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STABILITY OF FRAMES GENERATED BY NONLINEAR FOURIER ATOMS

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In this paper, we study the stability of two kinds of frames generated by nonlinear Fourier atoms. The first result is the Kadec type $\frac{1}{4}$-theorem. The second states that the nonlinear windowed Fourier atoms form a frame of $L^2(\mathbb{R})$.

Keywords: Nonlinear Fourier atoms; Kadec type $\frac{1}{4}$-theorem; Weyl–Heisenberg frame.

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1. Introduction

Quadrature signal processing is used in many fields of science and engineering, and quadrature signals are necessary to describe the processing and implementation that takes place in modern digital communication systems. It serves two purposes: to determine the parameters needed for the construction of a necessary model, and to confirm if the model constructed represents the physical phenomenon. Especially nowadays, with the development of science and technology, a large amount of data is waiting for further scientific exploration. Traditional data analysis methods such as Fourier analysis, based on the linear stationary assumption have been shown to be efficient for processing of linear and stationary data. However, data from real systems, either natural or man-made ones, are most likely to be both nonlinear and non-stationary. Many studies have shown that the traditional data analysis methods are not suitable for analyzing nonlinear and non-stationary data. Only in recent years have new methods been introduced to analyze non-stationary and nonlinear

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data. For example, wavelet analysis[8] and the Wigner–Ville distribution[22,23] are designed for linear but non-stationary data. Meanwhile, various nonlinear time series analysis methods are designed for nonlinear but stationary and deterministic systems.

Recently, Huang presented a new time-frequency algorithm for nonlinear and non-stationary signal analysis: *Hilbert–Huang Transform* (HHT).[12,13] By using the algorithm of *empirical mode decomposition* (EMD), any multi-component can be decomposed into a finite sum of *intrinsic mode functions* (IMFs), which are essentially mono-components. The notion of IMF defined by Huang plays a crucial role in the HHT algorithm. The original concept of IMFs is an engineering description: The local maximums and minimums take turn to occur, and between a pair of adjacent local extremes, the signal is monotone and passes through the zero only once, and is of the local symmetry, i.e. the mean of any adjacent pair of upper and lower envelopes is of zero value. Experiments show that IMFs behave nicely with Hilbert transform in the following sense[12,13]: Each term of the IMFs in the EMD, regarded as mono-component of the signal, is the real part of a complex-valued signal $f(t) = a(t)e^{i\theta(t)}$ satisfying the equation $\mathcal{H}(f)(t) = -if(t)$, where $\mathcal{H}(f)(t)$ is the Hilbert transform of $f(t)$ on the line,[21,25] defined by

$$\mathcal{H}(f)(t) = \frac{1}{\pi}\,\text{v.p.}\int_{-\infty}^{\infty} \frac{f(s)}{t-s}ds. \tag{1.1}$$

In Ref. 4, the authors show that these IMFs can be approximated by B-spline.

For a real-valued signal $f(t)$, there are infinitely many ways to write $f(t)$ as $a(t)\cos\theta(t)$. Gabor[10,18] first used the Hilbert transform to generate the *associated analytic signal* according to

$$f_a(t) = f(t) + i\mathcal{H}(f)(t) = a(t)e^{i\theta(t)}, \tag{1.2}$$

and then the original signal is the real part of the complex-valued function: $f(t) = \operatorname{Re} f_a(t) = a(t)\cos\theta(t)$. This amplitude-frequency modulation is unique and is called the *canonical modulation.* In such a way we obtain the one-to-one correspondence $f(t) \to (\rho(t), \theta(t))$, the latter being called the *canonical pair associated with* $f(t)$. With a canonical modulation, if $\theta'(t) \geq 0$, then $\theta'(t)$ is defined to be the *instantaneous frequency* of the complex signal $f_a(t)$, and also that of the associated real signal $f(t)$ (see, for example, Refs. 7 and 18).

