Adaptive Fourier Decomposition of Functions in the Orthogonal Rational System of Quaternionic Values *

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ABSTRACT. We study decompositions of functions in the Hardy spaces into linear combinations of the basic functions in the orthogonal rational systems $\{B_n(x)\}$ which can be obtained in the respective contexts through Gram-Schmidt orthogonalization process on shifted Cauchy kernels. Those lead to adaptive decompositions of quaternionic-valued signals of finite energy. This study is a generalization of the main result in [10, 11].

KEY WORDS: Hardy space, monogenic, adaptive decomposition, spherical harmonics, Fourier-Laplace series, greedy algorithm, Blaschke product, optimal approximation by rational functions

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1 Introduction

In [11] the authors studied adaptive decomposition of functions in the Hardy space $\mathcal{H}^2(\mathbb{D})$ (where $\mathbb{D}$ represents the unit disc in the complex plane) basing on the orthonormal rational function system, or Takenaka-Malmquist (TM) system

$$B_n(z) = \frac{\sqrt{1-|a_n|^2}}{1-\overline{a_n}z}\prod_{k=1}^{n-1}\frac{z-a_k}{1-\overline{a_k}z}, \quad n=1,2,\ldots,\ z\in\mathbb{D}. \tag{1}$$

The decomposition leads to an adaptive decomposition of a real-valued function (or signal) $f \in L^2(\partial\mathbb{D})$ into mono-components, where by definition, a *mono-component* is a signal that possesses an increasing analytic phase function (see [9]). The algorithm guarantees fast convergence and is considered as a realizable variation of greedy algorithm.

While in [10], the authors investigated intrinsic mono-component decomposition of functions in the Hardy space $\mathcal{H}^2(\mathbb{C}^+)$ (where $\mathbb{C}^+$ is the upper half complex plane) using the TM system

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in the upper half plane, it is

$$D_n(z) = \sqrt{\frac{\beta_n}{\pi}} \frac{1}{z - \overline{a_n}} \prod_{k=1}^{n-1} \frac{z - a_k}{z - \overline{a_k}}, \quad a_n = \alpha_n + \mathrm{i}\beta_n \in \mathbb{C}^+, \, n = 1, 2, \ldots, \, z \in \mathbb{C}^+. \tag{2}$$

This study is a counterpart corresponds to the case of unit disc in [11], which induces an adaptive expansion of a real-valued signal along the whole line. Both of the methods proposed by the authors in [10] and [11] are in some sense better than the using of the classic Fourier system.

In this paper, basing on quaternionic analysis, we study similar adaptive decompositions of functions in different Hardy spaces of quaternionic valued functions, viz. $\mathcal{H}^2(\mathbb{B}_4)$, $\mathcal{H}^2(\mathbb{R}^4 \setminus \mathbb{B}_4)$, $\mathcal{H}^2(\mathbb{R}^4_+)$ and $\mathcal{H}^2(\mathbb{R}^4_-)$. In each context there is a similar system $\{B_n\}_{n=1}^{\infty}$ generated by shifted Cauchy kernels. By adaptively choosing the parameters $\{a_n\}_{n=1}^{\infty}$ according to the given function we achieve fast decomposition in terms of energy such that

$$f = \sum_{n=1}^{\infty} B_n \langle f, B_n \rangle.$$

Invoking Sokhotskyi-Plemelj formula, we obtain adaptive decompositions of signals of finite energy that are not necessarily quaternionic monogenic, nor scalar-valued. Those, therefore, generalize the result in [10, 11] into four dimensions.

If f is a monogenic function in the Hardy space $\mathcal{H}^2$ inside the unit ball, then it has a non-tangential boundary value on the sphere that is in L^2. Inside the ball, f has a monogenic Taylor expansion. Restricted to the sphere the Taylor expansion reduces to the spherical harmonics expansion of the function. In the L^2 sense the the spherical harmonics expansion converges to the non-tangential boundary value of f. If f is an L^2 function defined on the sphere, say, being scalar-valued, then it has directly a Fourier-Laplace series in spherical harmonics, the Fourier-Laplace series coincides with the scalar part of the restriction of the Taylor expansion of the corresponding Hardy $\mathcal{H}^2$ function in the ball with the boundary value $f + Hf$ on the sphere, where H stands for the Hilbert transformation in the context ([12]). The proposed adaptive decomposition aims to obtain alternative spherical harmonics expansion that converge faster than the corresponding classical series expansion. In the unbounded domain cases the same idea can be proceeded to get fast series decompositions of the same kind, rather than the classical Fourier integral decomposition (Fourier inversion formula).

Like what is in greedy algorithm, our fastness statement is based on intuition and experience. Convergence rates are difficult to be established, for there is no smoothness conditions assumed for the boundary data. The existing convergence rate results ([2]) address the worst cases and are not sharp estimates. The current study is related to approximation to functions in the Hardy spaces by rational functions of a certain degree. This topic will be dealt with in a separate paper.

It is well known that conformal mappings in the higher dimensions are only Möbius transforms. Although Möbius transforms map the unit ball onto the unit ball, they themselves, are not monogenic. On the other hand, monogenic functions composed with Möbius transforms are no longer monogenic, unless being multiplied by a conformal weight function ([7]). Higher dimensional Möbius transforms therefore are very different from those in one complex variable, and that is why the analogous objects such as Blaschke products are not available. In fact, Blaschke products have not been defined in the higher dimensional spaces in the Clifford algebra setting. Note that if we let $a_n = 0$, then (1) gives a classical Blaschke product of $n-1$ order. This suggests that we may define Blaschke products to be the basic functions in the TM systems in individual contexts with the final parameter a_n being equal to zero. In other words, Blaschke products are those that are basic functions obtained through G-S orthogonalization process on shifted Cauchy kernels, with the ending parameter a_n being zero.

The writing plan of this paper is as follows. §2 provide the related notation and terminology, and some basic knowledge on quaternionic analysis. §3 is devoted to adaptive Fourier decomposition for functions in $\mathcal{H}^2(\mathbb{B}_4)$, basing on which, the adaptive decomposition for functions in $L^2(S^4)$ is given in §4. While in §5, we deal with the context $\mathcal{H}^2(\mathbb{R}^4_+)$ and $L^2(\mathbb{R}^3)$. In §6 we prove the same convergence rate for this higher dimension approximation. In §7 we point out that all the previous results can be transferred to the cases with underlying spaces in $\mathbb{R}^3$, which is convenient for us to present some numerical examples in §8.

2 Preliminaries

Through out the paper we work on the quaternions $\mathbb{H}$ (the only associative normed division algebra that extends the complex numbers) over the real numbers $\mathbb{R}$. To distinguish it from the complexified quaternions, we sometimes call it real-quaternions. A real-quaternion $x \in \mathbb{H}$ is of the form $x = \sum_{i=0}^{3} x_i e_i$, where $\operatorname{Re} x = x_0$ and $\operatorname{Im}_i x = x_i$ $(1 \le i \le 3)$ belong to $\mathbb{R}$, and the basis elements e_i $(0 \le i \le 3)$ satisfy

$$e_0^2 = e_0,\ e_i e_0 = e_0 e_i = e_i,\ e_i^2 = -1\,(1 \le i \le 3),$$

$$e_1 e_2 = e_3 = -e_2 e_1,\ e_2 e_3 = e_1 = -e_3 e_2,\ e_3 e_1 = e_2 = -e_1 e_3.$$

Basing on these relations one defines the quaternionic multiplication and addition by linearity and distributive law. The norm and the conjugate of x are defined respectively by $|x| = (\sum_0^3 x_i^2)^{\frac{1}{2}}$ and $\overline{x} = x_0 e_0 - \sum_{i=1}^{3} x_i e_i$, which obey $x\overline{x} = \overline{x}x = |x|^2$ (we often identify e_0 with the real identity 1). $x^{-1} = \overline{x}/|x|^2$ $(x \neq 0)$ gives the reverse of x.

For any $q_1, q_2, q_3 \in \mathbb{H}$, we have $|q_1 q_2| = |q_1||q_2|$, $\overline{q_1 q_2} = \overline{q_2}\,\overline{q_1}$ and $(q_1 q_2) q_3 = q_1 (q_2 q_3)$. Together with the multiplication law, real-quaternions $\mathbb{H}$ becomes a four-dimensional, normed division, associative but non-commutative algebra. It is easy to see that the embedding relations $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$ hold. As a vector space, or a topological space, $\mathbb{H}$ is identified with the four dimensional Euclidean space $\mathbb{R}^4$.

A function $f \in C^1(\Omega, \mathbb{H})$ is said to be left (right) $\mathbb{H}$-regular, or left (right) $\mathbb{H}$-monogenic in the open set $\Omega \subset \mathbb{R}^4$ if and only if there holds the Cauchy-Riemann-Fueter equation

$$Df = e_0 \frac{\partial f}{\partial x_0} + e_1 \frac{\partial f}{\partial x_1} + e_2 \frac{\partial f}{\partial x_2} + e_3 \frac{\partial f}{\partial x_3} = 0$$

$$\Big(fD = \frac{\partial f}{\partial x_0} e_0 + \frac{\partial f}{\partial x_1} e_1 + \frac{\partial f}{\partial x_2} e_2 + \frac{\partial f}{\partial x_3} e_3 = 0 \Big),$$

where $D = \sum_{i=0}^{3} e_i \frac{\partial}{\partial x_i}$ is the quaternionic Dirac operator. If f is both left and right $\mathbb{H}$-regular, then f is said to be $\mathbb{H}$-regular. Since $\overline{D}(Df) = \triangle f = (fD)\overline{D}$, where $\overline{D} = \frac{\partial}{\partial x_0} - \sum_{i=1}^{3} e_i \frac{\partial}{\partial x_i}$ and $\triangle = \sum_{i=0}^{3} \frac{\partial^2}{\partial x_i^2}$, it follows that any left (right) $\mathbb{H}$-regular function is always harmonic.

