A Note on Pointwise Convergence for Expansions in Surface Harmonics of Higher Dimensional Euclidean Spaces

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Abstract. We study the Fourier-Laplace series on the unit sphere of higher dimensional Euclidean spaces and obtain a condition for convergence of Fourier-Laplace series on the unit sphere. The result generalizes Carleson's Theorem to higher dimensional unit spheres.

§1. Introduction

We start with reviewing the basic notations and results. Let $f \in L^1([-\pi,\pi])$, then the Fourier coefficients c_k are all well-defined by

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dt, \qquad k \in \mathbf{Z}, \tag{1}$$

where $\mathbf{Z}$ denotes the set of all integers.

By $s_N(f)(x)$ we denote the partial sum

$$s_N(f)(x) = \sum_{|k|\le N} c_k e^{ikx}, \qquad x \in [-\pi,\pi],\ N \in \mathbf{N}_0, \tag{2}$$

of the Fourier series of f, where $\mathbf{N}_0$ denotes the set of all natural numbers.

Then we have,

$$s_N(f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_N(x-t)dt, \tag{3}$$

where

$$D_N(x) = \begin{cases} \frac{\sin(N+\frac{1}{2})x}{2\sin\frac{x}{2}} & \text{for } x \in [-\pi,\pi]\backslash\{0\}, \\ N+\frac{1}{2} & \text{for } x = 0, \end{cases}$$

is the *N-th Dirichlet kernel*.

Since $L^2([-\pi,\pi]) \subset L^1([-\pi,\pi])$, the Fourier coefficients of L^2 functions are also well-defined. The famous Carleson's Theorem is stated as follows.

Theorem 1.[Ca] If $f \in L^2([-\pi,\pi])$, then

$$s_N(f)(x) \to f(x) \qquad a.e. \quad x \in [-\pi,\pi], \text{ as } N \to +\infty.$$

L. Carleson proved this theorem in 1966. The next year, R.A. Hunt[Hu] further extended this result to $f \in L^p([-\pi,\pi])$, $1 < p < \infty$.

[1]2000 Mathematics Subject Classification: 42B05, 42A38, 30G35

[2]Key Words and Phrases: Spherical Harmonics, Fourier-Laplace Series, Carleson Theorem, Legendre Polynomials

One naturally asks what is the analogous result for the unit sphere Ω_n in the n-dimensional Euclidean space $\mathbf{R}^n$? For any $f \in L^2(\Omega_n)$, there is an associated *Fourier-Laplace series*:

$$f \sim \sum_{k=0}^{\infty} f_k, \tag{4}$$

where f_k is the homogeneous spherical harmonics of degree k. There has been literature for the study of convergence and summability of Fourier-Laplace series of various kinds on unit sphere of higher dimensional Euclidean spaces(see [Ro],[Ka],[WL]). However, except for the very lowest dimensional case, pointwise convergence, being the initial motivation of various summabilities, could be said to be very little known. The case $n = 2$ seems to be the only well studied case([Zy], [Ca]). Dirichlet ([Di]) gave the first detailed study on the case $n = 3$, on the so called Laplace series. Koschmieder([Ko]) studied the case $n = 4$. Roetman([Ro]) and Kalf([Ka]) considered the general cases, and, under certain conditions, reduced the convergence problem for $n = 2k + 2$ to $n = 2$; and $n = 2k + 3$ to $n = 3$. Among others, Meaney([Me]) addressed some related topics, including the L^p cases. In this note, we further study convergence of the series (4) in view of the classical Carleson's Theorem and the fundamental properties of Legendre polynomials. Based on the results obtained in [Ro] and [Ka], we further obtain a weaker condition that ensures the pointwise convergence of the Fourier-Laplace series of functions in Sobolev spaces. The result is a generalization of Carleson's Theorem to higher dimensional Euclidean spaces.

