Some Remarks on the Boundary Behaviors of the Hardy Spaces

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In memory of Jaime Keller

Abstract. Some estimates and boundary properties for functions in the Hardy spaces are given.

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1. Introduction

Let $\mathbb{D} = \{z = x + iy \in \mathbb{C} : |z| < 1\}$ be the open unit disc. The holomorphic Hardy space $\mathcal{H}^p(\mathbb{D})$ $(1 \le p < \infty)$ consists of all functions f that are holomorphic in $\mathbb{D}$ and satisfy

$$\|f\|_p = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty.$$

Getting close to the boundary of $\mathbb{D}$, singularities may happen for functions in $\mathcal{H}^p(\mathbb{D})$, where we have the well known estimate (cf. [3])

$$(1 - |z|)^{1/p} |f(z)| \le C_p \|f\|_p \quad \text{for } 1 \le p < \infty.$$

By using the density of the holomorphic polynomials (cf. [9]), or that of the Poisson integrals (cf. [5]), one can prove that

$$\lim_{|z| \to 1^-} (1 - |z|)^{1/p} |f(z)| = 0 \quad \text{for } 1 \le p < \infty,$$

which is more precise than the previous inequality near the boundary.

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In the case $p = 2$, $\mathcal{H}^p(\mathbb{D})$ is of particular importance. It is a Hilbert space with the inner product

$$\langle f, g\rangle = \int_0^{2\pi} f(e^{i\theta})\overline{g}(e^{i\theta})d\theta, \quad f, g \in \mathcal{H}^2(\mathbb{D}).$$

In a number of practical applications as the underlying space $\mathcal{H}^2(\mathbb{D})$ plays an important role (e.g., in signal processing, image processing and coding theory). Observing that for any function $f \in \mathcal{H}^2(\mathbb{D})$, we have

$$\langle f, \phi_a\rangle = \sqrt{1-|a|^2} f(a),$$

where $\phi_a(z) = \frac{\sqrt{1-|a|^2}}{1-\overline{a}z}$ is a unit vector of $\mathcal{H}^2(\mathbb{D})$ with the parameter $a \in \mathbb{D}$. By the aforementioned property, we get

$$\lim_{|a|\to 1^-} |\langle f, \phi_a\rangle| = 0,$$

which implies that there exists $a^\star \in \mathbb{D}$ such that $|\langle f, \phi_a^\star\rangle|$ attains the maximum value. This is crucial for the signal adaptive decomposition methods (as a variation and realization of greedy algorithm) introduced in [5, 8].

In this note, we give a generalization of the above result to higher dimensions, of which the special cases have been applied to the adaptive decomposition of functions of several variables ([6, 7]). Our method is a modification of the classic method (see [3, page 18]), which depends on some more delicate estimates. Before we state our main results, let us first have a quick review of some basic knowledge on Clifford algebra and Clifford analysis.

Let $e_1, \ldots, e_m$ be basic elements satisfying $e_ie_j + e_je_i = -2\delta_{ij}, i, j = 1, \ldots, n$, where δ_{ij} equals 1 if $i = j$ and 0 otherwise. Let $\mathbb{R}^{m+1} = \{x = x_0 + x_1e_1 + \cdots + x_me_m : x_i \in \mathbb{R}, 0 \le i \le m\}$ be identified with the usual $(m+1)$-dimensional Euclidean space. The real Clifford algebra generated by $e_1, \ldots, e_m$, denoted by $\mathscr{A}_m$, is an associative algebra in which each element is of the form $x = \sum_T x_Te_T$, where $x_T \in \mathbb{R}$, $e_T = e_{i_1}e_{i_2}\cdots e_{i_l}$ and $T = \{1 \le i_1 < i_2 < \cdots < i_l \le m\}$ runs over all ordered subsets of $\{1, \ldots, m\}$ and $x_\emptyset = x_0$, $e_\emptyset = e_0 = 1$. The norm and the conjugate of x are defined by $|x| = (\sum_T |x|_T^2)^{1/2}$ and $\overline{x} = \sum_T x_T\overline{e_T}$ respectively, where $\overline{e_T} = \overline{e_{i_l}}\cdots\overline{e_{i_2}}\,\overline{e_{i_1}}$ and $\overline{e_i} = -e_i$ for $i \neq 0$, $\overline{e_0} = e_0$. We have for any $x, y, z \in \mathscr{A}_m$, $\overline{xy} = \overline{y}\,\overline{x}$, $(xy)z = x(yz)$ and $|xy| \le 2^{m/2}|x||y|$.

