Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER Computers and Mathematics with Applications 51 (2006) 1471–1482
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Characterization of Analytic Phase Signals

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(Received February 2005; revised and accepted January 2006)

Abstract—In many cases, a real-valued signal $x(t)$ may be associated with a complex-valued signal $a(t)e^{i\theta(t)}$, the analytic signal associated with $x(t)$ with the characteristic properties $x(t) = a(t)\cos\theta(t)$ and $\mathcal{H}(a(\cdot)\cos\theta(\cdot))(t) = a(t)\sin\theta(t)$. Using such obtained amplitude-frequency modulation the instantaneous frequency of $x(t)$ at the time t_0 may be defined to be $\theta'(t_0)$, provided $\theta'(t_0) \geq 0$. The purpose of this note is to characterize, in terms of analytic functions, the unimodular functions $F(t) = C(t) + iS(t), C^2(t) + S^2(t) = 1$, a.e., that satisfy $\mathcal{H}C(t) = S(t)$. This corresponds to the case $a(t) \equiv 1$ in the above formulation. We show that a unimodular function satisfies the required condition if and only if it is the boundary value of a so called inner function in the upper-half complex plane. We also give, through an explicit formula, a large class of functions of which the parametrization $C(t) = \cos\theta(t)$ is available and the extra condition $\theta'(t) \geq 0$, a.e. is enjoyed. This class of functions contains Blaschke products in the upper-half complex plane as a proper subclass studied by Picinbono in [1]. © 2006 Elsevier Ltd. All rights reserved.

Keywords—Phase signal, Analytic signal, Intrinsic mode function, Instantaneous frequency, Blaschke product, Inner function.

1. INTRODUCTION

The core task of *time-frequency analysis* is to find a bivariate representation for a signal, called time-frequency distribution, that describes the energy density of the signal simultaneously in the time and the frequency domains. Based on such representation, we get to know how *frequency* of a signal changes with time.

The fundamental tool of time-frequency analysis is *Fourier spectrum analysis*. The basic view is that a general signal is a certain superposition of harmonic waves, of which each has a constant frequency. Fourier analysis plays a fundamental role in processing linear and stationary data.

The work was supported by research grant of the University of Macau No. RG092/04-05S/C118/QT/FST and Macao Sci. and Tech. Develop. Fund 051/2005/A.
Supported in part by NSFC under grants 10201034 and the Project-sponsored by SRF for ROCS, SEM.

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doi:10.1016/j.camwa.2006.01.007

However, most signals, either natural or man-made, are nonlinear and nonstationary. Fourier analysis cannot expose the time-varying property of frequency of nonstationary signals as Fourier transform is a univariate representation in the time domain or in the frequency domain separately, and thus does not posses the *time-frequency localization* property. This leads to the development of *windowed Fourier transform* and *wavelet transform*, which are bivariate representations of signals in time-frequency domain simultaneously, and offer limited time-frequency localization [2].

In the *atoms* view, the common idea of the traditional methods, including Fourier transform, windowed Fourier transform and wavelet transform, are to use some fixed time-frequency atoms to match a large variety of signals. It would be often the case that those fixed time-frequency atoms are not the *intrinsic components* of the signal under study, and thus, often leads to misleading results.

The ideal method of time-frequency analysis for nonlinear and nonstationary signals would be to adaptively decompose a signal into certain basic intrinsic components (atoms) which are called *monocomponents* [3,4], and, for those components, one can define meaningful *instantaneous frequency* and furthermore construct the *time-frequency distribution*.

The newly developed *HHT programme* by Huang [5–7] provides a practical approach to the above mentioned decomposition. It contains two steps. The first is the algorithm called *empirical mode decomposition* (EMD), which is a certain adaptive decomposition. Applying this algorithm, a multicomponent can be decomposed into a finite sum of *intrinsic mode functions* (IMFs). The next step is an application of Hilbert transformation to the obtained IMFs. Experiments showed that the IMFs behave nicely with Hilbert transformation and can be used to define meaningful instantaneous frequency. In such a way, a new kind of time-frequency distributions called *Hilbert amplitude spectrum* may be constructed, through representing the instantaneous amplitude and frequency as functions of time in three-dimensional plots, and thus, the amplitude can be contoured in the time-frequency plane [5].

The notion of IMFs used by Huang plays a crucial role in the HHT practice. An IMF does not have a precise mathematical definition but is only an engineering description. The occurrences of the local maximums and minimums take turn, and, between a pair of adjacent local extremes, the signal is monotone and passes through the zero once, and is of the local symmetry: the means over the adjacent upper and lower envelopes are all of the zero value.

