Transient Time-Frequency Distribution Based on Mono-component Decompositions

PEI DANG

Department of General Education, Macau University of Science and Technology, Macao (Via Hong Kong).

E-mail: peidang@gmail.com

TAO TIAN

Department of Mathematics, University of Macau, Macao (Via Hong Kong).

Email: fsttq@umac.mo

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In this paper we propose a new type of non-negative time-frequency distribution associated with mono-components in both the non-periodic and periodic cases, called *transient time-frequency distribution* (TTFD), and study its properties. The TTFD of a mono-component signal can be obtained directly through its analytic instantaneous frequency. The characteristic property of TTFD is its complete concentration along the analytic instantaneous frequency graph. For multi-components there are induced time-frequency distributions called *composing transient time-frequency distribution* (CTTFD). Each CTTFD is defined as the superposition of the TTFDs of the composing intrinsic mono-components in a suitable mono-components decomposition of the targeted multi-component. In studying the properties of TTFD and CTTFD the relations between the Fourier frequency and analytic instantaneous frequency are examined.

Keywords: time-frequency distribution of signal, Hilbert transform, analytic signal, Hardy space, Hardy-Sobolev space, instantaneous frequency, analytic phase derivaive

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1. Introduction

Distributions of signals in the time-frequency plane have been a central topic in signal analysis. This is especially the case when one is dealing with non-stationary or time-varying signals. One usually desires that a distribution has some good properties such as the marginal properties, the total energy property, positivity and so on.[12,13] The most common distribution models include the *Short-time Fourier transform* (STFT) and the *Wigner distribution* (WD). They both belong to the *Cohen class*.[5] The STFT is obtained by the Fourier transform of the signal $s(t')$ multiplied by a window function $\overline{g(t'-t)}$ centered around t, where $\overline{g(t'-t)}$ denotes the complex conjugation of $g(t'-t)$. The WD, among all the quadratic *time-*

frequency distributions (TFDs) with energetic interpretation, satisfies a quite large number of mathematical properties.[6,13,30] For example, the WD of a signal is always real-valued, and satisfies the marginal properties and total energy invariant. The instantaneous frequency and group delay can be obtained by the local average of, respectively, frequency and time of the WD. However, the WD is not always non-negative. There are many modified WDs attempting to achieve a positive distribution.[4,15,16] For a multi-component signal, the WD may produce the cross terms, which are always annoying in analyzing signals. Concentration of time-frequency distribution is one of the important properties to which intensive studies have been devoted.[6,7,26,27,28,3] The TTFD of each mono-component is of high concentration.

A TTFD of a signal is based on an adaptive mono-component decomposition: We first decompose a given signal into a sum of mono-components. Those composing mono-components have positive analytic instantaneous frequency, or instantaneous frequency, in brief. Meanwhile, the instantaneous frequency functions of the mono-components have no intersection. Then we define the TTFD for each of mono-components and take summation (see (3.5)). A TTFD satisfies many mathematical properties that are expected for a time-frequency distribution. Those include the correct total energy property, non-negative property, real-valuedness property, weak and strong finite support. We will discuss them in section 3. The TTFDs do not produce cross terms. They have good resolutions in both the time and frequency domains.

The organization of this paper is as follows. Section 2 first reviews the definition of analytic phase derivative, analytic instantaneous frequency and mono-component in both the non-periodic and periodic cases. We demonstrate by numerical example that signals with high positive Fourier frequency do not necessarily have positive analytic phase derivative (instantaneous frequency). In section 3 we define TTFDs in both the non-periodic and periodic cases and examine their properties. Section 4 gives examples of TTFDs. In Appendix we introduce the adaptive Fourier decomposition (AFD) of signals that TTFD depends on.

2. Analytic Phase Derivative and Instantaneous Frequency

Since the theories for the unit circle and the real line are similar, we will feel free to move from one context to the other.

For a given signal $s(t) = \rho(t)e^{i\varphi(t)},\ t \in \mathbb{R}$, and $s(t),\ \rho(t)$ and $\varphi(t)$ are differentiable, where $\rho(t) \geq 0$, differentiation of composed function gives

$$\rho'(t) = \rho(t)\mathrm{Re}\frac{s'(t)}{s(t)}, \qquad \varphi'(t) = \mathrm{Im}\frac{s'(t)}{s(t)}. \tag{2.1}$$