A function $f(x) = \sum_T f_T(x)e_T \in C^1(\Omega, \mathscr{A}_m)$ is said to be *left monogenic* in the open set $\Omega \subset \mathbb{R}^{m+1}$ if and only if it satisfies the generalized Cauchy-Riemann equation

$$Df = \sum_{i=0}^m e_i\frac{\partial f}{\partial x_i} = 0,$$

where the Dirac operator D is defined by $D = \frac{\partial}{\partial x_0} + \nabla = \sum_0^m e_i\frac{\partial}{\partial x_i}$. If f is left monogenic, then each component of f is a real-valued harmonic function. For more information about the monogenic function theory, see [2].

Let $\mathbb{B}^m(x, \rho) = \{y \in \mathbb{R}^{m+1} : |y-x| < \rho\}$ be the open ball in $\mathbb{R}^{m+1}$, which is centered at x and of radius ρ. For simplicity, we denote $\mathbb{B}^m = \mathbb{B}^m(0, 1)$.

The monogenic Hardy space $\mathcal{H}^p(\mathbb{B}^m)$ $(1 \le p < \infty)$, consists of all functions f that are left monogenic in $\mathbb{B}^m$ and satisfy

$$\|f\|_p = \sup_{0<r<1} \left(\int_{|\eta|=1} |f(r\eta)|^p dS \right)^{1/p} < \infty, \tag{1.1}$$

where dS is the area element of $\partial\mathbb{B}^m$. We prove that

Theorem 1.1. *If $f \in \mathcal{H}^p(\mathbb{B}^m)$ $(1 \le p < \infty)$, then*

$$(1-|x|)^{|\alpha|+\frac{m}{p}} |\partial^\alpha f(x)| \le C_{m,p,|\alpha|} \|f\|_p, \tag{1.2}$$

where $\alpha = (l_0, l_1, \ldots, l_m)$, $|\alpha| = \sum_{i=0}^m l_i$ and $\partial^\alpha = \partial_{x_0}^{l_0} \partial_{x_1}^{l_1} \cdots \partial_{x_m}^{l_m}$. Write $x = |x|\xi = r\xi$, then we have

$$\lim_{r\to 1^-} (1-|x|)^{|\alpha|+\frac{m}{p}} |\partial^\alpha f(x)| = 0 \tag{1.3}$$

uniformly in $|\xi| = 1$.

Corresponding to this, we also prove some propositions for the monogenic Hardy space $\mathcal{H}^p(\mathbb{R}_+^{m+1})$, which consists of all functions f that are left monogenic on the half space $\mathbb{R}_+^{m+1} = \{x = x_0 + \underline{x} \in \mathbb{R}^{m+1} : x_0 > 0, \underline{x} = x_1 e_1 + \cdots + x_m e_m \in \mathbb{R}^m\}$ and satisfy

$$\|f\|_p = \sup_{x_0>0} \left(\int_{\mathbb{R}^m} |f(x_0 + \underline{x})|^p d\underline{x} \right)^{1/p} < \infty, \tag{1.4}$$

where $d\underline{x} = dx_1 \cdots dx_m$. We note that for $f \in \mathcal{H}^p(\mathbb{R}_+^{m+1})$ $(1 \le p < \infty)$, the boundary values $f(\underline{x}) = \lim_{x_0 \to 0^+} f(x_0 + \underline{x})$ exist almost everywhere and comprise a function in $L^p(\mathbb{R}^m)$, of which the Poisson integral coincides with f ([4]).

Theorem 1.2. *Suppose $f \in \mathcal{H}^p(\mathbb{R}_+^{m+1})$ $(1 \le p < \infty)$, then*

$$x_0^{|\alpha|+\frac{m}{p}} |\partial^\alpha f(x)| \le C_{m,p,|\alpha|} \|f\|_p; \tag{1.5}$$

moreover,

$$\lim_{x_0\to 0^+} x_0^{|\alpha|+\frac{m}{p}} |\partial^\alpha f(x_0+\underline{x})| = \lim_{x_0\to +\infty} x_0^{|\alpha|+\frac{m}{p}} |\partial^\alpha f(x_0+\underline{x})| = 0 \tag{1.6}$$

holds uniformly with respect to $\underline{x} \in \mathbb{R}^m$, and

$$\lim_{|\underline{x}|\to +\infty} x_0^{|\alpha|+\frac{m}{p}} |\partial^\alpha f(x_0+\underline{x})| = 0 \tag{1.7}$$

holds uniformly in $x_0 > 0$.

