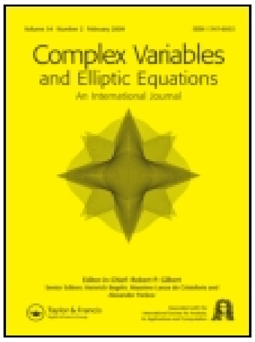

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An alternative proof of Fueter's theorem§

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In this article we establish an alternative proof of the generalized Fueter method presented in a former paper [Qian, T. and Sommen, F., 2003, Deriving harmonic functions in higher dimensional spaces. *Zeitschrift fur Analysis und ihre Anwendungen*, **22**(2), 275–288] leading to the construction of special harmonic and monogenic functions in higher dimensions. At the same time, we also obtain a generalization of this result.

Keywords: Dirac operators; Monogenic functions; Vekua systems

AMS Subject Classifications: 30G35; 32A25; 42B20

1. Introduction

It is a remarkable fact that the Cauchy–Riemann system in the plane generates monogenic functions. This was first observed by Fueter in [1] in the setting of quaternionic analysis.

Assume f to be holomorphic in an open set of the upper-half complex plane and substitute $f(z) = u(x, y) + iv(x, y)$ $(z = x + iy)$ where as usual $u = \operatorname{Re} f$, $v = \operatorname{Im} f$. Then, Fueter's theorem asserts that in the corresponding region the following relation holds:

$$D\Delta\left(u(q_0, |\underline{q}|) + \frac{\underline{q}}{|\underline{q}|}v(q_0, |\underline{q}|)\right) = 0$$

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§Dedicated to Professor Guochun Wen on the occasion of his 75th birthday.

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with

$$\underline{q} = q_1 i + q_2 j + q_3 k$$
$$D = \partial_{q_0} + i\partial_{q_1} + j\partial_{q_2} + k\partial_{q_3}$$
$$\Delta = \partial^2_{q_0} + \partial^2_{q_1} + \partial^2_{q_2} + \partial^2_{q_3}$$

and i, j, k are the basic elements of the Hamilton quaternionic space.

Let e_j, $j = 1, 2, \ldots, m$ be the generating basic elements of the 2^m-dimensional real linear associative but non-commutative Clifford algebra $\mathbb{R}_{0,m}$, with the multiplication rules

$$e_i e_j + e_j e_i = -2\delta_{ij}, \quad i, j = 1, 2, \ldots, m.$$

Any element $a \in \mathbb{R}_{0,m}$ may be written as

$$a = \sum_A a_A e_A, \quad a_A \in \mathbb{R},$$

where $e_A = e_{i_1} e_{i_2} \cdots e_{i_k}$, $A = \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, m\}$, $i_1 < \cdots < i_k$ and for $A = \emptyset$, $e_\emptyset = 1$ is the identity element of $\mathbb{R}_{0,m}$.

For $k = 0, 1, \ldots, m$ fixed, we call

$$\mathbb{R}^{(k)}_{0,m} = \left\{ a \in \mathbb{R}_{0,m}\colon \sum_{|A|=k} a_A e_A \right\},$$

the subspace of k-vectors and thus we have that

$$\mathbb{R}_{0,m} = \sum_{k=0}^{m} \oplus \mathbb{R}^{(k)}_{0,m}.$$

For $a \in \mathbb{R}_{0,m}$, thus we may write

$$a = \sum_{k=0}^{m} [a]_k,$$

where $[a]_k$ is the projection of a on $\mathbb{R}^{(k)}_{0,m}$.

The subspace $\sum_{k\ even}^{m} \oplus \mathbb{R}^{(k)}_{0,m}$, called 'even subalgebra' is denoted by $\mathbb{R}^{+}_{0,m}$.

The Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebra $\mathbb{R}_{0,m}$ by identifying $(x_1, x_2, \ldots, x_m)$ with the vector variable $\underline{x}$ given by

$$\underline{x} = \sum_{j=1}^{m} x_j e_j.$$

The first-order linear differential operator

$$\partial_{x_0} + \partial_{\underline{x}} = \partial_{x_0} + \sum_{j=1}^{m} e_j \partial_{x_j},$$

called the Cauchy–Riemann operator, splits the Laplace operator in $\mathbb{R}^{m+1}$:

$$\Delta = \partial_{x_0}^2 + \sum_{j=1}^{m} \partial_{x_j}^2 = (\partial_{x_0} + \partial_{\underline{x}})(\partial_{x_0} - \partial_{\underline{x}}).$$

A continuously differentiable $\mathbb{R}_{0,m}$-valued function g defined in some open set of $\mathbb{R}^{m+1}$ solution of the equation $(\partial_{x_0} + \partial_{\underline{x}})g = 0$ is called a left monogenic function (see e.g. [2,3]).

The operator $\partial_{\underline{x}}$ is called the Dirac operator in $\mathbb{R}^m$. For a differentiable k-vector-valued function $F_k = \sum_{|A|=k} e_A F_{k,A}$ and a differentiable $\mathbb{R}_{0,m}$-valued function g, we have that (see e.g. [4])

$$\partial_{\underline{x}}(F_k g) = (\partial_{\underline{x}} F_k)g + 2\sum_{j=1}^{m}[e_j F_k]_{k-1}\partial_{x_j} g + (-1)^k F_k(\partial_{\underline{x}} g).$$

This Leibniz rule, in the particular case of a differentiable scalar-valued function ϕ reads as:

$$\partial_{\underline{x}}(\phi\, g) = (\partial_{\underline{x}}\phi)g + \phi(\partial_{\underline{x}} g), \tag{1}$$

and for a vector-valued function $\underline{f} = \sum_{j=1}^{m} e_j f_j$:

$$\partial_{\underline{x}}(\underline{f} g) = (\partial_{\underline{x}}\underline{f})g - 2\sum_{j=1}^{m} f_j \partial_{x_j} g - \underline{f}(\partial_{\underline{x}} g). \tag{2}$$

In [5], Sce extended the Fueter's theorem to $\mathbb{R}_{0,m}$ for m odd, i.e. under the same assumptions on f, the function f

$$\Delta^{(m-1)/2}(u(x_0, r) + \underline{\omega}\, v(x_0, r)) \quad (r = |\underline{x}|,\ r\underline{\omega} = \underline{x}),$$

is left monogenic.

In [6], Sommen generalized the Sce's result: If m is an odd positive integer, then

$$\Delta^{k+((m-1)/2)}[(u(x_0, r) + \underline{\omega}\, v(x_0, r))P_k(\underline{x})]$$

is also left monogenic function, where $P_k(\underline{x})$ is a homogeneous left-monogenic polynomial of degree k in $\mathbb{R}^m$.

The Fueter's theorem has also been considered for m even (see [7,8]) and also for non-integer powers (see [7]).

The Fueter's theorem provides us with the so-called axial monogenic functions of degree k (see [9,10]), i.e. monogenic functions of the form

$$(A(x_0, r) + \underline{\omega}\, B(x_0, r))P_k(\underline{x}),$$

A and B being $\mathbb{R}$-valued and satisfying the Vekua-type system

$$\partial_{x_0} A - \partial_r B = \frac{2k+m-1}{r} B,$$
$$\partial_{x_0} B + \partial_r A = 0.$$

We split up $\mathbb{R}^m$ as $\mathbb{R}^m = \sum_{s=1}^{d} \oplus \mathbb{R}^{p_s}$, $\sum_{s=1}^{d} p_s = m$. Therefore, the vector variable $\underline{x}$ may be written as

$$\underline{x} = \sum_{s=1}^{d} \underline{x}^{(s)}, \quad \underline{x}^{(s)} = \sum_{j=1}^{p_s} e_j^{(s)} x_j^{(s)},$$

and the Dirac operator $\partial_{\underline{x}}$ as

$$\partial_{\underline{x}} = \sum_{s=1}^{d} \partial_{\underline{x}^{(s)}}, \quad \partial_{\underline{x}^{(s)}} = \sum_{j=1}^{p_s} e_j^{(s)} \partial_{x_j^{(s)}}.$$