The notion of intrinsic mode functions (IMFs) is closely related to monocomponents, and to instantaneous frequency and amplitude, see, for instance, [1,3,4,8]. Now, we recall some studies on instantaneous frequency. It is well accepted that instantaneous frequency $\omega(t)$ of a complex signal $x(t) = a(t)e^{i\theta(t)}$ is defined to be the derivative of the phase $\theta(t)$, that is,

$$\omega(t) = \theta'(t), \qquad \text{provided } \theta'(t) \geq 0, \quad t \in \mathbb{R}. \tag{1.1}$$

The mean frequency $\langle\omega\rangle = \int_{\mathbb{R}} \omega|\hat{x}(\omega)|^2\, d\omega$ of such a complex signal can be reduced to

$$\langle\omega\rangle = \int_{\mathbb{R}} \theta'(t)\,|x(t)|^2\, dt. \tag{1.2}$$

Based on this observation, Cohen [4] suggested that the derivative θ' of phase θ be treated as instantaneous frequency ω.

For a real-valued signal $x(t)$, there are infinitely many ways to write $x(t)$ into an amplitude-frequency modulation $a(t)\cos\theta(t)$. There, however, would exist at most one such modulation that gives rise to a qualified instantaneous amplitude and frequency. Hilbert transformation defined through the singular integral,

$$\mathcal{H}x(t) = v.p.\frac{1}{\pi}\int_{\mathbb{R}} \frac{x(s)}{t-s}\, ds, \tag{1.3}$$

may be used to construct the *analytic signal associated with* $x(t)$, denoted by $x_a(t)$, defined by

$$x_a(t) = x(t) + i\mathcal{H}x(t). \tag{1.4}$$

The analytic signal may be further written as $x_a(t) = a(t)e^{i\theta(t)}$, and in that case this construction singles out a special amplitude-frequency modulation $x(t) = \operatorname{Re} x_a(t) = a(t)\cos\theta(t)$, called the *canonical modulation* of $x(t)$. The associated function pair $(a(t), \theta(t))$, characterized by $\mathcal{H}(a(\cdot)\cos\theta(\cdot))(t) = a(t)\sin\theta(t)$ and $x(t) = a(t)\cos\theta(t)$, is called the *canonical pair associated with* $x(t)$. With a canonical modulation, if $\theta'(t) \geq 0$, then θ' is defined to be the *instantaneous frequency* of the complex signal $x_a(t)$, as well as of the original real signal $x(t)$. In general, through the canonical modulation one cannot get the instantaneous frequency, as a signal is often of multicomponents. For instance, the "instantaneous frequency" of the signal $x(t) = \cos t + \cos 2t$ obtained through its analytic signal has negative values. This suggests to decompose multi-components into a sum of monocomponents of which each has a meaningful instantaneous frequency. A large number of literature adress this problem, see [1,3,4,8,9]. For an arbitrary modulation $x(t) = a(t)\cos\theta(t)$, the associated complex-valued function $x_q(t) = a(t)e^{i\theta(t)}$ is called the *quadrature associated with the modulation pair* $(a(t), \theta(t))$. The key problem is: for a modulation $x(t) = a(t)\cos\theta(t)$, under what conditions the associated quadrature $x_q(t)$ coincides with the associated analytic signal $x_a(t)$? The relation $x_q(t) = x_a(t)$ is equivalent to

$$\mathcal{H}(a(\cdot)\cos\theta(\cdot))(t) = a(t)\sin\theta(t).$$

Some relations of this question with Bedrosian's and Nuttall's theorems are observed.

PROPOSITION 1.1. BEDROSIAN'S PRODUCT THEOREM. *(See [8].) Assume that $f(t)$ and $g(t)$ are complex-valued signals of finite energy. If*

(i) *$\hat{f}(\omega) = 0$ for $|\omega| > \alpha$ and $\hat{g}(\omega) = 0$ for $|\omega| < \beta$, where $\beta \geq \alpha \geq 0$, or*
(ii) *$\hat{f}(\omega) = 0$ for $\omega < -\alpha$ and $\hat{g}(\omega) = 0$ for $\omega > \beta$, where $\beta \geq \alpha \geq 0$, then*

$$\mathcal{H}(fg)(t) = -if(t)g(t) = f(t)\mathcal{H}g(t). \tag{1.5}$$

Assertion (i) has an important application to our question. It implies that if the spectrums of the amplitude $a(t)$ and that of $\cos\theta(t)$ are, respectively, of low-pass and high-pass and disjoint, i.e.,

$$\operatorname{supp}\hat{a}(\omega) \subset (-\omega_0, \omega_0) \quad \text{and} \quad \operatorname{supp}\widehat{\cos\theta(\cdot)}(\omega) \subset \mathbb{R}\setminus(-\omega_0, \omega_0), \qquad \omega_0 > 0,$$

then

$$\mathcal{H}\left(a(\cdot)\cos\theta(\cdot)\right)(t) = a(t)\mathcal{H}\cos\theta(t). \tag{1.6}$$

Under the assumptions on the spectrums, we are reduced to asserting conditions on θ such that

$$\mathcal{H}\cos\theta(t) = \sin\theta(t), \tag{1.7}$$

and, as consequence, $x_q(t) = x_a(t)$.

