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Half Dirichlet Problems and Decompositions of Poisson Kernels

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Abstract. Following the previous study on the unit ball of Delanghe et al, half-Dirichlet problems for the upper-half space are presented and solved. The solutions further lead to decompositions of the Poisson kernels, and the fact that the classical Dirichlet problems may be solved merely by using Cauchy transformation in the respective two contexts. We show that the only domains for which the half-Dirichlet problems are solvable in the same pattern are balls and half-spaces.

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1. Half-Dirichlet Problems on Surfaces

Let $\Phi(x)$ be a real-valued C^∞ function and let $\Phi(x) = 0$ represent a topologically closed, C^∞ and m-dimensional surface, Σ, in $\mathbf{R}^{m+1}$, the latter being identified with the linear subspace

$$\mathbf{R}^{0,m+1} = \{x = x_0\mathbf{e}_0 + \underline{x} \mid x_0 \in \mathbf{R}, \underline{x} = x_1\mathbf{e}_1 + \cdots + x_m\mathbf{e}_m\},$$

in the real Clifford algebra $\mathbf{R}_{0,m+1}$ generated by $\mathbf{e}_0, \mathbf{e}_1, \cdots, \mathbf{e}_m$, for $m \geq 1$, satisfying the properties

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i,\ \mathbf{e}_i^2 = -1,\ i, j = 0, 1, \cdots, m,\ i \neq j.$$

The complex Clifford algebra $\mathbf{C}_{m+1}$ over $\mathbf{C}^{m+1}$ generated by $\mathbf{e}_0, \mathbf{e}_1, \cdots, \mathbf{e}_m$ is defined to by

$$\mathbf{C}_{m+1} = \mathbf{R}_{0,m+1} \bigotimes \mathbf{C}.$$

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We adopt the notation of Clifford algebras from [5]. In particular, a general element $a \in \mathbf{C}_{m+1}$ has the form

$$a = \sum_A a_A \mathbf{e}_A,$$

where $a_A \in \mathbf{C}, A =< j_1, \cdots, j_l, 0 \leq j_1 < \cdots < j_l \leq m$, and where $\mathbf{e}_A = \mathbf{e}_{j_1} \cdots \mathbf{e}_{j_l}$ are the *reduced products* of basis elements. Furthermore $|a| = \left(\sum_A |a_A|^2\right)^{1/2}$ is the norm $a \in \mathbf{C}_{m+1}$. The conjugate $\overline{a}$ is defined to be the tensor product of the conjugation in $\mathbf{R}_{0,m+1}$ and the complex conjugation in $\mathbf{C}$. Of central importance is the fundamental solution of the Dirac operator (see below) in $\mathbf{R}^{m+1}$ denoted by $E(x)$; it has the expression

$$E(x) = \frac{1}{A_{m+1}} \frac{\overline{x}}{|x|^{m+1}},$$

where A_{m+1} is the area of the m-dimensional unit sphere in $\mathbf{R}^{m+1}$.

We introduce the functions

$$\alpha(x) = \frac{1}{2}\left(1 + i\frac{\partial\Phi(x)}{|\partial\Phi(x)|}\right), \quad \beta(x) = \frac{1}{2}\left(1 - i\frac{\partial\Phi(x)}{|\partial\Phi(x)|}\right),$$

where ∂ is the usual Dirac operator $\partial = \frac{\partial}{\partial x_0}\mathbf{e}_0 + \frac{\partial}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial}{\partial x_m}\mathbf{e}_m$, and i is the usual imaginary unit in the complex number system. The vector $\partial\Phi(x)/|\partial\Phi(x)|$ is a unit normal vector of the surface Σ to the point x on Σ, denoted by

$$n_x = \frac{\partial\Phi(x)}{|\partial\Phi(x)|}.$$

We assume that the surface is orientable and divides the whole space into two open regions of which at least one is simply-connected, denoted by Ω. Thus n_x is the well defined "outward" or "inward" pointing unit normal of Ω. In this note it is on $\overline{\Omega}$ that the Dirichlet problem and half-Dirichlet problems will be studied.