Let

$$r_s = \left|\underline{x}^{(s)}\right|, \ \underline{\omega}_s = \frac{\underline{x}^{(s)}}{r_s}, \quad s = 1, 2, \ldots, d.$$

Next, we consider the following harmonic multivector field

$$\underline{g}(x_0, r_1, \ldots, r_d) = (g_0(x_0, r_1, \ldots, r_d), \, g_1(x_0, r_1, \ldots, r_d), \ldots, g_d(x_0, r_1, \ldots, r_d))$$

i.e. $\underline{g}$ satisfies the Riesz system

$$\partial_{x_0} g_0 - \sum_{s=1}^{d} \partial_{r_s} g_s = 0,$$
$$\partial_{x_0} g_s + \partial_{r_s} g_0 = 0,$$
$$\partial_{r_s} g_j - \partial_{r_j} g_s = 0,$$

$s, j = 1, 2, \ldots, d, \ s \neq j.$

Looking for a version of Fueter's theorem in the poly-axial case, Qian and Sommen proved in [11] the following result: If p_s $(s = 1, 2, \ldots, d)$ are odd, then the function

$$\Delta^{(m-d)/2}\left(g_0(x_0, r_1, \ldots, r_d) + \sum_{j=1}^{d} \underline{\omega}_j \, g_j(x_0, r_1, \ldots, r_d) \right) \tag{3}$$

is left monogenic with respect to $\partial_{x_0} + \partial_{\underline{x}}$, where $\Delta = \partial_{x_0}^2 + \sum_{s=1}^{d} \Delta_{\underline{x}^{(s)}}$, $\Delta_{\underline{x}^{(s)}} = \sum_{j=1}^{p_s} \partial^2_{x_j^{(s)}}$.

The present article, extends the previous result as given in the following.

THEOREM 1 *Let $\underline{g}$, p_s $(s = 1, 2, \ldots, d)$ as above. Then*

$$\Delta^{k+((m-d)/2)}\left[\left(g_0(x_0, r_1, \ldots, r_d) + \sum_{j=1}^{d} \underline{\omega}_j\, g_j(x_0, r_1, \ldots, r_d)\right)\mathbf{P}_k(\underline{x})\right],$$

is also left monogenic with respect to $\partial_{x_0} + \partial_{\underline{x}}$, where $\mathbf{P}_k(\underline{x}) = \prod_{j=1}^{d} P_{k_j}(\underline{x}^{(j)})$, $k = \sum_{j=1}^{d} k_j$ and $P_{k_j}(\underline{x}^{(j)})$ is a homogeneous left-monogenic polynomial of degree k_j in $\mathbb{R}^{p_j}$ with values in $\mathbb{R}^{+}_{0,p_j}$.

In [11], the authors gave an elegant proof of the Fueter's theorem based on the fact that the function

$$g_0(x_0, r_1, \ldots, r_d) + \sum_{j=1}^{d} \underline{\omega}_j\, g_j(x_0, r_1, \ldots, r_d),$$

may be written locally as $(\partial_{x_0} - \partial_{\underline{x}})h(x_0, r_1, \ldots, r_d)$ for some scalar harmonic function in the $d+1$ variables $x_0, r_1, \ldots, r_d$. Therefore, function (3) is left monogenic if

$$\Delta^{(m-d+2)/2}h(x_0, r_1, \ldots, r_d) = 0,$$

see [11], Theorem 3.

The present article is not just an extension of the mentioned result in [11], but also proves it in a different way.