Nuttall's theorem [9] estimates the energy error when the quadrature signal is replaced by the analytic signal.

PROPOSITION 1.2. NUTTALL THEOREM.

$$\|x_a - x_q\|_2^2 \leq 2\int_{-\infty}^{0} \left|\hat{x}_q(\omega)\right|^2 d\omega. \tag{1.8}$$

In 1978, Vakman and Vainshtein [10] offered a point-wise estimate of the error.

PROPOSITION 1.3.

$$|x_a(t) - x_q(t)| \le \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{0} |\hat{x}_q(\omega)| \, d\omega. \tag{1.9}$$

The rest of this note is devoted to characterize the class of signals,

$$\{f(t) : |f(t)| = 1, \text{ a.e., and } \mathcal{H}(\mathrm{Re}\, f) = \mathrm{Im} f\}. \tag{1.10}$$

Following [1], complex signals $e^{i\theta(t)}$ in (1.10) are called *phase signals*. We sometimes call them *analytic phase signals* as well. A phase signal with the property $\theta'(t) \ge 0$ is called an *admissible phase signal*. In [1], it is asserted that signals of the form,

$$e^{i\theta(t)} = e^{i(\theta_0 + \omega t)} \prod_{i=1}^{\infty} \frac{t - z_k}{t - \overline{z_k}}, \tag{1.11}$$

are phase signals. Apart from the constant factor and the factor of linear phase, they are boundary values of Blaschke products of the upper-half complex plane (see Section 2). This note further pursues this topic. We give a characterization of the signal class (1.10) and obtain a class of admissible phase signals larger than the class (1.11) given in [1].

The studies cited in below are related but not necessary to understand the present article. In [11,12], we approach this question through Möbius transform on the unit disc. We started with functions of the form $e^{i\theta_a(t)} = \tau_a(e^{it})$, $t \in (-\infty, \infty)$, $a \in \mathbb{C}$, $|a| < 1$, that are boundary values of Möbius transforms $\tau_a(z)$,

$$\tau_a(z) = \frac{z - a}{1 - \overline{a}z}.$$

In [11], Qian shows that a strictly increasing function $\theta(t)$, $t \in [0, 2\pi]$ with the Lebesgue measure $m(\theta([0, 2\pi])) = 2\pi$ gives rise to an admissible phase signal $e^{i\theta(t)}$ if and only if $d\theta(t)$ is a harmonic measure on the circle, or, equivalently, $\theta(t) = \theta_a(t)$ for some $a \in \mathbb{C}$ with $|a| < 1$. We call the corresponding admissible phase functions *nonlinear Fourier atoms*. This result has a nonperiodic counterpart for strictly increasing functions $\Theta(s), s \in \mathbb{R}$, with $m(\Theta(\mathbb{R})) = 2\pi$. In [13], we studied some wavelet and time-frequency aspects of the periodic nonlinear Fourier atoms and proved that for a fixed a the nonlinear Fourier system $\{e^{in\theta_a(t)} : n \in \mathbb{Z}\}$ constitutes a Riesz basis in the space $L^2([0, 2\pi])$. We obtained the explicit decomposition,

$$\theta_a(t) = t + 2 \arctan \frac{|a| \sin(t - t_a)}{1 - |a| \cos(t - t_a)}, \tag{1.12}$$

where the first term of the right-hand-side is linear and the second is nonlinear and periodic. Such a decomposition is unique. In the theoretical paper, [14] Qian included detailed proofs and related results.

The writing plan is as follows. In Section 2, we construct concrete examples of admissible phase signals. Section 3 deals with the general theory. Section 4 presents visual examples. In Section 5, we draw the conclusion of this study.

2. CONSTRUCTIVE EXAMPLES

Denote by $\mathbb{C}^+$ the upper-half complex plane and $\mathbb{D}$ the unit disc both without boundary.

DEFINITION 1. *If $f(t)$ is a Lebesgue measurable function with the properties,*

1°. $|f| = 1$, *a.e. on* $\mathbb{R}$ *(i.e., a unimodular function on* $\mathbb{R}$*); and*

2°. *in writing $f(t) = c(t) + is(t)$, there holds $\mathcal{H}c(t) = s(t)$, where $c(t)$ and $s(t)$ are real-valued, then we say that f is an analytic phase signal (or phase signal, or phase function). If, in addition, f may be written as $f(t) = e^{i\theta(t)}$, where θ is differentiable almost everywhere and $\theta'(t) \ge 0$, a.e., then f is said to be an admissible phase signal or admissible phase function.*

Note that the terminology phase signal should not be mixed up with the terminology phase: a phase signal may have the form $e^{i\theta(t)}$, while $\theta(t)$ is the phase of the signal.

