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Mono-components for decomposition of signals

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SUMMARY

This note further carries on the study of the eigenfunction problem: Find $f(t)=\rho(t)e^{i\theta(t)}$ such that $Hf=-if$, $\rho(t)\geqslant 0$ and $\theta'(t)\geqslant 0$, a.e. where H is Hilbert transform. Functions satisfying the above conditions are called mono-components, that have been sought in time-frequency analysis. A systematic study for the particular case $\rho\equiv 1$ with demonstrative results in relation to Möbius transform and Blaschke products has been pursued by a number of authors. In this note, as a key step, we characterize a fundamental class of solutions of the eigenfunction problem for the general case $\rho\geqslant 0$. The class of solutions is identical to a special class of starlike functions of one complex variable, called circular H-atoms. They are building blocks of circular mono-components. We first study the unit circle context, and then derive the counterpart results on the line. The parallel case of dual mono-components is also studied. Copyright © 2006 John Wiley & Sons, Ltd.

KEY WORDS: analytic signal; instantaneous frequency; Hilbert transform; Möbius transform; mono-component; empirical mode decomposition; intrinsic mode functions; HHT (Hilbert–Huang transform); starlike functions

1. INTRODUCTION

In signal analysis one has been trying to understand, for a given signal, what are its instantaneous amplitude, instantaneous phase, and instantaneous frequency. A signal, denoted by $f(t)$, stands for a real-valued locally (Lebesgue) integrable function. A common approach to find the instantaneous objects is as follows. First, one introduces the associated analytic signal, $Af(t)=f(t)+iHf(t)$, where Hf is the Hilbert transform of f, being assumed to exist. Hilbert transform is formally defined by the principal value singular integral

$$Hf(t)=\text{p.v.}\,\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(s)}{t-s}\,ds$$

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which has the Fourier multiplier form

$$Hf(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi t}(-i\,\mathrm{sgn}(\xi))\hat{f}(\xi)\,d\xi$$

where Fourier transform is defined by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi t} f(t)\,dt$$

and sgn is the signum function that takes value 1 if $\xi > 0$; and -1 if $\xi < 0$.

Af may be written in the form $Af(t) = \rho(t)e^{i\theta(t)}$, with $\rho(t) \geqslant 0$, a.e. Consequently,

$$f(t) = \rho(t)\cos\theta(t) \tag{1}$$

Note that Af satisfies the relation

$$H(Af) = -iAf \tag{2}$$

Taking into account the relation $H^2 = -I$, where I stands for the identity operator, (2) is equivalent to

$$H(\rho(\cdot)\cos\theta(\cdot))(t) = \rho(t)\sin\theta(t) \tag{3}$$

With the uniquely determined modulation (1), one calls $\rho(t)$ and $\theta(t)$ the *instantaneous amplitude* and *instantaneous phase*, respectively, provided $\theta'(t) \geqslant 0$, or $\theta'(t) \leqslant 0$, a.e. Should the conditions be satisfied, then function $\theta'(t)$ is defined to be the qualified *instantaneous frequency*. Unfortunately, the requirements $\theta' \geqslant 0$ or $\theta' \leqslant 0$ are hardly met, and the definitions of instantaneous amplitude, phase and frequency via the associated analytic signal Af can be erroneous.

In Reference [1] we explore connections between eigenfunctions of Hilbert transformation and functions in Hardy H^p spaces. Denote by $\mathbf{S}$ for $\mathbf{S} = \mathbb{D}$ or $\mathbf{S} = \mathbb{C}^+$, the earlier being the open unit disc and the latter being the upper-half complex plane. In this notation $H_{\mathbf{S}}$ stands for $H_{\mathbb{C}^+}$ or $H_{\mathbb{D}}$, where $H_{\mathbb{C}^+}$ is the standard Hilbert transformation, H, on the line, and $H_{\mathbb{D}}$ is the circular Hilbert transformation, $\tilde{H}$, on the circle. The circular Hilbert transformation is defined through

$$\tilde{H}f(t) = \text{p.v.}\,\frac{1}{2\pi}\int_{-\pi}^{\pi} \cot\left(\frac{t-s}{2}\right) f(s)\,ds$$

with the Fourier multiplier form based on the Fourier expansion of $f(t)$:

$$\tilde{H}f(t) = \sum_{k=-\infty}^{\infty} -i\,\mathrm{sgn}(k)c_k e^{ikt}, \quad f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$$

The following result is proved in Reference [1, Theorems 3.2 and 4.3].