The notion of instantaneous frequency, however, is not valid for multi-components. For instance, the "instantaneous frequency" of the signal $f(t) = \cos t + \cos 2t$ obtained through its analytic signal has negative values. This suggests to decompose multi-components into the sum of mono-components to which meaningful instantaneous frequency may be defined. So far, there is no strict mathematical definition of mono-components. A large number of literature discuss this problem, see, for example, Refs. 2, 3, 7, 16 and 18. We know that if $x(t) = a(t)\cos\theta(t)$, then $f_q(t) = a(t)e^{i\theta(t)}$ is the associated quadrature. The key problem is: Under what conditions the quadrature $f_q(t)$ coincides with the associated analytic signal $f_a(t)$? The relation $f_q(t) = f_a(t)$ is equivalent to

$$\mathcal{H}(a(t)\cos\theta(t)) = a(t)\sin\theta(t). \tag{1.3}$$

This leads to the Bedrosian theorem[2]: if the spectrums of the amplitude $a(t)$ and that of $\cos\theta(t)$ are respectively of low-pass and high-pass and disjoint, then

$$\mathcal{H}(fg)(t) = f(t)\mathcal{H}g(t). \tag{1.4}$$

If, in addition,

$$\mathcal{H}\cos\theta(t) = \sin\theta(t), \tag{1.5}$$

then $f_q(t) = f_a(t)$, and the form $f(t) = a(t)\cos\theta(t)$ is the canonical representation of $f(t)$. Since it is expected that the spectrums of amplitude is lower than that of the unimodular part $e^{i\theta(t)}$, then the assertion $f_q(t) = f_a(t)$ is reduced to (1.5).

Nuttall theorem[16] states that the energy estimation of error when quadrature signal approximates to analytic signal. Vakman and Vainshtein[22] offered a pointwise estimation of the error between $f_a(t)$ and $f_q(t)$:

$$|f_a(t) - f_q(t)| \leq \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{0} \left|\hat{f}_q(\omega)\right| d\omega.$$

Based on the Bedrosian and Nuttall theorems, a natural question occurs: For an amplitude-frequency modulation signal $f(t) = \rho(t)\cos\theta(t)$, under what conditions on ρ and θ the associated quadrature signal $\rho(t)e^{i\theta(t)}$ becomes analytic? In Ref. 19, Qian proves that a strictly increasing function $\theta(t)$, $t \in [0, 2\pi]$ with $m(\theta([0, 2\pi])) = 2\pi$ gives rise to an analytic signal $e^{i\theta(t)}$ if and only if $d\theta(t)$ is a harmonic measure on the circle, and this result has a counterpart for strictly increasing functions $\Theta(s)$ with $m(\Theta(\mathbb{R})) = 2\pi$ on the whole real line. In Ref. 5, we explore some time-frequency aspects of the family of the new nonlinear Fourier atoms $\{e^{in\theta_a(t)} : n \in \mathbb{Z}\}$, $|a| < 1$, where $d\theta_a(t)$ is a harmonic measure, that is, the derivative of $\theta_a(t)$ is the Poisson kernel. We show that $\cos\theta_a(t)$ is of mono-component.[5] That essentially means that $\theta_a'(t) > 0$, the Hilbert transform of $\cos\theta_a(t)$ is $\sin\theta_a(t)$, and $\theta_a(t)$ can be decomposed into a sum of a linear part and a nonlinear but periodic part.

This paper contains 4 sections. In Sec. 2, we recall some properties of nonlinear Fourier atoms. In Sec. 3, we establish Kadec type $\frac{1}{4}$-theorem for the nonlinear Fourier atoms. Section 4 mainly concerns frame stability in relation to this new family of atoms.

2. Behavior of Nonlinear Fourier Atoms

An analytic signal is of the form

$$f(t) = \rho(t)e^{i\theta(t)},$$

where $\rho(t)$ and $\theta'(t)$ are the corresponding instantaneous amplitude and frequency of $f(t)$, which is the boundary value of an analytic function in the upper-half-complex plane. In Ref. 19 (see also Ref. 20), Qian introduces $\theta_a(t)$ defined by

$$e^{i\theta_a(t)} = \tau_a(e^{it}) = \frac{e^{it} - a}{1 - \bar{a}e^{it}} \tag{2.1}$$

through the Möbius transformation

$$\tau_a(z) = \frac{z - a}{1 - \bar{a}z} \tag{2.2}$$

that is a conformal mapping one-to-one and onto from the unit disc to itself under the condition $\tau(a) = 0$.