§2. Preliminaries

Referring the reader to Erdélyi([Er]), Müller([Mu]) and Roetman([Ro]) for details, we recall here some notations and main results for surface spherical harmonics that we shall need. Let $(x_1, \cdots, x_n)$ be the coordinates of a point of $\mathbf{R}^n$ with norm

$$|x|^2 = r^2 = x_1^2 + \cdots + x_n^2.$$

Then $x = r\xi$, where $\xi = (\xi_1, \cdots, \xi_n)$ is a point on the unit sphere Ω_n in $\mathbf{R}^n$. Denote by A_n the total surface area of Ω_n and by $d\omega_n$ the usual Hausdorff surface measure on the $(n-1)$-dimensional unit sphere,

$$A_n = \int_{\Omega_n} d\omega_n.$$

If $e_1, \cdots, e_n$ denote the orthonormal basis vectors in $\mathbf{R}^n$, then we can represent the points of Ω_n by

$$\xi = te_n + (1-t^2)^{\frac{1}{2}}\tilde{\xi}, \tag{5}$$

where $-1 \le t \le 1$, $t = \xi \cdot e_n$ and $\tilde{\xi}$ is a vector in the subspace $\mathbf{R}^{n-1}$ generated by $e_1, \cdots, e_{n-1}$. In the coordinates $(r, t, \tilde{\xi})$ the surface measure has the form

$$d\omega_n = (1-t^2)^{\lambda - \frac{1}{2}} dt d\omega_{n-1}, \tag{6}$$

where $\lambda = \frac{n-2}{2}$.

In accordance with (4), there associates a function $f \in L^2(\Omega_n)$ with a series of surface harmonics

$$S(f; n; \xi) \sim \sum_{k=0}^{\infty} Y_k(f; n; \xi), \tag{7}$$

where

$$Y_k(f;n;\xi) = \alpha_k(n) \int_{\Omega_n} P_k(n;\xi\cdot\eta) f(\eta) d\omega_n(\eta), \tag{8}$$

$P_k(n;s)$ are Legendre polynomials[Mu] defined by the generating relation

$$(1+x^2-2xs)^{-\lambda} = \sum_{k=0}^{\infty} c_k(n) x^k P_k(n;s),$$

where

$$c_k(n) = \frac{(n-2)N(n,k)}{2k+n-2}, \;\; \alpha_k(n) = \frac{N(n,k)}{A_n},$$

and

$$N(n,k) = \begin{cases} 1 & \text{for } k=0, \\ \frac{(2k+n-2)\Gamma(k+n-2)}{\Gamma(k+1)\Gamma(n-1)} & \text{for } k \geq 1. \end{cases}$$

The Legendre polynomials of dimension $n > 3$ are related to the Gegenbauer polynomials by $C_k^{\lambda}(s) = c_k(n) P_k(n;s)$.

In particular, we have

$$N(2,k) = 2; \;\; N(3,k) = 2k+1, \qquad k \in \mathbf{N}_0 \cup \{0\}; \tag{9}$$

and

$$P_k(2;t) = \cos(k \cos^{-1} t), \;\; t \in [-1,1], \tag{10}$$

being the well-known Chebyshev polynomial; and

$$P_k(3;t) = \frac{(-1)^k}{2^k k!} (\frac{d}{dt})^k (1-t^2)^k \tag{11}$$

being the ordinary Legendre polynomial. For $n \geq 3$, M$\ddot{u}$ller[Mu], p.15, gives that the Legendre polynomials are orthogonal polynomials in the sense

$$\int_{-1}^{1} P_k(n;t) P_l(n;t) (1-t^2)^{\frac{n-3}{2}} dt = \frac{A_n}{A_{n-1}} \cdot \frac{1}{N(n,k)} \cdot \delta_{kl}. \tag{12}$$

Let $S_N(f;n;\xi)$ denote the partial sum through the term with index N for the series (7). Then

$$S_N(f;n;\xi) = \int_{\Omega_n} f(\eta) \{ \sum_{k=0}^{N} \alpha_k P_k(n;\xi\cdot\eta) \} d\omega_n(\eta). \tag{13}$$

