} = D_{x_0}^2 f(tx_0),$$

and

$$\frac{d^3 g}{dt^3} = D_{x_0}^3 f(tx_0).$$

Since $Sf(x_0, tx_0) = 0$, we have

$$\left(\frac{dg}{dt}\right)^{-1} \frac{d^3 g}{dt^3} - \frac{3}{2}\left[\left(\frac{dg}{dt}\right)^{-1} \frac{d^2 g}{dt^2}\right]^2 = 0,$$

i.e.,

$$\frac{d}{dt}\left[\left(\frac{dg}{dt}\right)^{-1} \frac{d^2 g}{dt^2}\right] = \frac{1}{2}\left[\left(\frac{dg}{dt}\right)^{-1} \frac{d^2 g}{dt^2}\right]^2.$$

Set $v(t) = \left(\frac{dg}{dt}\right)^{-1} \frac{d^2 g}{dt^2} \in \Gamma_n^+$, then $\frac{dv}{dt} = \frac{1}{2} v^2$. Hence

$$v^{-1} \frac{dv}{dt} v^{-1} = \frac{1}{2}.$$

Differentiating

$$1 = v^{-1} v,$$

we have

$$0 = (dv^{-1}) v + v^{-1}(dv)$$

that gives

$$dv^{-1} = -v^{-1}(dv) v^{-1}.$$

Integrating the above equality from 0 to t, we obtain

$$v^{-1}(t) = v^{-1}(0) - \frac{1}{2} t.$$

Since

$$v^{-1}(0) = \left[\left(\frac{dg}{dt}\right)^{-1} \frac{d^2g}{dt^2}\right]^{-1}_{t=0} = [(D_{x_0}f)^{-1}D^2_{x_0}f]^{-1}_{t=0} \in I\!R,$$

we have $v^{-1}(t) \in I\!R$, and

$$v(t) = \frac{v(0)}{1 - \frac{1}{2}tv(0)},$$

i.e.,

$$\frac{d^2g}{dt^2} = \frac{v(0)}{1 - \frac{1}{2}tv(0)} \frac{dg}{dt}.$$

The above is

$$\frac{d^2g^\alpha}{dt^2} = \frac{v(0)}{1 - \frac{1}{2}tv(0)} \frac{dg^\alpha}{dt}, \qquad \alpha = 1, 2, \cdots\cdots, n.$$

Therefore

$$\frac{dg^\alpha}{dt}(t) = \frac{1}{(1 - \frac{t}{2}v(0))^2} \frac{dg^\alpha}{dt}(0), \qquad \alpha = 1, 2, \cdots\cdots, n.$$

Integrating the above equality, we have

$$g^\alpha(t) = \frac{t}{1 - \frac{1}{2}tv(0)} \frac{dg^\alpha}{dt}(0), \qquad \alpha = 1, 2, \cdots\cdots, n.$$

Noting $\frac{dg^\alpha}{dt}(0) = \sum_i x_0 \frac{\partial f}{\partial x_i}(0)$ and denoting $v(0)$ by $v_0(x_0)$, we have

$$f^\alpha(tx_0) - f^\alpha(0) = g^\alpha(t) = \frac{t}{1 - \frac{1}{2}tv_0(x_0)} \sum_i x_0 \frac{\partial f^\alpha}{\partial x^i}(0). \tag{13}$$

For $t = 1$, (13) becomes

$$f^\alpha(x_0) - f^\alpha(0) = \frac{1}{1 - \frac{1}{2}v_0(x_0)} \sum_i x_0^i \frac{\partial f^\alpha}{\partial x^i}(0). \tag{14}$$

For any $x_0 \in \Omega$, (14) is valid. Now replacing x_0 by tx_0 in (14), we obtain

$$f^\alpha(tx_0) - f^\alpha(0) = \frac{t}{1 - \frac{1}{2}v_0(tx_0)} \sum_i x_0^i \frac{\partial f^\alpha}{\partial x^i}(0). \tag{15}$$

Comparing (15) with (13), we have

$$v_0(tx_0) = tv(x_0).$$

The above implies that $v_0(x_0)$ is homogeneous of degree one, and hence $v_0(x_0) = \sum_i x^i d_i =< x, d >$. Thus

$$f^\alpha(x) - f^\alpha(0) = \frac{\sum_i x^i \frac{\partial f^\alpha}{\partial x^i}(0)}{1 - \frac{1}{2} < x, d >}.$$