The sketch of the proof of Theorem 1 is as follows: First to prove that

$$\Delta^{k+((m-d)/2)}\left[\left(g_0(x_0, r_1, \ldots, r_d) + \sum_{j=1}^{d} \underline{\omega}_j\, g_j(x_0, r_1, \ldots, r_d)\right)\mathbf{P}_k(\underline{x})\right],$$

has the form

$$\left(A_0(x_0, r_1, \ldots, r_d) + \sum_{j=1}^{d} \underline{\omega}_j\, A_j(x_0, r_1, \ldots, r_d)\right)\mathbf{P}_k(\underline{x}),$$

and then verify that A_j $(j = 0, 1, \ldots, d)$ satisfy the corresponding Vekua system for poly-axial monogenic functions of degree k.

2. Proof of Theorem 1

The proof of Theorem 1 is divided into several steps:

LEMMA 1 *Let $A_j(x_0, r_1, \ldots, r_d)$ $(j = 0, 1, \ldots, d)$ $\mathbb{R}$-valued, then the function*

$$\left(A_0(x_0, r_1, \ldots, r_d) + \sum_{j=1}^{d} \underline{\omega}_j\, A_j(x_0, r_1, \ldots, r_d)\right)\mathbf{P}_k(\underline{x}),$$

is left monogenic if the A_j $(j = 0, 1, \ldots, d)$ *are solutions of the system*

$$\partial_{x_0} A_0 - \sum_{j=1}^{d} \partial_{r_j} A_j = \sum_{j=1}^{d} \frac{2k_j + p_j - 1}{r_j} A_j,$$
$$\partial_{x_0} A_l + \partial_{r_l} A_0 = 0,$$
$$\partial_{r_l} A_j - \partial_{r_j} A_l = 0,$$

$l, j = 1, 2, \ldots, d,\ l \neq j.$

Proof Applying (1) and (2) yields

$$\partial_{\underline{x}}(A_0\, \mathbf{P}_k(\underline{x})) = \left(\sum_{l=1}^{d} \underline{\omega}_l\, \partial_{r_l} A_0 \right) \mathbf{P}_k(\underline{x})$$

$$\partial_{\underline{x}}(A_j\, \underline{\omega}_j\, \mathbf{P}_k(\underline{x})) = \big((\partial_{\underline{x}} A_j) \underline{\omega}_j + A_j (\partial_{\underline{x}}\, \underline{\omega}_j) \big) \mathbf{P}_k(\underline{x}) - 2 \frac{A_j}{r_j} \sum_{l=1}^{p_j} x_l^{(j)} \partial_{x_l^{(j)}} \mathbf{P}_k(\underline{x})$$
$$= \left(\sum_{l=1}^{d} \partial_{r_l} A_j\, \underline{\omega}_l\, \underline{\omega}_j - \frac{p_j - 1}{r_j} A_j \right) \mathbf{P}_k(\underline{x}) - 2 \frac{k_j}{r_j} A_j\, \mathbf{P}_k(\underline{x}).$$

Therefore,

$$(\partial_{x_0} + \partial_{\underline{x}}) \left(\left(A_0 + \sum_{j=1}^{d} \underline{\omega}_j\, A_j \right) \mathbf{P}_k(\underline{x}) \right) = \left(\partial_{x_0} A_0 - \sum_{j=1}^{d} \left(\partial_{r_j} A_j + \frac{2k_j + p_j - 1}{r_j} A_j \right) \right) \mathbf{P}_k(\underline{x})$$
$$+ \left(\sum_{j=1}^{d} (\partial_{x_0} A_j + \partial_{r_j} A_0)\, \underline{\omega}_j \right) \mathbf{P}_k(\underline{x})$$
$$+ \left(\sum_{l=1}^{d} \sum_{j=l+1}^{d} (\partial_{r_l} A_j - \partial_{r_j} A_l) \underline{\omega}_l\, \underline{\omega}_j \right) \mathbf{P}_k(\underline{x}),$$