One is interested in the convergence properties of $S_N(f;n;\xi)$ at ξ as N goes to infinity. Hold ξ fixed and write $\eta = t\xi + (1-t^2)^{\frac{1}{2}} \tilde{\eta}$, where $\tilde{\eta}$ is orthogonal to ξ. Let $\Omega(\xi)$ denote the unit ball in the $(n-1)$-dimensional space orthogonal to ξ. Equation (13) then yields

$$S_N(f;n;\xi) = \int_{-1}^{1} \{ \sum_{k=0}^{N} \alpha_k A_{n-1} P_k(n;t) \} \Phi_\xi(t) (1-t^2)^{\lambda - \frac{1}{2}} dt, \tag{14}$$

where

$$\Phi_\xi(t) = \frac{1}{A_{n-1}} \int_{\Omega(\xi)} f(t\xi + (1-t^2)^{\frac{1}{2}} \tilde{\eta}) d\omega_{n-1}(\tilde{\eta}) \tag{15}$$

is the average of f over the $(n-1)$-sphere of radius $(1-t^2)^{\frac{1}{2}}$ centered at $t\xi$ in the hyperplane orthogonal to ξ.

By [Mu] and [Ro], we have

$$S_N(f;2;\xi) = \int_{-1}^{1} D_N(t)\Phi_\xi(t)(1-t^2)^{-\frac{1}{2}}dt, \tag{16}$$

where

$$D_N(t) = \frac{\sin((N+\frac{1}{2})\cos^{-1}t)}{\pi\sin\frac{1}{2}\cos^{-1}t} \tag{17}$$

is a substitution of the Dirichlet kernel(see section 1 or [Zy]),

and if $n = 2l+2$, $l \in \mathbf{N}_0$,

$$S_N(f;2l+2;\xi) = \frac{2^{-l}}{\sqrt{\pi}\Gamma(l+\frac{1}{2})} \cdot \int_{-1}^{1} \frac{d^{l+1}}{dt^{l+1}}[\frac{1}{N+l}P_{N+l}(2;t) + \frac{1}{N+l+1}P_{N+l+1}(2;t)]\Phi_\xi(t)(1-t^2)^{l-\frac{1}{2}}dt; \tag{18}$$

$$S_N(f;3;\xi) = \int_{-1}^{1} K_N(t)\Phi_\xi(t)dt, \tag{19}$$

where

$$K_N(t) = \frac{1}{2}(P'_N(3;t) + P'_{N+1}(3;t)), \tag{20}$$

and if $n = 2l+3$, $l \in \mathbf{N}_0$,

$$S_N(f;2l+3;\xi) = \frac{2^{-l-1}}{\Gamma(l+1)} \cdot \int_{-1}^{1} \frac{d^{l+1}}{dt^{l+1}}[P_{N+l}(3;t) + P_{N+l+1}(3;t)]\Phi_\xi(t)(1-t^2)^{l}dt. \tag{21}$$

§3. Main Results

Let $n > 3$. We use $W^{[\frac{n-1}{2}]}([-1,1])$ for the Sobolev space

$$W^{[\frac{n-1}{2}]}([-1,1]) = \{g \in L^2([-1,1];d\mu(t)) | \frac{d^l}{dt^l}g \in L^{2-\mu}([-1,1];d\mu(t)), l = 1,2,\cdots,[\frac{n-1}{2}]\},$$

where $d\mu(t) = (1-t^2)^{-\frac{\mu}{2}}dt$, μ is defined by the relation $1-\mu = \text{n mod } 2$, i.e., μ equals to 0 or 1. This definition is also valid when n is 2 or 3, $(l = 0)$.