For each fixed $x \in \Sigma, \alpha(x)$ and $\beta(x)$ are hermitian orthogonal primitive idempotents in $\mathbf{C}_{m+1}$, i.e.

$$\alpha^2(x) = \alpha(x), \quad \beta^2(x) = \beta(x);$$
$$\alpha(x)\beta(x) = \beta(x)\alpha(x) = 0;$$
$$\overline{\alpha}(x) = \alpha(x), \quad \overline{\beta}(x) = \beta(x).$$

Moreover,

$$\alpha(x) + \beta(x) = 1.$$

The functions α and β give rise to Hardy-space projections as Fourier multiplier operators, acting on the frequency domain of the functions. The representations of those projections on the space domain are singular integrals. The study may be found in [4], [9], [10], and lately in [8]. In this note, however, use of those functions is made directly in the space domain of the functions.

The half Dirichlet problems with respect to α and β are formulated as follows. Given a boundary data $f \in C^\lambda(\Sigma), 0 < \lambda < 1$, where $C^\lambda(\Sigma)$ denotes the class of

Hölder continuous functions of degree λ, or $f \in L^p(\Sigma), 1 < p < \infty$, find $W(x)$ such that

$$\begin{cases} \partial W(x) = 0 & x \in \Omega \\ \alpha(x)W(x) = \alpha(x)f(x) & x \in \Sigma, \end{cases} \tag{1.1}$$

$$\begin{cases} \partial W(x) = 0 & x \in \Omega \\ \beta(x)W(x) = \beta(x)f(x) & x \in \Sigma. \end{cases} \tag{1.2}$$

The cases $p = 1$ and $p = \infty$ require more delicate analysis that will be omitted in this paper. The following two sections will be devoted to solving the half Dirichlet problems in the unit ball and in the upper-half space $\mathbf{R}^{m+1}$, respectively. We will also discuss the Dirichlet problems and the corresponding decompositions of the Poisson kernels in these contexts. In the third section we will show that balls and half-spaces are the only cases for which half Dirichlet problems have solutions of similar structure.

2. Half Dirichlet Problems in the Unit Ball

Denote the open unit ball centered at the origin by $B(1)$ whose closure is $\overline{B}(1)$. As boundary of $B(1)$, the unit sphere is denoted by S_m. The unit sphere consists of the points on the surface $\Phi(x) = 1$, where $\Phi(x) = |x|^2$. Consider the level surfaces $\Phi(x) = r \leq 1$ in the closed ball $\overline{B}(1)$. The idempotent functions α and β on the level surfaces are

$$\alpha(x) = \frac{1}{2}(1 + ix), \quad \beta(x) = \frac{1}{2}(1 - ix).$$

We have

$$\alpha(x)\beta(x) = \beta(x)\alpha(x) = 1 - |x|^2.$$

If, in particular, $x = r\omega$ with $r = 1$, i.e. x is on the unit sphere, then we have

$$\alpha(\omega)\beta(\omega) = \beta(\omega)\alpha(\omega) = 0.$$

The Cauchy transform of a given boundary data f is given by

$$C(f)(x) = \int_{S_m} \overline{C_x}(\omega)f(\omega)ds(\omega),$$

where

$$C(x,\omega) = \frac{1}{A_{m+1}}\omega\frac{x-\omega}{|x-\omega|^{m+1}}$$

is the Cauchy kernel on the sphere.

Throughout the paper we will adopt the inner product notation for the above integral, viz.

$$C(f) =< C_x, f_\Sigma,$$

while in general we define

$$< g, f_\Sigma = \int_\Sigma \overline{g}(\omega)f(\omega)ds(\omega).$$

To solve the half Dirichlet problem in relation to α, for instance, one considers the Cauchy integral $C(f) = C_{S_m}(f)$ of the function $2\alpha(\omega)f(\omega)$ inside the unit ball, where $f(\omega)$ is the boundary data given in (1.1), i.e.

$$\begin{aligned} C(2\alpha f)(x) &= \frac{1}{A_{m+1}} \int_{S_m} \frac{x-\omega}{|x-\omega|^{m+1}} \omega[2\alpha(\omega)f(\omega)]ds(\omega) \\ &= \frac{1}{A_{m+1}} \int_{S_m} \frac{x-\omega}{|x-\omega|^{m+1}} \omega(1+i\omega)f(\omega)ds(\omega). \end{aligned}$$