The differentiability, however, is not satisfied by general square integral signals. To make a general signal analyzable we work with the Hardy spaces in the upper and lower half planes, and perform the Hardy space decomposition $s = s^+ + s^-$, where $s^\pm = (1/2)(s \pm iHs)$, H is the Hilbert transform on the line. In fact, $s^\pm(t)$ are the non-tangential boundary limit functions of the Hardy-space-functions $s^{\pm}(z)$ given by

$$s^{\pm}(z) = \frac{\pm 1}{2\pi i}\int_{-\infty}^{\infty}\frac{s(u)}{u-z}du = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{it\omega}\chi_{\pm}(\omega)e^{-y\omega}\hat{s}(\omega)d\omega, \tag{2.2}$$

where $z = t + iy \in \mathbb{C}^{\pm}$. Note that the non-tangential boundary limits of $s^{+}(z)$, denoted by $s^{+}(t)$, is the $1/2$ multiple of the so called *analytic signal associated with* s, the latter being denoted by $As = s + iHs$. $s^{+}(t)$ is precisely defined except possibly a Lebesgue null sets E^{+}; and $s^{-}(t)$ is precisely define except possibly a Lebesgue null set E^{-}. Both $s^{+}(t)$ and $s^{-}(t)$ are a.e. non-zero. There has been an ample amount of studies on analytic signals. We refer the reader to Ref. 5, Ref. 31, Ref. 17 and the recent publications of Qian et al on mono-component.

Denote by $L_1^2(\mathbb{R})$ the Sobolev space on the line [29] defined by

$$L_1^2(\mathbb{R}) = \{s(t) \in L^2(\mathbb{R}) : s'(t) \in L^2(\mathbb{R})\}$$

with the norm

$$\sqrt{\|s\|_2^2 + \|s'\|_2^2},$$

where $s'(t)$ is the square-integrable distributional derivative of s being assumed to exist.

We can define a new type of derivative for $s \in L_1^2(\mathbb{R})$ (see Ref. 8):

Definition 2.1. *If* $s(t) = \rho(t)e^{i\varphi(t)} \in L_1^2(\mathbb{R})$, *then the Hardy-Sobolev derivatives of* $s(t)$, $\rho(t)$ *and* $\varphi(t)$ *are defined to be*

$$s^{*}(t) = {s^{+}}'(t) + {s^{-}}'(t), \tag{2.3}$$

$$\rho^{*}(t) = \begin{cases} \rho(t)\mathrm{Re}\left[\frac{{s^{+}}'(t)+{s^{-}}'(t)}{s^{+}(t)+s^{-}(t)}\right] & \textit{if } s^{+} \text{ and } s^{-} \text{ are defined, and } s^{+}(t)+s^{-}(t) \neq 0, \\ 0 & \textit{if } s^{+} \text{ or } s^{-} \text{ is not defined, or } s^{+}(t)+s^{-}(t) = 0, \end{cases} \tag{2.4}$$

and

$$\varphi^{*}(t) = \begin{cases} \mathrm{Im}\left[\frac{{s^{+}}'(t)+{s^{-}}'(t)}{s^{+}(t)+s^{-}(t)}\right] & \textit{if } s^{+} \text{ and } s^{-} \text{ are defined, and } s^{+}(t)+s^{-}(t) \neq 0, \\ 0 & \textit{if } s^{+} \text{ or } s^{-} \text{ is not defined, or } s^{+}(t)+s^{-}(t) = 0, \end{cases} \tag{2.5}$$

where

$${s^{\pm}}'(t) = \lim_{z\to t} {s^{\pm}}'(z), \quad z \in \mathbb{C}^{\pm}.$$

We note that the notation $s(t)$ is different from the notation $s^{+}(t) + s^{-}(t)$. The former, in fact, represents an equivalent class of square-integrable functions.The notation $s^{+}(t)+s^{-}(t)$, however, represents a precise function that is uniquely defined in the complementary set of a Lebesgue null set on which either s^{+} or s^{-} is not defined. $s^{+} + s^{-}$ is a precise representative of the equivalent class s. The same is for the notations s' and s^*. The certainty of $s^+(t)+s^-(t)$ and ${s^+}'+{s^-}'$ are needed in defining instantaneous frequency and the related time varying quantities. We call the Hardy-Sobolev derivative of $\varphi(t)$ the *H-S frequency* of signal $s(t)$. By the Definition 2.1, the H-S frequency of $s^+(t)=\rho^+(t)e^{i\varphi^+(t)}$ is

$$\varphi^{+*}(t)=\begin{cases}\mathrm{Im}\left[\frac{{s^+}'(t)}{s^+(t)}\right] & \text{if } \ s^+(t)\neq 0,\\ 0 & \text{if } \ s^+(t)=0 \quad \text{or not defined,}\end{cases} \tag{2.6}$$