The proof of Theorem 3 is now complete. □

4. Möbius Transformation

Denote the group of Möbius transformations of $I\!R^n$ by $M\ddot{o}b(n)$ and the group of linear fractional transformations of $I\!R^n$ by $Fra(n)$.
For $f \in M\ddot{o}b(n)$, from Vahlen's theorem, we have

$$f = (ax + b)(cx + d)^{-1},$$

where $a, b, c, d \in \mathcal{A}_n$, $a, d \in \Gamma_n^+ \cup \{0\}$, $b, c \in \Gamma^- \cup \{0\}$, and $ad^* - bc^* = 1$, $ab^*, bd^*, dc^*, ca^* \in I\!R^n$.
For $ab^* \in I\!R^n$, and $b \in \Gamma_n$, it is proved that $b^*(ab^*)b$ is in $I\!R^n$ ([4,7]). It implies $b^*a \in I\!R^n$. Similarly, d^*b, c^*d, a^*c are in $I\!R^n$.
We can have the following theorem by a direct computation (see also [7]).

Theorem 4. If $f \in M\ddot{o}b(n)$, then $Sf(\lambda, x) \equiv 0$, for any $\lambda \in I\!R^n$ and any $x \in I\!R^n$.

Based on Theorem 4 and 3 we obtain: If $f \in M\ddot{o}b(n)$, and if there exists a point $x_0 \in I\!R^n$ such that $D_\lambda f(x_0) \wedge D_\lambda^2 f(x_0) = 0$ for any $\lambda \in I\!R^n$, then $f \in M\ddot{o}b(n) \cap Fra(n)$.
In comparison with Theorem 3 and Theorem 4, we wish to pose the following

5. Injective Criterion

In this section we will prove a injective theorem.

Theorem 6. Suppose that $\Omega \in I\!R^n$ is a convex domain, whose diameter is $\delta \leq \infty$. Let $f : \Omega \to I\!R^n$ be an immersion, satisfying

i) $Sf(\lambda, x) \in I\!R$, for any $\lambda \in I\!R^n$ and any $x \in \Omega$;

ii) $Sf(\lambda, x) \leq \frac{\pi^2}{\delta^2}$, for any $\lambda \in S^{n-1}, x \in \Omega$.

Then f is an embedding.

Proof. Suppose that the theorem is not true, then there exist two points x_1 and $x_2 \in \Omega$ such that $x_1 \neq x_2$, and $f(x_1) = f(x_2)$.
Set $g : [0,1] \to I\!R^n, g(t) = f((1-t)x_1 + tx_2) - f(x_1)$. Obviously, $g(0) = g(1) = 0$. We have

$$\dot{g}(t) = \frac{dg}{dt} = \sum_i (x_2^i - x_1^i)\frac{\partial f}{\partial x^i} = \|x_2 - x_1\| \sum_i \lambda^i \frac{\partial f}{\partial x_i} = \|x_2 - x_1\| D_\lambda f,$$

where $\lambda = \frac{x_2 - x_1}{\|x_2 - x_1\|} \in S^{n-1}$, and

$$\ddot{g}(t) = \frac{d^2 f}{dt^2} = \|x_2 - x_1\|^2 \sum_{i,j} \lambda^i \lambda^j \frac{\partial^2 f}{\partial x^i \partial x^j} = \|x_2 - x_1\|^2 D_\lambda^2 f.$$

Set

$$A = \dot{g}^{-1}\ddot{g} = \|x_2 - x_1\|(D_\lambda f)^{-1} D_\lambda^2 f. \tag{22}$$

Then we have

$$\frac{dA}{dt} - \frac{1}{2}A^2 = \|X_2 - X_1\|^2 Sf(\lambda, x), \qquad x = (1-t)x_1 + tx_2. \tag{23}$$

Assume that G is the solution of the ordinary differential equation

$$\frac{dG}{dt} = \frac{1}{2}AG, \qquad G(0) = 1.$$

By the theory of ordinary differential equations ([2]), the solution G is unique and $G(t) \neq 0$, $t \in [0,1]$. Therefore we have

$$\frac{d(G\bar{G})}{dt} = -\frac{1}{2}(AG\bar{G} + G\bar{G}\bar{A}), \tag{24}$$

$$\frac{d^2(G\bar{G})}{dt^2} = -\frac{1}{2}\left[2\|x_2 - x_1\|^2 Sf(\lambda, x)G\bar{G} - AG\bar{G}\bar{A}\right]. \tag{25}$$