Set

$$W^\alpha(x) = C(2\alpha f)(x).$$

Write $x = r\xi$, we have

(i) $W^\alpha(x)$ is (left-) monogenic in $B(1)$;

(ii)

$$\begin{aligned} \lim_{r\to 1-} W^\alpha(r\xi) &= W(\xi) \qquad \text{(as definition)} \\ &= \frac{1}{2}\left[2\alpha(\xi)f(\xi) + \mathcal{H}(2\alpha f)(\xi)\right], \end{aligned}$$

where $\mathcal{H}$ is the *Hilbert transformation* on $C^\lambda(S_m)$ and $L^p(S_m)$. (ii) is the so called Plemelj-Sokhotzki formula. The Hilbert transform of a general function f on the sphere is defined to be the principal value integral

$$\mathcal{H}(f)(x) = p.v. \frac{2}{A_{m+1}} \int_{S_m} \frac{\xi-\omega}{|\xi-\omega|^{m+1}} \omega f(\omega)ds(\omega).$$

The fact that $\mathcal{H}$ maps $C^\lambda(S_m)$ to $C^\lambda(S_m)$ as a bounded operator is traced back to [11]; and that $\mathcal{H}$ maps $L^p(S_m)$ to $L^p(S_m)$, is based, for $p = 2$, on the Plancherel theorem on the sphere; for $p \neq 2$ we refer to [2] or [4]. The validity of the Plemelj-Sokhotzki formula for functions in L^p is a consequence of the boundedness of $\mathcal{H}$ in the L^p spaces (see [12] or [14]).

Note that the use of the terminology Hilbert transformation is not uniform among analysts. Some authors call the above defined $\mathcal{H}$ the *Cauchy singular integral* on the sphere. They instead use the terminology Hilbert transformation for the mapping that maps the real part to the imaginary part of the boundary value of a (left-) monogenic function in Ω (see, for instance, [1]). On the upper-half space the two concepts coincide but this does not happen for general domains including balls.

Now consider the function $\alpha(x)W^\alpha(x)$. Taking the limit to the boundary, we have

$$\begin{aligned} \lim_{r\to 1-} \alpha(x)W^\alpha(x) &= \alpha(\xi)W^\alpha(\xi) \\ &= \alpha^2(\xi)f(\xi) + \alpha(\xi)\mathcal{H}(\alpha f)(\xi) \\ &= \alpha(\xi)f(\xi) + \alpha(\xi)\mathcal{H}(\alpha f)(\xi). \end{aligned}$$

But, as

$$(1+i\xi)(\xi-\omega)\omega(1+i\omega) = 0, \tag{2.1}$$

we obtain

$$\alpha(\xi)\mathcal{H}(\alpha f)(\xi) = 0.$$

Consequently,

$$\lim_{r\to 1-} \alpha(x)W^{\alpha}(x) = \alpha(\xi)f(\xi)$$

Therefore, W^{α} solves the problem (1.1). Similarly,

$$W^{\beta}(x) = C(2\beta f)(x)$$

solves the problem (1.2).

The above solutions W^{α} and W^{β} to the problems (1.1) and (1.2), respectively, give rise to the solutions of the classical Dirichlet problem: Given boundary data $f \in C^{\lambda}(\Sigma), 0 < \lambda < 1$ or $f \in L^{p}(\Sigma), 1 < p < \infty$, find $U(x)$ such that

$$\begin{cases} \Delta U(x) = 0 & x \in B(1) \\ U|_{S_m}(x) = f(x) & x \in S_m, \end{cases} \tag{2.2}$$

We recall the following facts.

(i) W^{α} and W^{β} are left-monogenic in $B(1)$; and

(ii) For any (left-) monogenic function f in the open set $\Omega \subset \mathbf{R}^{m+1}$, the function $xf(x)$ is harmonic in Ω (see, for instance, [5]).

