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Direct Sum Decomposition of $L^2(\mathbf{R}_1^n)$ into Subspaces Invariant under Fourier Transformation

Ming-gang Fei and Tao Qian

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ABSTRACT. *Denote by* $\mathbf{R}_1^n$ *the real-linear span of* $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n$*, where* $\mathbf{e}_0 = 1, \mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}, 1 \leq i, j \leq n$. *Under the concept of left-monogeneity defined through the generalized Cauchy-Riemann operator we obtain the direct sum decomposition of* $L^2(\mathbf{R}_1^n)$, $n > 1$,

$$L^2(\mathbf{R}_1^n) = \sum_{k=-\infty}^{\infty} \bigoplus \Omega^k ,$$

where Ω^k *is the right-Clifford module of finite linear combinations of functions of the form* $R(x)h(|x|)$*, where, for* $d = n+1$*, the function* R *is a* k*- or* $-(d+|k|-2)$*-homogeneous left-monogenic function, for* $k > 0$ *or* $k < 0$*, respectively, and* h *is a function defined in* $[0, \infty)$ *satisfying a certain integrability condition in relation to* k*, the spaces* Ω^k *are invariant under Fourier transformation. This extends the classical result for* $n = 1$*. We also deduce explicit Fourier transform formulas for functions of the form* $R(x)h(r)$ *refining Bochner's formula for spherical* k*-harmonics.*

1. Introduction

Fourier analysis in Euclidean spaces is intimately connected with the action of the group of rotations, as well as that of the groups of translations and dilations. The related study not only has fruitful results by itself, but also stimulates elegant generalizations to abstract harmonic analysis on groups. This note will concentrate in the rotation aspect. Among

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the special features of the theory is the invariance of Fourier transformation on certain subspaces of the square integrable functions, defined through radial functions, spherical harmonics and Bessel functions. The latter is regarded as the symmetric property of Fourier transformation [8, 3].

In the one-dimensional Euclidean space a function may be decomposed into a sum of an even function and an odd function. It is easy to verify that the Fourier transform of an even function is still an even function, and that of an odd function is still an odd function (We suppress the words "square-integrable" in front of "function" when we discuss Fourier transform of the function). In the two-dimensional space the same kind of decomposition is induced by the Fourier series expansion of the function restricted on the unit circle. The space of square-integrable functions has the direct sum decomposition (see Section 2 for details)

$$L^2(\mathbf{R}^2) = \sum_{k=-\infty}^{\infty} \bigoplus \Omega^k ,$$

where for each k, $k \in \mathbf{Z}$, the set for integers, the closed subspaces Ω^k is the totality of the square-integrable functions of the form $g(r)e^{ik\theta},\ x = re^{i\theta} \in \mathbf{R}^2$. It is proved that for any k, the space Ω^k is invariant under Fourier transformation in $\mathbf{R}^2$. In the spaces $\mathbf{R}^n,\ n > 2$, the kind of direct sum decomposition is achieved through the spherical harmonics decomposition of square-integrable functions on the unit sphere, being reads as (see Section 2 for details)

$$L^2(\mathbf{R}_1^n) = \sum_{k=0}^{\infty} \bigoplus \mathcal{N}_k ,$$

where $\mathcal{N}_k$, $k \geq 0$, are closed subspaces consisting of the functions of the form $H_k(x')f_0(r)$, $x = rx', r = |x|$, and H_k a spherical harmonics of degree k. It is shown that Fourier transformation in $\mathbf{R}^n$ preserves the subspaces. That is, if $f \in \mathcal{N}_k$ with $f(x) = H_k(x')f_0(r)$, then $\hat{f}(x) = H_k(x')g_0(r)$. It is further proved that g_0 is determined by f_0 and the index k through a Bochner's formula (see Section 2).

The described results for $n = 2$ and $n > 2$ are not quite the same. In fact, the result for $n = 2$ is finer, while for $n > 2$ something is missing. What is missing in $n > 2$ is a finer decomposition, like the decomposition of spherical harmonics into spherical analytic functions, given by

$$\cos k\theta = \frac{1}{2}(z^k + z^{-k}), \quad \sin k\theta = \frac{1}{2i}(z^k - z^{-k}) ,$$

where $z = e^{i\theta}, k = 1, 2, \ldots$. By virtue of Clifford algebras, analytic functions are generalized to left- (or right-) monogenic functions. The counterpart decompositions of spherical harmonics on the sphere then has ground to be established [7, 5, 2]. The task of this note is to fill in the obvious gap, extending the direct sum decomposition to $k < 0$ by involving spherical monogenics. This presents exactly the same direct sum decomposition in $\mathbf{R}^n$ as in the $\mathbf{R}^2$ case. In particular, through a generalized Bochner's formula in the Clifford monogenics context, we show that the Fourier transformation invariance property still holds in the finer direct sum decomposition with an explicit formula representation. The results contribute better understanding to the symmetric property of Fourier transformation in Euclidean spaces, as well as to Bocnher's formula. The kind of decomposition is of particular interests in practice, as functions are decomposed into components of different phases. In some recent work useful sampling results in Bessel functions are established based on this decomposition [4].