The investigation starts from a recall of Plemelj theorem. The theorem says that if $f(t)$ is a function in $L^p(\mathbb{R})$, $1 < p < \infty$, then the Cauchy integral,

$$F(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t)}{t-z}\,dt$$

is a well defined analytic function in $\mathbb{C}^+$, and

$$\lim_{s\to 0^+} F(t_0 + is) = \frac{1}{2} f(t_0) + i\mathcal{H}\left(\frac{1}{2f}\right)(t_0), \text{ a.e.}$$

Based on this observation, we look for phase functions with nontangential boundary values from analytic functions in $\mathbb{C}^+$.

DEFINITION 2. *Let $f(z)$ be an analytic function in $\mathbb{C}^+$ with a.e., nontangential boundary limits on $\mathbb{R}$. If the boundary value function is a phase signal, then $f(z)$ is said to be a grand analytic function.*

Note that not all analytic functions in $\mathbb{C}^+$ are grand analytic. For instance, the function e^{-z^2} is analytic in the whole complex plane with the boundary value e^{-x^2} on $\mathbb{R}$, that is, not a phase signal, and thus e^{-z^2} is not grand analytic. In view of the maximal modulus theorem, we would look for grand analytic functions from those that map $\mathbb{C}^+$ to $\mathbb{D}$, and $\mathbb{R}$ to $\partial\mathbb{D}$. In the rest of the section, we give some fundamental conformal mappings that give rise to admissible phase functions.

(i) *Cayley transform*

$$\kappa(z) = \frac{i-z}{i+z},$$

or, more generally, all the conformal mappings from $\mathbb{C}^+$ to $\mathbb{D}$ of the form,

$$\nu(z) = -\frac{z-z_0}{z-\overline{z_0}}, \qquad \mathrm{Im} z_0 > 0.$$

The boundary value of Cayley transform κ is

$$e^{i\theta(t)} = \frac{i-t}{i+t} = \frac{1-t^2}{1+t^2} + i\frac{2t}{t^2+1} = \cos(2\arctan t) + i\sin(2\arctan t)$$

and

$$\theta(t) = 2\arctan t, \qquad \frac{1}{2\pi}\theta'(t) = \frac{1}{\pi}\frac{1}{1+t^2} = P_1(t) > 0,$$

the Poisson kernel of $\mathbb{R}$ at $y = 1$. Theorem 3.1 in Section 3 guarantees that $e^{2i\arctan t}$ is an admissible phase function.

Owing to the multiplication rule proved in Section 3 any finite products of functions of the form ν, or infinite products with certain conditions, give rise to admissible phase functions. They are Blaschke products in $\mathbb{C}^+$ studied in [1].

(ii) *Periodized Möbius transform*

For any $a \in \mathbb{D}, a \neq 0$, a Möbius transform $\tau_a(z), \tau_a(a) = 0$, is of the form,

$$\tau_a(z) = e^{i\alpha}\frac{z-a}{1-\overline{a}z}, \qquad a \in \mathbb{R}.$$

The corresponding boundary value function is

$$e^{i\theta_a(t)} = \tau_a(e^{it}) = e^{i\alpha}\frac{e^{it}-a}{1-\overline{a}e^{it}}, \qquad t \in [-\pi, \pi].$$

Denote $\iota(z) = e^{iz}$. The composed mappings $\tau_a \circ \iota$, in $\mathbb{C}^+$, are called *periodized Möbius transforms.* It is easy to compute [11] that

$$\frac{1}{2\pi}\theta'_a(t) = \frac{1}{2\pi}\frac{1-|a|^2}{1-2|a|\cos(t-\alpha)+|a|^2} = p_a(t) > 0,$$

called the *periodized Poisson kernel*. The functions $\tau_a \circ \iota$ are grand analytic, and the boundary values $e^{i\theta_a(t)}$ are admissible phase signals.

In general, we may form $2L$-periodized Möbius transforms $e^{i\theta_a(\pi z/L)}$ which are admissible phase functions. if $a = 0$, then the corresponding Möbius transform is of the form $\tau_0(z) = e^{i\alpha}z$, and the corresponding phase function reduces to $e^{i(\theta_0+\omega t)}$, $\theta_0 \in \mathbb{R}$, $\omega \geq 0$.