Then we have our main theorem,

Theorem 2. *Let $\Phi_\xi(t) \in W^{[\frac{n-1}{2}]}([-1,1])$, if $\Phi_\xi(1) = \lim_{t\to 1}\Phi_\xi(t)$ exists, then*

$$\lim_{N\to\infty} S_N(f;n;\xi) = \Phi_\xi(1).$$

If, in particular, f is continuous at ξ, then

$$\lim_{N\to\infty} S_N(f;n;\xi) = f(\xi).$$

Proof. Define on $-1 \le t \le 1$

$$\Psi_\xi^\mu(t) = \frac{(-1)^l\Gamma(\frac{\mu}{2})2^{-l}}{\Gamma(l+1-\frac{\mu}{2})}(1-t^2)^{\frac{\mu}{2}}\frac{d^l}{dt^l}[\Phi_\xi(t)(1-t^2)^{l-\frac{\mu}{2}}], \tag{22}$$

By integration by parts, the partial sums of (18) and (21) reduce to

$$S_N(f;2l+2;\xi) = \int_{-1}^1 D_{N+l}(t)\Psi_\xi^1(t)(1-t^2)^{-\frac{1}{2}}dt \tag{23}$$

and

$$S_N(f;2l+3;\xi) = \int_{-1}^1 K_{N+l}(t)\Psi_\xi^0(t)dt. \tag{24}$$

Now we distinguish two cases.

a) **n even**. Let $n = 2l+2$, $l \in \mathbf{N}_0$. From (22), we have

$$\begin{aligned}
\Psi_\xi^1(t) &= \frac{(-1)^l\Gamma(\frac{1}{2})}{2^l\Gamma(l+\frac{1}{2})}(1-t^2)^{\frac{1}{2}}\frac{d^l}{dt^l}[\Phi_\xi(t)(1-t^2)^{l-\frac{1}{2}}]\\
&= \frac{(-1)^l\Gamma(\frac{1}{2})}{2^l\Gamma(l+\frac{1}{2})}(1-t^2)^{\frac{1}{2}}\{\Phi_\xi(t)\frac{d^l}{dt^l}(1-t^2)^{l-\frac{1}{2}} + \sum_{j=1}^l C_l^j\Phi_\xi^{(j)}(t)\frac{d^{l-j}}{dt^{l-j}}(1-t^2)^{l-\frac{1}{2}}\}\\
&= \Phi_\xi(t)t^l + (1-t^2)^{\frac{1}{2}}\sum_{j=1}^l C_l^j\Phi_\xi^{(j)}(t)(1-t^2)^{j-\frac{1}{2}}P_{l-j}(t)\\
&= \Phi_\xi(t)t^l + (1-t^2)^{\frac{1}{2}}\sum_{j=1}^l \Phi_\xi^{(j)}(t)(1-t^2)^{j-\frac{1}{2}}Q_{l-j}(t),
\end{aligned}$$

where $P_{l-j}(t)$ and $Q_{l-j}(t)$ are polynomials of degree $\le l-j$.

Then (23) becomes

$$\begin{aligned}
S_N(f;2l+2;\xi) &= \int_{-1}^1 D_{N+l}(t)\Phi_\xi(t)t^l(1-t^2)^{-\frac{1}{2}}dt\\
&\quad + \int_{-1}^1 D_{N+l}(t)\sum_{j=1}^l \Phi_\xi^{(j)}(t)(1-t^2)^{j-\frac{1}{2}}Q_{l-j}(t)dt\\
&= \frac{1}{\pi}\int_0^\pi \frac{\sin(N+l+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\Phi_\xi(\cos\theta)(\cos\theta)^l d\theta\\
&\quad + \frac{2}{\pi}\sum_{j=1}^l\int_0^\pi \sin(N+l+\frac{1}{2})\theta\Phi_\xi^{(j)}(\cos\theta)(\sin\theta)^{2j-1}Q_{l-j}(\cos\theta)\cos\frac{1}{2}\theta d\theta.
\end{aligned}$$