2. Preliminaries

We will be working with $\mathbf{R}_1^n$, the real-linear span of $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n$, where $\mathbf{e}_0$ is identical with 1 and $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}$. The real- $(n+1)$-dimensional linear space $\mathbf{R}_1^n$ is embedded into the real-Clifford algebra $\mathbf{R}^{(n)}$ and complex-Clifford algebra $\mathbf{C}^{(n)}$ generated by $\mathbf{e}_1, \ldots, \mathbf{e}_n$ over the real and complex number fields, respectively. A typical element in $\mathbf{R}_1^n$ is denoted by $x = x_0 + \underline{x}$, where $x_0 \in \mathbf{R}$ and $\underline{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n \in \mathbf{R}^n$, $x_j \in \mathbf{R}$, $j = 1, 2, \ldots, n$. We usually write $x = rx'$, where $r = |x|$. A typical element in the complex-Clifford algebra $\mathbf{C}^{(n)}$ is

$$x = \sum_{S=\emptyset \ \text{or} \ (j_1,\ldots,j_l)} x_S \mathbf{e}_S \ ,$$

where $1 \le j_1 < \ldots j_l \le n$, $1 \le l \le n$, $x_S \in \mathbf{C}$, $\mathbf{e}_S = \mathbf{e}_{j_1} \ldots \mathbf{e}_{j_l}$, $\mathbf{e}_\emptyset = \mathbf{e}_0$. Functions to be studied in this note are assumed to be $\mathbf{R}_1^n$-variable and complex-Clifford algebra-valued. A general function is of the form $f(x) = \sum_S f_S(x)\mathbf{e}_S$, and the component functions f_S are complex-valued. Left- and right-monogenic functions are introduced via the generalized Cauchy-Riemann operator $D = \frac{\partial}{\partial x_0}\mathbf{e}_0 + \frac{\partial}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial}{\partial x_n}\mathbf{e}_n$: A function f with continuous first order derivatives is said to be *left-monogenic* or *right-monogenic* if $Df = 0$ or $fD = 0$ in its domain, respectively. In this note we only concern left-monogenic functions. The theory for right-monogenic functions is parallel. The Cauchy kernel is $E(x) = \frac{1}{A_n}\frac{\bar{x}}{|x|^{n+1}}$, $x \in \mathbf{R}_1^n \setminus \{0\}$, where A_n is the surface area of the n-dimensional unit sphere Σ_n in $\mathbf{R}_1^n$ and $A_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$. The Cauchy kernel is both left- and right-monogenic. Note that there exist Cauchy's Theorem and Cauchy's formula in this setting [1] or [2]. There exists also a Taylor and Laurent series theory for left-monogenic functions. For $n = 1$ we write $\mathbf{R}_1^1 = \mathbf{R}^2 = \mathbf{C} = \mathbf{R}^{(1)}$, the space of complex numbers, where left- and right- monogenic functions reduce to holomorphic functions. The symbols $\mathbf{Z}$ and $\mathbf{N}$ denote the sets of all integers and natural numbers, respectively. We denote by d the dimension $n+1$ of the linear space $\mathbf{R}_1^n$.

The Fourier transform of any function in $L^1(\mathbf{R}_1^n)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbf{R}_1^n} e^{-2\pi i x\cdot\xi} f(x)\, dx \ .$$

The inner product in $L^2(\Sigma_n)$ is defined by

$$(f, g) = \int_{\Sigma_n} \sum_S f_S(x')\overline{g_S(x')}\, dx' \ ,$$

where dx' denotes the n-dimensional Lebesgue area measure on Σ_n and $\overline{g_S(x')}$ the complex conjugate of $g_S(x')$.

The following function spaces are important to this study.