Theorem 1.1
The function $f(t) = \rho(t)(c(t) + is(t))$, with $\rho \geqslant 0$ and $\rho \in L^p(\mathbf{S}), 1 \leqslant p \leqslant \infty$, $c^2 + s^2 = 1$, is the boundary value of a function in $H^p(\mathbf{S})$ if and only if $H_{\mathbf{S}}(\rho c) = \rho s$ modulo constants.

Note that when $\mathbf{S} = \mathbb{C}^+$ and $p = \infty$ the Hilbert transformation takes the distribution sense. The theorem will be recalled in the proofs of our main results below.

In References [1–4] a systematic study on the unimodular case $\rho \equiv 1$ is carried out. In this paper we extend the study to the general non-unimodular case. We found that the well-established theory of starlike functions in one complex variable best fits to our need. Boundary values of starlike functions provide easily accessible circular mono-components. We now introduce the related notation and terminology.

Let f be an eigenfunction of the circular or non-circular Hilbert transformation $H_{\mathbf{S}}$. Then $H_{\mathbf{S}} f = kf$, $k \in \mathbb{C}$. Since $H_{\mathbf{S}}^2 f = k^2 f = -f$, we obtain $k = \pm i$, where i is the complex imaginary unit. In below a condition like $g \geqslant 0$, a.e. will be briefly written as $g \geqslant 0$.

Definition 1.1
A function f is said to be an $H_{\mathbf{S}}$-eigenfunction if $H_{\mathbf{S}} f = -\mathrm{i} f$; and a dual $H_{\mathbf{S}}$-eigenfunction if $H_{\mathbf{S}} f = \mathrm{i} f$. An $H_{\mathbf{S}}$-eigenfunction f is called an **S**-mono-component if with the form $f(t) = \rho(t)\mathrm{e}^{\mathrm{i}\theta(t)}$ it satisfies $\rho(t) \geqslant 0$ and $\theta'(t) \geqslant 0$; and, a dual $H_{\mathbf{S}}$-eigenfunction f is called a dual **S**-mono-component if with the form $f(t) = \rho(t)\mathrm{e}^{\mathrm{i}\theta(t)}$ it satisfies $\rho(t) \geqslant 0$ and $\theta'(t) \leqslant 0$.

In the sequel, we simply call $H_{\mathbb{C}^+}$-eigenfunctions, dual $H_{\mathbb{C}^+}$-eigenfunctions, $\mathbb{C}^+$-mono-components and dual $\mathbb{C}^+$-mono-components as *H-eigenfunctions*, *dual H-eigenfunctions*, *mono-components* and *dual mono-components*, respectively; and, we call $H_{\mathbb{D}}$-eigenfunctions, dual $H_{\mathbb{D}}$-eigenfunctions, $\mathbb{D}$-mono-components and dual $\mathbb{D}$-mono-components as *circular H-eigenfunctions*, *dual circular H-eigenfunctions*, *circular mono-components* and *dual circular mono-components*, respectively.

Very often, we investigate $\mathrm{Re}\, f$ instead of f, and, with the form $f(t) = \rho(t)\mathrm{e}^{\mathrm{i}\theta(t)}$, we have $\mathrm{Re}\, f = \rho(t)\cos\theta(t)$. In the case, we have, $H_{\mathbf{S}} f = \mp \mathrm{i} f$ if and only if $H_{\mathbf{S}}(\rho(\cdot)\cos\theta(\cdot))(t) = \pm\rho(t)\sin\theta(t)$. We correspondingly call the real part $\rho(t)\cos\theta(t)$ *a real H-eigenfunction*, or *a real mono-component*, etc. If there will be no confusion, then we suppress '*real*', and still call it *a H-eigenfunction*, or *a mono-component*, etc. The same convention is valid for the circular case.

If a signal is not **S**-mono-component or a dual **S**-mono-component, then it is called a **S**-*multi-component*, or simply *multi-component*. Signals are usually multi-components. In [5] Huang proposed a practical algorithm, called Empirical Mode Decomposition, to decompose a signal into a sum

$$f(t) = \sum \rho_i(t)\cos\theta_i(t) \tag{4}$$

where each entry of the sum is expected to be a mono-component or a dual mono-component. He also obtained numerically rapid convergence. However, the algorithm suffers for it does not always result in the desired decomposition in terms of mono- and dual mono-components. A mathematical theory providing exact mathematical concepts and approximation methods is desired.