Being analogous with complex analysis of one variable, we have the Cauchy integral formula (on bounded or unbounded domain) in the quaternionic analysis, the following is a simple version:

Lemma 2.1. *[13] Let $\Omega \subset \mathbb{R}^4$ be a bounded domain with smooth boundary $\partial\Omega$. Suppose f is left $\mathbb{H}$-regular in Ω and continuous in $\overline{\Omega} = \Omega \cup \partial\Omega$, if x is a point in Ω, then*

$$f(x) = \frac{1}{2\pi^2} \int_{\partial\Omega} \frac{(q-x)^{-1}}{|q-x|^2} (n(q)dS) f(q),$$

where $n(q)$ is the outward normal unit to $\partial\Omega$ at the point q, dS stands for the surface area element on $\partial\Omega$.

The theory of quaternionic analysis was built up by the Swiss mathematician R. Fueter et al. in 1930s, now it has been widely applied to many fields (such as physics, image processing etc.). For more information about quaternion, quaternionic analysis and their higher dimensional analogues, we refer the reader to [1, 3, 6, 13].

Denote by $\mathbb{B}_4$ the unit ball $\{x \in \mathbb{R}^4 : |x| < 1\}$ in $\mathbb{R}^4$, and S^4 the unit sphere $\{x \in \mathbb{R}^4 : |x| = 1\}$. The Hardy space $\mathcal{H}^2(\mathbb{B}_4)$ consists of all functions f that are left $\mathbb{H}$-regular in $\mathbb{B}_4$ and satisfy

$$\|f\| = \sup_{0<r<1} \Big(\frac{1}{2\pi^2} \int_{|\eta|=1} |f(r\eta)|^2 dS \Big)^{1/2} < \infty,$$

where dS is the area element of S^4. For any $f \in \mathcal{H}^2(\mathbb{B}_4)$, the non-tangential boundary values of f on S^4 exist and belong to $L^2(S^4)$. Moreover, the Cauchy integral formula holds for all functions in $\mathcal{H}^2(\mathbb{B}_4)$, and $\mathcal{H}^2(\mathbb{B}_4)$ is a Hilbert space with the inner product defined by

$$\langle f, g \rangle = \frac{1}{2\pi^2} \int_{|\eta|=1} \overline{g(\eta)} f(\eta) dS, \quad f, g \in \mathcal{H}^2(\mathbb{B}_4).$$

The above concepts can be built in arbitrary n-dimensional space ($n \geq 2$), see, e.g. [4] for more details.

3 Fast convergent decomposition of functions in $\mathcal{H}^2(\mathbb{B}_4)$

In what follows we assume $\{a_n\}_{n=1}^{\infty}$ is a sequence of quaternionic numbers in $\mathbb{B}_4$. Set

$$\alpha_n(x) = \alpha_{\{a_n\}}(x) = (1-|a_n|^2)^{\frac{3}{2}} \frac{\overline{1-\overline{a_n}x}}{|1-\overline{a_n}x|^4}, \quad n \in N_+.$$

Which is sometimes written as α_n, or $\alpha_{\{a_n\}}$ for short. It is easy to show that $\alpha_{\{a\}}(x) \in \mathcal{H}^2(\mathbb{B}_4)$ if any only if $|a| \leq 1$, moreover, we have

Proposition 3.1. *For any function $f \in \mathcal{H}^2(\mathbb{B}_4)$ and any quaternion $a \in \mathbb{B}_4$, there holds*

$$\langle f, \alpha_{\{a\}} \rangle = (1-|a|^2)^{\frac{3}{2}} f(a).$$

Proof. By definition and the Cauchy integral formula for $\mathcal{H}^2(\mathbb{B}_4)$,

$$\begin{aligned}\langle f, \alpha_{\{a\}}\rangle &= \frac{1}{2\pi^2}\int_{|\eta|=1}(1-|a|^2)^{\frac{3}{2}}\frac{1-\overline{a}\eta}{|1-\overline{a}\eta|^4}f(\eta)dS \\ &= (1-|a|^2)^{\frac{3}{2}}\frac{1}{2\pi^2}\int_{|\eta|=1}\frac{\overline{\eta-a}}{|\eta-a|^4}(\eta dS)f(\eta) \\ &= (1-|a|^2)^{\frac{3}{2}}f(a).\end{aligned}$$

□

From the above property, we immediately arrive at

$$\|\alpha_n(x)\|^2 = \langle\alpha_n, \alpha_n\rangle = (1-|a_n|^2)^{\frac{3}{2}}\alpha_n(a_n) = 1.$$

Another proof of this equality is to use the spherical coordinates transform, or an alternative method proposed in [5]. When a_n $(n \in N_+)$ are distinct from each other, α_n $(n \in N_+)$ are mutually linear independent in $\mathcal{H}^2(\mathbb{B}_4)$. By Gram-Schmidt orthogonalization process, they can be orthogonalized by setting

$$\begin{aligned}\beta_1(x) &= \beta_{\{a_1\}}(x) = \alpha_1(x), \\ \beta_n(x) &= \beta_{\{a_1,\ldots,a_n\}}(x) = \alpha_n(x) - \sum_{i=1}^{n-1}\beta_i(x)\frac{\langle\alpha_n, \beta_i\rangle}{\langle\beta_i, \beta_i\rangle}, \quad n \geq 2.\end{aligned}$$

Thus $\{B_n\} = \{B_{\{a_1,\ldots,a_n\}}\} = \{\frac{\beta_n}{\|\beta_n\|}\}$ becomes an orthonormal system.

But if at least two of the parameters are the same, for example, a_2 equals a_1, then obviously $\beta_2(x) = \beta_{\{a_1,a_1\}}(x) = 0$. At this case, we interpret B_2 as $\lim_{\rho\to 0^+} B_{\{a_1,b\}}$, where $b = a_1 + (\rho\cos\theta, \rho\sin\theta\cos\varphi, \rho\sin\theta\sin\varphi\cos\psi, \rho\sin\theta\sin\varphi\sin\psi)$ and $\theta, \varphi \in [0,\pi]$, $\psi \in [0, 2\pi]$. More precisely,

$$\begin{aligned}\lim_{\rho\to 0^+} B_{\{a_1,b\}} &= \lim_{\rho\to 0^+}\frac{\beta_{\{a_1,b\}}}{\|\beta_{\{a_1,b\}}\|} \\ &= \lim_{\rho\to 0^+}\frac{\beta_{\{a_1,b\}} - \beta_{\{a_1,a_1\}}}{\sqrt{\langle\beta_{\{a_1,b\}} - \beta_{\{a_1,a_1\}}, \beta_{\{a_1,b\}} - \beta_{\{a_1,a_1\}}\rangle}} \\ &= \lim_{\rho\to 0^+}\frac{\frac{\beta_{\{a_1,b\}} - \beta_{\{a_1,a_1\}}}{\rho}}{\sqrt{\langle\frac{\beta_{\{a_1,b\}} - \beta_{\{a_1,a_1\}}}{\rho}, \frac{\beta_{\{a_1,b\}} - \beta_{\{a_1,a_1\}}}{\rho}\rangle}} \\ &= \frac{\nabla_{\vec{n}}\beta_{\{a_1,y\}}|_{y=a_1}}{\|\nabla_{\vec{n}}\beta_{\{a_1,y\}}|_{y=a_1}\|} \\ &= \frac{\nabla_{\vec{n}}\alpha_{\{y\}}|_{y=a_1} - \beta_{\{a_1\}}\frac{\langle\nabla_{\vec{n}}\alpha_{\{y\}}|_{y=a_1}, \beta_{\{a_1\}}\rangle}{\langle\beta_{\{a_1\}}, \beta_{\{a_1\}}\rangle}}{\|\nabla_{\vec{n}}\alpha_{\{y\}}|_{y=a_1} - \beta_{\{a_1\}}\frac{\langle\nabla_{\vec{n}}\alpha_{\{y\}}|_{y=a_1}, \beta_{\{a_1\}}\rangle}{\langle\beta_{\{a_1\}}, \beta_{\{a_1\}}\rangle}\|}\end{aligned}$$

where $\nabla_{\vec{n}}\alpha_{\{y\}} = \frac{\partial\alpha_{\{y\}}}{\partial y_0}\cos\theta + \frac{\partial\alpha_{\{y\}}}{\partial y_1}\sin\theta\cos\varphi + \frac{\partial\alpha_{\{y\}}}{\partial y_2}\sin\theta\sin\varphi\cos\psi + \frac{\partial\alpha_{\{y\}}}{\partial y_3}\sin\theta\sin\varphi\sin\psi$ is the directional derivative of $\alpha_{\{y\}}$. In other words, when $a_2 = a_1$, B_2 is interpreted as the orthonormalization of $\alpha_{\{a_1\}}$ and $\nabla_{\vec{n}}\alpha_{\{y\}}|_{y=a_1}$, it contains one quaternionic parameter a_1 and three real parameters θ, φ, ψ.

We further note that as a function of y, $\alpha_{\{y\}}$ satisfies $\alpha_{\{y\}}\overline{D} = 0$, which implies that $\frac{\partial \alpha_{\{y\}}}{\partial y_0}$, $\frac{\partial \alpha_{\{y\}}}{\partial y_1}$, $\frac{\partial \alpha_{\{y\}}}{\partial y_2}$ and $\frac{\partial \alpha_{\{y\}}}{\partial y_3}$ are linear dependent in $\mathcal{H}^2(\mathbb{B}_4)$, hence, if the multiplicity of any quaternionic parameter a_n (we call the cardinal number of the set $\{a_k : a_k = a_n, k \leq n\}$ the multiplicity of a_n and denote it by $m(a_n)$) is larger than 4, then the second order partial derivatives of $\alpha_{\{y\}}$ at the point a_n should be involved in the orthogonalization process. In general, when $m(a_n) > \sum_{k=0}^{n-1}\binom{k+2}{2} = \binom{n+2}{3}$, then the n-th order partial derivatives of $\alpha_{\{y\}}$ at the point a_n must be appear.