Note that $\theta_a(t)$ is defined on the unit circle and its derivative is the Poisson kernel (see Refs. 11 or 19)

$$\theta'_a(t) = p_a(t) = \frac{1 - |a|^2}{1 - 2|a|\cos(t - t_a) + |a|^2}.$$

The function θ_a may be continuously extended to the whole real line with the property $\theta_a(t + 2\pi) = \theta_a(t) + 2\pi$ whose derivative $p_a(t)$ is continuous and 2π-periodic. The corresponding period functions $e^{i\xi\theta_a(t)}$, $\xi > 0$, except for the trivial case $a = 0$ corresponding to $e^{i\xi t}$ of the linear phase ξt, are not included in the general form of Picinbono.[18] Indeed, the derivatives of the phases of the signals in Ref. 18 are not periodic. The atomic case of Picinbono was studied in Ref. 19.

Let $a = |a|e^{it_a}$. We then obtain by a directly computation

$$e^{i\theta_a(t)} = e^{it}\frac{A(t)}{\bar{A}(t)},$$

where $A(t) = 1 - |a|e^{i(t_a - t)}$. By noting that

$$\operatorname{Arg} A(t) = \arctan\frac{|a|\sin(t - t_a)}{1 - |a|\cos(t - t_a)},$$

we get the explicit expression

$$\theta_a(t) = t + 2\arctan\frac{|a|\sin(t - t_a)}{1 - |a|\cos(t - t_a)}. \tag{2.3}$$

Note that the first part is linear and the second part is periodic, and the decomposition is unique.

It is interesting to note that the signal $\cos\theta_a(t)$ is a mono-component with frequency modulation. To see this, we need to show that its Hilbert transform is the corresponding sine function $\sin\theta_a(t)$.

The circular Hilbert transform $\tilde{\mathcal{H}}$[11,25] of a function $f = \sum_k c_k e^{ikt} \in L^2([0, 2\pi])$ is defined by

$$\tilde{\mathcal{H}}f(t) = -i\sum_k \operatorname{sgn}(k)c_k e^{ikt}. \tag{2.4}$$

It has a singular integral expression

$$\tilde{\mathcal{H}} f(t) = \frac{1}{2\pi} \text{ v.p.} \int_{-\pi}^{\pi} \cot\left(\frac{t-s}{2}\right) f(s) ds, \quad \text{a.e.} \tag{2.5}$$

In accordance with the Bedrosian theorem, it has been well accepted that if $\mathcal{H}(a(t)\cos\phi(t)) = a(t)\sin\phi(t)$ and $\phi'(t) \geq 0$, then meaningful instantaneous amplitudes and frequencies may be defined through the amplitude-frequency modulation signal $s(t) = a(t)\cos\phi(t)$. In the case, we regard $s(t)$ as of mono-component.[3] The following theorem states that $\cos\theta_a(t)$ is a mono-component with constant amplitude.

Proposition 2.1. (*see Refs.* 5 *and* 20) (i) *Treating* $\cos\theta_a(t)$ *as a function defined on the unit circle, we have*

$$\tilde{\mathcal{H}} \cos\theta_a(t) = \sin\theta_a(t), \quad \mathcal{H}\sin\theta_a(t) = -\cos\theta_a(t) + a, \quad \text{and} \tag{2.6}$$

(ii) *treating* $\cos\theta_a(t)$ *as a* 2π*-periodic function on the whole real line, we have*

$$\mathcal{H}\cos\theta_a(t) = \sin\theta_a(t), \quad \textit{and} \quad \mathcal{H}\sin\theta_a(t) = -\cos\theta_a(t). \tag{2.7}$$

We provide an explicit representation for the function $\cos\theta_a(t)$. By (2.3), we have that

$$\begin{aligned}\cos\theta_a(t) = {} & \cos\left(2\arctan\frac{|a|\sin(t-t_a)}{1-|a|\cos(t-t_a)}\right)\cos t \\ & - \sin\left(2\arctan\frac{|a|\sin(t-t_a)}{1-|a|\cos(t-t_a)}\right)\sin t.\end{aligned}$$

Through a direct computation, we have

$$\cos\theta_a(t) = \frac{\cos t - 2|a|\cos t_a + |a|^2\cos(t-2t_a)}{1+|a|^2-2|a|\cos(t-t_a)}.$$