Definition 1. Let $k \in N$. Denote by

(i) M_k^+ the space of left-monogenic homogeneous polynomials in $\mathbf{R}_1^n$ of degree k. An arbitrary element of it, denoted by P_k, is called a left-inner monogenics of degree k.
(ii) M_k^- the space of left-monogenic homogeneous functions in $\mathbf{R}_1^n \setminus \{0\}$ of degree $-(d+k-1)$. An arbitrary element of it, denoted by Q_k, is called a *left-outer monogenics of degree k*.
(iii) $\mathcal{M}_k^+$ and $\mathcal{M}_k^-$ the spaces consisting of the restrictions to the unit sphere Σ_n of, respectively, the functions in M_k^+ and M_k^-. The elements of $\mathcal{M}_k^+$ and $\mathcal{M}_k^-$ are called *spherical monogenics,* or *surface spherical monogenics.*
(iv) $\mathcal{H}_k$ the space of surface spherical harmonics of degree k in $\mathbf{R}_1^n$.

For the lowest dimension $n = 1$ we recall the following result [8].
For $k \in \mathbf{Z}$, let

$$\Omega^k = \left\{ g \in L^2(\mathbf{R}_1^1) : g(z) = f(r)e^{ik\theta} \text{ for some measurable function } f(r) \right.$$
$$\left. \text{satisfying} \int_0^\infty |f(r)|^2 r\, dr < \infty \right\}.$$

We have the following.

Proposition 1. *The direct sum decomposition*

$$L^2(\boldsymbol{R}_1^1) = \sum_{k=-\infty}^{\infty} \bigoplus \Omega^k \tag{2.1}$$

holds in the sense that:
(a) *The subspaces Ω^k are closed.*
(b) *The subspaces Ω^k are mutually orthogonal, $k \in \mathbf{Z}$.*
(c) *Every function of $L^2(\boldsymbol{R}_1^1)$ is a limit of finite linear combinations of functions in $\bigcup_{k=-\infty}^{\infty} \Omega^k$.*
(d) *Fourier transformation maps each subspace Ω^k into itself.*

Furthermore, we have the following.

Proposition 2. *For any $f \in \Omega^k$ of the form $f(z) = f_0(r)e^{ik\theta}$, where $z = re^{i\theta}$, then $\hat{f}(\omega) = F_0(R)e^{ik\phi}$, where $\omega = \mathrm{Re}^{i\phi}$,*

$$F_0(R) = 2\pi i^{-k} \int_0^\infty f_0(r) J_k(2\pi Rr) r\, dr \,,$$

where $J_k(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{it\sin\theta} e^{-ik\theta}\, d\theta$ is the Bessel function of order k, $k \in \mathbf{Z}$.

For the spaces $\mathbf{R}_1^n$, $n > 1$, the result is not quite the same. There holds [8]

$$L^2(\mathbf{R}_1^n) = \sum_{k=0}^{\infty} \bigoplus \mathcal{N}_k \,, \tag{2.2}$$

where $\mathcal{N}_k$, $k \geq 0$, is the right-Clifford module of finite linear combinations of functions of the form $H(x)f(r)$, where f is a function, defined on $[0, \infty)$, satisfying $\int_0^\infty |f(r)|^2 r^{d+2k-1}\, dr < \infty$ and H a *solid harmonics of degree k.* Moreover, for $f \in \mathcal{N}_k$

of the form $f(x) = H_k(x)f_0(r)$, where $x = rx'$ and H_k is a k-harmonics, there holds Bochner's formula in terms of spherical harmonics

$$\hat{f}(x) = H_k(x)g_0(r) ,$$

where

$$g_0(r) = 2\pi i^{-k} r^{-[(d+2k-2)/2]} \int_0^\infty f_0(s) J_{(d+2k-2)/2}(2\pi r s) s^{(d+2k)/2} \, ds ,$$

and

$$J_k(t) = \frac{(t/2)^k}{\Gamma[(2k+1)/2]\Gamma\left(\frac{1}{2}\right)} \int_{-1}^1 e^{its}\left(1 - s^2\right)^{(2k-1)/2} ds$$

is a form of Bessel function of order k, $k \in \mathbf{N}$.

Thanks to Clifford algebra, for any $k \in N$, there holds the decomposition

$$\mathcal{H}_k = \mathcal{M}_k^+ \bigoplus \mathcal{M}_{k-1}^- . \tag{2.3}$$

Note that $\mathcal{M}_0^-$ is a one-dimensional space generated by the Cauchy kernel function E. The results (2.3) predates the reference [2]. It was established in four dimensions (the quaternionic space) in the classical article of A. Sudbery [7], and independently extended to all dimensions by F. Sommen in [6].