and we call the H-S frequency of $s^+(t)$ as the *analytic phase derivative* of $s^+(t)$. In the other word, if a signal is the non-tangential boundary value function of an analytic function in $H^2(\mathbb{C}^+)$, then the terminology H-S frequency is identical with analytic phase derivative. If and only if the analytic phase derivative of an analytic signal is a non-negative measurable function, then we say that the signal has *analytic instantaneous frequency*, or, in brief, *instantaneous frequency*, given by the analytic phase derivative. If a complex-valued signal is an analytic signal possessing instantaneous frequency, then it is called a *mono-component signal.* If a complex valued signal is not a mono-component, then it is called a *multi-component.* We also note that if a complex-valued signal $s=\rho e^{i\varphi}$ has smooth ρ and φ, then $s'=s^*, \rho'=\rho^*$ and $\varphi'=\varphi^*$.[8] For $s=s^+$ we have been using the notation $\varphi'(t)$ for φ^* as they coincide if the classical derivative exists.[9]

The Fourier frequency of $s(t)$ is the variable ω in its Fourier representation

$$s(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{it\omega}\hat{s}(\omega)d\omega, \tag{2.7}$$

where the Fourier transform $\hat{s}$ of s is given by

$$\hat{s}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-it\omega}s(t)dt. \tag{2.8}$$

The quantities $|\hat{s}(\omega)|^2$ and $|s(t)|^2$ are, respectively, the densities at the Fourier frequency ω and at the time t. The mean of a measurable function $g(t)$ with respect to $s(t)$ is defined by

$$\langle g(t)\rangle_s=\int_{-\infty}^{\infty}g(t)|s(t)|^2dt,$$

and the mean of a measurable function $h(\omega)$ with respect to $s(t)$ is given by

$$\langle h(\omega)\rangle_s=\int_{-\infty}^{\infty}h(\omega)|\hat{s}(\omega)|^2d\omega,$$

provided that those integrals are defined.

In particular,

$$\langle \omega^n\rangle_s=\int_{-\infty}^{\infty}\omega^n|\hat{s}(\omega)|^2d\omega, \quad n=1,2,... \tag{2.9}$$

In signal analysis, the relationship between the Fourier frequency and phase derivative has been devoted much attention (see Ref. 5, Ref. 10.) For $s(t)\ =$

$\rho(t)e^{i\varphi(t)}$, the following equations depict the relation between two kinds of frequency (see Ref. 8). For $s \in L_1^2(\mathbb{R})$,

$$\langle\omega\rangle_s = \int_{-\infty}^{\infty} \varphi^*(t)\rho^2(t)dt, \tag{2.10}$$

$$\langle\omega^2\rangle_s = \int_{-\infty}^{\infty} \rho^{*2}(t)dt + \int_{-\infty}^{\infty} \varphi^{*2}(t)\rho^2(t)dt, \tag{2.11}$$

The relations (2.10) and (2.11) are generalizations of the classical results on smooth functions (see Ref. 5):

$$\langle\omega\rangle_s = \int_{-\infty}^{\infty} \varphi'(t)\rho^2(t)dt, \tag{2.12}$$

$$\langle\omega^2\rangle_s = \int_{-\infty}^{\infty} \rho'^2(t)dt + \int_{-\infty}^{\infty} \varphi'^2(t)\rho^2(t)dt. \tag{2.13}$$

For a periodic signal $f(e^{it}) = \frac{1}{\sqrt{2\pi}}\sum_{l=-\infty}^{+\infty} c_l e^{ilt}$, where c_l's are given by

$$c_l = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} f(e^{it})e^{-ilt}dt, \quad l = 0, \pm 1, \pm 2, \cdots,$$

we define the corresponding *H-S frequency*, *analytic phase derivative*, instantaneous frequency and *mono-component* in a similar way (see Ref. 8). We use the same notations as for the non-periodic case. However, the *Fourier frequency* of $f(e^{it})$ is regarded as integers l's in the Fourier series expansion of $f(e^{it})$.