In particular, when a is a real number less than 1, $\cos\theta_a(t)$ can be simplified into

$$\cos\theta_a(t) = \frac{\left(1+|a|^2\right)\cos t - 2|a|}{1+|a|^2-2|a|\cos t}.$$

With the notation $\theta_a'(t) = p_a(t)$, $p_a(t)$ being the Poisson kernel, we have $p_0 = 1$. There holds the estimates for p_a:

$$\frac{1-|a|}{1+|a|} \leq p_a(t) \leq \frac{1+|a|}{1-|a|}. \tag{2.8}$$

Define

$$L^2_{p_a}([0,2\pi]) = \left\{ f : [0,2\pi] \to \mathbb{C} : \int_0^{2\pi} |f(t)|^2 p_a(t) dt < \infty \right\}.$$

It is a Hilbert space equipped with the inner product

$$\langle f, g\rangle_{p_a} = \int_0^{2\pi} f(t)\overline{g(t)} p_a(t) dt$$

and the norm $\|f\|_{p_a} = \big(\int_0^{2\pi} |f(t)|^2 p_a(t)dt\big)^{\frac{1}{2}}$. We note that for $a = 0$, the space $L^2_{p_a}([0, 2\pi])$ reduces to the classical Hilbert space $L^2([0, 2\pi])$ with the usual norm $\|f\| = \big(\int_0^{2\pi} |f(t)|^2 dt\big)^{\frac{1}{2}}$. From (2.8), we have equivalent relationship between the two norms for $f \in L^2_a([0, 2\pi])$ (also for $f \in L^2([0, 2\pi])$)

$$\frac{1-|a|}{1+|a|}\ \|f\|^2 \le \|f\|^2_{p_a} \le \frac{1+|a|}{1-|a|}\ \|f\|^2. \tag{2.9}$$

Note that all the spaces $L^2_{p_a}([0, 2\pi])$, $|a| < 1$, are identical as function sets with different but equivalent norms. Through change of variable the classical Carleson's Theorem reduces to the assertion (also see Ref. 19) that for any $f \in L^2_{p_a}([0, 2\pi])$,

$$f(t) = \sum_n c_n^a(f) e^{in\theta_a(t)}, \qquad \text{a.e.}$$

The identity of the function sets then implies that the last equality also holds for functions in $f \in L^2([0, 2\pi])$. That is, the standard square integrable functions can be approximated by the nonlinear Fourier atoms with the weighted Fourier coefficients $c_n^a(f)$.

3. Kadec Type $\frac{1}{4}$-Theorem for Nonlinear Fourier Atoms

The notion of frame has been introduced by Duffin and Schaeffer.[9] A sequence of distinct vectors $\{\phi_n : n \in \mathbb{Z}\}$ belongs to a separable Hilbert space H is said to be a frame if there exist positive constants A and B such that

$$A\|f\|^2 \le \sum_{n\in\mathbb{Z}}^{\infty} |(f, \phi_n)|^2 \le B\|f\|^2$$

for every $f \in H$. The numbers A and B are called the lower and upper bounds of the frame.

If the sequence $\{\phi_n : n \in \mathbb{Z}\}$ is a (Schauder) basis as well as a frame in H, then $\{\phi_n : n \in \mathbb{Z}\}$ is called a Riesz basis.

The fundamental stability criterion for Riesz basis, historically the first, is due to Paley and Wiener.[17] We formulate it as follows (see Ref. 9, Chap. 1, Theorem 13):

Proposition 3.1. *Let $\{\phi_n : n \in \mathbb{Z}\}$ be an orthonormal basis for a separable Hilbert space H and let $\{\psi_n : n \in \mathbb{Z}\}$ be "close" to $\{\phi_n : n \in \mathbb{Z}\}$ in the sense that*

$$\left\| \sum_{n\in\mathbb{Z}} c_n(\phi_n - \psi_n) \right\|_H \le \lambda \tag{3.1}$$

for some constant λ, $0 \le \lambda < 1$, where $\{c_n : n \in \mathbb{Z}\}$ is an arbitrary sequence satisfying $\sum_{n\in\mathbb{Z}} |c_n|^2 \le 1$. Then $\{\psi_n : n \in \mathbb{Z}\}$ is a Riesz basis for H.

In the context of an orthonormal Fourier basis, Kadec proved the so-called $\frac{1}{4}$-theorem in his celebrated paper,[14] see also Ref. 24.