The space of solid harmonics of degree k, being isomorphic to the space of spherical k-harmonics, $\mathcal{H}_k$, has the finite dimension, say, γ_k. Let $\{H^{(1)}, H^{(2)}, \ldots, H^{(\gamma_k)}\}$ be an orthonormal basis of $\mathcal{H}_k$, where the inner product is inherited from that of $L^2(\Sigma_n)$. For any $f \in \mathcal{N}_k$, $k \in N$,

$$f(x) = \sum_{j=1}^{\gamma_k} H^{(j)}(x) f_j(r) = \sum_{j=1}^{\gamma_k} Y^{(j)}(x') f_j(r) r^k ,$$

where $Y^{(j)} \in \mathcal{H}_k$ and f_j are functions defined on $[0, \infty)$, $j = 1, 2, \ldots, \gamma_k$. From (2.3),

$$f(x) = \sum_{j=1}^{\gamma_k} \left(g_j^+(x') + g_j^-(x')\right) f_j(r) r^k ,$$

where $g_j^+ \in \mathcal{M}_k^+$, $g_j^- \in \mathcal{M}_{k-1}^-$. Therefore,

$$f(x) = \sum_{j=1}^{\gamma_k} P_j(x) f_j(r) + \sum_{j=1}^{\gamma_k} Q_j(x) g_j(r) , \tag{2.4}$$

where $P_j \in M_k^+$, $Q_j \in M_{k-1}^-$, and $g_j(r) = f_j(r) r^{d+2k-2}$.

Definition 2. Define

(i) Ω^k, $k \geq 0$, to be the right-Clifford module of finite linear combinations of functions of the form $P(x)f(r)$, where f is a function, defined on $[0, \infty)$, satisfying $\int_0^\infty |f(r)|^2 r^{d+2k-1} \, dr < \infty$ and P a left-inner monogenic function in $\mathbf{R}_1^n$ homogeneous of degree k.

(ii) $\Omega^{-k}, k > 0$, to be the right-Clifford module of finite linear combinations of functions of the form $Q(x)g(r)$, where g is a function, defined on $[0, \infty)$, satisfying $\int_0^\infty |g(r)|^2 r^{-(d+2k-3)}\, dr < \infty$ and Q is a left-outer monogenic function in $\mathbf{R}_1^n \setminus \{0\}$, homogeneous of degree $-(d + |k| - 2)$.

In (2.4), since $f_j(r)$ satisfies $\int_0^\infty |f(r)|^2 r^{d+2k-1}\, dr < \infty$, with $g_j(r) = f_j(r) r^{d+2k-2}$, we have $\int_0^\infty |g_j(r)|^2 r^{-(d+2k-3)}\, dr < \infty$. Therefore, we have

$$\begin{aligned} \mathcal{N}_0 &= \Omega^0\,, \\ \mathcal{N}_k &= \Omega^k \bigoplus \Omega^{-k},\ k \in N\,. \end{aligned}$$

We note that for $k \in \mathbf{N}$, the space Ω^{-k} corresponds to the space M_{k-1}^-.

3. Main Results

The following two theorems extend the classical $n = 1$ case (Proposition 1) to any $n \in \mathbf{N}$.

Theorem 1. *The direct sum decomposition*

$$L^2(\mathbf{R}_1^n) = \sum_{k=-\infty}^{\infty} \bigoplus \Omega^k$$

holds in the sense that:

(a) *The subspaces Ω^k are closed.*

(b) *The subspaces Ω^k are mutually orthogonal, $k \in \mathbf{Z}$.*

(c) *Every function of $L^2(\mathbf{R}_1^n)$ is a limit of finite linear combinations of functions in $\bigcup_{k=-\infty}^{\infty} \Omega^k$.*

By the direct sum decompositions (2.2) and (2.3), this theorem can be easily checked.

Theorem 2. *Fourier transformation maps each subspaces Ω^k, $k \in \mathbf{Z}$, into itself.*

Proof. For $k \geq 0$, since left-inner monogenic functions are harmonic, Bochner's formula on Fourier transform of functions in $\mathcal{N}_k$ implies that the subspaces Ω^k, $k \geq 0$, are invariant under Fourier transformation.