which gives the desired result. ∎

Lemma 2 *Let* $h(x_0, r_1, \ldots, r_d)$ *be a scalar function. Then*

(i) $\partial_{r_s}^2 [D_{r_s}(\mu)\{h\}] = D_{r_s}(\mu)\{\partial_{r_s}^2 h\} - 2\mu D_{r_s}(\mu + 1)\{h\}$,
(ii) $\partial_{r_s}[D_{r_s}(\mu - 1)\{h/r_s\}] = D^{r_s}(\mu)\{h\}$,
(iii) $D^{r_s}(\mu)\{\partial_{r_s} h\} = \partial_{r_s}[D_{r_s}(\mu)\{h\}]$,
(iv) $D_{r_s}(\mu)\{\partial_{r_s} h\} - \partial_{r_s} D^{r_s}(\mu)\{h\} = (2\mu/r_s)\, D^{r_s}(\mu)\{h\}$.

where $D_{r_s}(\mu)\{h\} = ((1/r_s)\, \partial_{r_s})^{\mu}\{h\}$ *and*

$$D^{r_s}(0)\{h\} = h,$$
$$D^{r_s}(1)\{h\} = \partial_{r_s}\left(\frac{h}{r_s} \right),$$
$$D^{r_s}(\mu)\{h\} = \partial_{r_s}\Big(\frac{D^{r_s}(\mu - 1)\{h\}}{r_s} \Big), \quad \mu \geq 2,$$

$s = 1, \ldots, d.$

Proof To prove (i), we use mathematical induction. When $\mu = 1$, we have

$$\partial_{r_s}^2[D_{r_s}(1)\{h\}] = \frac{\partial_{r_s}^3 h}{r_s} - 2\frac{\partial_{r_s}^2 h}{r_s^2} + 2\frac{\partial_{r_s} h}{r_s^3}$$
$$= D_{r_s}(1)\{\partial_{r_s}^2 h\} - 2D_{r_s}(2)\{h\},$$

as desired.

Now, we proceed to show that when the case (i) holds for a positive integer μ, then (i) also holds for $\mu + 1$. Indeed,

$$\begin{aligned}\partial_{r_s}^2[D_{r_s}(\mu+1)\{h\}] &= D_{r_s}(1)\{\partial_{r_s}^2[D_{r_s}(\mu)\{h\}]\} - 2D_{r_s}(2)\{D_{r_s}(\mu)\{h\}\}\\ &= D_{r_s}(1)\Big\{D_{r_s}(\mu)\{\partial_{r_s}^2 h\} - 2\mu\, D_{r_s}(\mu+1)\{h\}\Big\} - 2D_{r_s}(\mu+2)\{h\}\\ &= D_{r_s}(\mu+1)\{\partial_{r_s}^2 h\} - 2(\mu+1)\, D_{r_s}(\mu+2)\{h\},\end{aligned}$$

where, we have used the mathematical induction hypothesis on μ.

(ii) comes easily from the definition of $D^{r_s}(\mu)\{h\}$. Next, using (ii), we get

$$D^{r_s}(\mu)\{\partial_{r_s} h\} = \partial_{r_s}\left[D_{r_s}(\mu-1)\left\{\frac{\partial_{r_s} h}{r_s}\right\}\right] = \partial_{r_s}[D_{r_s}(\mu)\{h\}].$$