We now establish the Kadec type $\frac{1}{4}$-theorem for the nonlinear Fourier atoms.

Theorem 3.1. *If* $\{\lambda_n : n \in \mathbb{Z}\}$ *is a sequence of real numbers for which*

$$|\lambda_n - n| \leq L < \frac{1}{4}, \quad \forall\, n \in \mathbb{Z}, \tag{3.2}$$

then $\{e^{i\lambda_n\theta_a(t)} : n \in \mathbb{Z}\}$ *forms a Riesz basis for* $L^2_a([0, 2\pi])$ *and for* $L^2([0, 2\pi])$*.*

Proof. Let the sequence $\{c_n : n \in \mathbb{Z}\}$ satisfy $\sum_{n\in\mathbb{Z}} |c_n|^2 \leq 1$. From the proof of Kadec type $\frac{1}{4}$-theorem[14,24] we know that there exists a constant λ, $0 \leq \lambda < 1$, such that

$$\left\| \sum_{n\in\mathbb{Z}} c_n(e^{int} - e^{i\lambda_n t}) \right\| \leq \lambda < 1$$

holds for the standard orthonormal Fourier basis $\{e^{int} : n \in \mathbb{Z}\}$ and for the sequence $\{\lambda_n : n \in \mathbb{Z}\}$ satisfying the inequality (3.2).

On the other hand, by the change variable we have

$$\left\| \sum_{n\in\mathbb{Z}} c_n(e^{in\theta_a(t)} - e^{i\lambda_n\theta_a(t)}) \right\|_{p_a} = \left\| \sum_{n\in\mathbb{Z}} c_n(e^{int} - e^{i\lambda_n t}) \right\| \leq \lambda < 1.$$

This shows that the nonlinear Fourier atoms $\{e^{in\theta_a(t)} : n \in \mathbb{Z}\}$ satisfy the Paley–Wiener criterion (3.1) for the Hilbert space $L^2_a([0, 2\pi])$. Therefore, $\{e^{i\lambda_n\theta_a(t)} : n \in \mathbb{Z}\}$ forms a Riesz basis for $L^2_a([0, 2\pi])$.

By (2.9), we finally get that $\{e^{i\lambda_n\theta_a(t)} : n \in \mathbb{Z}\}$ forms a Riesz basis for $L^2([0, 2\pi])$. The proof of Theorem 3.1 is complete. □

It then follows that the nonlinear Fourier atoms $\{e^{i\lambda_n\theta_a(t)} : n \in \mathbb{Z}\}$ forms a Riesz basis for $L^2([0, 2\pi])$ under "sufficiently small" perturbations of the integers. Accordingly, every function f in $L^2([0, 2\pi])$ will have a unique nonlinear Fourier series expansion

$$f(t) = \sum_{n\in\mathbb{Z}} c_n e^{i\lambda_n\theta_a(t)}$$

with $\sum_{n\in\mathbb{Z}} |c_n|^2 < \infty$. We now investigate the frame aspect of nonlinear Fourier atoms $\{e^{i\lambda_n\theta_a(t)} : n \in \mathbb{Z}\}$. From the general definition of frame, a system $\{e^{i\lambda_n\theta_a(t)} : n \in \mathbb{Z}\}$ becomes a frame in $L^2([0, 2\pi])$ with the lower and upper bounds A and B provided that

$$A \int_0^{2\pi} |\phi(t)|^2 dt \leq \sum_{n\in\mathbb{Z}} \left| \int_0^{2\pi} \phi(t) e^{-i\lambda_n\theta_a(t)} dt \right|^2 \leq B \int_0^{2\pi} |\phi(t)|^2\, dt$$

for every $\phi \in L^2([0, 2\pi])$.

A direct calculation gives the relationship of frames of the spaces $L^2([0, 2\pi])$ and $L^2_a([0, 2\pi])$, as follows.