As an example, we calculate the explicit form of $B_2(x)$ when $a_2 \neq a_1$:

$$\begin{aligned} \langle \alpha_2, \beta_1 \rangle &= (1-|a_1|^2)^{\frac{3}{2}}\alpha_2(a_1) \\ &= [(1-|a_1|^2)(1-|a_2|^2)]^{\frac{3}{2}}\frac{\overline{1-\overline{a_2}a_1}}{|1-\overline{a_2}a_1|^4}, \end{aligned}$$

$$\begin{aligned} \beta_2(x) &= \alpha_2(x) - \beta_1(x)\frac{\langle \alpha_2, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} \\ &= (1-|a_2|^2)^{\frac{3}{2}}\Big(\frac{\overline{1-\overline{a_2}x}}{|1-\overline{a_2}x|^4} - (1-|a_1|^2)^3\frac{\overline{1-\overline{a_1}x}}{|1-\overline{a_1}x|^4}\frac{\overline{1-\overline{a_2}a_1}}{|1-\overline{a_2}a_1|^4}\Big), \end{aligned}$$

and

$$\begin{aligned} \|\beta_2\|^2 &= \langle \alpha_2 - \beta_1\frac{\langle \alpha_2, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle}, \alpha_2 - \beta_1\frac{\langle \alpha_2, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle}\rangle \\ &= \|\alpha_2\|^2 - \overline{\langle \alpha_2, \beta_1 \rangle}\langle \alpha_2, \beta_1 \rangle - \langle \beta_1, \alpha_2 \rangle\langle \alpha_2, \beta_1 \rangle + |\langle \alpha_2, \beta_1 \rangle|^2 \\ &= 1 - |\langle \alpha_2, \beta_1 \rangle|^2 \\ &= 1 - \Big[\frac{(1-|a_1|^2)(1-|a_2|^2)}{|1-\overline{a_2}a_1|^2}\Big]^3. \end{aligned}$$

Hence $\frac{\beta_2(x)}{\|\beta_2\|}$ gives $B_2(x)$.

The rest of this section will be devoted to the adaptive approximation of $f \in \mathcal{H}^2(\mathbb{B}_4)$ by the system $\{B_n\}$.

Given a function $f \in \mathcal{H}^2(\mathbb{B}_4)$, suppose f can be expanded into

$$f(x) = \sum_{n=1}^{\infty} B_n(x)c_n,$$

then the coefficients c_n are given by

$$\begin{aligned} c_1 &= \langle f, B_{\{a_1\}}\rangle \\ &= (1-|a_1|^2)^{\frac{3}{2}} f(a_1), \end{aligned}$$

$$\begin{aligned} c_n &= \langle f, B_n\rangle \\ &= \big\langle f, \frac{\alpha_n - \sum_{i=1}^{n-1}\beta_i \frac{\langle \alpha_n,\beta_i\rangle}{\langle \beta_i,\beta_i\rangle}}{\|\beta_n\|}\big\rangle \\ &= \frac{\big\langle f - \sum_{i=1}^{n-1}\beta_i \frac{\langle f,\beta_i\rangle}{\langle \beta_i,\beta_i\rangle}, \alpha_n\big\rangle}{\|\beta_n\|}. \end{aligned}$$

Set

$$f_n(x) = f(x) - \sum_{i=1}^{n-1}\beta_i(x)\frac{\langle f,\beta_i\rangle}{\langle \beta_i,\beta_i\rangle} = f(x) - \sum_{i=1}^{n-1} B_i(x)\langle f, B_i\rangle,$$

if $m(a_n) = 1$, then

$$\langle f, B_n\rangle = \frac{(1-|a_n|^2)^{\frac{3}{2}} f_n(a_n)}{\|\beta_n\|}, \tag{3}$$

and

$$\begin{aligned} \|\beta_n\|^2 &= \big\langle \alpha_n - \sum_{i=1}^{n-1}\beta_i\frac{\langle \alpha_n,\beta_i\rangle}{\langle \beta_i,\beta_i\rangle}, \alpha_n - \sum_{i=1}^{n-1}\beta_i\frac{\langle \alpha_n,\beta_i\rangle}{\langle \beta_i,\beta_i\rangle}\big\rangle \\ &= \langle \alpha_n,\alpha_n\rangle - \sum_{i=1}^{n-1}\frac{\overline{\langle \alpha_n,\beta_i\rangle}}{\langle \beta_i,\beta_i\rangle}\langle \alpha_n,\beta_i\rangle - \sum_{i=1}^{n-1}\langle \beta_i,\alpha_n\rangle\frac{\langle \alpha_n,\beta_i\rangle}{\langle \beta_i,\beta_i\rangle} + \sum_{i=1}^{n-1}\frac{|\langle \alpha_n,\beta_i\rangle|^2}{\langle \beta_i,\beta_i\rangle} \\ &= 1 - \sum_{i=1}^{n-1}\frac{|\langle \alpha_n,\beta_i\rangle|^2}{\langle \beta_i,\beta_i\rangle}. \end{aligned} \tag{4}$$

Otherwise, if $m(a_n) > 1$, $\langle f, B_n\rangle$ and $\|\beta_n\|$ will be taken in the limit sense as before.

Lemma 3.1. *Let $a_1, \ldots, a_{n-1} \in \mathbb{B}_4$ be fixed, $a = |a|\xi = r\xi$, then*

$$\lim_{r\to 1^-}\|\beta_{\{a_1,\ldots,a_{n-1},a\}}\| = 1$$

uniformly in $|\xi| = 1$.

Proof. From (4) it suffices to show that $\lim_{r\to 1^-}|\langle \alpha_{\{a\}},\beta_i\rangle|^2 = 0$ whenever $i \le n-1$, this follows once we show that $\lim_{r\to 1^-}|\langle \alpha_{\{a\}},\alpha_i\rangle|^2 = 0$, due to β_i is the linear combination of $\alpha_1,\ldots,\alpha_i$. Note that when $r \to 1^-$, a must be different from a_i $(i \le n-1)$, hence $|\langle \alpha_{\{a\}},\alpha_i\rangle|^2 = \big[\frac{(1-|a_i|^2)(1-|a|^2)}{|1-\bar{a}a_i|^2}\big]^3$ if $m(a_i) = 1$, or the linear combination of the partial derivatives of $\big[\frac{(1-|a_i|^2)(1-|a|^2)}{|1-\bar{a}a_i|^2}\big]^3$ with respect to a_i if $m(a_i) > 1$. In both cases it is clear that $\lim_{r\to 1^-}|\langle \alpha_{\{a\}},\alpha_i\rangle|^2 = 0$, the lemma follows. □

Lemma 3.2. *Assumptions are as in Lemma 3.1, then*

$$\lim_{r\to 1^-} |\langle f, B_{\{a_1,\ldots,a_{n-1},a\}}\rangle| = 0$$

uniformly in $|\xi| = 1$.

Proof. From the formula (3) and Lemma 3.1, it is equivalent to show that for any function $g \in \mathcal{H}^2(\mathbb{B}_4)$ there holds

$$\lim_{r\to 1^-} (1-r^2)^3 |g(a)|^2 = \lim_{r\to 1^-} |\langle g, B_{\{a\}}\rangle|^2 = 0.$$

It suffices to prove that

$$\lim_{r\to 1^-} \|g - B_{\{a\}}\langle g, B_{\{a\}}\rangle\| = \|g\|,$$

due to

$$\begin{aligned}\|g - B_{\{a\}}\langle g, B_{\{a\}}\rangle\|^2 &= \big\langle g - B_{\{a\}}\langle g, B_{\{a\}}\rangle, g - B_{\{a\}}\langle g, B_{\{a\}}\rangle\big\rangle \\ &= \|g\|^2 - |\langle g, B_{\{a\}}\rangle|^2.\end{aligned}$$

Let $P_\rho(\eta,\mu) = \frac{1}{2\pi^2}\frac{1-\rho^2}{|\rho\eta-\mu|^4}$ be the Poisson kernel, where $\rho \in (0,1)$ and $|\eta| = |\mu| = 1$. For any $\varepsilon > 0$, we can choose ρ sufficiently close to 1 so that

$$\begin{aligned}\|g\| &\geq \|g - B_{\{a\}}\langle g, B_{\{a\}}\rangle\| \\ &\geq \|P_\rho * (g - B_{\{a\}}\langle g, B_{\{a\}}\rangle)\|_{L^2(S^4)} \\ &\geq \|P_\rho * g\|_{L^2(S^4)} - |\langle g, B_{\{a\}}\rangle| \|P_\rho * B_{\{a\}}\|_{L^2(S^4)} \\ &\geq (1-\varepsilon)\|g\| - \|g\| \|P_\rho * B_{\{a\}}\|_{L^2(S^4)}, \end{aligned} \tag{5}$$

where

$$\begin{aligned}P_\rho * B_{\{a\}} &= \frac{1}{2\pi^2}\int_{|\mu|=1} \frac{1-\rho^2}{|\rho\eta-\mu|^4}(1-r^2)^{\frac{3}{2}}\frac{\overline{1-\overline{a}\mu}}{|1-\overline{a}\mu|^4} dS \\ &= B_{\{a\}}(\rho\eta).\end{aligned}$$

Hence,

$$\begin{aligned}\|P_\rho * B_{\{a\}}\|^2_{L^2(S^4)} &= \frac{1}{2\pi^2}\int_{|\eta|=1} \frac{(1-r^2)^3}{|1-\overline{a}\rho\eta|^6} dS \\ &= \Big(\frac{1-r^2}{1-\rho^2 r^2}\Big)^3 \frac{1}{2\pi^2}\int_{|\eta|=1}\Big(\frac{1-\rho^2r^2}{|1-\overline{a}\rho\eta|^2}\Big)^3 dS \\ &= \Big(\frac{1-r^2}{1-\rho^2 r^2}\Big)^3 \|B_{\{\rho a\}}\|^2 \\ &= \Big(\frac{1-r^2}{1-\rho^2 r^2}\Big)^3. \end{aligned} \tag{6}$$