Now we consider Ω^{-k}, $k \in \mathbf{N}$. Since $L^1(\mathbf{R}_1^n) \cap L^2(\mathbf{R}_1^n)$ is dense in $L^2(\mathbf{R}_1^n)$, without loss of generality, we may assume $f \in \Omega^{-k}$ and $f \in L^1(\mathbf{R}_1^n) \cap L^2(\mathbf{R}_1^n)$, of the form $f(u) = Q(u) f_0(\rho) = Y(u') \rho^{-(d+k-2)} f_0(\rho)$ with $Y \in \mathcal{M}_{k-1}^-$, where $\rho = |u|$ and $u = \rho u'$. With $r = |x|$ and $x = rx'$, we have

$$\begin{aligned} \hat{f}(x) &= \int_{\mathbf{R}_1^n} e^{-2\pi i x\cdot u} f(u)\, du \\ &= \int_0^\infty \left\{ \int_{\Sigma_n} e^{-2\pi i r\rho x'\cdot u'} Y\left(u'\right) du' \right\} f_0(\rho) \rho^{-(d+k-2)+(d-1)}\, d\rho \\ &= \int_0^\infty \left\{ \int_{\Sigma_n} e^{-2\pi i r\rho x'\cdot u'} Y\left(u'\right) du' \right\} f_0(\rho) \rho^{-(k-1)}\, d\rho\,. \end{aligned}$$

Since $Y \in \mathcal{M}_{k-1}^-$, by (2.3), we have $Y \in \mathcal{H}_k$. Recalling the proof of (2.2) (Lemma 2.18, Chapter IV, [8]), there exists a function φ, defined on $[0, \infty)$, such that

$$\begin{aligned}
\hat{f}(x) &= \int_0^\infty \left\{ \int_{\Sigma_n} e^{-2\pi r\rho x'\cdot u'} Y(u')\,du' \right\} f_0(\rho)\rho^{-(k-1)}\,d\rho \\
&= \int_0^\infty \left(\varphi(r\rho)Y(x')\right) f_0(\rho)\rho^{-(k-1)}\,d\rho \\
&= Y(x')\left\{ \int_0^\infty f_0(\rho)\rho^{-(k-1)}\varphi(r\rho)\,d\rho \right\} \\
&= Q(x)\left\{ r^{d+k-2}\int_0^\infty f_0(\rho)\rho^{-(k-1)}\varphi(r\rho)\,d\rho \right\}.
\end{aligned}$$

Let $g_0(r) = r^{d+k-2}\int_0^\infty f_0(\rho)\rho^{-(k-1)}\varphi(r\rho)\,d\rho$. Since $f \in L^2(\mathbf{R}_1^n)$, by the Plancherel Theorem, $\|\hat{f}\|_2 < \infty$, where $\hat{f} = Q(x)g_0(r)$ and $Q(x) \in M_{k-1}^-$. With $\|Q\|_\Sigma = (\int_{\Sigma_n}|Q(x')|^2\,dx')^{1/2}$ and $\|Q\|_\Sigma < \infty$, we have

$$\begin{aligned}
\|\hat{f}\|_2^2 &= \int_{\mathbf{R}_1^n} |g(r)Q(x)|^2\,dx \\
&= \left(\int_0^\infty \left|g(r)r^{-(d+2k-3)}\right| dr \right) \|Q\|_\Sigma^2 < \infty .
\end{aligned}$$

So, $\int_0^\infty |g(r)|^2 r^{-(d+2k-3)}\,dr < \infty$. This shows $\hat{f} \in \Omega^{-k}, k \in \mathbf{N}$. □

When $k \geq 0$, the space Ω^k, being isomorphic to M_k^+, has the dimension $\alpha_k = C_{d+k-2}^k$ [2]. Let $\{P^{(1)}, P^{(2)}, \ldots, P^{(\alpha_k)}\}$ be an orthonormal basis of the space. A general function $F(x)$ in Ω^k can be uniquely written in the form $\sum_{j=0}^{\alpha_k} P^{(j)}(x)f_j(r)$, and

$$\|F\|_2^2 = \int_{\mathbf{R}_1^n} |F(x)|^2\,dx = \sum_{j=1}^{\alpha_k}\int_0^\infty |f_j(r)|^2 r^{d+2k-1}\,dr . \tag{3.1}$$

Similarly, when $k \in \mathbf{N}$, the space Ω^{-k}, being isomorphic to M_{k-1}^-, has the dimension $\beta_k = C_{d+k-3}^{k-1}$ [2]. Let $\{Q^{(1)}, Q^{(2)}, \ldots, Q^{(\beta_k)}\}$ be an orthonormal basis of the space. Any typical function $G(x)$ in Ω^{-k} can be uniquely written in the form $\sum_{j=0}^{\beta_k} Q^{(j)}(x)g_j(r)$, and

$$\|G\|_2^2 = \int_{\mathbf{R}_1^n} |G(x)|^2\,dx = \sum_{j=1}^{\beta_k}\int_0^\infty |g_j(r)|^2 r^{-(d+2k-3)}\,dr . \tag{3.2}$$