The mean of Fourier frequency and higher moments of Fourier frequency with respect to $f(e^{it})$ are given by

$$\langle l\rangle_f \triangleq \sum_{l=-\infty}^{\infty} l|c_l|^2 \quad \text{and} \quad \langle l^n\rangle_f \triangleq \sum_{l=-\infty}^{\infty} l^n|c_l|^2, \qquad n \in \mathbf{Z}^+. \tag{2.14}$$

The mean of time and duration with respect to $f(e^{it})$ are defined as (see Ref. 14)

$$\langle t\rangle_f \triangleq \arg\int_0^{2\pi} e^{it}|f(e^{it})|^2dt \quad \text{and} \quad \sigma_{t,f}^2 \triangleq 1 - \int_0^{2\pi} \cos(t - \langle t\rangle_f)|f(e^{it})|^2dt.$$

There are counterpart results on the relation between the Fourier frequency and H-S frequency (see Ref. 8), as follows.

Theorem 2.1. *Assume* $f(e^{it}) = \rho(t)e^{i\varphi(t)} \in L_1^2([0, 2\pi))$. *There hold*

$$\langle l\rangle_f = \int_0^{2\pi} \varphi^*(t)\rho^2(t)dt,$$

$$\langle l^2\rangle_f = \int_0^{2\pi} \rho^{*2}(t)dt + \int_0^{2\pi} \varphi^{*2}(t)\rho^2(t)dt.$$

If, in particular, s or f in (2.9) or (2.14) is an analytic signal, then the corresponding relations become

$$\int_0^{\infty} \omega|\hat{s}(\omega)|^2d\omega = \int_{-\infty}^{\infty} \varphi'(t)|s(t)|^2dt$$

and

$$\sum_{l=0}^{\infty} l|c_l|^2 = \int_0^{2\pi} \varphi'(t)\rho^2(t)dt.$$

The positivity of the left-hand-sides of the above relations might tacitly hint that the analytic phase derivative function is non-negative. But that is not the case. Below is the L^2-version of a distributional counter example given by Cohen.[5]

Example 2.1. Let $s(t) = -\frac{6}{5\sqrt{2\pi}} \int_M^{M+\frac{1}{M}} e^{it\omega}d\omega + \frac{1}{\sqrt{2\pi}} \int_{2M}^{2M+\frac{1}{M}} e^{it\omega}d\omega \in L^2(\mathbb{R}),\ M$ be an arbitrary positive number. Obviously s is an analytic signal and thus with non-zero Fourier spectra only at positive frequencies. Then

$$s(t) = \frac{-6e^{it(M+\frac{1}{M})} + 6e^{itM} + 5e^{it(2M+\frac{1}{M})} - 5e^{i2tM}}{i5\sqrt{2\pi}t} = \rho(t)e^{i\varphi(t)}$$

$$s'(t) = \frac{-6it(M+\frac{1}{M})e^{it(M+\frac{1}{M})} + 6itMe^{itM} + 5it(2M+\frac{1}{M})e^{it(2M+\frac{1}{M})} - 10itMe^{i2tM}}{i5\sqrt{2\pi}t^2} - \frac{-6e^{it(M+\frac{1}{M})} + 6e^{itM} + 5e^{it(2M+\frac{1}{M})} - 5e^{i2tM}}{i5\sqrt{2\pi}t^2}$$

$$\varphi'(t) = \mathrm{Im}\left(\frac{s'(t)}{s(t)}\right) = \frac{172M + \frac{61}{M} - (180M + \frac{60}{M})\cos tM}{122 - 120\cos tM},$$

Let $M > \frac{\sqrt{2}}{4}$. Then at $t = \frac{2k\pi}{M}, k \in \mathbf{Z}$, we have $\varphi'(t) < 0$; and at $t = \frac{2k\pi+\pi}{M}, k \in \mathbf{Z}$, we have $\varphi'(t) > 0$. Since the phase derivative is a continuous function it is positive and negative in adjacent intervals.

Figure 2.1 shows the graphs of the analytic frequencies, $\xi = \varphi'(t)$, for $M = 10, 100, 1000, 10000$.

Signals in practice are real-valued. If $s(t) \in L^2_1(\mathbb{R})$ is real-valued, by the Hardy decomposition, $s(t) = s^+(t) + s^-(t)$, and $s^-(t) = \overline{s^+(t)}$, then $s(t) = 2\mathrm{Re}s^+(t)$. Hence We can concentrate in $s^+(t)$ instead of $s(t)$. $s^+(t)$ has non-zero spectra only for the positive frequencies. As shown in the above example, s^+ does not always possess instantaneous frequency. What we could do is to decompose s^+ into a sum of mono-components.[18] Fourier series is the classical example of such decomposition. Expansions in Takenaka-Malmquist (TM) systems as generalizations of Fourier series belong to the same category.[1,2,11]