Finally to obtain (iv) we use (i) and (ii), respectively. In fact,

$$\begin{aligned}D_{r_s}(\mu)\big\{\partial_{r_s} h\big\} - \partial_{r_s} D^{r_s}(\mu)\{h\} &= D_{r_s}(\mu)\{\partial_{r_s} h\} - \partial_{r_s}^2\left[D_{r_s}(\mu-1)\left\{\frac{h}{r_s}\right\}\right]\\ &= D_{r_s}(\mu)\{\partial_{r_s} h\} - D_{r_s}(\mu-1)\left\{\partial_{r_s}^2\left\{\frac{h}{r_s}\right\}\right\} + 2(\mu-1)\, D_{r_s}(\mu)\left\{\frac{h}{r_s}\right\}\\ &= 2\mu\, D_{r_s}(\mu)\left\{\frac{h}{r_s}\right\} = \frac{2\mu}{r_s}\, D^{r_s}(\mu)\{h\},\end{aligned}$$

and this completes the proof. ■

LEMMA 3 *Let $h(x_0, r_1, \ldots, r_d)$ be a scalar function harmonic in the $d+1$ variables $x_0, r_1, \ldots, r_d$. Then*

(i) $$\partial_{x_0}^2 \prod_{s=1}^{d} D_{r_s}(\mu_s)\{h\} + \sum_{j=1}^{d} \partial_{r_j}^2 \prod_{s=1}^{d} D_{r_s}(\mu_s)\{h\} = -2\sum_{j=1}^{d} \mu_j \prod_{s=1,\, s\neq j}^{d} D_{r_s}(\mu_s) D_{r_j}(\mu_j+1)\{h\},$$

(ii) $$\partial_{x_0}^2 \prod_{s=1,\, s\neq c}^{d} D_{r_s}(\mu_s) D^{r_c}(\mu_c)\{h\} + \sum_{j=1}^{d} \partial_{r_j}^2 \prod_{s=1,\, s\neq c}^{d} D_{r_s}(\mu_s) D^{r_c}(\mu_c)\{h\}$$

$$= -2 \sum_{j=1,\, j\neq c}^{d} \mu_j \prod_{s=1,\, s\neq c,j}^{d} D_{r_s}(\mu_s) D^{r_c}(\mu_c) D_{r_j}(\mu_j+1)\{h\}$$

$$-2\mu_c \prod_{s=1,\, s\neq c}^{d} D_{r_s}(\mu_s) D^{r_c}(\mu_c+1)\{h\}.$$

Proof Using Lemma 2 and the assumption on h, we can prove that

$$\partial_{x_0}^2 \prod_{s=1}^{d} D_{r_s}(\mu_s)\{h\} + \sum_{j=1}^{d} \partial_{r_j}^2 \prod_{s=1}^{d} D_{r_s}(\mu_s)\{h\} = \prod_{s=1}^{d} D_{r_s}(\mu_s)\{\partial_{x_0}^2 h\}$$
$$+ \sum_{j=1}^{d} \prod_{s=1,\, s\neq j}^{d} D_{r_s}(\mu_s)\Big\{D_{r_j}(\mu_j)\{\partial_{r_j}^2 h\} - 2\mu_j D_{r_j}(\mu_j+1)\{h\}\Big\}$$
$$= \prod_{s=1}^{d} D_{r_s}(\mu_s)\{\partial_{x_0}^2 h\} + \sum_{j=1}^{d} \prod_{s=1}^{d} D_{r_s}(\mu_s)\{\partial_{r_j}^2 h\} - 2\sum_{j=1}^{d} \mu_j \prod_{s=1,\, s\neq j}^{d} D_{r_s}(\mu_s) D_{r_j}(\mu_j+1)\{h\}$$
$$= -2\sum_{j=1}^{d} \mu_j \prod_{s=1,\, s\neq j}^{d} D_{r_s}(\mu_s) D_{r_j}(\mu_j+1)\{h\}.$$