Since $\Phi_\xi(t) \in W^{[\frac{n-1}{2}]}([-1,1])$, then

$$\Phi_\xi(\cos\theta) \in L^2([0,\pi]) \text{ and } \Phi_\xi^{(j)}(\cos\theta) \in L^1([0,\pi]),\ j = 1,2,\cdots,l.$$

Further,

$$\Phi_\xi(\cos\theta)(\cos\theta)^l \in L^2([0,\pi])$$

and

$$\Phi_\xi^{(j)}(\cos\theta)(\sin\theta)^{2j-1}Q_{l-j}(\cos\theta)\cos\frac{1}{2}\theta \in L^1([0,\pi]),\ j = 1,2,\cdots,l.$$

Therefore, using Carleson's Theorem for the first part of the above expression and using Riemann-Lebesgue Lemma for the second part, we have

$$\begin{aligned}\lim_{N\to\infty} S_N(f;2l+2;\xi) &= \Phi_\xi(\cos 0)(\cos 0)^l + 0 \\ &= \Phi_\xi(1).\end{aligned}$$

b) **n odd**. Let $n = 2l+3$, $l \in \mathbf{N}_0$. From (22), we have

$$\Psi_\xi^0(t) = \frac{(-1)^l}{2^l\Gamma(l+1)}\frac{d^l}{dt^l}[\Phi_\xi(t)(1-t^2)^l].$$

Let $G_\xi(t) = \Phi_\xi(t)(1-t^2)^l$, then (24) becomes

$$S_N(f;2l+3;\xi) = \frac{(-1)^l}{2^{l+1}\Gamma(l+1)}\int_{-1}^1 [P'_{N+l}(3;t) + P'_{N+l+1}(3;t)]G_\xi^{(l)}(t)dt.$$

Since $\Phi_\xi(t) \in W^{[\frac{n-1}{2}]}$, i.e. $\frac{d^k}{dt^k}\Phi_\xi(t) \in L^2([-1,1])$, $k = 0,1,\cdots,l+1$.

Then

$$\frac{d^k}{dt^k}G_\xi(t) \in L^2([-1,1]),\ k = 0,1,\cdots,l+1.$$

Thus, we can integrate the above integral by parts to obtain

$$\begin{aligned}S_N(f;2l+3;\xi) &= \frac{(-1)^l}{2^{l+1}\Gamma(l+1)}\{[P_{N+l}(3;t) + P_{N+l+1}(3;t)]G_\xi^{(l)}(t)|_{-1}^1 \\ &\quad - \int_{-1}^1 [P_{N+l}(3;t) + P_{N+l+1}(3;t)]G_\xi^{(l+1)}(t)dt\} \\ &= \Phi_\xi(1) - \frac{(-1)^l}{2^{l+1}\Gamma(l+1)}\int_{-1}^1 [P_{N+l}(3;t) + P_{N+l+1}(3;t)]G_\xi^{(l+1)}(t)dt.\end{aligned}$$

So, the assertion of the theorem follows if we can show

$$\int_{-1}^1 |P_m(3;t)G_\xi^{(l+1)}(t)|dt \to 0, \text{ as } m \to \infty.$$

From (12) we have

$$\int_{-1}^1 |P_m(3;t)|^2 dt = \frac{2}{2m+1},\ m \in \mathbf{N}_0.$$

By Hölder's inequality, we have

$$\begin{aligned}\int_{-1}^{1}|P_m(3;t)G_\xi^{(l+1)}(t)|dt &\le (\int_{-1}^{1}|P_m(3;t)|^2dt)^{\frac{1}{2}}\cdot(\int_{-1}^{1}|G_\xi^{(l+1)}(t)|^2dt)^{\frac{1}{2}}\\ &= \|G_\xi^{(l+1)}\|_{L^2}\cdot\sqrt{\frac{2}{2m+1}}.\end{aligned}$$