Now consider a typical TM system consisting of modified Blaschke products $\{B_n\}_{n=1}^{\infty}$,

$$B_n(z) = B_{\{a_1,...,a_n\}} = \frac{\sqrt{1-|a_n|^2}}{1-\overline{a}_n z} \prod_{k=1}^{n-1} \frac{z-a_k}{1-\overline{a}_k z}, \quad a_1, ..., a_n \in \mathbb{D},$$

with the condition

$$\sum_{n=1}^{\infty}(1-|a_n|) = \infty.$$

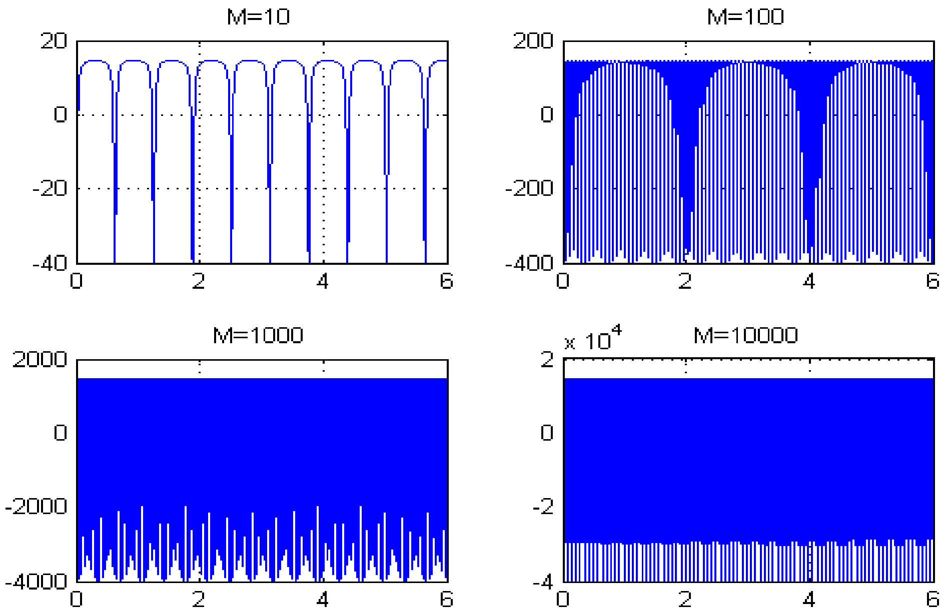

Fig. 2.1.

Note that the TM system is an orthogonal system. It may be easily shown that by writing

$$B_n(e^{it}) = \rho_n(t)e^{i\varphi_n(t)},$$

there follows

$$\varphi_n'(t) < \varphi_{n+1}'(t), \quad t \in [0, 2\pi]. \tag{2.15}$$

$B_n(e^{it})$ may or may not be a mono-component, but $B_n(e^{it})$ is always a *pre-mono-component* in the sense that there exists a positive number $M > 0$ such that $e^{iMt}B_n(e^{it})$ is a mono-component. If $a_1 = 0$, then B_1 is a mono-component, and, consequently, all B_n are mono-components. In Appendix we give an account of adaptive mono-component decomposition by using the TM system, called *adaptive Fourier decomposition* (AFD).

3. Transient Time-Frequency Distribution Associated with Adaptive Mono-Component Decompositions

In this section we give the definition of $TTFD$ and its properties in two cases: non-periodic and periodic signals. Except the domains of TTFD and the global average property, they have the same definition and properties.

Through an adaptive mono-component decomposition a signal $s(t)$ may be decomposed into a sum with a regular manner, such as increase of their instantaneous frequencies as given in (2.15), and of fast convergence, namely

$$s(t) = \sum_{k=1}^{\infty} s_k(t) \tag{3.1}$$

or

$$s(t) = \sum_{k=1}^{n} s_k(t) + r_n(t), \tag{3.2}$$

where $s_k(t)$'s are mono-components, and r_n's are the remainder in H^2. It is desirable, as what we have in the AFD case, the first few mono-components added up is quite close to the given signal in energy. Under AFD the decomposition (3.1) is orthogonal.