(iii) *Pseudo-periodized Möbius transform*

By composing the one to one and onto conformal mappings $\mathbb{C}^+ \to \mathbb{C}^+$

$$\mu_{b,c,d,e}(z) = \frac{bz+c}{dz+e}, \qquad b,c,d,e \in \mathbb{R}, \quad be-cd > 0,$$

with Möbius transforms τ_a we obtain *pseudo-periodized Möbius transforms* $\tau_a \circ \mu_{b,c,d,e}(z) = \tau_a(e^{i((bz+c)/(dz+e))})$. Note that the $2L$-periodized Möbius transform is the particular case with $\mu(z) = \pi z/L$. A direct computation shows that, with

$$e^{i\theta(t)} = e^{i((bt+c)/(dt+e))},$$

we have

$$\theta'(t) = \frac{be-cd}{(dt+e)^2} > 0, \qquad \text{if } dt+e \neq 0.$$

Hence, the boundary value is an admissible phase function.

In particular, this class includes those of the form $e^{i\theta(t)} = e^{i((bt+c)/(dt+e))}$ that do not fall into the scopes of (i) and (ii).

Besides composition one can also use multiplication, as already mentioned, to construct phase signals (see Section 3). The rule asserts that products of any finite many or countably infinite many of phase functions or admissible phase functions are still phase functions or admissible phase functions, respectively, provided that the involved infinite products converge.

Constructed from the elementary phase signals in (i), (ii), and (iii), using composition and multiplication, functions of the following type are admissible phase signals.

$$e^{i\theta_0}\prod_{k=1}^{\infty}\frac{z_k-z}{z-\overline{z}_k}\exp\left(i\sum_{k=1}^{\infty}\theta_{w_k}\left(\frac{b_kz+c_k}{d_kz+e_k}\right)\right),$$

where θ_0 is a real constant, $z_k, k = 1,2,\dots$, are complex numbers in the upper-half complex plane, $w_k, k = 1,2,\dots$, are complex numbers in the unit disc, and b_k, c_k, d_k, e_k are real numbers satisfying $b_ke_k - c_kd_k > 0$ for each k.

Note that the phase signals identified in [1] form a proper subclass of the class defined by the above form. Indeed, the periodic and pseudo-periodic factors are from the so called *singular functions* in complex analysis [14].

3. GENERAL THEORY

The constructions given in Section 2 are based on a general theory. The basic references of this section are [15–18]. The Nevanlina class is the class of analytic functions, f, in $\mathbb{C}^+$, and $\log|f|$ has the least harmonic majorant the Poisson integral of a Radon measure on $\mathbb{R}$. Since functions in the Nevanlina class have harmonic majorants, Fatou's theorem on nontangential boundary limits of harmonic functions asserts that they have nontangential boundary limits, a.e. These observations suggest to look for grand analytic functions from the Nevanlina class.

Functions in the Nevanlina class have the following characteristic property.

If F belongs to the Nevanlina class, then

$$F(z) = \frac{B(z)G(z)S_1(z)}{S_2(z)},$$

where B is a Blaschke product, $G(z)$ is an *outer function*, S_1 and S_2 are *singular functions* (see the definitions below).

Inner functions are defined to be bounded analytic functions in $\mathbb{C}^+$ with unimodular boundary values on $\mathbb{R}$. A function F is an inner function if and only if it has the factorization $F = BS$, where B is a Blaschke product and S is a singular function.

An *outer function*, G, in $\mathbb{C}^+$, can be expressed as

$$G(z) = e^{u(z)+iv(z)}$$

and (3.1)

$$u(z) = P_y * (\log h)(x),$$

where $h \geq 0$, $\int_{\mathbb{R}}(|\log h(t)|/(1+t^2))\,dt < \infty$, P_y is the Poisson kernel, and v is a harmonic conjugate function of u. A *singular function* S in $\mathbb{C}^+$ has the representation,

$$S(z) = e^{i(\omega_0 z + \int_{\mathbb{R}}(1/(\lambda - z) - \lambda/(\lambda^2+1))\,dm(\lambda))}, \tag{3.2}$$

where $\omega_0 > 0, dm(\lambda)$ is a positive Borel measure such that $(1/(\lambda^2+1))\,dm(\lambda)$ is finite.

In below we present two theorems. Their proofs are outlined. For details see [14].

THEOREM 3.1. *Let $F = C + iS$ be an inner function, where C and S are real-valued. Then, its nontangential boundary value is an analytic phase function, that is $\mathcal{H}C = S$ in the distribution sense on the real line.*