When r is close to 1, (5) and (6) give

$$\|g\| \geq (1-2\varepsilon)\|g\|,$$

which proves the lemma. ☐

Another Proof. Denote $V_r = \frac{\pi^2}{2}(1-r)^4$ the volume of the ball $\mathbb{B}_4(a, 1-r) = \{x : |x-a| < 1-r\}$, write $x = |x|\eta = \rho\eta$, note that

$$|x-a| \geq ||x| - |a|| = |\rho - r|,$$

and

$$\begin{aligned} |x-a| &= |\rho\eta - r\xi| \\ &= |r(\eta - \xi) - (r-\rho)\eta| \\ &\geq r|\eta - \xi| - |r - \rho| \\ &\geq r|\eta - \xi| - |x - a|, \end{aligned}$$

so $x \in \mathbb{B}_4(a, 1-r)$ implies

$$\begin{cases} 2r - 1 < \rho < 1, \\ |\eta - \xi| < 2(1-r)/r. \end{cases}$$

Hence, when r is sufficiently close to 1, we have

$$\begin{aligned} |\langle g, B_{\{a\}}\rangle| &= |(1-r^2)^{3/2} g(a)| \\ &= (1-r^2)^{3/2} \Big| V_r^{-1} \int_{\mathbb{B}_4(a,1-r)} g(x)dx \Big| \\ &\leq (1-r^2)^{3/2} \Big(V_r^{-1} \int_{\mathbb{B}_4(a,1-r)} |g(x)|^2 dx \Big)^{1/2} \\ &\leq (1-r^2)^{3/2} \Big(V_r^{-1} \int_{2r-1}^{1} \rho^3 \int_{|\eta-\xi|<2(1-r)/r} |g(\rho\eta)|^2 dS d\rho \Big)^{1/2} \\ &\leq (1-r^2)^{3/2} \Big(V_r^{-1} 2(1-r) \sup_{0<\rho<1} \int_{|\eta-\xi|<2(1-r)/r} |g(\rho\eta)|^2 dS \Big)^{1/2} \\ &\lesssim \Big(\int_{|\eta-\xi|<2(1-r)/r} \sup_{0<\rho<1} |g(\rho\eta)|^2 dS \Big)^{1/2}. \end{aligned}$$

Note that $\sup_{0<\rho<1} |g(\rho\eta)| \in L^2(S^4)$ whenever $g \in \mathcal{H}^2(\mathbb{B}_4)$, and the measure of the set $\{\eta : |\eta - \xi| < 2(1-r)/r\}$ tends to 0 as $r \to 1^-$, the lemma follows by the absolute continuity of the Lebesgue integral. $\square$

Lemma 3.2 guarantees that we can choose $a_n \in \mathbb{B}_4$ at each step such that

$$|\langle f, B_{\{a_1,\ldots,a_{n-1},a_n\}}\rangle|^2 = \sup\{|\langle f, B_{\{a_1,\ldots,a_{n-1},a\}}\rangle|^2 : a \in \mathbb{B}_4\}, \tag{7}$$

which will make the expansion converges in a fast way in the sense of energy.

Theorem 3.1. *Under the above selection criterion (7), we have*

$$\|\sum_{n=1}^{N} B_n \langle f, B_n\rangle - f\| \to 0 \quad (N \to \infty). \tag{8}$$

Proof. Bessel inequality gives

$$\sum_{n=1}^{\infty} |\langle f, B_n\rangle|^2 \leq \|f\|^2,$$

which asserts that there exists $g \in \mathcal{H}^2(\mathbb{B}_4)$ s.t.

$$\sum_{n=1}^{\infty} B_n\langle f, B_n\rangle = g$$

holds in the sense of $\mathcal{H}^2(\mathbb{B}_4)$. If (8) does not hold, then

$$h = f - g \neq 0,$$

so there exists a point $b \in \mathbb{B}_4 \setminus \bigcup_{i=1}^{\infty}\{a_i\}$ s.t.

$$|\langle h, B_{\{b\}}\rangle| = \delta > 0.$$

Set

$$f_N = f - \sum_{n=1}^{N-1} B_n\langle f, B_n\rangle, \quad r_N = -\sum_{n=N}^{\infty} B_n\langle f, B_n\rangle.$$

Since when N is large enough

$$|\langle r_N, B_{\{b\}}\rangle| \leq \|r_N\|\|B_{\{b\}}\| = \|r_N\| = \big(\sum_{n=N}^{\infty} |\langle f, B_n\rangle|^2\big)^{1/2} < \delta/2, \tag{9}$$

we have

$$|\langle f_N, B_{\{b\}}\rangle| = |\langle h - r_N, B_{\{b\}}\rangle| > |\langle h, B_{\{b\}}\rangle| - |\langle r_N, B_{\{b\}}\rangle| > \delta/2.$$

Note that

$$\begin{aligned}
\langle f_N, B_{\{b\}}\rangle &= \Big\langle f - \sum_{n=1}^{N-1} B_n\langle f, B_n\rangle, B_{\{b\}}\Big\rangle \\
&= \Big\langle f, B_{\{b\}} - \sum_{n=1}^{N-1} B_n\langle B_{\{b\}}, B_n\rangle\Big\rangle \\
&= \Big\langle f, B_{\{b\}} - \sum_{n=1}^{N-1} \beta_n\frac{\langle B_{\{b\}}, \beta_n\rangle}{\langle \beta_n, \beta_n\rangle}\Big\rangle \\
&= \langle f, B_{\{a_1,\ldots,a_{N-1},b\}}\rangle\|\beta_{\{a_1,\ldots,a_{N-1},b\}}\|,
\end{aligned}$$

we get

$$|\langle f, B_{\{a_1,\ldots,a_{N-1},b\}}\rangle| = \frac{|\langle f_N, B_{\{b\}}\rangle|}{\|\beta_{\{a_1,\ldots,a_{N-1},b\}}\|} > |\langle f_N, B_{\{b\}}\rangle| > \delta/2.$$

On the other hand, we know from (9) that

$$|\langle f, B_N\rangle| = |\langle f, B_{\{a_1,\ldots,a_{N-1},a_N\}}\rangle| < \delta/2,$$

so

$$|\langle f, B_{\{a_1,\ldots,a_{N-1},a_N\}}\rangle| < |\langle f, B_{\{a_1,\ldots,a_{N-1},b\}}\rangle|.$$

By the maximal selection criterion (7) we should not have chosen a_N but b at the N-th step, it is a contradiction. □

4 Fast convergent decomposition of functions in $L^2(S^4)$

For convenience, Throughout this article we use $\operatorname{Im} x$ to denote the vector part of any quaternion x, i.e., $\operatorname{Im} x = x - \operatorname{Re} x$. Given a function $f(\xi)$, which is defined on S^4, assume $f \in L^2(S^4)$, without lost of generality, we further assume that f is real-valued (otherwise we handle with f component-wisely). If a function $F(x) \in \mathcal{H}^2(\mathbb{B}_4)$ satisfies

$$\lim_{r\to 1^-} \operatorname{Re}(F(r\xi)) = f(\xi), \quad \text{a.e. on } S^4,$$

then the adaptive decomposition of $F(x)$ according to section 3 will lead to the adaptive decomposition of $f(\xi)$. Follows the idea proposed in [1] and [12], such F can be constructed explicitly (not necessarily uniquely) by

$$F(x) = T(f)(x) = \int_{|\omega|=1} P(x,\omega) f(\omega) d\omega + \int_{|\omega|=1} Q(x,\omega) f(\omega) d\omega, \quad |x| < 1,$$

where $P(x,\omega) = \frac{1}{2\pi^2}\frac{1-|x|^2}{|x-\omega|^4}$ is the Poisson kernel, and

$$\begin{aligned} Q(x,\omega) &= \operatorname{Im}\Big(\int_0^1 t^2 (\overline{D}P)(tx,\omega) x dt\Big) \\ &= \Big(\frac{1}{2\pi^2}\int_0^1 \frac{4t^2(1-t^2|x|^2)}{|tx-\omega|^6} dt\Big) \operatorname{Im}(\overline{\omega}x) \\ &= \frac{1}{2\pi^2}\Big(\frac{(3+|x|^2)(3-\operatorname{Re}(\overline{\omega}x)) - 8}{|x-\omega|^4} - \frac{\arctan \frac{\sqrt{|x|^2 - (\operatorname{Re}(\overline{\omega}x))^2}}{1-\operatorname{Re}(\overline{\omega}x)}}{\sqrt{|x|^2 - (\operatorname{Re}(\overline{\omega}x))^2}}\Big) \frac{\operatorname{Im}(\overline{\omega}x)}{|x|^2 - (\operatorname{Re}(\overline{\omega}x))^2} \end{aligned}$$

is the Cauchy-type harmonic conjugate of the Poisson kernel on the unit sphere. Similar to [12], we can prove that T is a bounded operator from $L^2(S^4)$ to $\mathcal{H}^2(\mathbb{B}_4)$.

5 The case when the variables are in the whole range

Sometimes we may write $x \in \mathbb{R}^4$ as $x = x_0 + \underline{x}$ for convenience, where $\underline{x} \in \mathbb{R}^3$. Denote the half space $\{x \in \mathbb{R}^4 : \operatorname{Re} x > 0\}$ by $\mathbb{R}^4_+$, functions lies in the Hardy space $\mathcal{H}^2(\mathbb{R}^4_+)$ satisfy $Df = 0$ in $\mathbb{R}^4$ and

$$\sup_{x_0>0} \Big(\frac{1}{2\pi^2}\int_{\mathbb{R}^3} |f(x_0+\underline{x})|^2 d\underline{x}\Big)^{1/2} < \infty.$$

The inner product on $\mathcal{H}^2(\mathbb{R}^4_+)$ is defined by

$$\langle f, g\rangle = \frac{1}{2\pi^2}\int_{\mathbb{R}^3} \overline{g(\underline{y})} f(\underline{y}) d\underline{y}, \quad f, g \in \mathcal{H}^2(\mathbb{R}^4_+),$$

and the norm

$$\|f\| = \Big(\frac{1}{2\pi^2}\int_{\mathbb{R}^3} |f(\underline{y})|^2 d\underline{y}\Big)^{1/2} = \langle f, f\rangle^{1/2}.$$

Suppose $f(\underline{x})$ is a real-valued function, $f(\underline{x}) \in L^2(\mathbb{R}^3)$, consider the Cauchy integral of f:

$$F(x) = C(f)(x) = \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \frac{\overline{\underline{y} - x}}{|\underline{y} - x|^4} f(\underline{y})(-1)d\underline{y}, \quad x \in \mathbb{R}^4_+,$$

We have $F(x) \in \mathcal{H}^2(\mathbb{R}^4_+)$, and the well known Sokhotskyi-Plemelj formula gives

$$\lim_{x_0 \to 0^+} F(x_0 + \underline{x}) = \frac{1}{2} f(\underline{x}) + \frac{1}{2} H(f)(\underline{x}),$$

where $H(f) = \sum_{i=1}^3 e_i R_i(f)$, and

$$R_i(f)(\underline{x}) = \frac{1}{\pi^2} p.v. \int_{\mathbb{R}^3} \frac{x_i - y_i}{|\underline{x} - \underline{y}|^4} f(\underline{y}) d\underline{y}$$

is the i-th Riesz transform of f. Hence, the adaptive decomposition of $f(\underline{x})$ is now turned to the approximation of $F(x)$.