We therefore have that $\alpha(x)W^{\alpha}(x)$ and $\beta(x)W^{\beta}(x)$ both are harmonic in $B(1)$. Hence

$$U(x) = \alpha(x)W^{\alpha}(x) + \beta(x)W^{\beta}(x) \tag{2.3}$$

is harmonic in $B(1)$. Moreover,

$$\begin{aligned} \lim_{r\to 1-} U(r\xi) &= \alpha(\xi)W^{\alpha}(\xi) + \beta(\xi)W^{\beta}(\xi) \\ &= \alpha(\xi)f(\xi) + \beta(\xi)f(\xi) \\ &= f(\xi). \end{aligned}$$

Consequently, $U(x)$ solves the Dirichlet problem (2.2).

The solutions W^{α} and W^{β} to the problems (1.1) and (1.2) also give rise to a decomposition of the Poisson kernel on the sphere. Note that the solution of (2.2) is given by

$$U(x) = \int_{S_m} P(x,\omega)f(\omega)ds(\omega),$$

where

$$P(x,\omega) = \frac{1}{A_{m+1}}\frac{1-|x|^2}{|x-\omega|^{m+1}}, \quad x \in B(1),\ \xi \in S_m,$$

is the Poisson kernel on the sphere. The solutions W^{α} and W^{β} now motivate to define the functions

$$C_x^{\alpha}(\omega) = \frac{2}{A_{m+1}}\alpha(x)\frac{x-\omega}{|x-\omega|^{m+1}}\omega\alpha(\omega)$$

and

$$C_x^{\beta}(\omega) = \frac{2}{A_{m+1}}\beta(x)\frac{x-\omega}{|x-\omega|^{m+1}}\omega\beta(\omega).$$

Due to the simple relation

$$(1+ix)(x-\omega)\omega(1+i\omega)+(1-ix)(x-\omega)\omega(1-i\omega)=2(1-|x|^2),$$

we obtain the decomposition

$$P(x,\omega)=C_x^\alpha(\omega)+C_x^\beta(\omega), \tag{2.4}$$

and hence the solutions $\alpha(x)W^\alpha(x)$ and $\beta(x)W^\beta(x)$ are given, respectively, by

$$\alpha(x)W^\alpha(x)=\int_{S_m} C_x^\alpha(\omega)f(\omega)ds(\omega) \tag{2.5}$$

and

$$\beta(x)W^\beta(x)=\int_{S_m} C_x^\beta(\omega)f(\omega)ds(\omega). \tag{2.6}$$

Remarks

(i) The solutions of (1.1) and (1.2) for the unit ball case are already discussed in the paper [7].

(ii) Formula (2.3), together with (2.5) and (2.6), may be written as

$$U(x)=\alpha(x)\int_{S_m}\overline{C_x}(\omega)(2\alpha f)(\omega)ds(\omega)+\beta(x)\int_{S_m}\overline{C_x}(\omega)(2\beta f)(\omega)ds(\omega),$$

thus indicating the fact that the classical Dirichlet problem (2.2) for the unit ball may be solved by using the Cauchy transformation only.

(iii) It is based on the splitting (2.4) that we obtain the decomposition (2.3). Indeed, apart from the decomposition (2.4) all the other results may already be found in [7]. The latter paper uses the splitting

$$P(x,\omega)=P^\alpha(x,\omega)+P^\beta(x,\omega),$$

where

$$P^\alpha(x,\omega)=\alpha(x)P(x,\omega),\quad P^\beta=\beta(x)P(x,\omega).$$

The observation that the Dirichlet problem for Δ in $B(1)$ can thus be solved by using the Cauchy transformation was not explicitly made, although it is implicitly presented in [7] Theorem 3.2 (i).

(iv) In [6], it is proved that the unique solution to the problem (2.2) reads

$$U(x)=F_1(x)+xF_2(x), \tag{2.7}$$

where

$$F_1(x)=<S_x(\omega),f(\omega)_{S_m}$$

and

$$F_2(x)=<S_x(\omega),\overline{\omega}f(\omega)_{S_m},$$

and $S_x(\omega)$ is the Szegö kernel for the ball. Note that for the ball, $S_x(\omega)=C_x(\omega)$. We thus have that F_1 and F_2 both are (left-) monogenic in $B(1)$. If f is square-integrable, then F_1, F_2 belong to the Hardy space $H^2(B(1))$ with

$$\lim_{r\to 1-}F_1(r\xi)=\mathbf{P}f(\xi)$$

and

$$\lim_{r\to 1-} F_2(r\xi) = \mathbf{P}(\overline{\omega} f)(\xi),$$

where $\mathbf{P}$ is the orthogonal projection operator of $L^2(S_m)$ onto $H^2(S_m)$.