PROOF. By using Cayley transform the bounded analytic function F in $\mathbb{C}^+$ is transformed to the corresponding bounded analytic function, f, in $\mathbb{D}$. The boundary value of F on $\mathbb{R}$ is accordingly transformed to the boundary value of f on $\partial\mathbb{D}$. Below, we will make no difference in notation between analytic functions in $\mathbb{C}^+$ and their boundary values on $\mathbb{R}$. Since f belongs to the Hardy $H^\infty(\mathbb{D})$ space, it is the Cauchy integral of its boundary value function f. The Plemelj formula can be applied to the Cauchy integral, to result

$$f\left(e^{it}\right) = \left(\frac{1}{2}\right) f\left(e^{it}\right) + i\left(\frac{1}{2}\right)\tilde{\mathcal{H}}(f)\left(e^{it}\right)$$

or

$$f\left(e^{it}\right) = i\tilde{\mathcal{H}}f\left(e^{it}\right),$$

where $\tilde{\mathcal{H}}$ is the *circular Hilbert transform* on the unit circle. Substituting $f(e^{it})$ with $c(t) + is(t)$, we obtain

$$\tilde{\mathcal{H}}c(t) = s(t), \qquad \tilde{\mathcal{H}}s(t) = -c(t).$$

Now through Cayley transform again these relations are transformed back to $\mathbb{R}$. Therefore, we have $\mathcal{H}C = S$, $\mathcal{H}S = -C$. Note that for bounded functions Hilbert transformation is taken to be of the distribution sense, through harmonic representations of distributions (see [19]).

The converse to the above theorem also holds.

THEOREM 3.2. *Let $F = C+iS$ be unimodular, where C and S are real-valued, satisfying $\mathcal{H}C = S$ in the distribution sense. F is then the nontangential boundary value of an inner function in $\mathbb{C}^+$.*

PROOF. The Poisson integral U of the bounded function C is a harmonic representation of C which is bounded in $\mathbb{C}^+$. Let V be any harmonic conjugate of U. Due to the theory of distributional Hilbert transform [19] and the assumption of the theorem, the function V, modula a constant, is a harmonic representation of S. To conclude the theorem it suffices to show that V is bounded. The functions U, V are transformed, by Cayley transform, to u, v, in the unit disc $\mathbb{D}$, so that $u + iv$ is analytic in $\mathbb{D}$. The distributional boundary values C, S are transformed to c, s that are the distributional boundary values of u, v respectively. The advantage of the disc is that on the disc the analytic function $u + iv_1$ obtained by integrating the boundary function c against the Schwarz kernel is in the Hardy $H^2(\mathbb{D})$ space. A distributional argument asserts that the boundary value v_1 and s defer at most by a constant and thus the boundary value v_1 is bounded. As a function in H^2 with a bounded boundary value, the function $u + iv_1$ is further asserted to be a bounded analytic function in $\mathbb{D}$, and, since v and v_1 defer at most by a constant, $u + iv$ is a bounded analytic function in $\mathbb{D}$, too. Using Cayley transform again, $U + iV$ is bounded analytic in $\mathbb{C}^+$.

4. VISUAL EXAMPLES

In this section, we show some concrete examples.

(i) The case of Blaschke products in $\mathbb{C}^+$

Choosing z_0 to be i, $3 + 3i$ and $1/2 + i/2$ in the conformal mapping,

$$\nu(z) = -\frac{z - z_0}{z - \overline{z}_0},$$

and multiplying the three obtained analytic functions, we get the admissible phase signal $e^{i\theta(t)}$ with

$$\theta(t) = 2\arctan t + 2\arctan\left(\frac{t}{3} - 1\right) + 2\arctan(2t - 1).$$

Figure 1 illustrates the plots of the signal,

$$x(t) = \cos\left(2\arctan t + 2\arctan\left(\frac{t}{3} - 1\right) + 2\arctan(2t - 1)\right),$$

its phase and its instantaneous frequency.

(ii) The case of Periodic Möbius transform

Let

$$a = \frac{1}{2} \quad \text{and} \quad a = \frac{1}{3}$$

in the Möbius transform,

$$\tau_a(z) = \frac{z - a}{1 - \overline{a}z},$$

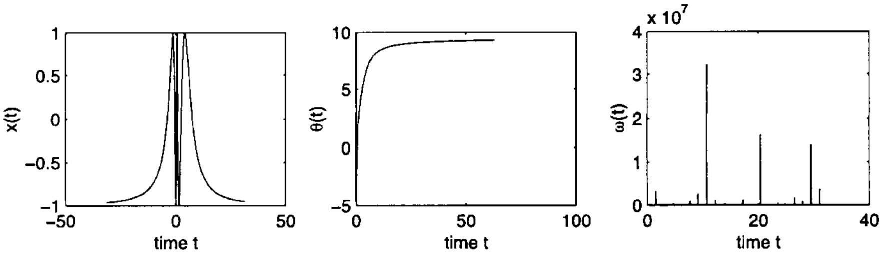
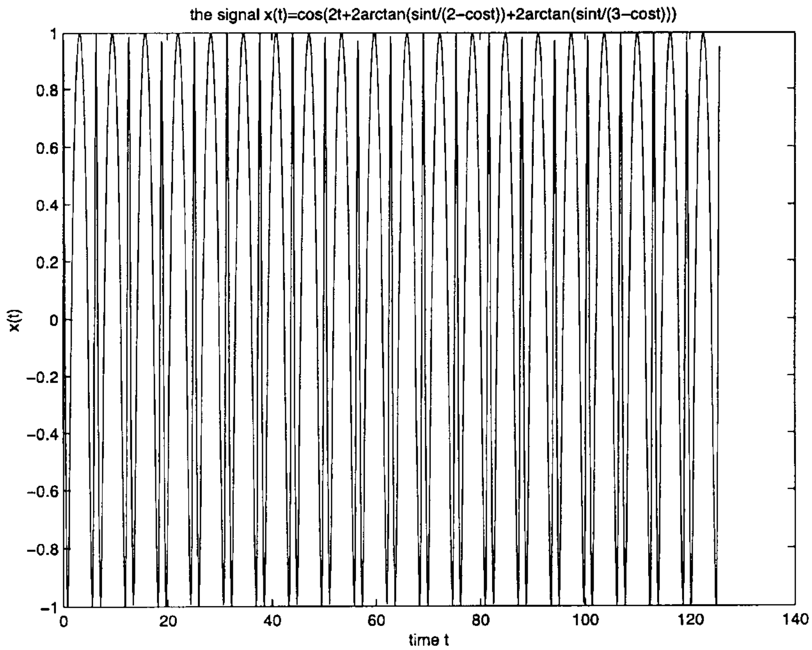