The *transient time-frequency distribution* (TTFD) of a mono-component signal $s(t) = \rho(t)e^{i\varphi(t)}$ is defined as

$$P(t,\xi) = \rho^2(t)\delta_M(\xi - \varphi'(t)), \quad (t,\xi) \in \mathbb{R} \times [-\frac{1}{2M}, +\infty) \tag{3.3}$$

where

$$\delta_M(\xi - \varphi'(t)) = \begin{cases} M & \text{if } \ \xi \in [\varphi'(t) - \frac{1}{2M}, \varphi'(t) + \frac{1}{2M}], \\ 0 & \text{if } \ \xi \notin [\varphi'(t) - \frac{1}{2M}, \varphi'(t) + \frac{1}{2M}], \end{cases} \tag{3.4}$$

where M is a large enough positive number. The following results will all hold for $M = \infty$, when the L^2-function δ_M becomes the distributional Dirac (generalized) function. The reason for making M to be a finite number is for the mathematical convenience and practical applications. An induced time-frequency distribution of a multi-component to be given below is dependent of its mono-component decomposition, and thus is not unique. Assume that a multi-component square-integrable analytic signal s has a fast and orthogonal mono-component decomposition given by (3.1), then the corresponding *composing-transient-time-frequency distribution* (CTTFD) is defined to be

$$P(t,\xi) = \sum_{k=1}^{\infty} P_k(t,\xi) = \sum_{k=1}^{\infty} \rho_k^2(t)\delta_M(\xi - \varphi_k'(t)), \quad (t,\xi) \in \mathbb{R} \times [-\frac{1}{2M}, \infty), \tag{3.5}$$

where $P_k(t,\xi)$ is the TTFD of the mono-component s_k.

Note that in the definition of TTFD the frequency stands for the instantaneous frequency. We can induce a CTTFD for a general signal s in L^2. To this end we first project s into the corresponding Hardy space, that is to get s^+, and then adaptively decompose s^+ into a sum of mutually orthogonal mono-components with fast convergence. Then we form the corresponding CTTFD of s^+. A candidate of CTTFD of s is one of the CTTFDs of s^+.

3.1. *The correct total energy property*

Let $s \in H^2$. For an orthogonal mono-component decomposition $s(t) = \sum_{k=1}^{\infty} s_k(t)$, we have

$$\int\int P(t,\xi)dtd\xi = \int |s(t)|^2 dt.$$

In fact,

$$\int\int P(t,\xi)dtd\xi = \int\int \sum_{k=1}^{\infty} \rho_k^2(t)\delta_M(\xi - \varphi_k'(t))dtd\xi \tag{3.6}$$

$$= \sum_{k=1}^{\infty} \int \rho_k^2(t) \int_{\varphi_k'(t)-\frac{1}{2M}}^{\varphi_k'(t)+\frac{1}{2M}} M d\xi dt \tag{3.7}$$

$$= \sum_{k=1}^{\infty} \int \rho_k^2(t)dt = \int |s(t)|^2 dt. \tag{3.8}$$

The last equality is due to the orthogonality of the composing mono-components.

A signal $s(t)$ in H^2 can have many different orthogonal mono-component decompositions and accordingly have many different CTTFDs. This property shows that the energies of different CTTFDs are the same, being equal to the energy of the signal.

3.2. *The non-negative property*

For a mono-component signal $s(t)$, its corresponding TTFD satisfies

$$P(t,\xi) \geq 0, \quad t, \ \xi \in \mathbb{R}.$$

A CTTFD also satisfies this property. Since TTFD or CTTFD represent the energy distribution of a signal in the time-frequency plane, it is natural to require the non-negativity property. However, for the time-frequency distribution in the Cohen class, the non-negativity property can not be compatible with some most desirable properties.[13]

3.3. *The real-valuedness property*

For a signal $s(t)$, we see that its corresponding TTFD or CTTFD is real-valued.

3.4. *Weak and strong finite support*

For a mono-component signal $s(t)$, if it does not start until t_1, we know $s(t) = 0,\ t < t_1$, thus $\rho(t) = 0$ when $t < t_1$. Hence the TTFD of $s(t)$ equals to zero for $t < t_1$. Similarly, if the signal stops after time t_2, we also obtain the TTFD equals to zero for $t > t_2$. This is to say TTFD satisfies the weak finite support property, formulated as

$$P(t,\xi) = 0 \quad \text{for } t \text{ outside } (t_1,t_2) \quad \text{if } s(t) \text{ is zero outside } (t_1,t_2).$$

Suppose a mono-component signal $s(t)$ just stops for a short time, or even a single point, and then starts again. Then $P(t,\xi)$ is zero for that time duration or at the particular point, too. So TTFD satisfies the strong finite support property. But, obviously, neither of the weak finite support, nor the strong finite support property is enjoyed by CTTFD, in general.