For $f \in L^2(\mathbf{R}_1^n)$, from Theorem 2, there exists a unique sequence $(f^{(k)})_{k\in\mathbf{Z}}$ such that

$$f = \sum_{k=-\infty}^{\infty} f^{(k)},$$

where $f^{(k)} \in \Omega^k,\ k \in \mathbf{Z}$. By the orthogonality of Ω^k, the Plancherel Theorem reads

$$\|f\|_2^2 = \sum_{k=-\infty}^{\infty} \|f^{(k)}\|_2^2 .$$

Using (3.1) and (3.2), we further write, for $k \geq 0$, $\|f^{(k)}\|_2^2 = \sum_{j=1}^{\alpha_k} \int_0^\infty |f_j^{(k)}(r)|^2 r^{d+2k-1}\, dr$; and, for $k < 0$, $\|f^{(k)}\|_2^2 = \sum_{j=1}^{\beta_{|k|}} \int_0^\infty |g_j^{(|k|)}(r)|^2 r^{-(d+2|k|-3)}\, dr$.

In the proof of Theorems 1 and 2 we know little about the function φ, so we did not get the explicit representation of $g_0(r)$ in terms of f_0. In below we will concentrate in obtaining such explicit formulae. When $k \geq 0$, $\Omega^k \subset \mathcal{N}_k$, and hence any function f in Ω^k is also in $\mathcal{N}_k$. Thus, Bochner's formula on harmonic polynomials can be used: For $f(x) = H_k(x) f_0(r)$, where H_k is either a k-harmonics or k-monogenics, we have $\hat{f}(x) = H_k(x) g_0(r)$, where

$$g_0(r) = 2\pi i^{-k} r^{-[(d+2k-2)/2]} \int_0^\infty f_0(s) J_{(d+2k-2)/2}(2\pi r s) s^{(d+2k)/2}\, ds . \tag{3.3}$$

Next we consider the case Ω^{-k}, $k \in \mathbf{N}$. We need more information on bases of Ω^{-k}.

Definition 3. Let

$$\omega_0(x) = E(x)$$
$$\omega_{l_1 \ldots l_k}(x) = (-1)^k \frac{\partial}{\partial x_{l_1}} \cdots \frac{\partial}{\partial x_{l_k}} E(x) ,$$

where $(l_1, \ldots, l_k) \in \{1, 2, \ldots, n\}^k$, $k \in \mathbf{N}$.

It is deduced in [2] that $\{\omega_{l_1 \ldots l_k} : (l_1, \ldots, l_k) \in \{1, 2, \ldots, n\}^k\}$ is a basis of M_k^-. For $x \in \mathbf{R}_1^n \setminus \{0\}$,

$$\begin{aligned}
\omega_{l_1 \ldots l_k}(x) &= \frac{(-1)^k}{A_n} \frac{\partial}{\partial x_{l_1}} \cdots \frac{\partial}{\partial x_{l_k}} \frac{\bar{x}}{|x|^{n+1}} \\
&= \frac{1}{n-1} \frac{(-1)^{k+1}}{A_n} \bar{D} \left[\frac{\partial}{\partial x_{l_1}} \cdots \frac{\partial}{\partial x_{l_k}} \frac{1}{|x|^{n-1}} \right] \\
&= \bar{D} \left[\frac{H_k^{(l_1 \ldots l_k)}(x)}{|x|^{n+2k-1}} \right] \\
&= \frac{1}{|x|^{n+2k+1}} \left[|x|^2 \bar{D} H_k^{(l_1 \ldots l_k)}(x) - (n+2k-1) \bar{x} H_k^{l_1 \ldots l_k}(x) \right] \\
&= \frac{H_{k+1}^{(l_1 \ldots l_k)}(x)}{|x|^{n+2k+1}} ,
\end{aligned}$$

where $H_k^{(l_1 \ldots l_k)}(x)$ and $H_{k+1}^{(l_1 \ldots l_k)}(x)$ are polynomials of homogeneity k and $k+1$, respectively. We will show that both of them are harmonic.

Lemma 1. *Let $G(x) = \frac{P(x)}{|x|^{n+2k-1}}$, $x \in \mathbf{R}_1^n \setminus \{0\}$, where $P(x)$ is a homogeneous polynomial of degree k, then*

$$\Delta G(x) = \frac{\Delta P(x)}{|x|^{n+2k-1}} ,$$

where Δ is the Laplacian for $n+1$ variables $x_0, x_1, \ldots, x_n$.