Notice that the decomposition (2.7) was obtained by using the following decomposition of $P(x,\omega)$:

$$P(x,\omega) = \overline{S_x}(\omega) + x\overline{S_x}(\omega)\overline{\omega}, \quad x \in B(1), \omega \in S_m.$$

Since for the ball $S_x(\omega) = C_x(\omega)$, this again shows that the classical Dirichlet problem may be solved by merely using the Cauchy transformation.

(v) Notice that (2.7) generalizes to $\mathbf{R}^{m+1}$ the following result concerning the Dirichlet problem for the unit disc in the complex plane $\mathbf{C}$ (see [2]).

Given $f \in L^2(S_1)$, the solution u to

$$\begin{cases} \Delta u(x) = 0 & x \in B(1) \\ u|_{S_1}(x) = f(x) & x \in S_1, \end{cases} \tag{2.8}$$

is given by

$$u(z) = h(z) + \overline{H(z)},$$

where

$$h(z) = (Sf)(z)$$

and

$$H(z) = zS(\overline{z}\overline{f})$$

are both holomorphic in $B(1)$, where S is the Szegö transform on $L^2(S_1)$.

3. Half Dirichlet Problems in the Upper Half Space

Throughout this section we take $\Omega = \mathbf{R}^{m+1}_+$, where

$$\mathbf{R}^{m+1}_+ = \{(x_0, \underline{x}) \in \mathbf{R}^{m+1} : x_0 > 0\}.$$

The boundary of Ω is $\Sigma = \mathbf{R}^m$. Note that the vector $\overline{\mathbf{e}}_0$ is the unit normal of the surface $\mathbf{R}^m$ outward pointing with respect to $\mathbf{R}^{m+1}_+$. The Cauchy kernel on $\mathbf{R}^{m+1}_+$ is

$$C_x(y) = \frac{1}{A_{m+1}} \overline{\mathbf{e}}_0 \frac{x-\underline{y}}{|x-\underline{y}|^{m+1}}, \quad x \in \mathbf{R}^{m+1}_+, \underline{y} \in \mathbf{R}^m.$$

The Cauchy transformation on $L^p(\mathbf{R}^m), 1 < p < \infty$, is given by

$$Cf(x) =< C_x, f = \frac{1}{A_{m+1}} \int_{\mathbf{R}^m} \frac{x-\underline{y}}{|x-\underline{y}|^{m+1}} \overline{e}_0 f(\underline{y}) d\underline{y}.$$

It enjoys the following properties:

(i) $Cf \in H^p(\mathbf{R}^{m+1}_+)$, the Hardy space on the upper-half space, for $f \in L^p(\mathbf{R}^m)$; (ii) (Plemelj-Sokhotzki)

$$\begin{aligned} \lim_{x_0\to 0+} Cf(x_0,\underline{x}) &= (Cf)^+(\underline{x}) \\ &= \frac{1}{2}[f(\underline{x}) + \mathcal{H}f(\underline{x})], \end{aligned}$$

where $\mathcal{H}$ is the Hilbert transformation on $\mathbf{R}^m$. Note that in the present case $\mathcal{H}$ maps the real part to the imaginary part of the boundary value of a (left-) monogenic function in the upper-half space $\mathbf{R}^{m+1}_+$. The boundedness of the Hilbert transformation is referred to that of the Riesz transformations ([14]).

We now solve the corresponding half Dirichlet problems in the upper-half space with the idempotent functions

$$\sigma^\pm = \frac{1}{2}(1 \pm i\overline{\mathbf{e}}_0).$$

Note that they are just the functions $\alpha(x)$ and $\beta(x)$ defined in §1 and they are constant functions. The half Dirichlet problems are posed as follows.