3.5. *Marginal properties*

For a mono-component signal $s(t)=\rho(t)e^{i\varphi(t)}$, its TTFD is

$$P(t,\xi)=\rho^2(t)\delta_M(\xi-\varphi'(t)).$$

We have the time marginal distribution

$$P_1(t)=\int P(t,\xi)d\xi=\int \rho^2(t)\delta_M(\xi-\varphi'(t))d\xi=\rho^2(t)=|s(t)|^2. \tag{3.9}$$

Since the analytic instantaneous frequency does not have a direct relation with the Fourier frequency, Fourier frequency marginal distribution condition cannot be expected to hold. TTFD, however, has its own frequency marginal distribution condition.

Denote by

$$\tau^2(\xi)=P_2(\xi)=\int P(t,\xi)dt=\int \rho^2(t)\delta_M(\xi-\varphi'(t))dt=M\int_{I_\xi}\rho^2(t)dt, \tag{3.10}$$

where $I_\xi=\{t:|\xi-\varphi'(t)|\le \frac{1}{2M}\}$, we have

$$\int \tau^2(\xi)d\xi=\int |s(t)|^2dt=\|s\|^2 \tag{3.11}$$

and

$$\int \xi\tau^2(\xi)d\xi=\int_{-\infty}^{\infty}\varphi'(t)|s(t)|^2dt. \tag{3.12}$$

In comparison with the relation (2.12) we see that $\tau^2(\xi)$ as density for ξ plays the same role as $|\hat{s}(\omega)|^2$ for the Fourier frequency. The validity of (3.11) is a consequence of the correct total energy property (3.6). To see (3.12) we have

$$\begin{aligned}\int \xi\tau^2(\xi)d\xi&=\int\rho^2(t)dt\int\xi\delta_M(\xi-\varphi'(t))d\xi\\&=\int\rho^2(t)dtM\int_{\varphi'(t)-1/(2M)}^{\varphi'(t)+1/(2M)}\xi d\xi\\&=\int\varphi'(t)\rho^2(t)dt.\end{aligned}$$

The relations (3.9) and (3.11), as well as (3.12) as reference, show that TTFD for mono-components satisfies the marginal properties.

Let $s \in H^2$ be a multi-component with a mono-component decomposition (3.1). Denote by $\tilde{s} = (s_1, ..., s_n, ...), \tilde{\varphi}' = (\varphi_1', ..., \varphi_n', ...)$, and $\tilde{\tau} = (\tau_1, ..., \tau_n, ...)$. Then we can show

$$\int |\tilde{s}(t)|^2 dt = \int |\tilde{\tau}(\xi)|^2 d\xi \tag{3.13}$$

and

$$\int \langle \widetilde{\varphi'}(t), \tilde{s}(t) \bigotimes \overline{\tilde{s}}(t) \rangle dt = \int \xi |\tilde{\tau}(\xi)|^2 d\xi, \tag{3.14}$$

where in (3.13) $|\tilde{s}|$ and $|\tilde{\tau}|$ are understood as the l^2 norms of respectively the sequences $\tilde{s}$ and $\tilde{\tau}$, and, in (3.14), the notation $\bigotimes$ is for direct product between two sequences. The proofs of (3.13) and (3.14) are through the reduction to the mono-components as given in (3.11) and (3.12).

3.6. *Global Averages*

The global average value of any function of time and frequency with respect to the time-frequency distribution is to be calculated and denoted by

$$\langle g(t, \xi) \rangle_P = \int \int g(t, \xi) P(t, \xi) dt d\xi.$$

In the following we consider *the means of time and frequency*, *the second moments of time and frequency*, *the global standard deviation*, and *the covariance* for TTFDs of mono-components and CTTFDs of multi-components.

Means of Time and Frequency

The means of time and frequency with respect to TTFD for mono-component signal $s(t) = \rho(t)e^{i\varphi(t)}$ are given, respectively, by

$$\langle t \rangle_P = \int \int tP(t, \xi) dt d\xi, \quad \langle \xi \rangle_P = \int \int \xi P(t, \xi) dt d\xi.$$

Computation gives

$$\langle t \rangle_P = \int \int tP(t, \xi) dt d\xi = \int \int t\rho^2(t) \delta_M(\xi - \varphi'(t)) dt d\xi = \int t\rho^2(t) dt = \langle t \rangle_s,$$

$$\begin{aligned} \langle \xi \rangle_P &= \int \int \xi P(t, \xi) dt d\xi = \int \int \xi \rho^2(t) \delta_M(\xi - \varphi'(t)) dt d\xi \\ &= \int \rho^2(t) \int_{\varphi'(t) - \frac{1}{2M}}^{\varphi'(t) + \frac{1}{2M}} \xi \delta_M(\xi - \varphi'(t)) d\xi dt \\ &= \int \varphi'(t) \rho^2(t) dt \\ &= \langle \varphi'(t) \rangle_s \\ &= \langle \omega \rangle_s. \end{aligned}$$