Similarly we set

$$\alpha_n(x) = \alpha_{\{a_n\}}(x) = (2\text{Re}(a_n))^{\frac{3}{2}} \frac{\overline{x + \overline{a_n}}}{|x + \overline{a_n}|^4}, \quad a_n \in \mathbb{R}^4_+, n \in N_+,$$

then $\alpha_n(x) \in \mathcal{H}^2(\mathbb{R}^4_+)$ and $\|\alpha_n(x)\|^2 = 1$ for all n, they can also be orthogonalized by Gram-Schmidt orthogonalization process (if the multiplicities of some parameters are larger than 1, then we treat with it in the limit sense as before), which leads to an orthonormal system $\{B_n\} = \{B_{\{a_1,\ldots,a_n\}}\}$. Given a function $f \in \mathcal{H}^2(\mathbb{R}^4_+)$, we have

$$\begin{aligned} \langle f, B_{\{a\}} \rangle &= \frac{1}{2\pi^2} \int_{\mathbb{R}^3} (2a_0)^{\frac{3}{2}} \frac{\underline{y} + \overline{a}}{|\underline{y} + \overline{a}|^4} f(\underline{y}) d\underline{y} \\ &= (2a_0)^{\frac{3}{2}} \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \frac{\overline{\underline{y} - a}}{|\underline{y} - a|^4} f(\underline{y})(-1) d\underline{y} \\ &= (2a_0)^{\frac{3}{2}} f(a). \end{aligned} \tag{10}$$

Corresponding to Lemma 3.2, The following propositions hold.

Proposition 5.1.

$$\lim_{a_0 \to 0^+} |\langle f, B_{\{a\}} \rangle| = 0 \quad \textit{uniformly in } \underline{a} \in \mathbb{R}^3.$$

We would like to write down the proof for completeness.

Proof. It suffices to show that

$$\lim_{a_0 \to 0^+} \|f - B_{\{a\}} \langle f, B_{\{a\}} \rangle\| = \|f\|.$$

Let $P_t(\underline{x}) = \frac{1}{\pi^2} \frac{t}{(|\underline{x}|^2 + t^2)^2}$ be the Poisson kernel on half plane. For any $\varepsilon > 0$, we can choose t sufficiently close to 0 so that

$$\begin{aligned} \|f\| &\geq \|f - B_{\{a\}} \langle f, B_{\{a\}} \rangle\| \\ &\geq \|P_t * (f - B_{\{a\}} \langle f, B_{\{a\}} \rangle)\|_{L^2(\mathbb{R}^3)} \\ &\geq (1 - \varepsilon)\|f\| - \|f\| \|P_t * B_{\{a\}}\|_{L^2(\mathbb{R}^3)}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} P_t * B_{\{a\}} &= \frac{1}{\pi^2}\int_{\mathbb{R}^3}\frac{t}{(|\underline{x}-\underline{y}|^2+t^2)^2}B_{\{a\}}(\underline{y})d\underline{y} \\ &= B_{\{a\}}(t+\underline{x}). \end{aligned}$$

Hence,

$$\begin{aligned} \|P_t * B_{\{a\}}\|^2_{L^2(\mathbb{R}^3)} &= \frac{1}{2\pi^2}\int_{\mathbb{R}^3}(2a_0)^3\frac{d\underline{x}}{(|\underline{x}-\underline{a}|^2+(t+a_0)^2)^3} \\ &= \Big(\frac{a_0}{t+a_0}\Big)^3\frac{1}{2\pi^2}\int_{\mathbb{R}^3}\Big(\frac{2(t+a_0)}{|\underline{x}-\underline{a}|^2+(t+a_0)^2}\Big)^3 d\underline{x} \\ &= \Big(\frac{a_0}{t+a_0}\Big)^3\|B_{\{t+a\}}\|^2 \\ &= \Big(\frac{a_0}{t+a_0}\Big)^3. \end{aligned} \tag{12}$$

When a_0 is close to 0, (11) and (12) yield

$$\|f\| \geq (1-2\varepsilon)\|f\|,$$

which proves the lemma. $\Box$

Another Proof. Denote $V_{a_0} = \frac{\pi^2 a_0^4}{2}$ the volume of the ball $\mathbb{B}_4(a, a_0)$, then

$$\begin{aligned} |\langle f, B_{\{a\}}\rangle| &= |(2a_0)^{3/2}f(a)| \\ &= (2a_0)^{3/2}\Big|V_{a_0}^{-1}\int_{\mathbb{B}_4(a,a_0)} f(x_0+\underline{x})dx\Big| \\ &\leq (2a_0)^{3/2}\Big(V_{a_0}^{-1}\int_{\mathbb{B}_4(a,a_0)} |f(x_0+\underline{x})|^2dx\Big)^{1/2} \\ &\leq (2a_0)^{3/2}\Big(V_{a_0}^{-1}\int_0^{2a_0}\int_{|\underline{x}-\underline{a}|\leq a_0} |f(x_0+\underline{x})|^2d\underline{x}dx_0\Big)^{1/2} \\ &\leq (2a_0)^{3/2}\Big(2a_0V_{a_0}^{-1}\sup_{x_0\in(0,2a_0)}\int_{|\underline{x}-\underline{a}|\leq a_0} |f(x_0+\underline{x})|^2d\underline{x}\Big)^{1/2} \\ &\lesssim \Big(\int_{|\underline{x}-\underline{a}|\leq a_0}\sup_{x_0\in(0,2a_0)} |f(x_0+\underline{x})|^2d\underline{x}\Big)^{1/2} \\ &\leq \Big(\int_{|\underline{x}-\underline{a}|\leq a_0}\sup_{x_0>0} |f(x_0+\underline{x})|^2d\underline{x}\Big)^{1/2} \end{aligned}$$

Note that $\sup_{x_0>0}|f(x_0+\underline{x})| \in L^2(\mathbb{R}^3)$ and $|\{\underline{x} : |\underline{x}-\underline{a}| \leq a_0\}| \to 0$ (as $a_0 \to 0^+$), by the absolute continuity of the Lebesgue integral we have

$$\lim_{a_0\to 0^+}\int_{|\underline{x}-\underline{a}|\leq a_0}\sup_{x_0>0}|f(x_0+\underline{x})|^2d\underline{x} = 0. \qquad \Box$$

Proposition 5.2.

$$\lim_{a_0\to+\infty} |\langle f, B_{\{a\}}\rangle| = 0 \quad \textit{uniformly in } \underline{a} \in \mathbb{R}^3.$$

Proof. Denote $C_{a_0} = \frac{\pi^2}{2}(\frac{a_0}{2})^4$ the volume of the ball $\mathbb{B}_4(a, a_0/2)$, then

$$\begin{aligned}
|\langle f, B_{\{a\}}\rangle| &= |(2a_0)^{3/2} f(a)| \\
&= (2a_0)^{3/2}\Big|C_{a_0}^{-1}\int_{\mathbb{B}_4(a,a_0/2)} f(x_0+\underline{x})dx\Big| \\
&\leq (2a_0)^{3/2}\Big(C_{a_0}^{-1}\int_{\mathbb{B}_4(a,a_0/2)} |f(x_0+\underline{x})|^2 dx\Big)^{1/2} \\
&\leq (2a_0)^{3/2}\Big(C_{a_0}^{-1}\int_{a_0/2}^{3a_0/2}\int_{\mathbb{R}^3} |f(x_0+\underline{x})|^2 d\underline{x}dx_0\Big)^{1/2} \\
&\leq (2a_0)^{3/2}\Big(a_0 C_{a_0}^{-1}\sup_{x_0\in(a_0/2,3a_0/2)}\int_{\mathbb{R}^3} |f(x_0+\underline{x})|^2 d\underline{x}\Big)^{1/2} \\
&\lesssim \Big(\int_{\mathbb{R}^3}\sup_{x_0\in(a_0/2,3a_0/2)} |f(x_0+\underline{x})|^2 d\underline{x}\Big)^{1/2}
\end{aligned}$$

holds uniformly in $\underline{a} \in \mathbb{R}^3$, and

$$\sup_{x_0\in(a_0/2,3a_0/2)} |f(x_0+\underline{x})| \leq \sup_{x_0>0} |f(x_0+\underline{x})| \in L^2(\mathbb{R}^3)$$

due to the maximal inequality. Hence, the proposition follows by the Lebesgue dominated convergence theorem once we prove that

$$\lim_{a_0\to+\infty}\sup_{x_0\in(a_0/2,3a_0/2)} |f(x_0+\underline{x})| = 0.$$

It is known that

$$|f(x_0+\underline{x})| \lesssim x_0^{-3/2}\|f\|,$$

which can also be proved by similar discussions. Hence

$$\sup_{x_0\in(a_0/2,3a_0/2)} |f(x_0+\underline{x})| \lesssim a_0^{-3/2}\|f\|,$$

and the right hand side tends to zero uniformly in $\underline{x}$ as $a_0 \to +\infty$. □

Proposition 5.3.

$$\lim_{|\underline{a}|\to+\infty} |\langle f, B_{\{a\}}\rangle| = 0 \quad \textit{uniformly in } a_0 > 0.$$

Proof. In fact, by the above two propositions and the formula (10), we just need to prove that

$$\lim_{|\underline{a}|\to+\infty} |f(a_0+\underline{a})| = 0 \quad \text{uniformly in } a_0 \in [c,d] \subset \mathbb{R}.$$