The task would be two fold. The first is to establish a bank of mono- and dual mono-components. The second is to find rapid approximation to signals by linear combinations of mono- and dual mono-components. The present paper addresses the first. Along with the results previously obtained in References [1–4], in this note we are to characterize a class of easily accessible mono- and dual mono-components. They are the signals for which instantaneous amplitude, phase and frequency may be well defined, and, they are constructive units of the decomposition (4). In below we first provide a survey on what have been achieved in this direction.

In References [3,4] we establish the theory of non-linear Fourier atoms $e^{i\theta_a(t)}$, $0 \leqslant t \leqslant 2\pi$, where a is any complex number in $\mathbb{D}$, and θ_a is an absolutely continuous and strictly increasing function with $\theta_a(2\pi) - \theta_a(0) = 2\pi$, and $\theta'_a(t)$ is the Poisson kernel for the unit disc at the point a, and therefore positive. The function θ_a is defined through a typical Möbius transform τ_a sending a to zero:

$$\tau_a(z) = \frac{z-a}{1-\bar{a}z}, \quad e^{i\theta_a(t)} = \frac{e^{it}-a}{1-\bar{a}e^{it}} \tag{5}$$

It was shown that $e^{i\theta_a}$ is a circular H-eigenfunction that is equivalent to $\tilde{H}\cos\theta_a(t) = \sin\theta_a(t)$ modulo constants. Note that when $a=0$, $e^{i\theta_a(t)} = e^{it}$. The finite product of k copies of e^{it} is e^{ikt}. A generalized Fourier series and weighted Fourier transform theory are studied in Reference [4]. This simplest unimodular case, viz. $\rho \equiv 1$, is further extended to finite products of non-linear Fourier atoms corresponding to finite Blaschke products, as given in Reference [1].

One can introduce two types of mono- and dual mono-components on the real line based on finite Blaschke products on the circle. One is periodic extensions of the functions on $[0, 2\pi]$ inherited from the finite Blaschke products on the circle; and the other is images of those functions under Cayley transformation (see Section 3). The latter type was previously studied in Reference [2] based on a different approach. Apart from the systematic study in References [1,3,4], some related aspects in wavelet theory are developed in References [6,7]. We cite the following spectrum results for the two types of mono-components [6]. They will be recalled in Section 2.

Viewing $e^{i\theta_a(t)}$ as a periodic function on the line, we have [6]

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\theta_a(t)} e^{-i\xi t}\, dt = -\sqrt{2\pi}a\delta(\xi) + \frac{\sqrt{2\pi}(1-|a|^2)}{\bar{a}} \sum_{k=1}^{\infty} \bar{a}^k \delta(\xi - k) \tag{6}$$

On the other hand, denoting by $e^{i\phi_a(s)}$ the image of the non-linear Fourier atom $e^{i\theta_a(t)}$ under Cayley transform, we have [6]

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\phi_a(t)} e^{-i\xi t}\, dt = -\sqrt{2\pi}\delta(\xi) + \frac{2\sqrt{2\pi}(1-|a|)}{(1+|a|)} e^{-(1-|a|)/(1+|a|)\xi} H(\xi) \tag{7}$$

where $H(\xi)$ is the Heaviside function.

We note that in either of the two cases the spectrum contains non-trivial impulse at the origin. This prevents from direct use of Bedrosian's Theorem [8] in deducing mono- or dual mono-components $\rho(t)e^{i\theta_a(t)}$ or $\rho(t)e^{i\phi_a(t)}$ with general $\rho \geqslant 0$.

In below we give some remarks on dual mono-components.

When expending $f \in L^2([0, 2\pi])$ into its Fourier series

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos kt + b_k \sin kt$$

or its complex Fourier series

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$$

the entries $\sin kt = \cos(\pi/2 - kt)$ and e^{-ikt}, $k > 0$, are dual circular mono-components. These can be verified directly, or derived from Theorem 1.2 (see below). They are also dual mono-components on the line if they are considered as periodic functions (see Section 3). The following result allows us to merely concentrate to the non-dual case.

Theorem 1.2
$\rho(t)e^{i\theta(t)}$ is a (circular) mono-component if and only if $\rho(t)e^{-i\theta(t)}$ is a dual (circular) mono-component.