Theorem 3.2. *Suppose that the system* $\{e^{i\lambda_n t} : n \in \mathbb{Z}\}$ *is a frame in* $L^2([0, 2\pi])$ *with the lower and upper bounds* A *and* B*. Then, the system* $\{e^{i\lambda_n\theta_a(t)} : n \in \mathbb{Z}\}$ *is also a frame in* $L^2_a([0, 2\pi])$ *with the same bounds, and vice versa.*

Combining Theorem 3.2 with Ref. 24, Chap. 4, Theorem 13, there holds:

Theorem 3.3. *Suppose that the system* $\{e^{i\lambda_n\theta_a(t)} : n \in \mathbb{Z}\}$ *is a frame in* $L_a^2([0, 2\pi])$. *Then there exists a positive constant* L *with the property that* $\{e^{i\mu_n\theta_a(t)} : n \in \mathbb{Z}\}$ *is a frame in* $L_a^2([0, 2\pi])$ *whenever* $|\lambda_n - \mu_n| \le L$ *for every* n. *In the case,* $\{e^{i\mu_n\theta_a(t)} : n \in \mathbb{Z}\}$ *is also a frame in* $L^2([0, 2\pi])$.

We shall now show that Kadec type $\frac{1}{4}$-theorem can be improved. A better estimate L is given by Balan for the standard Fourier basis. See Refs. 1 and 6 for the relevant historical notes and further references therein.

Theorem 3.4. *Suppose the system* $\{e^{i\lambda_n\theta_a(t)} : n \in \mathbb{Z}\}$ *is a frame in* $L_a^2([0, 2\pi])$ *with lower and upper bounds* A *and* B. *Set*

$$L = \frac{1}{4} - \frac{1}{\pi}\arcsin\left(\frac{1}{\sqrt{2}}\left(1 - \sqrt{\frac{A}{B}}\right)\right).$$

If a real sequence $\{\mu_n : n \in \mathbb{Z}\}$ *satisfies* $|\mu_n - \lambda_n| \le \lambda < L$ *for every* $n \in \mathbb{N}$, *then the system* $\{e^{i\mu_n\theta_a(t)} : n \in \mathbb{Z}\}$ *is a frame in* $L_a^2([0, 2\pi])$ *with lower and upper bounds*

$$A\left(1 - \sqrt{\frac{A}{B}}(1 - \cos\lambda\pi + \sin\lambda\pi)\right)^2 \quad \textit{and} \quad B(2 - \cos\lambda\pi + \sin\lambda\pi)^2.$$

Furthermore, $\{e^{i\mu_n\theta_a(t)} : n \in \mathbb{Z}\}$ *is a frame in* $L^2([0, 2\pi])$ *whenever* $|\lambda_n - \mu_n| \le \lambda < L$ *for every* n *with lower and upper bounds*

$$A\left(1 - \sqrt{\frac{A}{B}}(1 - \cos\lambda\pi + \sin\lambda\pi)\right)^2 \frac{1 - |a|}{1 + |a|}$$

and

$$B(2 - \cos\lambda\pi + \sin\lambda\pi))^2 \frac{1 + |a|}{1 - |a|}$$

respectively.

Proof. By Theorem 3.2, we know that if the system $\{e^{i\lambda_n\theta_a(t)} : n \in \mathbb{Z}\}$ is a frame in $L_a^2([0, 2\pi])$ with the lower and upper bounds A and B, then the system $\{e^{i\lambda_n t} : n \in \mathbb{Z}\}$ is a frame in $L^2([0, 2\pi])$ with the same bounds. Applying Theorem 1 in Ref. 1, the system $\{e^{i\mu_n t} : n \in \mathbb{Z}\}$ is a frame in $L^2([0, 2\pi])$ with the lower and upper bounds

$$A\left(1 - \sqrt{\frac{A}{B}}(1 - \cos\lambda\pi + \sin\lambda\pi)\right)^2$$

and $B(2 - \cos\lambda\pi + \sin\lambda\pi)^2$. In the case, Theorem 3.2 shows that the system $\{e^{i\mu_n\theta_a(t)} : n \in \mathbb{Z}\}$ is a frame in $L_a^2([0, 2\pi])$ with the lower and upper bounds

$$A\left(1 - \sqrt{\frac{A}{B}}(1 - \cos\lambda\pi + \sin\lambda\pi)\right)^2 \quad \text{and} \quad B(2 - \cos\lambda\pi + \sin\lambda\pi)^2.$$

The last assertion in the theorem can be deduced from the inequalities (2.9). □

4. Weyl–Heisenberg Frames for Nonlinear Fourier Atoms

For any function $f \in L^1(\mathbb{R})$, the weighted Fourier transform is defined through the 2π-periodized function p_a by

$$\hat{f}^a(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-i\xi\theta_a(t)} p_a(t)\, dt. \tag{4.1}$$