Since

$$\begin{aligned}
|f(a_0+\underline{a})| &= \frac{1}{2\pi^2}\Big|\int_{\mathbb{R}^3}\frac{\overline{a_0+\underline{a}-\underline{y}}}{|a_0+\underline{a}-\underline{y}|^4}f(\underline{y})d\underline{y}\Big| \\
&\lesssim \int_{\mathbb{R}^3}\frac{|f(\underline{y})|}{(|\underline{y}-\underline{a}|^2+a_0^2)^{3/2}}d\underline{y} \\
&\lesssim \int_{|\underline{y}|>N}\frac{|f(\underline{y})|}{(|\underline{y}-\underline{a}|^2+c^2)^{3/2}}d\underline{y}+\int_{|\underline{y}|\le N}\frac{|f(\underline{y})|}{(|\underline{y}-\underline{a}|^2+c^2)^{3/2}}d\underline{y} \\
&= I_1+I_2,
\end{aligned}$$

by Hölder's inequality,

$$\begin{aligned}
I_1 &\le \Big(\int_{|\underline{y}|>N}\frac{1}{(|\underline{y}-\underline{a}|^2+c^2)^3}d\underline{y}\Big)^{1/2}\Big(\int_{|\underline{y}|>N}|f(\underline{y})|^2d\underline{y}\Big)^{1/2} \\
&\le \Big(\int_{\mathbb{R}^3}\frac{1}{(|\underline{y}|^2+c^2)^3}d\underline{y}\Big)^{1/2}\Big(\int_{|\underline{y}|>N}|f(\underline{y})|^2d\underline{y}\Big)^{1/2} \\
&\lesssim \Big(\int_{|\underline{y}|>N}|f(\underline{y})|^2d\underline{y}\Big)^{1/2}.
\end{aligned}$$

Because $f(\underline{y})\in L^2(\mathbb{R}^3)$, I_1 is small provided N is large enough. With N fixed,

$$I_2\lesssim\frac{1}{|\underline{a}|^3}\int_{|\underline{y}|\le N}|f(\underline{y})|d\underline{y}\to 0\quad(|\underline{a}|\to+\infty)$$

due to $f(\underline{y})$ is integrable on $\{\underline{y}:|\underline{y}|\le N\}$. That proves the proposition. □

Remark: In fact, for any fixed $a_1,\dots,a_{n-1}$, we have

$$\lim_{a_0\to 0^+}|\langle f,B_{\{a_1,\dots,a_{n-1},a\}}\rangle|=\lim_{a_0\to+\infty}|\langle f,B_{\{a_1,\dots,a_{n-1},a\}}\rangle|=0\quad\text{uniformly in } \underline{a}\in\mathbb{R}^3,$$

and

$$\lim_{|\underline{a}|\to+\infty}|\langle f,B_{\{a_1,\dots,a_{n-1},a\}}\rangle|=0\quad\text{uniformly in } a_0>0.$$

We conclude from the above properties that the maximal selection criterion

$$|\langle f,B_{\{a_1,\dots,a_{n-1},a_n\}}\rangle|^2=\sup\{|\langle f,B_{\{a_1,\dots,a_{n-1},a\}}\rangle|^2:a\in\mathbb{R}^4_+\}\tag{13}$$

that corresponds to (7) could be carried out to adaptively approximate f, and Theorem 3.1 also holds in this setting.

6 The rate of fast convergent decomposition

In this section we deal with the convergence rate of the adaptive decomposition discussed above. For convenience, we use the unified notation $\mathcal{H}^2(\Omega)$ to denote $\mathcal{H}^2(\mathbb{B}_4)$ or $\mathcal{H}^2(\mathbb{R}^4_+)$. For a sequence of points $\{a_n\}$ in Ω, we correspondingly set $B_{\{a_n\}}=\alpha_{\{a_n\}}(x)=\alpha_n(x)\in\mathcal{H}^2(\Omega)$

with $\|\alpha_n(x)\| = 1$ as before, and $\{B_n\} = \{B_{\{a_1,\ldots,a_n\}}\}$ is their orthonormalization (when the multiplicity $m(a_i) > 1$ for some i, we replace $\alpha_i(x)$ by its partial derivatives with respect to a_i in the orthogonalization process).

Given a function $f \in \mathcal{H}^2(\Omega)$, according to the maximal selection criterion (7) or (13), there holds

$$f = \sum_{n=1}^{\infty} B_n \langle f, B_n \rangle = \sum_{n=1}^{\infty} B_{\{a_1,\ldots,a_n\}} \langle f, B_{\{a_1,\ldots,a_n\}} \rangle$$

in the following sense

$$\|f_N\| = \|f - \sum_{n=1}^{N-1} B_n \langle f, B_n \rangle\| \to 0 \quad (N \to \infty).$$

In addition, we assume that f lies in the function class

$$\mathcal{H}^2(\Omega, M) := \Big\{ f \in \mathcal{H}^2(\Omega) : f = \sum_{k=1}^{\infty} \alpha_{\{b_k\}}(x) c_k, \ \sum_{k=1}^{\infty} |c_k| \le M < \infty \text{ and } \ b_k \in \Omega, k = 1, 2, \ldots \Big\},$$

which actually implies

$$\|f_1\| = \|f\| \le \sum_{k=1}^{\infty} |c_k| \cdot \|\alpha_{\{b_k\}}\| = \sum_{k=1}^{\infty} |c_k| \le M.$$

Theorem 6.1. *Under the above assumption, we have*

$$\|f_N\| \le \frac{M}{\sqrt{N}}.$$

Lemma 6.1. *[2] Let $\{d_m\}_{m=l}^{\infty}$ be a sequence of non-negative numbers satisfying the inequalities*

$$d_1 \le A, \quad d_{m+1} \le d_m(1 - d_m/A), \quad m = 1, 2, \ldots.$$

Then we have for each m

$$d_m \le A/m.$$

Proof of Theorem 6.1 We have

$$\begin{aligned} \|f_{N+1}\|^2 &= \|f_N - B_N \langle f, B_N \rangle\|^2 \\ &= \|f_N\|^2 - |\langle f, B_N \rangle|^2, \end{aligned} \tag{14}$$

and

$$\begin{aligned} |\langle f, B_N \rangle| &= \frac{|\langle f_N, \alpha_N \rangle|}{\|\beta_N\|} \\ &= \sup_{a_n \in \Omega} \frac{|\langle f_N, \alpha_{\{a_n\}} \rangle|}{\|\beta_{\{a_1,\ldots,a_n\}}\|} \\ &\ge \sup_{a_n \in \Omega} |\langle f_N, \alpha_{\{a_n\}} \rangle|. \end{aligned} \tag{15}$$

Since $f \in \mathcal{H}^2(\Omega, M)$, there exists a sequence $\{b_k\}$ in Ω such that $f = \sum_{k=1}^{\infty} \alpha_{\{b_k\}} c_k$, then

$$\begin{aligned} \|f_N\|^2 &= |\langle f_N, f_N \rangle| \\ &= |\langle f_N, f \rangle| \\ &= |\langle f_N, \sum_{k=1}^{\infty} \alpha_{\{b_k\}} c_k \rangle| \\ &\leq M \sup_{k \geq 1} |\langle f_N, \alpha_{\{b_k\}} \rangle|. \end{aligned} \tag{16}$$

Combine (14)–(16), we obtain

$$\|f_{N+1}\|^2 \leq \|f_N\|^2 (1 - \frac{\|f_N\|^2}{M^2}),$$

from this and Lemma 6.1 one could easily see that $\|f_N\| \leq M/\sqrt{N}$. □

7 The case when underlying space lies in $\mathbb{R}^3$

In this section, we point out the fact that we could build up an analogous theory in the unit ball $\mathbb{B}_3$ and the half space $\mathbb{R}^3_+$, where $\mathbb{B}_3 = \{x = x_0 + \underline{x} = x_0 + x_1 e_1 + x_2 e_2 \in \mathbb{R}^3 : |x| < 1\}$ and $\mathbb{R}^3_+ = \{x = x_0 + \underline{x} \in \mathbb{R}^3 : x_0 > 0\}$. A function f defined on $\mathbb{B}_3$ ($\mathbb{R}^3_+$), taking values in $\mathbb{H}$, is called a left monogenic function if it satisfies

$$\mathcal{D}f = e_0 \frac{\partial f}{\partial x_0} + e_1 \frac{\partial f}{\partial x_1} + e_2 \frac{\partial f}{\partial x_2} = 0$$

in its domain. If in addition, f satisfies

$$\|f\|^2 = \frac{1}{4\pi} \sup_{0<r<1} \int_{\eta \in S^3} |f(r\eta)|^2 dS < \infty \quad \Big(\frac{1}{4\pi} \sup_{x_0 > 0} \int_{\mathbb{R}^2} |f(x_0 + \underline{x})|^2 d\underline{x} < \infty \Big),$$

here S^3 is the boundary of $\mathbb{B}_3$, then f belongs to the Hardy space $\mathcal{H}^2(\mathbb{B}_3)$ ($\mathcal{H}^2(\mathbb{R}^3_+)$). The Cauchy formula for this setting is

$$f(x) = \frac{1}{4\pi} \int_{\partial\Omega} \frac{(q-x)^{-1}}{|q-x|} (n(q) dS) f(q), \quad x \in \Omega,\ f \in \mathcal{H}^2(\Omega),\ \Omega = \mathbb{B}_3\ (\mathbb{R}^3_+).$$

For more details about this function class and their generalizations, see e.g. [4].

The following operator

$$T(f)(x) = F(x) = \int_{\omega \in S^3} P(x, \omega) f(\omega) d\omega + \int_{\omega \in S^3} Q(x, \omega) f(\omega) d\omega, \quad x \in \mathbb{B}_3,$$

where $P(x, \omega) = \frac{1}{4\pi} \frac{1-|x|^2}{|x-\omega|^3}$ is the Poisson kernel, and

$$\begin{aligned} Q(x, \omega) &= \mathrm{Im}\Big(\int_0^1 t(\overline{D}P)(tx, \omega) x dt \Big) \\ &= \Big(\frac{1}{4\pi} \int_0^1 \frac{3t(1-t^2|x|^2)}{|tx-\omega|^5} dt \Big) \mathrm{Im}(\overline{\omega} x) \\ &= \frac{1}{4\pi} \Big(\frac{(3+|x|^2)(3-\mathrm{Re}(\overline{\omega}x)) - 8}{|x-\omega|^3} - 1 \Big) \frac{\mathrm{Im}(\overline{\omega}x)}{|x|^2 - (\mathrm{Re}(\overline{\omega}x))^2}, \end{aligned}$$

maps a real-valued function $f(\omega) \in L^2(S^3)$ to a function $F(x) \in \mathcal{H}^2(\mathbb{B}_3)$, and the real part of the boundary values of F coincides with f.