Proof
Assume that $f(t) = \rho(t)e^{i\theta(t)}$ is a mono-component. We have

$$H(\rho(\cdot)\cos\theta(\cdot))(t) = \rho(t)\sin\theta(t)$$

and, since $H^2 = -I$,

$$H(\rho(\cdot)\sin\theta(\cdot))(t) = -\rho(t)\cos\theta(t)$$

They can be re-written as

$$H(\rho(\cdot)\cos(-\theta(\cdot)))(t) = -\rho(t)\sin(-\theta(t)), \quad H(\rho(\cdot)\sin(-\theta(\cdot)))(t) = \rho(t)\cos(-\theta(t))$$

The last two relations are equivalent to

$$H(\rho(\cdot)e^{-i\theta(\cdot)})(t) = i\rho(t)e^{-i\theta(t)}$$

Therefore, $\rho(t)e^{-i\theta(t)}$ is a dual H-eigenfunction. Since $\rho \geqslant 0, -\theta' \leqslant 0$, it is a dual mono-component. The argument is reversible. For the circular case we replace H by $\tilde{H}$. The proof is complete. □

We show that for $k > 0$, $\sin kt$ is a dual (circular) mono-component. In fact, Theorem 1.2 implies that ie^{-ikt} is a dual (circular) mono-component. Therefore, $\sin kt = \mathrm{Re}(ie^{-ikt})$ is a dual (circular) mono-component. In general, $f = u + iv$ is a dual (circular) eigenfunction if and only if $H_S u = -v$.

The writing plan of the paper is as follows. Section 2 is devoted to our main results in relation to starlike functions. In Section 3 we deal with mono-components on the line.

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2. BOUNDARY VALUES OF STARLIKE FUNCTIONS

This section deals with the circular case. In below, a connected and open set of the complex plane $\mathbb{C}$ is called a *domain*. A function f is said to be *univalent* if it takes different values at different points. Our definition for starlike domains, and therefore that for starlike functions, takes a narrower sense, that is, starlike with respect to the pole $z = 0$.

Definition 2.1
A domain Ω is said to be starlike if $0 \in \Omega$, and $tz \in \Omega$, $0 < t < 1$, whenever $z \in \Omega$. A univalent and holomorphic function $f : \mathbb{D} \to f(\mathbb{D})$ is said to be starlike if $f(\mathbb{D})$ is starlike and $f(0) = 0$.

Closely related are *convex domains* and *convex functions*.

Definition 2.2
A domain Ω is said to be convex, if $0 \in \Omega$, and $tz_1 + (1-t)z_2 \in \Omega$, $0 < t < 1$, whenever $z_1, z_2 \in \Omega$. A univalent and holomorphic function $f : \mathbb{D} \to f(\mathbb{D})$ is said to be convex, if $f(\mathbb{D})$ is convex and $f(0) = 0$.

Clearly, a convex domain is a starlike domain, and a convex function is a starlike function. The Taylor expansion of a starlike function is of the form

$$g(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots, \quad |z| < 1 \tag{8}$$

We denote by S the class of univalent and holomorphic functions in $\mathbb{D}$ having the Taylor expansion

$$g(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots, \quad |z| < 1 \tag{9}$$

The totality of starlike functions in S is denoted by S^*, and the totality of convex functions in S is denoted by C. It may be shown that C is a proper subclass of S^*, and S^* is a proper subclass of S. We call functions in S^* *normalized starlike functions*; and those in C *normalized convex functions*. There has been a deep study with fruitful results on the classes C, S^* and S. Among literature on starlike functions we refer to References [9–13]. The most striking feature of the subtle analysis on the classes C, S^* and S would be its connections with Bieberbach conjecture (1916) whose final and celebrated proof was given by de Branges in 1984 [9]. In this note we will specify some connections between the mentioned study and the H-eigenfunction problem. We first introduce some concepts.

Definition 2.3
Let $\rho(t)$ and $\theta(t)$, $0 \leqslant t \leqslant 2\pi$, be absolutely continuous, $\rho \geqslant 0$, and

$$\int_0^{2\pi} \rho(t) \mathrm{e}^{\mathrm{i}\theta(t)}\, \mathrm{d}t = 0 \tag{10}$$

With the above properties, a function $f(t) = \rho(t)\mathrm{e}^{\mathrm{i}\theta(t)}$ is called a circular H-atom, if f is a circular mono-component satisfying $\theta(2\pi) - \theta(0) = 2\pi$; and, a dual circular H-atom, if f is a dual circular mono-component satisfying $\theta(2\pi) - \theta(0) = -2\pi$.