Remark 1.3. Similar discussions as in Section 2 will show that Theorem 1.1 (resp. Theorem 1.2) holds for the harmonic Hardy space $H^p(\mathbb{B}^m)$ (resp. $H^p(\mathbb{R}_+^{m+1})$) for $1 < p < \infty$, where by definition, a function f lies in $H^p(\mathbb{B}^m)$ (resp. $H^p(\mathbb{R}_+^{m+1})$) means that f is harmonic in $\mathbb{B}^m$ (resp. $\mathbb{R}_+^{m+1}$) and (1.1) (resp. (1.4)) holds. But for the case $p = 1$, (1.3) (resp. (1.6) and (1.7)) may not hold for $H^1(\mathbb{B}^m)$ (resp. $H^1(\mathbb{R}_+^{m+1})$). For example, $f(x_0, x_1) = \frac{x_0}{x_0^2+x_1^2} \in H^1(\mathbb{R}_+^2)$, but $x_0 f(x_0, x_1)$ does not uniformly tend to zero as $x_0 \to 0^+$.

2. Proof of the Theorems

Proof of Theorem 1.1. From Cauchy's estimate (cf. [1]) we know that

$$|\partial^\alpha f(x)| \le C_{m,|\alpha|}(1-|x|)^{-|\alpha|} \max_{y\in\partial\mathbb{B}^m(x,\frac{1-|x|}{2})} |f(y)|,$$

hence

$$(1-|x|)^{|\alpha|+\frac{m}{p}}|\partial^\alpha f(x)| \le C_{m,|\alpha|} \max_{y\in\partial\mathbb{B}^m(x,\frac{1-|x|}{2})} (1-|y|)^{\frac{m}{p}}|f(y)|.$$

So, to prove (1.2) and (1.3), it is enough to show that

$$(1-|x|)^{\frac{m}{p}}|f(x)| \le C_{m,p}\|f\|_p \tag{2.1}$$

and

$$\lim_{r\to 1^-}(1-|x|)^{\frac{m}{p}}|f(x)| = 0 \tag{2.2}$$

for $1 \le p < \infty$.

Denote by $V_r = C_m(1-r)^{m+1}$ the volume of the ball $\mathbb{B}^m(x, 1-r)$, write $y = |y|\eta = \rho\eta$, note that

$$|y-x| \ge ||y|-|x|| = |\rho - r|,$$

and

$$\begin{aligned} |y-x| &= |\rho\eta - r\xi| \\ &= |r(\eta-\xi) - (r-\rho)\eta| \\ &\ge r|\eta-\xi| - |r-\rho| \\ &\ge r|\eta-\xi| - |y-x|, \end{aligned}$$

so $y \in \mathbb{B}^m(x, 1-r)$ implies

$$\begin{cases} 2r-1 < \rho < 1, \\ |\eta-\xi| < 2(1-r)/r. \end{cases}$$

Hence, for $1 \le p < \infty$, we have

$$\begin{aligned} &|(1-r)^{m/p}f(x)| \\ &\quad= (1-r)^{m/p}\Big|V_r^{-1}\int_{\mathbb{B}^m(x,1-r)} f(y)dy\Big| \\ &\quad\le (1-r)^{m/p}\Big(V_r^{-1}\int_{\mathbb{B}^m(x,1-r)} |f(y)|^p dy\Big)^{1/p} \\ &\quad\le (1-r)^{m/p}\Big(V_r^{-1}\int_{2r-1}^1 \rho^m \int_{|\eta-\xi|<2(1-r)/r} |f(\rho\eta)|^p dS d\rho\Big)^{1/p} \\ &\quad\le (1-r)^{m/p}\Big(V_r^{-1}2(1-r)\sup_{0<\rho<1}\int_{|\eta-\xi|<\frac{2(1-r)}{r}} |f(\rho\eta)|^p dS\Big)^{1/p} \qquad (2.3) \\ &\quad\le (1-r)^{m/p}\Big(V_r^{-1}2(1-r)\sup_{0<\rho<1}\int_{|\eta|=1} |f(\rho\eta)|^p dS\Big)^{1/p} \\ &\quad= C_{m,p}\|f\|_p. \end{aligned}$$

(2.1) is now proved. On the other hand,

$$(2.3) \leq C_{m,p}\Big(\int_{|\eta-\xi|<\frac{2(1-r)}{r}} \sup_{0<\rho<1} |f(\rho\eta)|^p dS\Big)^{1/p}.$$