Proof. Consecutively taking partial derivatives, we have, for any $i = 0, 1, \ldots, n$,

$$\frac{\partial}{\partial x_i}G(x) = \left(\frac{\partial}{\partial x_i}P(x)\right)\frac{1}{|x|^{n+2k-1}} + P(x)[-(n+2k-1)]\frac{x_i}{|x|^{n+2k+1}},$$

$$\begin{aligned}\frac{\partial^2}{\partial x_i^2}G(x) &= \left(\frac{\partial^2}{\partial x_i^2}P(x)\right)\frac{1}{|x|^{n+2k-1}} + \left(\frac{\partial}{\partial x_i}P(x)\right)[-(n+2k-1)]\frac{x_i}{|x|^{n+2k+1}}\\ &\quad + \left(\frac{\partial}{\partial x_i}P(x)\right)[-(n+2k-1)]\frac{x_i}{|x|^{n+2k+1}}\\ &\quad + P(x)[-(n+2k-1)]\left[\frac{1}{|x|^{n+2k+1}} - (n+2k+1)\frac{x_i^i}{|x|^{n+2k+3}}\right]\\ &= \left(\frac{\partial^2}{\partial x_i^2}P(x)\right)\frac{1}{|x|^{n+2k-1}} - 2(n+2k-1)\frac{1}{|x|^{n+2k+1}}x_i\frac{\partial}{\partial x_i}P(x)\\ &\quad - (n+2k-1)\frac{P(x)}{|x|^{n+2k+1}} + \frac{(n+2k-1)(n+2k+1)}{|x|^{n+2k+3}}x_i^2P(x).\end{aligned}$$

Then

$$\begin{aligned}\Delta G(x) &= [\Delta P(x)]\frac{1}{|x|^{n+2k-1}} - 2(n+2k-1)\frac{1}{|x|^{n+2k+1}}\left[x_0\frac{\partial}{\partial x_0}P(x) + \cdots + x_n\frac{\partial}{\partial x_n}P(x))\right]\\ &\quad - (n+1)(n+2k-1)\frac{P(x)}{|x|^{n+2k+1}} + \frac{(n+2k-1)(n+2k+1)}{|x|^{n+2k+3}}(x_0^2 + \cdots + x_n^2)P(x)\\ &= \frac{\Delta P(x)}{|x|^{n+2k-1}} - \frac{2(n+2k-1)}{|x|^{n+2k+1}}\left[\left(x_0\frac{\partial}{\partial x_0}P(x) + \cdots + x_n\frac{\partial}{\partial x_n}P(x)\right) - kP(x)\right].\end{aligned}$$

Since $P(x)$ is homogeneous of degree k, by Euler's formula, we have that

$$\sum_{i=0}^{n} x_i\frac{\partial P(x)}{\partial x_i} = kP(x),$$

i.e., $$\left(x_0\frac{\partial}{\partial x_0}P(x) + \cdots + x_n\frac{\partial}{\partial x_n}P(x)\right) - kP(x) = 0.$$

Therefore, we get that

$$\Delta G(x) = \frac{\Delta P(x)}{|x|^{n+2k-1}}.$$

□

Corollary 1. *Functions $H_k^{(l_1\ldots l_k)}(x)$ and $H_{k+1}^{(l_1\ldots l_k)}(x)$ are harmonic.*

Proof. Set

$$g_{l_1\ldots l_k}(x) = \frac{H_k^{(l_1\ldots l_k)}(x)}{|x|^{n+2k-1}}.$$

Since $\omega_{l_1\ldots l_k}$ is left-monogenic in $\mathbf{R}_1^n \setminus \{0\}$, it follows from $\Delta = D\bar{D}$ that

$$\omega = \bar{D}g \quad \text{and} \quad \Delta g_{l_1\ldots l_k}(x) = D\omega_{l_1\ldots l_k}(x) = 0,\ x \in \mathbf{R}_1^n \setminus \{0\}.$$

Therefore, $g_{l_1\ldots l_k}$ is harmonic in $\mathbf{R}_1^n \setminus \{0\}$. From Lemma 1 we conclude that $H_k^{(l_1\ldots l_k)}(x)$ is harmonic. Since $\omega_{l_1\ldots l_k}$ is harmonic, the lemma implies that $H_{k+1}^{(l_1\ldots l_k)}(x)$ is harmonic. □

The corollary can also be established by noting that if $P(x)$ is left monogenic then $xP(x)$ is harmonic and then applying Kelvin inversion. That $xP(x)$ is harmonic was first deduced by A. Sudbery in [7] for quaternionic case and was extended to higher-dimensional cases by J. Ryan in [5].