Owing to the assumption of $\Phi_\xi(t)$, we have $G_\xi^{(l+1)}(t)\in L^2([-1,1])$, then

$$\lim_{m\to\infty}\int_{-1}^{1}|P_m(3;t)G_\xi^{(l+1)}(t)|dt=0.$$

Thus,

$$\lim_{N\to\infty}S_N(f;2l+3;\xi)=\Phi_\xi(1).\qquad\Box$$

Remark 1. The above proof of Theorem 2 is also valid for $n=2$ and, in fact, directly reduced to Carleson's Theorem. It is observed that for $n=2$, i.e., $l=0$. In the first part of Theorem 2, the average $\Phi_\xi(t)$ becomes simply evaluation at two endpoints of the interval $(-\cos^{-1}t,\cos^{-1}t)$,

$$\Phi_\xi(t)=\frac{1}{2}[f(\theta_\xi+\cos^{-1}t)+f(\theta_\xi-\cos^{-1}t)],$$

where θ_ξ is the angle between ξ and e_1. The required Sobolev space reduces to L^2 space. From the condition of Theorem 2, let $t=\cos\theta$, the Dirichlet kernel is just the same as the one in the complex plane, and $\Phi_\xi\in L^2([0,\pi])$ if and only if $\frac{1}{2}[f(\theta_\xi+\theta)+f(\theta_\xi-\theta)]\in L^2([0,\pi])$. In particular, if $\xi=1$, Theorem 2 reduces to the classical Carleson's Theorem.

Remark 2. By the result of R.A. Hunt[Hu], we can obviously extend the first part of Theorem 2, which n is an even number, to L^p cases, $1<p<\infty$.

Remark 3. We prefer to impose the condition on the average of f, but not on f, since the former is weaker than the latter. By the definition of $\Phi_\xi(t)$ and the Whitney's extension theorem(see [Wh] or [Ro]), the continuity property of $\Phi_\xi(t)$ can be inherited from f. But the L^2-bounded property can not. In general, $f\in L^p(\Omega_n)$, $p\ge1$, implies $\Phi_\xi(t)\in L^p([-1;1];(1-t^2)^{\lambda-\frac{1}{2}}dt)$, in fact, by Jensen's Inequality, since x^p, $p\ge1$, is a convex function when $x\ge0$,

$$\begin{aligned}\int_{-1}^{1}|\Phi_\xi(t)|^p(1-t^2)^{\lambda-\frac{1}{2}}dt &= \int_{-1}^{1}|\int_{\Omega(\xi)}f(t\xi+(1-t^2)^{\frac{1}{2}}\tilde\eta)d\omega_{n-1}(\tilde\eta)/A_{n-1}|^p(1-t^2)^{\lambda-\frac{1}{2}}dt\\ &\le \int_{-1}^{1}(\int_{\Omega(\xi)}|f(t\xi+(1-t^2)^{\frac{1}{2}}\tilde\eta)|d\omega_{n-1}(\tilde\eta)/A_{n-1})^p(1-t^2)^{\lambda-\frac{1}{2}}dt\\ &\le \int_{-1}^{1}\int_{\Omega(\xi)}|f(t\xi+(1-t^2)^{\frac{1}{2}}\tilde\eta)|^pd\omega_{n-1}(\tilde\eta)/A_{n-1})(1-t^2)^{\lambda-\frac{1}{2}}dt\\ &= \int_{\Omega_n}|f(\eta)|^pd\omega_n(\eta).\end{aligned}$$

In particular, when $n=3$, for any $p\ge1$, $f\in L^p(\Omega_n)$ implies $\Phi_\xi(t)\in L^p([-1;1])$ since $\lambda-\frac{1}{2}=0$ in the case. Note that, $\Phi_\xi(t)\in L^p([-1;1])$ implies $\Phi_\xi(t)\in L^p([-1;1];(1-t^2)^{\lambda-\frac{1}{2}}dt)$ for any $p\ge1$, but not vice versa.

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