Note that as a function of η, $\sup_{0<\rho<1}|f(\rho\eta)| \in L^p(\partial\mathbb{B}^m)$, and the measure of the set $\{\eta : |\eta-\xi| < 2(1-r)/r\}$ tends to zero as $r \to 1^-$, (2.2) follows by the absolute continuity of the Lebesgue integral. □

Proof of Theorem 1.2. By Cauchy's estimate we have

$$|\partial^\alpha f(x)| \leq C_{m,|\alpha|} x_0^{-|\alpha|} \max_{y\in\partial\mathbb{B}^m(x,x_0/2)} |f(y)|,$$

hence

$$x_0^{|\alpha|+\frac{m}{p}} |\partial^\alpha f(x)| \leq C_{m,|\alpha|} \max_{y\in\partial\mathbb{B}^m(x,x_0/2)} y_0^{m/p}|f(y)|.$$

So, the proof of (1.5) and (1.6) is now reduced to the proof of the following

$$x_0^{m/p}|f(x)| \leq C_{m,p}\|f\|_p \tag{2.4}$$

and

$$\lim_{x_0\to 0^+} x_0^{m/p}|f(x)| = \lim_{x_0\to+\infty} x_0^{m/p}|f(x)| = 0 \tag{2.5}$$

for $1 \leq p < \infty$. Once these have been proved, the proof of (1.7) will be reduced to the proof of

$$\lim_{|\underline{x}|\to+\infty} |f(x_0+\underline{x})| = 0 \tag{2.6}$$

uniformly with respect to $x_0 \in [a,b] \subset (0,+\infty)$.

Denote by $V_{x_0} = C_m x_0^{m+1}$ the volume of the ball $\mathbb{B}^m(x, \frac{x_0}{2})$, then for $1 \leq p < \infty$,

$$\begin{aligned}
x_0^{m/p}|f(x)| &= x_0^{m/p}\Big|V_{x_0}^{-1}\int_{\mathbb{B}^m(x,\frac{x_0}{2})} f(y_0+\underline{y})dy\Big| \\
&\leq x_0^{m/p}\Big(V_{x_0}^{-1}\int_{\mathbb{B}^m(x,\frac{x_0}{2})} |f(y_0+\underline{y})|^p dy\Big)^{1/p} \quad (2.7)\\
&\leq x_0^{m/p}\Big(V_{x_0}^{-1}\int_{\frac{x_0}{2}}^{\frac{3x_0}{2}}\int_{\mathbb{R}^m} |f(y_0+\underline{y})|^p d\underline{y}dy_0\Big)^{1/p} \quad (2.8)\\
&\leq x_0^{m/p}\Big(x_0 V_{x_0}^{-1}\sup_{y_0>0}\int_{\mathbb{R}^m} |f(y_0+\underline{y})|^p d\underline{y}\Big)^{1/p} \\
&= C_{m,p}\|f\|_p,
\end{aligned}$$

so (2.4) is verified.

On the other hand, when x_0 is small,

$$
\begin{aligned}
(2.7) &\leq x_0^{m/p}\Big(V_{x_0}^{-1}\int_{\frac{x_0}{2}}^{\frac{3x_0}{2}}\int_{|\underline{y}-\underline{x}|\leq\frac{x_0}{2}}|f(y_0+\underline{y})|^p d\underline{y}dy_0\Big)^{1/p}\\
&\leq x_0^{m/p}\Big(x_0V_{x_0}^{-1}\sup_{y_0\in(\frac{x_0}{2},\frac{3x_0}{2})}\int_{|\underline{y}-\underline{x}|\leq\frac{x_0}{2}}|f(y_0+\underline{y})|^p d\underline{y}\Big)^{1/p}\\
&\leq C_{m,p}\Big(\int_{|\underline{y}-\underline{x}|\leq\frac{x_0}{2}}\sup_{y_0\in(\frac{x_0}{2},\frac{3x_0}{2})}|f(y_0+\underline{y})|^p d\underline{y}\Big)^{1/p}\\
&\leq C_{m,p}\Big(\int_{|\underline{y}-\underline{x}|\leq\frac{x_0}{2}}\sup_{y_0>0}|f(y_0+\underline{y})|^p d\underline{y}\Big)^{1/p}.
\end{aligned}
$$