The following extends the classical Bochner's formula to homogeneous monogenic functions of negative degrees.

Theorem 3. *Let $f \in \Omega^{-k}$ of the form $f(x) = Q(x)f_0(|x|)$, $Q(x) \in M_{k-1}^-$. Then*

$$\hat{f}(x) = Q(x)g_0(|x|)\,,$$

where, with $r = |x|$,

$$g_0(r) = 2\pi i^{-k} r^{(d+2k-2)/2} \int_0^\infty f_0(s) J_{(d+2k-2)/2}(2\pi rs) s^{-[(d+2k-4)/2]}\, ds\,, \tag{3.4}$$

where

$$J_k(t) = \frac{(t/2)^k}{\Gamma[(2k+1)/2]\Gamma\left(\frac{1}{2}\right)} \int_{-1}^1 e^{its}\left(1-s^2\right)^{(2k-1)/2} ds$$

is the Bessel function of order k.

Proof. From [2], the $-(d+k-2)$-homogeneous functions $\omega_{l_1\ldots l_k}$, $(l_1,\ldots,l_{k-1}) \in \{1,2,\ldots,n\}^{k-1}$, form a basis of M_{k-1}^-. As a function in M_{k-1}^-, the function Q has the form

$$Q(x) = \frac{H_k(x)}{|x|^{d+2k-2}}\,,$$

where H_k is a Clifford-valued k-homogeneous polynomial. Invoking the second assertion of Corollary 1, with k being replaced by $k-1$, we obtain that H_k is harmonic.

Consequently,

$$f(x) = Q(x)f_0(r) = H_k(x)f_0(r)r^{-(d+2k-2)} = H_k(x)F_0(r) \in \mathcal{N}_k\,.$$

Bochner's formula on Fourier transform of functions in $\mathcal{N}_k$ (Theorem 3.10, Chapter IV, [8]) gives

$$\hat{f}(x) = H_k(x)G_0(|x|) = Q(x)|x|^{d+2k-2}G_0(|x|) = Q(x)g_0(|x|)\,,$$

and

$$\begin{aligned} g_0(r) &= r^{d+2k-2}G_0(r) \\ &= r^{d+2k-2} 2\pi i^{-k} r^{-[(d+2k-2)/2]} \int_0^\infty F_0(s) J_{(d+2k-2)/2}(2\pi rs) s^{(d+2k)/2}\, ds \\ &= 2\pi i^{-k} r^{(d+2k-2)/2} \int_0^\infty f_0(s) J_{(d+2k-2)/2}(2\pi rs) s^{-[(d+2k-4)/2]}\, ds\,, \end{aligned}$$

where

$$J_k(t) = \frac{(t/2)^k}{\Gamma[(2k+1)/2]\Gamma\left(\frac{1}{2}\right)} \int_{-1}^1 e^{its}\left(1-s^2\right)^{(2k-1)/2} ds$$

is the Bessel function of order k. □

The formulas (3.3) and (3.4) together provide a refinement of Bochner's formula with spherical harmonics replaced by spherical monogenics. The formulas (3.3) and (3.4) can be unified into one formula by using the signum function.

Let $f \in \Omega^k, k \in \mathbf{Z}$, and $f(x) = R(x)h(r)$, where if $k \geq 0$, then $R(x) \in M_k^+$; and, if $k < 0$, then $R(x) \in M_{|k|-1}^-$. Then we have

$$\hat{f}(x) = R(x)H(r) ,$$

and, with $c_k = (d + 2|k| - 2)/2, k \in \mathbf{Z}$,

$$H(r) = 2\pi i^{-k} r^{-\operatorname{sgn}(k)c_k} \int_0^\infty f_0(s) J_{c_k}(2\pi rs) s^{1+\operatorname{sgn}(k)c_k} \, ds ,$$

where $\operatorname{sgn}(k)$ is the signum function that takes the value $+1$, -1 or 0 for $k > 0, k < 0$ or $k = 0$.

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Faculty of Science and Technology, The University of Macau, P.O. Box 3001, Macao (via Hong Kong)
e-mail: ya47402@umac.mo

Faculty of Science and Technology, The University of Macau, P.O. Box 3001, Macao (via Hong Kong)
e-mail: fsttq@umac.mo