As a consequence of Theorem 1.2, the following result addresses the symmetry property between circular and dual circular H-atoms.

Theorem 2.1
$\rho(t)\mathrm{e}^{\mathrm{i}\theta(t)}$ is a circular H-atom if and only if $\rho(t)\mathrm{e}^{-\mathrm{i}\theta(t)}$ is a dual circular H-atom.

The following results are contained in [10, Section 1, Chapter 10]. If $f(z)$ is holomorphic, and it univalently maps $\mathbb{D}$ into a simply connected region Q whose boundary is a bounded rectifiable closed Jordan curve, then f continuously extends to $\bar{\mathbb{D}}$ such that on $\partial\mathbb{D}$ it is absolutely continuous with

$$\frac{\mathrm{d}f(\mathrm{e}^{\mathrm{i}t})}{\mathrm{d}t} = \mathrm{i}\mathrm{e}^{\mathrm{i}t}f'(\mathrm{e}^{\mathrm{i}t}), \quad \text{a.e.}$$

where $f'(\mathrm{e}^{\mathrm{i}t})$ is the non-tangential boundary value of $f'(z)$ in $\mathbb{D}$. If, moreover, $f(z)$ is starlike, then both $\rho(t)$ and $\theta(t)$ are absolutely continuous.

For practical reasons we only concern such ideal starlike functions. The importance of starlike functions lies on the following Theorem.

Theorem 2.2
$\rho(t)\mathrm{e}^{\mathrm{i}\theta(t)}$, $0 \leqslant t \leqslant 2\pi$, is a circular H-atom if and only if it is the boundary value $f(\mathrm{e}^{\mathrm{i}t})$ of a starlike function $f(z)$ whose boundary is a bounded rectifiable closed Jordan curve.

Proof
We first assume that $f(\mathrm{e}^{\mathrm{i}t}) = \rho(t)\mathrm{e}^{\mathrm{i}\theta(t)}$ is a circular H-atom. Owing to Theorem 1.1, it is the boundary value of a function, $f(z)$, in $H^\infty(\mathbb{D})$. Since $f(\mathrm{e}^{\mathrm{i}t})$ is absolutely continuous, and $\theta(t)$ is non-decreasing, moving from $\theta(0)$ to $\theta(0)+2\pi$, the *argument principle* implies that f is univalent. The non-decreasing property of θ implies that $f(\mathbb{D})$ is starlike with the pole zero. Through Cauchy's formula, condition (10) implies that $f(0)=0$. We thus conclude that $f(z)$ is a starlike function with the required properties.

Now assume that $f(\mathrm{e}^{\mathrm{i}t}) = \rho(t)\mathrm{e}^{\mathrm{i}\theta(t)}$ is the boundary value of a starlike function $f(z)$, where $f(\mathbb{D})$ is a starlike domain with the pole zero whose boundary is a bounded rectifiable closed Jordan curve. Obviously, $f(z)$ is in $H^\infty(\mathbb{D})$. Theorem 1.1 then asserts that its boundary value is a circular H-eigenfunction. Owing to the results in Reference [10] recalled before the statement of the theorem, both ρ and θ are absolutely continuous. As the boundary of a starlike domain, the quantity $\arg(f(\mathrm{e}^{\mathrm{i}t})) = \theta(t)$ is non-decreasing, and its derivative is non-negative. This implies that the angle $\theta(t)$ increasingly goes from $\theta(0)$ to $\theta(0)+2\pi$ as t goes increasingly from 0 to 2π. Condition (10) is a consequence of Cauchy's formula and $f(0)=0$. We thus conclude that $f(\mathrm{e}^{\mathrm{i}t})$ is a circular H-atom. The proof is complete. □

It is noted that, since $f(z) = a_1 z + a_2 z^2 + \cdots$, in the second part of the proof the fact that f is a circular H-eigenfunction can also be derived from the Fourier multiplier expression of the circular Hilbert transformation. That is,

$$\tilde{H}f(\mathrm{e}^{\mathrm{i}(\cdot)})(t) = \sum_{k=1}^{\infty} -\mathrm{i}\,\mathrm{sgn}(k) a_i \mathrm{e}^{\mathrm{i}kt} = -\mathrm{i}f(\mathrm{e}^{\mathrm{i}t})$$

We note that in complex analysis the normalized starlike functions with respect to the pole ∞ are of the form

$$f(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots \tag{11}$$

This is mainly for a geometrically symmetric theory for starlike functions with respect to the pole ∞. In particular, with form (11), when $z=\mathrm{e}^{\mathrm{i}t}$ goes along the unit circle in the anticlockwise direction, then $f(\mathrm{e}^{\mathrm{i}t})$ goes along the boundary of $f(\mathbb{D})$ anticlockwise as well. For the theory of dual mono-component we, however, adopt the following definition that is analytically symmetric, and works well with Hilbert transform.