Similarly (ii) can be proved in the same way. In fact,

$$\partial_{x_0}^2 \prod_{s=1,\, s\neq c}^{d} D_{r_s}(\mu_s) D^{r_c}(\mu_c)\{h\} + \sum_{j=1}^{d} \partial_{r_j}^2 \prod_{s=1,\, s\neq c}^{d} D_{r_s}(\mu_s) D^{r_c}(\mu_c)\{h\}$$
$$= \prod_{s=1,\, s\neq c}^{d} D_{r_s}(\mu_s) D^{r_c}(\mu_c)\{\partial_{x_0}^2 h\} + \sum_{j=1,\, j\neq c}^{d} \prod_{s=1,\, s\neq c}^{d} D_{r_s}(\mu_s) D^{r_c}(\mu_c)\{\partial_{r_j}^2 h\}$$
$$- 2 \sum_{j=1,\, j\neq c}^{d} \mu_j \prod_{s=1,\, s\neq c,j}^{d} D_{r_s}(\mu_s) D^{r_c}(\mu_c) D_{r_j}(\mu_j+1)\{h\}$$
$$+ \prod_{s=1,\, s\neq c}^{d} D_{r_s}(\mu_s)\left\{\partial_{r_c}^3\left[D_{r_c}(\mu_c-1)\left\{\frac{h}{r_c}\right\}\right]\right\}.$$

As,

$$\partial_{r_c}^3\left[D_{r_c}(\mu_c-1)\left\{\frac{h}{r_c}\right\}\right] = \partial_{r_c}\left[D_{r_c}(\mu_c-1)\left\{\partial_{r_c}^2\left\{\frac{h}{r_c}\right\}\right\}\right] - 2(\mu_c-1)\,\partial_{r_c} D_{r_c}(\mu_c)\left\{\frac{h}{r_c}\right\}$$
$$= D^{r_c}(\mu_c)\{\partial_{r_c}^2 h\} - 2\mu_c D^{r_c}(\mu_c+1)\{h\},$$

we get that

$$\partial_{x_0}^2 \prod_{s=1,\, s\neq c}^{d} D_{r_s}(\mu_s) D^{r_c}(\mu_c)\{h\} + \sum_{j=1}^{d} \partial_{r_j}^2 \prod_{s=1,\, s\neq c}^{d} D_{r_s}(\mu_s) D^{r_c}(\mu_c)\{h\}$$
$$= \prod_{s=1,\, s\neq c}^{d} D_{r_s}(\mu_s) D^{r_c}(\mu_c)\{\partial_{x_0}^2 h\} + \sum_{j=1}^{d} \prod_{s=1,\, s\neq c}^{d} D_{r_s}(\mu_s) D^{r_c}(\mu_c)\{\partial_{r_j}^2 h\}$$
$$- 2 \sum_{j=1,\, j\neq c}^{d} \mu_j \prod_{s=1,\, s\neq c,j}^{d} D_{r_s}(\mu_s) D^{r_c}(\mu_c) D_{r_j}(\mu_j+1)\{h\}$$

$$- 2\mu_c \prod_{s=1,\ s\neq c}^{d} D_{r_s}(\mu_s) D^{r_c}(\mu_c + 1)\{h\}$$

$$= -2 \sum_{j=1,\ j\neq c}^{d} \mu_j \prod_{s=1,\ s\neq c,j}^{d} D_{r_s}(\mu_s) D^{r_c}(\mu_c) D_{r_j}(\mu_j + 1)\{h\}$$

$$- 2\mu_c \prod_{s=1,\ s\neq c}^{d} D_{r_s}(\mu_s) D^{r_c}(\mu_c + 1)\{h\},$$

and we have our result. ■

LEMMA 4 *Let $h(x_0, r_1, \ldots, r_d)$ be a scalar function harmonic in the $d+1$ variables $x_0, r_1, \ldots, r_d$. Then*

$$\Delta^{\mu}[h\,\mathbf{P}_k(\underline{x})] = \left(\sum \frac{\mu!}{\mu_1! \cdots \mu_d!} \prod_{s=1}^{d} d_{p_s,k_s}(\mu_s) \prod_{s=1}^{d} D_{r_s}(\mu_s)\{h\}\right) \mathbf{P}_k(\underline{x})$$