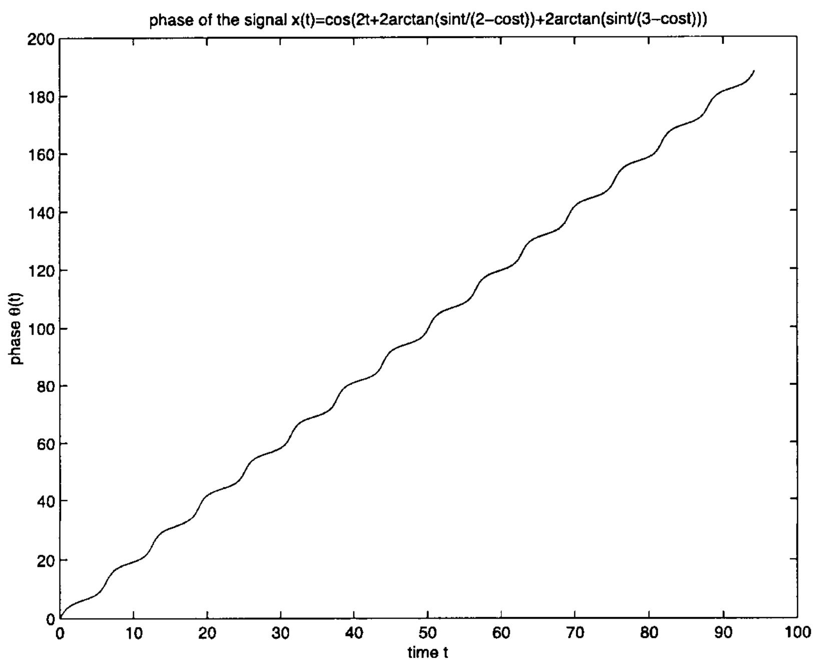
Figure 1. Left: the plot of $x(t) = \cos(2\arctan t + 2\arctan(t/3 - 1) + 2\arctan(2t - 1))$. Middle: the phase of the signal $x(t)$. Right: the instantaneous frequency of the signal $x(t)$.

and multiply the obtained analytic functions. By using the decomposition (1.12), we get the following signal,

$$x(t) = \cos\left(2t + 2\arctan\frac{\sin t}{2-\cos t} + 2\arctan\frac{\sin t}{3-\cos t}\right).$$



(a). The plot of the signal $x(t) = \cos(2t + 2\arctan(\sin t/(2-\cos t)) + 2\arctan(\sin t/(3-\cos t)))$.



(b). The phase of the signal $x(t)$.

Figure 2.