Definition 2.4
A function $f(z)$ is said to be starlike with respect to the pole ∞ if $f(1/z)$ is starlike (with respect to the pole zero).

With this definition we have the counterpart result for dual circular H-atoms.

Theorem 2.3
$\rho(t)e^{i\theta(t)}$, $0 \leqslant t \leqslant 2\pi$, is a dual circular H-atom if and only if $\rho(t)e^{i\theta(t)}$, $0 \leqslant t \leqslant 2\pi$, is the boundary value $f(e^{it})$ of a starlike function $f(z)$ with respect to the pole ∞, whose boundary is a bounded rectifiable closed Jordan curve.

Example 2.1 (*The Circle Family*)
The simplest example would be the circle family. Any fractional-linear transformation

$$w = f(z) = \frac{az}{cz + d}$$

that maps $\mathbb{D}$ into a disc $f(\mathbb{D}) \ni 0$, $f(0) = 0$, with the consistent orientation as t rotates from 0 to 2π under the parametrization $z = e^{it}$, will give rise to a circular H-atom. We now form this family in a systematic way using Möbius transform. The Möbius transform $\tau_a(z) = (z - a)/(1 - \bar{a}z)$ has the power series expansion

$$\tau_a(z) = -a + b_1 z + b_2 z + \cdots$$

where $b_1 = 1 - |a|^2 > 0$. We construct

$$f_a(z) = \frac{1}{b_1}(\tau_a(z) + a) = \frac{z}{1 - \bar{a}z} \tag{12}$$

This function is in the class C. It maps discs in $\mathbb{D}$ into discs. The images $f_a(\mathbb{D}_r)$, $\mathbb{D}_r = r\mathbb{D}$, $0 < r < 1$, are discs not centred at $z = 0$ if $a \neq 0$. Indeed,

$$f_a(re^{it}) = \frac{r}{\sqrt{1 - 2r|a|\cos(t - t_a) + |a|^2 r^2}} e^{i(t - \arg(1 - r|a|e^{i(t - t_a)}))}$$

where $a = |a|e^{it_a}$. It follows from Theorem 2.2 that for every fixed $r : 0 < r < 1$, the function $f_a(re^{it})$ is a circular H-atom. The mapping can be extended to $r : 1 \leqslant r < 1/|a|$, and the diameter of the disc $f(\mathbb{D})$ passing through 0 is divided by 0 into two parts with lengths, respectively, $r/(1 - r|a|)$ and $r/(1 + r|a|)$. So, the closer the number $r|a|$ to 1, the closer the pole zero to the boundary of the image circle.

One can similarly formulate the ellipse family and the Casimire curve family.

As a consequence of the *argument principle* finite products of circular and dual circular H-atoms are multi-valent functions. We have the following

Theorem 2.4
Finite products of circular and dual circular H-atoms are, respectively, circular mono-components and dual circular mono-components.

Proof
Products of finite many starlike functions is a function in H^∞. Therefore, their boundary values are circular H-eigenfunctions (Theorem 1.1). The argument of the boundary value of

such a product is the sum of the arguments of the boundary values of the factor starlike functions, and therefore is non-decreasing and absolutely continuous. Hence, finite products of circular H-atoms are circular mono-components. For dual circular H-atoms the proof is similar. □

The established theory on the classes S, S^* and C provides a source of starlike functions with a great variety. The basic references are [9–13]. Reference [13], in particular, provides many working examples. We briefly recall, without proof, some results in the literature that may have significant impacts to our study.

(i) It may be shown that if $f(D)$ is starlike, then $f(D_r)$ is starlike for all $r \in (0,1)$. In Example 2.1 on the circle family we assert this fact from the property of fractional-linear transformations. It, however, holds in general. This implies that when $z = re^{it}$ traces out the circle $|z| = r$ anticlockwise, then the complex number $f(z) = \rho e^{i\theta}$ must also traces out a complete circle anticlockwise. It follows that

$$\frac{\partial}{\partial t} \arg\{f(z)\} = \frac{\partial \theta}{\partial t} \geqslant 0$$

This latter condition implies

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant 0, \quad z \in \mathbb{D}$$

This turns to be a sufficient condition for starlike domains as well.