Given $u \in L^p(\mathbf{R}^m), 1 < p < \infty$, find W in $\mathbf{R}^{m+1}_+$ such that (1.1) and (1.2) hold respectively for $\alpha(x) = \sigma^+$ and $\beta(x) = \sigma^-$, where $f = u, \Omega = \mathbf{R}^{m+1}_+, \Sigma = \mathbf{R}^m$.

We claim that the half Dirichlet problems are solved by

$$W^\pm(x) = C(2\sigma^\pm u)(x).$$

Indeed, we have $\partial_x W^\pm(x) = 0$, and, as a matter of fact, $W^\pm \in H^2(\mathbf{R}^{m+1}_+)$. As for the boundary conditions, let us show that

$$\lim_{x_0 \to 0+} \sigma^+ W^+(x_0, \underline{x}) = \sigma^+ u(\underline{x}),$$

the case for W^- being similar.

The boundedness of the Riesz transforms imply the Plemelj-Sokhotzki formula. We therefore have

$$\lim_{x_0 \to 0+} W^+(x_0, \underline{x}) = \frac{1}{2}[2\sigma^+ u(\underline{x}) + \mathcal{H}(2\sigma^+ u)(x)].$$

Notice that for $\underline{x}, \underline{y} \in \mathbf{R}^m$,

$$(1 + i\overline{\mathbf{e}}_0)(\underline{x} - \underline{y})\overline{\mathbf{e}}_0(1 + i\overline{\mathbf{e}}_0) = 0, \tag{3.1}$$

whence

$$\sigma^+ \mathcal{H}(\sigma^+ u) = 0.$$

Together with the fact that $\sigma^+ 2 = \sigma^+$, we get

$$\lim_{x_0 \to 0+} \sigma^+ W^+(x_0, \underline{x}) = \sigma^+ u(\underline{x}).$$

Analogously,

$$\lim_{x_0 \to 0+} \sigma^- W^-(x_0, \underline{x}) = \sigma^- u(\underline{x}).$$

Now we consider a similar decomposition of the Poisson kernel in terms of the Cauchy kernel and the corresponding solution of the classical Dirichlet problem.

Define, for $(x, \underline{y}) \in \mathbf{R}^{m+1}_+ \times \mathbf{R}^m$, the kernel functions

$$C_x^\pm(\underline{y}) = \frac{2}{A_{m+1}} \sigma^\pm \frac{x - \underline{y}}{|x - \underline{y}|^{m+1}} \overline{\mathbf{e}}_0 \sigma^\pm.$$

A straightforward computation shows that

$$4x_0 = (1 + i\overline{\mathbf{e}}_0)(x - \underline{y})\overline{\mathbf{e}}_0(1 + i\overline{\mathbf{e}}_0) + (1 - i\overline{\mathbf{e}}_0)(x - \underline{y})\overline{\mathbf{e}}_0(1 - i\overline{\mathbf{e}}_0),$$

which then leads to the following decomposition of the Poisson kernel $P(x,\underline{y})$ for $\mathbf{R}_+^{m+1}$:

$$P(x,\underline{y}) = \frac{2}{A_{m+1}} \frac{x_0}{|x-\underline{y}|^{m+1}} = C_x^+(\underline{y}) + C_x^-(\underline{y}), x \in \mathbf{R}_+^{m+1}; y \in \mathbf{R}^m$$

Consequently, for $u \in L^p(\mathbf{R}^m)$ given,

$$\begin{aligned} \int_{\mathbf{R}^m} P(x,\underline{y})u(\underline{y})d\underline{y} &= \int_{\mathbf{R}^m} C_x^+(\underline{y})u(\underline{y})d\underline{y} + \int_{\mathbf{R}^m} C_x^-(\underline{y})u(\underline{y})d\underline{y} \\ &= \sigma^+W^+(x) + \sigma^-W^-(x). \end{aligned}$$

As σ^+W^+ and σ^-W^- both are harmonic in $\mathbf{R}_+^{m+1}$, and

$$\lim_{x_0 \to 0+} \left(\sigma^+W^+(x_0,\underline{x}) + \sigma^-W^-(x_0,\underline{x})\right) = u(\underline{x}),$$

it follows that the (unique) solution to the classical Dirichlet problem

$$\begin{cases} \Delta U(x) = 0 & x \in \mathbf{R}_+^{m+1} \\ U|_{\mathbf{R}^m}(x) = u(x) & x \in \mathbf{R}^m, \end{cases} \tag{3.2}$$

is given by

$$U(x) = \sigma^+W^+(x) + \sigma^-W^-(x).$$

This shows that the Dirichlet problem may be solved by using the Cauchy transformation.