Through change of variable, we have the relation

$$\hat{f}^a(\xi) = (f \circ \theta_a^{-1})\hat{}\,(\xi).$$

Since $f \in L^2(\mathbb{R})$ if and only if $f \circ \theta_a^{-1} \in L^2(\mathbb{R})$, the last relation enables us to define the corresponding weighted Fourier transform for functions in $L^2(\mathbb{R})$. Note that when $a = 0$, it reduces to the standard Fourier transform

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-i\xi t} dt.$$

In distribution sense, we can check that the weighted Fourier transform of $\cos\theta_a(t)$ is $\frac{1}{2}\left(\delta(\xi - 1) + \delta(\xi + 1)\right)$. The inverse formula of (4.1) takes the form

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}^a(\xi) e^{i\xi\theta_a(t)} d\xi. \tag{4.2}$$

Define

$$L^2_{p_a}(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} \, : \, \int_{\mathbb{R}} |f(t)|^2 p_a(t) dt < \infty \right\}.$$

Then, it is a Hilbert space equipped with the inner product

$$\langle f, g \rangle_{p_a} = \int_{\mathbb{R}} f(t)\overline{g(t)} p_a(t) dt$$

and norm $\|f\|_{p_a} = \left(\int_{\mathbb{R}} |f(t)|^2 p_a(t) dt\right)^{\frac{1}{2}}$. The Plancherel's theorem now takes the form $\|f\|^2_{p_a} = \|\hat{f}^a\|^2$, where $\|\cdot\|_{p_0} = \|\cdot\|$. Furthermore, we have the equivalence of the two norms

$$\frac{1-|a|}{1+|a|}\|\cdot\|^2 \le \|\cdot\|^2_{p_a} \le \frac{1+|a|}{1-|a|}\|\cdot\|^2. \tag{4.3}$$

Let $I = [0, 2\pi]$ and let $L^2_{p_a}(I) = \{f : f \in L^2_{p_a}(\mathbb{R}),\ \text{supp } f \subset I\}$ and let χ_I be the characteristic function of I with value 1 on I and 0 elsewhere. It is clear that

$$\sum_{n \in \mathbb{Z}} \chi_I(t + 2\pi n) = 1, \quad \text{for a.e. } t \in \mathbb{R}. \tag{4.4}$$

We have the following theorem (see also Ref. 19).

Theorem 4.1. *For $k \in \mathbb{Z}$, define*

$$e^a_k(t) := \frac{1}{\sqrt{2\pi}} e^{ik\theta_a(t)} \chi_I(t). \tag{4.5}$$

Then, the system $\{e^a_k : k \in \mathbb{Z}\}$ is a complete orthonormal basis for $L^2_{p_a}(I)$.

Proof. Since the function $\theta_a(t)$ strictly increases satisfying $\theta_a^{-1}(t+2\pi n) = \theta_a^{-1}(t) + 2\pi n$, we have from (4.4),

$$\sum_{n\in\mathbb{Z}} \chi_I(\theta_a^{-1}(t+2\pi n)) = 1, \quad \text{for a.e. } t \in \mathbb{R}. \tag{4.6}$$

It follows that

$$\begin{aligned}
\langle e_m^a, e_n^a\rangle_{p_a} &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{im\theta_a(t)} e^{-in\theta_a(t)} \chi_I(t) p_a(t) dt \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \chi_I(\theta^{-1}(x)) e^{i(m-n)x}\, dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{k\in\mathbb{Z}} \chi_I(\theta_a^{-1}(x+2\pi k)) e^{i(m-n)x}\, dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)x}\, dx \\
&= \delta_{mn}.
\end{aligned}$$

This shows that the system $\{e_k^a : k \in \mathbb{Z}\}$ is orthonormal. For completeness, suppose that $f \in L^2_{p_a}(I)$ and $\langle f, e_k^a\rangle_{p_a} = 0$ for all $k \in \mathbb{Z}$. Noting that $f\chi_I = f$ since $\operatorname{supp} f \subset I$, and changing variables, we obtain for all $k \in \mathbb{Z}$,

$$\begin{aligned}
0 = \langle f, e_k^a\rangle_{p_a} &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(t) e^{-ik\theta_a(t)} \chi_I(t) p_a(t)\, dt \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(t) e^{-ik\theta_a(t)} p_a(t)\, dt \\
&= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \sum_{n\in\mathbb{Z}} f(\theta_a^{-1}(t+2\pi n)) e^{-ikt}\, dt.
\end{aligned}$$