(ii) It may be shown that a function is convex in $\mathbb{D}$ if and only if $1 + z(f''(z))/(f(z))$ has a positive real part in $\mathbb{D}$. As a consequence, $f(\mathbb{D}_r)$, $0<r<1$, is also convex. Based on this it may be shown that $f(z)$ is convex if and only if $F(z) = zf'(z)$ is starlike. Therefore, a convex function $f(z)$ has the formula

$$f(z) = \int_0^z \frac{F(\zeta)}{\zeta} \, d\zeta$$

where $F(z)$ is a starlike function. The last relation also gives rise to a representation formula for all convex functions (see (iv) below).

(iii) If f and g are in class S^*, then their weighted product $f^\alpha g^\beta$, $\alpha + \beta = 1$, $0 \leqslant \alpha, \beta \leqslant 1$, is in S^*.

If f and g are in the class C with the expansions

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

then their Hadamard product (also called Hadamard convolution)

$$(f * g)(z) = \sum_{n=1}^{\infty} a_a b_n z^n$$

is in C.

If f and g are in the class S^*, then the modified Hadamard product

$$(f \otimes g)(z) = \sum_{n=1}^{\infty} \frac{a_a b_n}{n} z^n$$

is in S^*.

(iv) If $P(z)$ is holomorphic with positive real part then there holds Herglotz's formula:

$$P(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \, d\alpha(t)$$

where $\alpha(t)$ is a non-decreasing function satisfying

$$\int_0^{2\pi} d\alpha(t) = 1 \quad \text{and} \quad \alpha(t) = \tfrac{1}{2}[\alpha(t+0) + \alpha(t-0)] \tag{13}$$

There is a one-to-one relationship between the functions $P(z)$ and $\alpha(t)$.

Based on Herglotz's formula one has the representation formula for starlike functions: a function f is starlike in $\mathbb{D}$ if and only if

$$f(z) = z \exp\left(2\int_0^{2\pi} \log \frac{1}{1 - e^{-it}z} \, d\alpha(t)\right)$$

where α is a non-decreasing function satisfying (13). Theoretically, the formula provides all starlike functions with the pole zero.

(v) It is an interesting fact that if $f(z)$ is in S, then for small enough $r > 0$ the image $f(r\mathbb{D})$ is starlike, and therefore $f(rz)$ is in S^*. One can show that there exists a positive number, $R_{ST} = (e^{\pi/2} - 1)/(e^{\pi/2} + 1) \approx 0.65579$, called *radius of starlikeness*, such that whenever $r \leqslant R_{ST}$ the image $f(r\mathbb{D})$ is starlike for all $f \in S$. The number R_{ST} is sharp in the sense that if $r > R_{ST}$, then there exists a function $f \in S$ such that $f(r\mathbb{D})$ is not starlike.

For the class S there is also a sharp constant, $R_{CV} = 2 - \sqrt{3} \approx 0.26\cdots$, called *radius of convexity*, such that whenever $r \leqslant R_{CV}$ the set $f(r\mathbb{D})$ is convex for all $f \in S$.

3. MONO-COMPONENTS ON THE LINE

It is the identical relationship given in Theorem 2.2 between circular H-atoms and certain starlike functions that motivates the definition of circular H-atoms. There is no counterpart concepts on the line. In this section we will induce mono-components and dual mono-components on the line based on those obtained on the circle.

Theorem 3.1
Assume that $\tilde{f}(t) = \rho(t)e^{i\theta(t)}$, $0 \leqslant t < 2\pi$, where $\rho \in L^p([0, 2\pi))$, $1 \leqslant p \leqslant \infty$. Then,

(i) for $1 \leqslant p \leqslant \infty$, $\tilde{f}(t)$ is a (dual) circular mono-component if and only if $f(t) = \rho(t)e^{i\theta(t)}$, $-\infty < t < \infty$, is a (dual) mono-component on the line, where ρ and θ are extended to satisfy $\rho(t + 2\pi) = \rho(t)$ and $\theta(t + 2\pi) = \theta(t) + 2\pi$.