While the operator

$$C(f)(x) = F(x) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{\overline{\underline{y} - x}}{|\underline{y} - x|^3} f(\underline{y})(-1) d\underline{y}, \quad x \in \mathbb{R}^3_+,$$

sends a real-valued function $f(\underline{x}) \in L^2(\mathbb{R}^2)$ to a function $F(x) \in \mathcal{H}^2(\mathbb{R}^3_+)$, and the real part of the boundary values of F coincides with $\frac{1}{2}f$.

Hence, the adaptive decomposition of $f \in L^2(S^3)$ $(L^2(\mathbb{R}^2))$ could be obtained from the adaptive decomposition of $F \in \mathcal{H}^2(\mathbb{B}_3)$ $(\mathcal{H}^2(\mathbb{R}^3_+))$. For this, we use the orthonormal basis generated by

$$\alpha_{\{a\}}(x) = (1 - |a|^2) \frac{\overline{1 - \overline{a}x}}{|1 - \overline{a}x|^3} \in \mathcal{H}^2(\mathbb{B}_3), \quad a, x \in \mathbb{B}_3$$

$$\Big(\alpha_{\{a\}}(x) = (2\mathrm{Re}\, a) \frac{\overline{x + \overline{a}}}{|x + \overline{a}|^3} \in \mathcal{H}^2(\mathbb{R}^3_+), \quad a, x \in \mathbb{R}^3_+\Big).$$

For $f \in \mathcal{H}^2(\mathbb{B}_3)$ $(\mathcal{H}^2(\mathbb{R}^3_+))$, we have

$$\langle f, \alpha_{\{a\}} \rangle = (1 - |a|^2) f(a) \quad ((2\mathrm{Re}\, a) f(a)).$$

With the above settings, we could get the results parallel to §3, §5 and §6 after similar discussions, we omit the details.

8 Some numerical experiments

In order to make our results visible in $\mathbb{R}^3$, here we work on the space $\mathcal{H}^2(\mathbb{B}_3)$ $(\mathcal{H}^2(\mathbb{R}^3_+))$.

Example 1: Let $f(x) = \frac{\partial}{\partial x_0}(\frac{1-0.5\overline{x}}{|1-0.5x|^3}) = \frac{(1-0.75x_0+0.25x)(1-0.5\overline{x})}{|1-0.5x|^5}$, then $f \in \mathcal{H}^2(\mathbb{B}_3)$ and $\|f\|^2 = 32/9 \approx 3.5556$. By the maximal selection criterion, the adaptive decomposition of f by the 5-th partial sum of the system $\{B_n\}$ is given by:

$$f(x) \sim S_5(f, x) = \sum_{n=1}^{5} B_n(x) c_n = \sum_{n=1}^{5} B_n(x) \langle f, B_n \rangle, \quad x \in \mathbb{B}_3,$$

with the parameters $a_1 = (0.6458, -0.0000, -0.0000)$, $a_2 = (-0.0721, 0.0013, 0.0019)$, $a_3 = (0.6456, 0.0000, -0.0000)$, $a_4 = (0.2969, 0.0001, 0.0001)$, $a_5 = (0.8448, 0.0002, 0.0001)$, and the coefficients $c_1 = \langle f, B_1 \rangle = (1.8779, 0, 0, 0)$, $c_2 = (-0.1176, 0.0001, 0.0001, 0)$, $c_3 = (0.0897, -0.0001, -0.0001, -0.0000)$, $c_4 = (-0.0850, 0.0000, 0.0000, 0.0000)$, $c_5 = (0.0046, 0.0000, -0.0000, -0.0000)$. The energy of S_5 is $\sum_{n=1}^{5} |c_n|^2 \approx 3.5555 \approx \|f\|^2$.

On the sphere S^3, set $F(\theta, \varphi) = f(\cos(\theta), \sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi))$, $\theta \in [0, \pi], \varphi \in [0, 2\pi]$, we get the figure of $\mathrm{Re}(F)$ (Figure 1), similarly, we can get the figures of $\mathrm{Re}(S_5)$ (Figure 2), $\mathrm{Re}(F - S_5)$ (Figure 3) and $|F - S_5|$ (Figure 4) on S^3.

Let us compare this with the spherical monogenic expansion of f, on S^3, $f(\omega) = \sum_{k=0}^{\infty} P_k(f, \omega)$, $|\omega| = 1$, where P_k is the inner spherical monogenic of order k. Figure 5 shows the expansion of $\mathrm{Re}(F)$ up to $k=10$, Figure 6 reflects the difference between $\mathrm{Re}(F)$ and $\mathrm{Re}\sum_{k=0}^{10} P_k(f,\omega)$. The energy of $\sum_{k=0}^{10} P_k(f,\omega)$ equals $\sum_{k=0}^{10}\|P_k\|^2 = 1864043/524288 \approx 3.5554$. Compare these with S_5, we can see that our method works well for f than the approximation by P_k.

Example 2: Let $g(x) = \frac{(\overline{x}+1)(x-3x_0-2)}{|\overline{x}+1|^5}$, then $g \in \mathcal{H}^2(\mathbb{R}^3_+)$ and $\|g\|^2 = 0.375$. The adaptive decomposition of g by the 5-th partial sum of the system $\{B_n\}$ is $g(x) \sim S_5 = \sum_{n=1}^{5} B_n(x)c_n$, $x \in \mathbb{R}^3_+$, with the parameters $a_1 = (0.5000, 0.0005, 0.0008)$, $a_2 = (3.1482, -0.0053, 0.0008)$, $a_3 = (0.3264, 0.0006, 0.0009)$, $a_4 = (3.2086, -0.0058, 0.0007)$, $a_5 = (0.3044, 0.0006, 0.0008)$, and the coefficients $c_1 = \langle f, B_1\rangle = (-0.5926, 0.0003, 0.0005, 0)$, $c_2 = (0.1180, -0.0000, -0.0002, -0.0000)$, $c_3 = (0.0827, -0.0000,\ 0.0001, -0.0000)$, $c_4 = (-0.0454, 0.0001, 0.0002, 0.0000)$, $c_5 = (-0.0261, -0.0000, -0.0001, 0.0000)$. $\|S_5\|^2 = \sum_{n=1}^{5}|c_n|^2 \approx 0.3747 = 99.92\%\|g\|^2$.

The real part of the boundary values of g is $g(\underline{x}) = \frac{x_1^2+x_2^2-2}{(x_1^2+x_2^2+1)^{5/2}}$, $\underline{x} = (x_1, x_2) \in \mathbb{R}^2$. Since $g(\underline{x})$ is very small when $|\underline{x}|$ is large, we focus our attention on the rectangular domain $\mathcal{Q} = [-1,1]^2 = [-1,1]\times[-1,1]$. The figure of $g(\underline{x})$, and the figures of the boundary values of $\mathrm{Re}(S_5)$, $g(\underline{x}) - \mathrm{Re}(S_5)$ and $|g - S_5|$ on $\mathcal{Q}$ are respectively shown in Figure 7-10.

On $\mathbb{B}_2 = \{\underline{x} \in \mathbb{R}^2 : |\underline{x}| < 1\} \subset \mathcal{Q}$, $g(\underline{x})$ admits the Taylor expansion

$$g(\underline{x}) = T_5(g,\underline{x}) + o(|\underline{x}|^8) = -2 + 6|\underline{x}|^2 - \frac{45}{4}|\underline{x}|^4 + \frac{35}{2}|\underline{x}|^6 - \frac{1575}{64}|\underline{x}|^8 + o(|\underline{x}|^8), \quad |\underline{x}| < 1.$$

Figure 11 shows T_5 on $[-0.7, 0.7]^2 \subset \mathbb{B}_2$, Figure 12 shows the error (remainder term) $g(\underline{x}) - T_5$ on $[-0.7, 0.7]^2$. We could see that Taylor series works bad on the endpoints, and in general it may not convergent when $|\underline{x}|$ is large.

On $\mathcal{Q}$, $g(\underline{x})$ could be expanded into Fourier series:

$$g(\underline{x}) = \sum_{m,n=-\infty}^{+\infty} c_{m,n} e^{i\pi(mx_1+nx_2)}, \quad \underline{x} \in \mathcal{Q},$$

where

$$c_{m,n} = \frac{1}{4}\int_{\mathcal{Q}} f(\underline{y}) e^{-i\pi(my_1+ny_2)} d\underline{y}.$$

$F_5(g,\underline{x}) = \sum_{m,n=-4}^{4} c_{m,n} e^{i\pi(mx_1+nx_2)}$ and $g(\underline{x}) - F_5$ are shown in Figure 13-14. Compare F_5 with the adaptive expansion by S_5, we could see that Fourier series is non-stationary and it does not converge to $g(\underline{x})$ on $\mathbb{R}^2 \setminus \mathcal{Q}$, because $g(\underline{x})$ is not periodic.