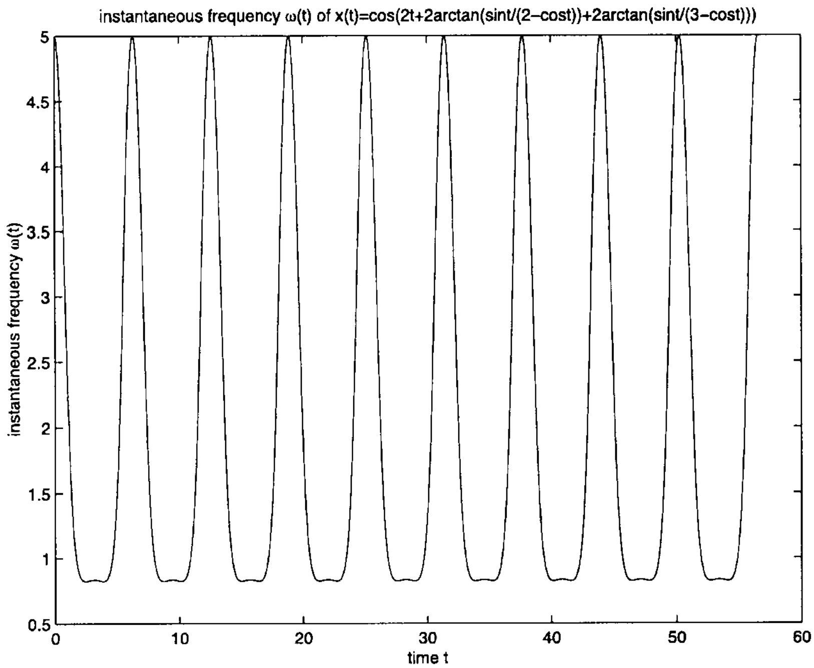
Figure 2 is the plots of the signal,

$$x(t) = \cos\left(2t + 2\arctan\frac{\sin t}{2-\cos t} + 2\arctan\frac{\sin t}{3-\cos t}\right),$$

its phase and its instantaneous frequency.

(iii) The case of Pseudo-periodized Möbius transform

We study the admissible phase signal $x(t) = \cos((2t+1)/(t+1))$. Figure 3 is the plots of the signal $x(t)$, its phase and its instantaneous frequency.



(c). The instantaneous frequency of the signal $x(t)$.

Figure 2. (cont.)

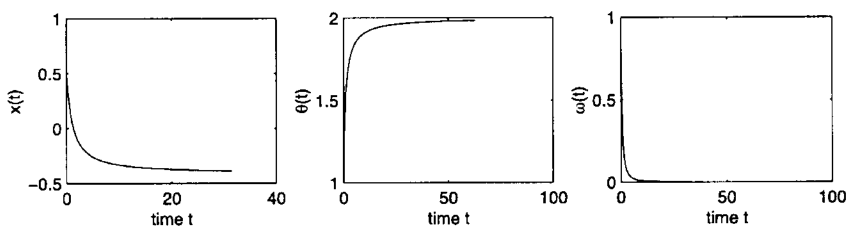


Figure 3. Left: the plot of the admissible phase signal $x(t) = \cos((2t+1)/(t+1))$. Middle: the phase of the signal $x(t)$. Right: the instantaneous frequency of the signal $x(t)$.

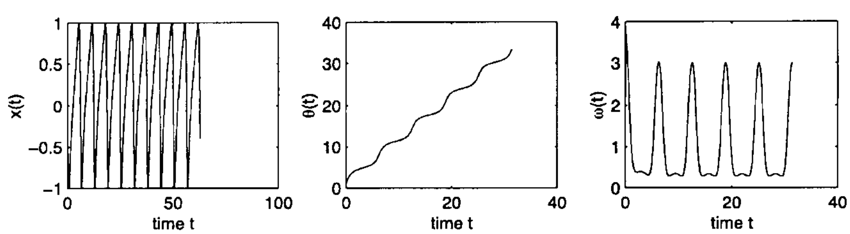


Figure 4. Left: the plot of the signal $x(t) = \cos(t + 2\arctan(\sin t/(2-\cos t)) + ((2t+1)/(t+1)))$. Middle: the phase of the signal $x(t)$. Right: the instantaneous frequency of the signal $x(t)$.

(iv) Mixed case

The product of the admissible phase signals $e^{i(t+2\arctan(\sin t/(2-\cos t)))}$ and $e^{i((2t+1)/(t+1))}$ is a new admissible phase signal,

$$e^{i\theta(t)} = e^{i(t+2\arctan(\sin t/(2-\cos t))+((2t+1)/(t+1)))}.$$

Figure 4 illustrates the plots of the signal,

$$x(t) = \cos\left(t + 2\arctan\frac{\sin t}{2-\cos t} + \frac{2t+1}{t+1}\right),$$

its phase and its instantaneous frequency.

5. CONCLUSION

We proved that a necessary and sufficient condition for unimodular functions $f(t)$ to satisfy

$$\mathcal{H}(\mathrm{Re} f) = \mathrm{Im} f$$

in the distribution sense is that f is the boundary value of an inner function in the upper-half-complex plane. The class of such functions includes a subclass with the parametrization,

$$f(t) = \cos\theta(t) + i\sin\theta(t),$$

satisfying

$$\theta'(t) \geq 0.$$

For signals in the mentioned subclass meaningful instantaneous frequencies can be defined. Some examples of such parameterized functions are constructed that contains the class studied in [1] as a proper subclass. Based on Bedrosian's theorem nonunimodular solutions of the singular integral equation,

$$\mathcal{H}(\mathrm{Re} f) = \mathrm{Im} f,$$

may be deduced that are closely related to adaptive decomposition of nonstationary and nonlinear signals in time-frequency analysis.

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