As in the unit ball case we now cite the decomposition of the Poisson kernel and the corresponding decomposition of the solution of the Dirichlet problem (3.2) in relation to the Szegö kernel in the upper half space.

For the half-space case the Szegö kernel, as in the ball case, is the same as the Cauchy kernel (see [3]):

$$S_x(\underline{y}) = \frac{1}{A_{m+1}} \overline{\mathbf{e}}_0 \frac{x-\underline{y}}{|x-\underline{y}|^{m+1}} = C_x(\underline{y}).$$

Due to the elementary relation

$$(x-\underline{y})\overline{\mathbf{e}}_0 + \overline{\mathbf{e}}_0 \left((x-\underline{y})\overline{\mathbf{e}}_0\right)\mathbf{e}_0 = 2x_0,$$

we obtain

$$P(x,\underline{y}) = \overline{S_x(\underline{y})} + \overline{\mathbf{e}}_0 \overline{S_x(\underline{y})} \mathbf{e}_0. \tag{3.3}$$

Correspondingly, we have the decomposition of the solution of the Dirichlet problem:

$$U(x) = F_1 + \overline{\mathbf{e}}_0 F_2, \quad x \in \mathbf{R}_+^{m+1} \tag{3.4}$$

where

$$F_1(x) =< S_x, u, \quad F_2(x) =< S_x, \mathbf{e}_0 u.$$

Note that for $u \in L^2(\mathbf{R}^m)$, F_1 and F_2 both belong to the Hardy space $H^2(\mathbf{R}_+^{m+1})$. The decomposition (3.4) was already obtained in [6].

4. Half Dirichlet Problems for General Domains

Let the boundary surface Σ of a general domain Ω be given by $\Phi(x) = 0$. We recall that the primitive idempotents are defined by

$$\alpha(x) = \frac{1}{2}\left(1 + i\frac{\partial\Phi(x)}{|\partial\Phi(x)|}\right), \quad \beta(x) = \frac{1}{2}\left(1 - i\frac{\partial\Phi(x)}{|\partial\Phi(x)|}\right),$$

where the vector $\partial\Phi(x)/|\partial\Phi(x)|$ is a unit normal vector to the surface Σ at the point $x \in \Sigma$, denoted by

$$n_x = \frac{\partial\Phi(x)}{|\partial\Phi(x)|}.$$

For the ball case $\Phi(x) = \sum_{k=0}^{m} x_k 2 - 1$ and for the upper half space case $\Phi(x) = x_0$. For these cases $n_x = \frac{x}{|x|}$ and $n_x = \overline{\mathbf{e}}_0$, respectively. In the former two sections the crucial relations for developing the theories in the two contexts are (2.1) and (3.1), respectively. If the half Dirichlet problems (1.1) and (1.2) were solvable in the same pattern, then the following relation should hold

$$(1 + in_x)(x - y)n_y(1 + in_y) = 0.$$

Through simple computation, taking into account $n_y^2 = -1$, there then would hold

$$[(x - y)n_y + n_x(x - y)] + i[n_x(x - y)n_y - (x - y)] = 0,$$

or

$$(x - y)n_y + n_x(x - y) = 0 \quad and \quad n_x(x - y)n_y - (x - y) = 0.$$

By multiplying n_x to both sides of the second equation, it becomes the same as the first, and they further reduce to the relations

$$< x - y, n_y + < x - y, n_x = 0, \quad and \quad (x - y) \wedge n_y - (x - y) \wedge n_x = 0,$$

or

$$(x - y) \perp (n_x + n_y) \quad and \quad (x - y) \parallel (n_x - n_y).$$

The last two conditions imply that the surface must be a sphere or an m-dimensional hyperplane, which are exactly the two cases studied in the previous two sections. We thus could not expect to have an analogous theory on the surfaces other than spheres and half-spaces.

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