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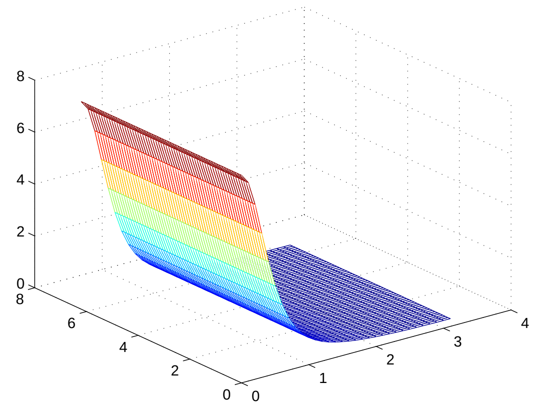

Figure 1: Re(F)

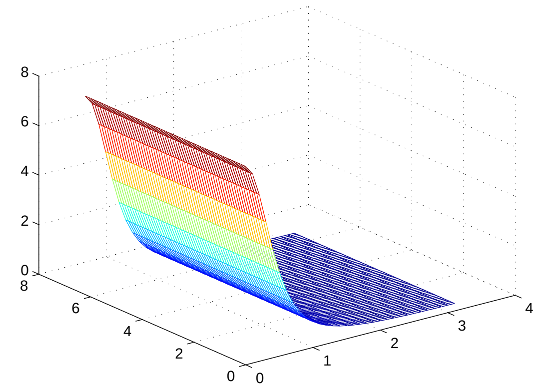

Figure 2: Re(S_5) on S^3

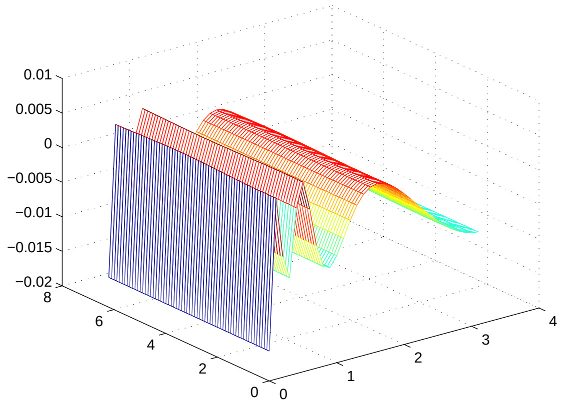

Figure 3: Re$(F - S_5)$ on S^3

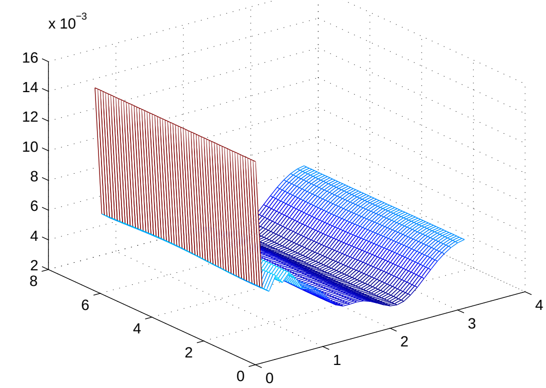

Figure 4: $|F - S_5|$ on S^3

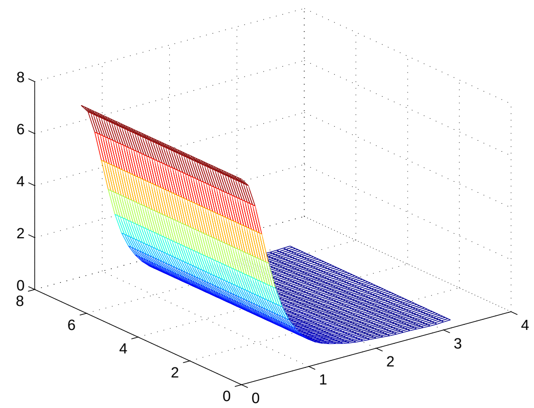

Figure 5: $\mathrm{Re}\sum_{k=0}^{10} P_k(f,\omega)$

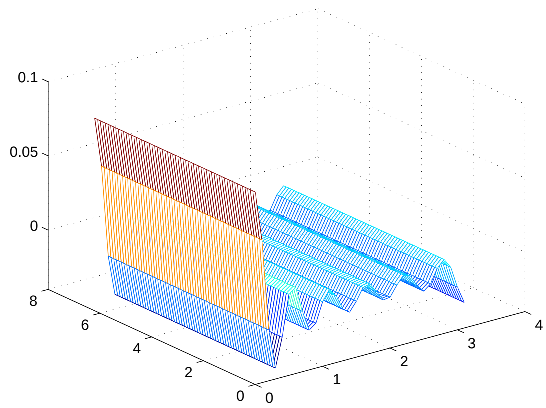

Figure 6: $\mathrm{Re}(F - \sum_{k=0}^{10} P_k(f,\omega))$

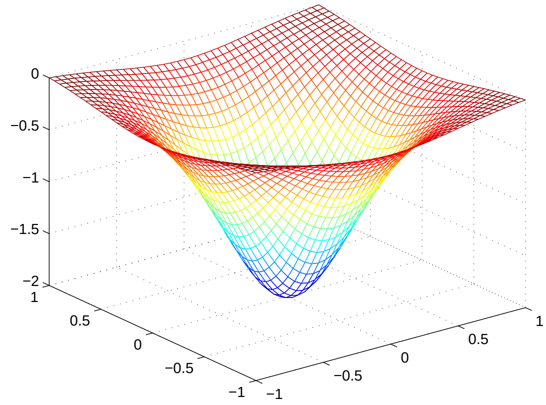

Figure 7: $g(\underline{x})$ on $\mathcal{Q}$

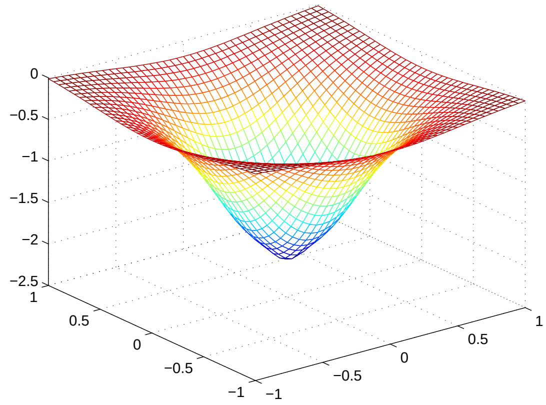

Figure 8: $\mathrm{Re}(S_5)$ on $\mathcal{Q}$

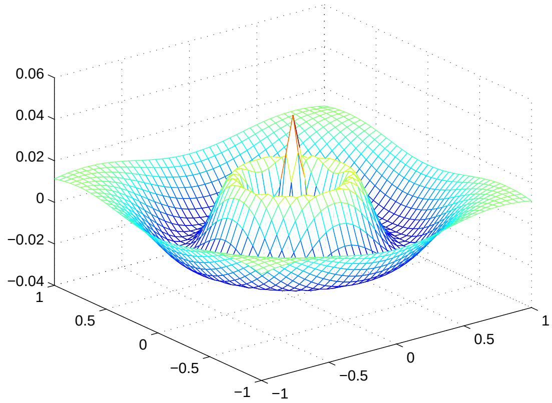

Figure 9: $g(\underline{x}) - \mathrm{Re}(S_5)$ on $\mathcal{Q}$

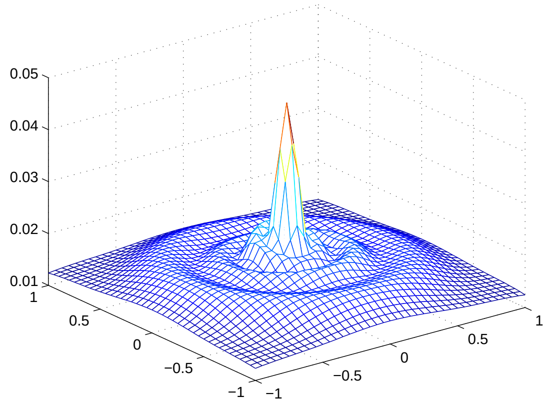

Figure 10: $|g - S_5|$ on $\mathcal{Q}$

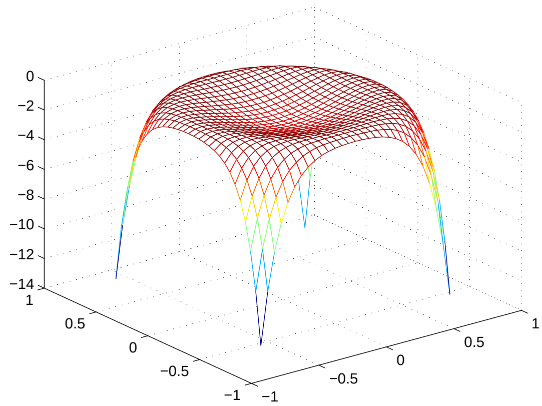

Figure 11: T_5 on $[-0.7, 0.7]^2$

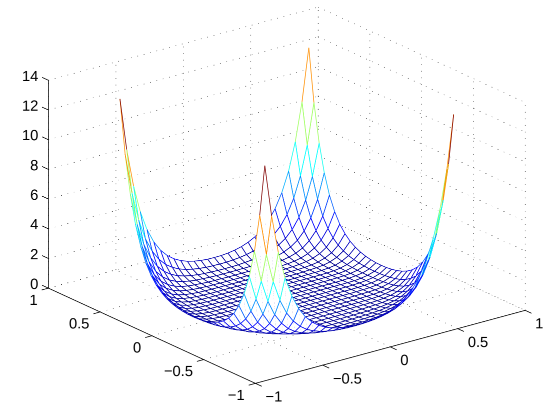

Figure 12: $g(\underline{x}) - T_5$ on $[-0.7, 0.7]^2$

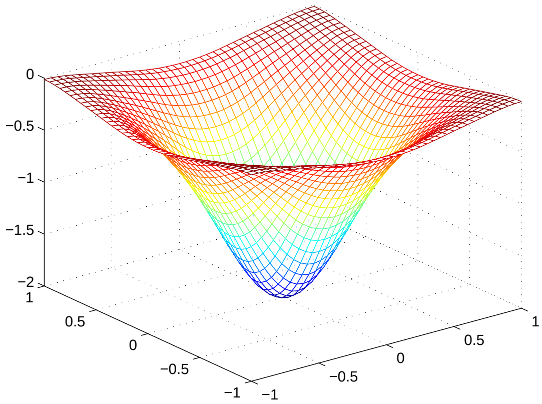

Figure 13: F_5 on $\mathcal{Q}$

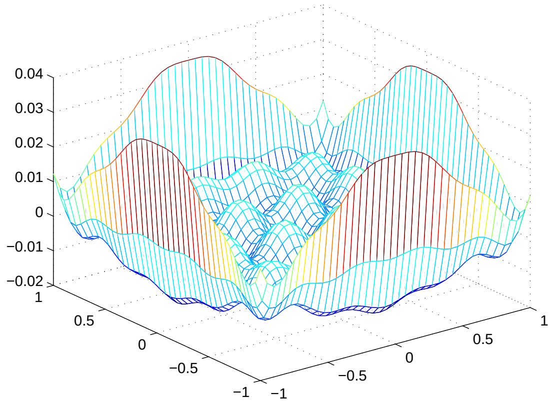

Figure 14: $g(\underline{x}) - F_5$ on $\mathcal{Q}$

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