By the standard completeness result for Fourier series, $\sum_{n\in\mathbb{Z}} f(\theta_a^{-1}(t+2\pi n))$ is 0 in $L^2(I)$, hence is 0 a.e. on $[0, 2\pi]$. But this is 2π-periodic and thus is 0 a.e. on $\mathbb{R}$. Since

$$\operatorname{supp} f \subset I \quad \text{and} \quad \sum_{n\in\mathbb{Z}} \chi_I\big(\theta_a^{-1}(t+2\pi n)\big) = 1,$$

the functions $f\big(\theta_a^{-1}(t+2\pi n)\big)$, $n \in \mathbb{Z}$, have a.e. disjoint supports, and hence each must be 0 a.e. on $\mathbb{R}$. Letting $n = 0$, we see that $f(\theta_a^{-1}(t)) = 0$, and $f = 0$ a.e. on I, and the completeness follows. □

To define the Weyl–Heisenberg frame, we introduce the operators of *modulation* E_β^a and *translation* T_α for function $f \in L^2(\mathbb{R})$ by

$$E_\beta^a f(t) := e^{i\beta\theta_a(t)} f(t), \quad \beta \in \mathbb{R}$$

and

$$T_\alpha f(t) := f(t - 2\pi\alpha), \quad \alpha \in \mathbb{R}.$$

For simplicity, we also introduce the following auxiliary function:

$$G(t) := \sum_{n\in\mathbb{Z}} |g(t-2\pi n)|^2.$$

Theorem 4.2. *Let $g \in L^2(\mathbb{R})$ and let $\operatorname{supp} g \subset [0, 2\pi]$. If there exist constants A and B such that $A \le G(t) \le B$ a.e., then $\{E_m^a T_n g\}_{m,n\in\mathbb{Z}}$ is a Weyl–Heisenberg frame for $L^2(\mathbb{R})$ with frame bounds A and B.*

Proof. Let $f \in L^2(\mathbb{R})$. Fix n, and observe that the function $f \cdot T_n\bar{g}$ is supported in $I_n = \{t + 2\pi n : t \in I\}$. It follows from condition $A \le G(t) \le B$, a.e., that g is bounded, so $f \cdot T_n\bar{g} \in L^2(I_n)$. In view of Theorem 4.1, the collection of functions $\{e_m^a(t) : m \in \mathbb{Z}\}$ is an orthonormal basis for $L^2_{p_a}(I_n)$. Therefore,

$$\sum_{m\in\mathbb{Z}} \left|\left\langle f \cdot T_n\bar{g}, e_m^a \right\rangle_{p_a}\right|^2 = \int_{\mathbb{R}} |f(t)|^2 |g(t-2\pi n)|^2 p_a(t)\, dt.$$

We further deduce that

$$\begin{aligned}
\sum_{m,n\in\mathbb{Z}} \left|\left\langle f, E_m^a T_n g \right\rangle_{p_a}\right|^2 &= \sum_{m,n\in\mathbb{Z}} \left|\left\langle f \cdot T_n\overline{g}, e_m^a \right\rangle_{p_a}\right|^2 \\
&= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} |f(t)|^2 |g(t-2\pi n)|^2 p_a(t)\, dt \\
&= \int_{\mathbb{R}} |f(t)|^2 G(t) p_a(t)\, dt.
\end{aligned}$$

It follows from condition $A \le G(t) \le B$, a.e., that

$$A\|f\|_{p_a}^2 \le \sum_{m,n\in\mathbb{Z}} \left|\left\langle f, E_m^a T_n g \right\rangle_{p_a}\right|^2 \le B\|f\|_{p_a}^2.$$

Finally the relationship (4.3) of the norms yields

$$A\|f\|^2 \le \sum_{m,n\in\mathbb{Z}} \left|\left\langle f, E_m^a T_n g \right\rangle\right|^2 \le B\|f\|^2. \qquad \square$$

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