Note that as a function of $\underline{y}$, $\sup_{y_0>0}|f(y_0+\underline{y})|\in L^p(\mathbb{R}^m)$ and the measure of the set $\{\underline{y}:|\underline{y}-\underline{x}|\leq\frac{x_0}{2}\}$ tends to zero as $x_0\to 0^+$, by the absolute continuity of the Lebesgue integral we have

$$\lim_{x_0\to 0^+} x_0^{m/p}|f(x)|=0.$$

When x_0 is large,

$$
\begin{aligned}
(2.8) &\leq x_0^{m/p}\Big(x_0V_{x_0}^{-1}\sup_{y_0\in(\frac{x_0}{2},\frac{3x_0}{2})}\int_{\mathbb{R}^m}|f(y_0+\underline{y})|^p d\underline{y}\Big)^{1/p}\\
&\leq C_{m,p}\Big(\int_{\mathbb{R}^m}\sup_{y_0\in(\frac{x_0}{2},\frac{3x_0}{2})}|f(y_0+\underline{y})|^p d\underline{y}\Big)^{1/p}
\end{aligned}
$$

holds uniformly with respect to $\underline{x}\in\mathbb{R}^m$, and

$$\sup_{y_0\in(\frac{x_0}{2},\frac{3x_0}{2})}|f(y_0+\underline{y})|\leq\sup_{y_0>0}|f(y_0+\underline{y})|\in L^p(\mathbb{R}^m)\quad\text{for } 1\leq p<\infty.$$

Also, from (2.4) we know that

$$\sup_{y_0\in(\frac{x_0}{2},\frac{3x_0}{2})}|f(y_0+\underline{y})|\leq x_0^{-m/p}C_{m,p}\|f\|_p,$$

which implies

$$\lim_{x_0\to+\infty}\sup_{y_0\in(\frac{x_0}{2},\frac{3x_0}{2})}|f(y_0+\underline{y})|=0$$

holds uniformly with respect to $\underline{y}\in\mathbb{R}^m$. By the Lebesgue's dominated convergence theorem we have

$$\lim_{x_0\to+\infty} x_0^{m/p}|f(x)|=0.$$

Now we proceed to prove (2.6). Since

$$
\begin{aligned}
&|f(x_0+\underline{x})| \\
&\quad = \frac{\Gamma(\frac{m+1}{2})}{\pi^{\frac{m+1}{2}}}\Big|\int_{\mathbb{R}^m} \frac{x_0}{(|\underline{x}-\underline{y}|^2+x_0^2)^{\frac{m+1}{2}}} f(\underline{y})d\underline{y}\Big| \\
&\quad \leq b\frac{\Gamma(\frac{m+1}{2})}{\pi^{\frac{m+1}{2}}}\int_{\mathbb{R}^m} \frac{|f(\underline{y})|d\underline{y}}{(|\underline{x}-\underline{y}|^2+a^2)^{\frac{m+1}{2}}} \\
&\quad = C_m\Big(\int_{|\underline{y}|>N} \frac{|f(\underline{y})|d\underline{y}}{(|\underline{x}-\underline{y}|^2+a^2)^{\frac{m+1}{2}}} + \int_{|\underline{y}|\leq N} \frac{|f(\underline{y})|d\underline{y}}{(|\underline{x}-\underline{y}|^2+a^2)^{\frac{m+1}{2}}}\Big) \\
&\quad = C_m(I_1+I_2),
\end{aligned}
$$

by Hölder's inequality,

$$
\begin{aligned}
I_1 &\leq \Big(\int_{|\underline{y}|>N} (|\underline{x}-\underline{y}|^2+a^2)^{\frac{-(m+1)p'}{2}} d\underline{y}\Big)^{1/p'} \Big(\int_{|\underline{y}|>N} |f(\underline{y})|^p d\underline{y}\Big)^{1/p} \\
&\leq \Big(\int_{\mathbb{R}^m} (|\underline{y}|^2+a^2)^{\frac{-(m+1)p'}{2}} d\underline{y}\Big)^{1/p'} \Big(\int_{|\underline{y}|>N} |f(\underline{y})|^p d\underline{y}\Big)^{1/p} \\
&\leq C_{m,p}\Big(\int_{|\underline{y}|>N} |f(\underline{y})|^p d\underline{y}\Big)^{1/p},
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{p'}=1$. Because $f(\underline{y})\in L^p(\mathbb{R}^m)$, I_1 is small provided N is large enough. With N fixed,

$$
I_2 \leq \frac{C_m}{|\underline{x}|^{m+1}}\int_{|\underline{y}|\leq N} |f(\underline{y})|d\underline{y} \to 0 \quad (|\underline{x}|\to+\infty),
$$

due to $f(\underline{y})$ is integrable on $\{\underline{y} : |\underline{y}|\leq N\}$, that proves (2.6).

The proof of Theorem 1.2 is now complete. □

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