(ii) for $1 \leqslant p < \infty$, the function

$$\frac{1}{(s^2+1)^{1/p}} \rho(2 \arctan s) \in L^p(\mathbb{R})$$

and, if $\tilde{f}(t)$ is a (dual) circular mono-component, then

$$F(s)=\frac{1}{(s^2+1)^{1/p}}\rho(2\arctan s)e^{i(\theta(2\arctan s)+(2/p)\arccos(-s/\sqrt{s^2+1})-(2\pi/p))}, \quad -\infty<s<\infty$$

is a (dual) mono-component on the line.

(iii) for $p=\infty$, $\tilde{f}(t)$ is a (dual) circular mono-component if and only if

$$F(s)=\rho(2\arctan s)e^{i\theta(2\arctan s)}, \quad -\infty<s<\infty$$

is a (dual) mono-component on the line.

The proof of (i) of the theorem is based on the following lemma.

Lemma 3.1

Let $\tilde{f}\in L^p([-\pi,\pi))$, $1\leqslant p\leqslant\infty$, and f be the 2π-periodic extension of $\tilde{f}$ to the real line. Then Hf is 2π-periodic, and, restricted in $[-\pi,\pi)$, $Hf=\tilde{H}\tilde{f}$, where Hf is defined by

$$Hf(t)=\lim_{\epsilon\to 0, N\to\infty}\frac{1}{\pi}\int_{\epsilon<|t-s|<(2N+1)\pi}\frac{f(s)}{t-s}\,ds$$

Proof

It may be easily shown (also see Reference [4] or [1] or [6])

$$\begin{aligned} Hf(t)&=\frac{1}{\pi}\lim_{\epsilon\to 0, N\to\infty}\int_{(-\pi,\pi)\cap\{|x-t|>\epsilon\}}\left(\sum_{k=-N}^{N}\frac{1}{t-x-2k\pi}\right)f(x)\,dx\\ &=\frac{1}{2\pi}\lim_{\epsilon\to 0}\int_{(-\pi,\pi)\cap\{|x-t|>\epsilon\}}\cot\left(\frac{t-x}{2}\right)f(x)\,dx\\ &=\tilde{H}\tilde{f}(t), \quad \text{a.e.} \end{aligned}$$

□

Proof of Theorem 3.1

We only prove the mono-component case. The dual case is similar.

(i) Assume that $\tilde{f}$ is a circular mono-component. Then Lemma 3.1 implies that the periodically extended $f(t)$ is an H-eigenfunction. Since the extended θ is non-decreasing, $f(t)$ is a mono-component. The argument is reversible. We thus complete the proof of (i).

(ii) and (iii) The $L^p([-\pi,\pi])$ condition and the circular mono-component condition together guarantee that the function $\rho(t)e^{i\theta(t)}$ is the boundary value of a function in $H^p(\mathbb{D})$ (Theorem 1.1). Under the Cayley transformation $\kappa:\mathbb{C}^+\to\mathbb{D}$,

$$z=\kappa(w)=\frac{i-w}{i+w}$$

and the corresponding boundary relation

$$e^{it}=\frac{i-s}{i+s}, \quad s=\tan\frac{t}{2}$$

the function $F(w) = (1/(w+\mathrm{i})^{2/p}) f(\kappa(w)) \in H^p(\mathbb{C}^+)$ (see Reference [14] or [1]), and therefore its boundary value is an H-eigenfunction (Theorem 1.1). The boundary value of the induced weight factor $1/(w+\mathrm{i})^{2/p}$ is

$$\frac{1}{(s+\mathrm{i})^{2/p}} = \frac{1}{(s^2+1)^{1/p}} \mathrm{e}^{\mathrm{i}[(2/p)\arccos(-s/\sqrt{s^2+1})-(2\pi/p)]}$$

with the frequency

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{2}{p}\arccos\left(\frac{-s}{\sqrt{s^2+1}}\right) - \frac{2\pi}{p}\right) = \frac{2}{p}\,\frac{1}{1+s^2} \geqslant 0$$

The frequency of the $f(\kappa(s))$ part is

$$\frac{\mathrm{d}}{\mathrm{d}s}(\theta(2\arctan s)) = \theta'(2\arctan s)\frac{2}{1+s^2} \geqslant 0$$

Putting them together, the frequency of $F(s)$ is non-negative. Hence, $F(s)$ is a mono-component. For $p=\infty$ the argument is reversible. The proof is complete. □

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