And from (22), (24) and (25), we have

$$\frac{d^2(gG\bar{G}\bar{g})}{dt^2} = 2\frac{dg}{dt}G\bar{G}\frac{d\bar{g}}{dt} + \frac{1}{2}gAG\bar{G}\bar{A}\bar{g} - \left(\frac{dg}{dt}G\bar{G}\bar{A}\bar{g} + gAG\bar{G}\frac{d\bar{g}}{dt}\right)$$
$$-\|x_2 - x_1\|^2 Sf(\lambda, x) gG\bar{G}\bar{g}. \tag{26}$$

We have, from (26)

$$\frac{d^2(gG\bar{G}\bar{g})}{dt^2} + \|x_2 - x_1\|^2 Sf(\lambda, x) gG\bar{G}\bar{g} =$$

$$= (\sqrt{2}\frac{dg}{dt}G - \frac{1}{\sqrt{2}}gAG)\overline{(\sqrt{2}\frac{dg}{dt}G - \frac{1}{\sqrt{2}}gAG)}. \tag{27}$$

Taking the scalar parts of two sides of (27), we obtain

$$\frac{d^2[gG\bar{G}\bar{g}]_0}{dt^2} + \|x_2 - x_1\|^2 Sf(\lambda, x)[gG\bar{G}\bar{g}]_0 = \|\sqrt{2}\frac{dg}{dt}G - \frac{1}{\sqrt{2}}gAG\|^2 \geq 0. \tag{28}$$

Since δ is the diameter of Ω and $x_1, x_2 \in \Omega$, $\|x_2 - x_1\| < \delta$, there exists $\epsilon > 0$ such that

$$1 + 2\epsilon \leq \frac{\delta}{\|x_2 - x_1\|}. \tag{29}$$

Let $u = \sin \frac{t+\epsilon}{1+2\epsilon}\pi$, $t \in [0,1]$, then $u(t) > 0$, $t \in [0,1]$. Let further $p = \frac{\pi^2}{(1+2\epsilon)^2}$, then

$$\frac{d^2u}{dt^2} + pu = 0. \tag{30}$$

For convenience, we denote $[gG\bar{G}\bar{g}]_0$ by h. Consider

$$\frac{d}{dt}\left[\frac{dh}{dt}h - \frac{1}{u}\frac{du}{dt}h^2\right] = \frac{d^2h}{dt^2}h - \frac{1}{u}\frac{d^2u}{dt^2}h^2 + \left(\frac{dh}{dt} - \frac{1}{u}\frac{du}{dt}h\right)^2. \tag{31}$$

Substituting (28) and (30) into (31), we have

$$\frac{d}{dt}\left[\frac{dh}{dt}h - \frac{1}{u}\frac{du}{dt}h^2\right] \geq ph^2 - \|x_2 - x_1\|^2 Sf(\lambda, x)h^2 + (\frac{dh}{dt} - \frac{1}{u}\frac{du}{dt}h)^2. \tag{32}$$

Owing to the assumption $Sf(\lambda, x) \leq \frac{\pi^2}{\delta^2}$ and (29), we have

$$\frac{d}{dt}\left[\frac{dh}{dt}h - \frac{1}{u}\frac{du}{dt}h^2\right] \geq (\frac{dh}{dt} - \frac{1}{u}\frac{du}{dt}h)^2 \geq 0.$$

Integrating the two sides of the above inequality from 0 to 1, we obtain

$$0 \geq \int (\frac{dh}{dt} - \frac{1}{u}\frac{du}{dt}h)^2 \geq 0.$$

So

$$\frac{dh}{dt} - \frac{1}{u}\frac{du}{dt}h \equiv 0. \tag{33}$$

Hence, we have

$$\frac{d}{dt}(\frac{h}{u}) = \frac{1}{u}\left(\frac{dh}{dt} - \frac{1}{u}\frac{du}{dt}h\right) \equiv 0.$$

Therefore, we have $\frac{h}{u}$ = constant, h = constant $\cdot\, u$. But $h(0) = 0$ and $u(0) \neq 0$, so the constant$= 0$. That is $h \equiv 0, [gG\bar{G}\bar{g}]_0 \equiv 0$. It implies that $\|gG\| \equiv 0$. One the other hand we have $G(t) \neq 0$, for any $t \in [0, 1]$, thus $g(t) \equiv 0$. It is contradictory to the fact that f is an immersion. That ends the proof. □

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