$$\Delta^{\mu}[\underline{\omega}_j\, h\,\mathbf{P}_k(\underline{x})] = \underline{\omega}_j \left(\sum \frac{\mu!}{\mu_1! \cdots \mu_d!} \prod_{s=1}^{d} d_{p_s,k_s}(\mu_s) \prod_{s=1,\ s\neq j}^{d} D_{r_s}(\mu_s)\, D^{r_j}(\mu_j)\{h\}\right) \mathbf{P}_k(\underline{x}),$$

where the summation runs over all possible $\mu_1, \ldots, \mu_d \in \mathbb{N}_0$ such that

$$\sum_{s=1}^{d} \mu_s = \mu,$$

and

$$d_{p_s,k_s}(\mu_s) = (2k_s + p_s - 1)(2k_s + p_s - 3) \cdots (2k_s + p_s - (2\mu_s - 1))$$
$$d_{p_s,k_s}(0) = 1.$$

Proof The proof follows by induction using Lemma 3, and the following identities

$$\Delta[h\,\mathbf{P}_k(\underline{x})] = \left(\partial_{x_0}^2 h + \sum_{s=1}^{d} \partial_{r_s}^2 h + \sum_{s=1}^{d} \frac{2k_s + p_s - 1}{r_s}\, \partial_{r_s} h\right) \mathbf{P}_k(\underline{x})$$

$$\Delta[\underline{\omega}_j\, h\,\mathbf{P}_k(\underline{x})] = \underline{\omega}_j \left(\partial_{x_0}^2 h + \sum_{s=1}^{d} \partial_{r_s}^2 h + \sum_{s=1,\ s\neq j}^{d} \frac{2k_s + p_s - 1}{r_s}\, \partial_{r_s} h + (2k_j + p_j - 1)\partial_{r_j}\left\{\frac{h}{r_j}\right\}\right) \mathbf{P}_k(\underline{x}),$$

which are valid for any scalar function h. ■

Remark If, in addition to the assumption in Lemma 4, we assume that $p_1, p_2, \ldots, p_d$ are odd, then

$$\Delta^{k+((m-d+2)/2)}[h\,\mathbf{P}_k(\underline{x})] = \Delta^{k+((m-d+2)/2)}[\underline{\omega}_j\, h\,\mathbf{P}_k(\underline{x})] = 0.$$

Indeed, in that case, all the terms in the expansions of Lemma 4 are zero. In fact, if there is a non-zero term in those expansions, then we have $2k_s + p_s - (2\mu_s - 1) \geq 2$

for $s = 1, 2, \ldots, d$. Since $\sum_{s=1}^{d} p_s = m$, $\sum_{s=1}^{d} k_s = k$ and $\sum_{s=1}^{d} \mu_s = \mu = k + ((m-d+2)/2)$, adding up the previous inequality together produces the false relation $2 \leq 0$.

Proof of Theorem 1 From Lemma 4 we have

$$\Delta^{k+((m-d)/2)}\left[\left(g_0 + \sum_{j=1}^{d} \underline{\omega}_j\, g_j\right)\mathbf{P}_k(\underline{x})\right] = \left(A_0 + \sum_{j=1}^{d} \underline{\omega}_j\, A_j\right)\mathbf{P}_k(\underline{x}), \tag{4}$$

with

$$A_0 = (2k+m-d)!! \prod_{s=1}^{d} D_{r_s}\left(k_s + \frac{p_s - 1}{2}\right)\{g_0\}$$
$$A_j = (2k+m-d)!! \prod_{s=1,\ s\neq j}^{d} D_{r_s}\left(k_s + \frac{p_s - 1}{2}\right) D^{r_j}\left(k_j + \frac{p_j - 1}{2}\right)\{g_j\},$$

$j = 1, 2, \ldots, d$.

Now, taking into account Lemma 2 and the fact that $\underline{g}$ satisfy the Riesz system, it is easy to check that (4) satisfy the Vekua system for poly-axial monogenic functions of degree k. ■

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