On the other hand, first integrating with respect to the t variable, we have

$$\begin{aligned}\langle \xi \rangle_P &= \int \xi \int_{I_\xi} \rho^2(t)dt \\ &= \int \xi |\tau(\xi)|^2 d\xi \\ &= \langle \xi \rangle_\tau .\end{aligned}$$

We can see that for mono-component signals the means of time and frequency with respect to the TTFD are exactly those with respect to the signal itself, that is natural and desired for any reasonable time-frequency distribution.

For multi-components we work with CTTFD. By using the notation in the subsection 3.5 for marginal properties, we have

$$\langle t \rangle_P = \int t|\tilde{s}(t)|^2 dt \triangleq \langle t \rangle_{\tilde{s}} \tag{3.15}$$

and

$$\langle \xi \rangle_P = \int \langle \widetilde{\varphi'}(t), \tilde{s}(t) \bigotimes \overline{\tilde{s}}(t) \rangle dt = \int \xi |\tilde{\tau}(\xi)|^2 d\xi . \tag{3.16}$$

But, in general, except for the mono-component case,

$$\langle t \rangle_P \neq \langle t \rangle_s$$

and

$$\langle \xi \rangle_P \neq \langle \varphi'(t) \rangle_s .$$

Second Moments of Time and Frequency

For mono-components the means of square time and frequency are formulated as:

$$\langle t^2 \rangle_P = \int\int t^2 P(t,\xi) dt d\xi = \int\int t^2 \rho^2(t) \delta_M(\xi - \varphi'(t)) dt d\xi = \int t^2 \rho^2(t) dt = \langle t^2 \rangle_s ,$$

$$\begin{aligned}\langle \xi^2 \rangle_P &= \int\int \xi^2 P(t,\xi) dt d\xi = \int\int \xi^2 \rho^2(t) \delta_M(\xi - \varphi'(t)) dt d\xi \\ &= \int \rho^2(t) \int_{\varphi'(t)-\frac{1}{2M}}^{\varphi'(t)+\frac{1}{2M}} \xi^2 \delta_M(\xi - \varphi'(t)) d\xi dt \\ &= \int {\varphi'}^2(t) \rho^2(t) dt + \frac{1}{12M^2} \int \rho^2(t) dt \\ &= \langle {\varphi'}^2(t) \rangle_s + \frac{1}{12M^2} \int |s(t)|^2 dt .\end{aligned}$$

Thus, the mean of square time with respect to the TTFD of a mono-component is equal to that with respect to signal itself. The mean of square frequency with respect to the TTFD is quite close to the mean of square instantaneous frequency with respect to the signal itself if the positive M is large enough.

For multi-components we accordingly have

$$\langle t^2\rangle_P = \int t^2|\tilde{s}(t)|^2 dt = \langle t^2\rangle_{\tilde{s}} \tag{3.17}$$

and

$$\langle \xi^2\rangle_P = \int \xi^2|\tilde{\tau}(\xi)|^2 d\xi \tag{3.18}$$

$$= \int \langle \widetilde{\varphi'^2}(t), \tilde{s}(t) \bigotimes \overline{\tilde{s}}(t)\rangle dt + \frac{1}{12M^2}\int |\tilde{s}(t)|^2 dt, \tag{3.19}$$

where $\widetilde{\varphi'^2} = ({\varphi'_1}^2, ..., {\varphi'_n}^2, ...)$. They usually are not identical with $\langle t^2\rangle_s$ and $\langle {\varphi'}^2(t)\rangle_s$.

Global Standard Deviation

We first deal with mono-components. The following two formulas exhibit the relationships between the duration and bandwidth with respect to the TTFD and those with respect to the mono-component signal.

$$\begin{aligned}\sigma^2_{t,P} \triangleq \langle (t-\langle t\rangle_P)^2\rangle_P &= \int\int (t-\langle t\rangle_P)^2 P(t,\xi)dtd\xi \\ &= \int\int t^2 P(t,\xi)dtd\xi - 2\langle t\rangle_P \int\int tP(t,\xi)dtd\xi + \langle t\rangle_P^2 \int\int P(t,\xi)dtd\xi \\ &= \langle t^2\rangle_s - 2\langle t\rangle_s\langle t\rangle_s + \langle t\rangle_s^2\int |s|^2 dt = \langle (t-\langle t\rangle_s)^2\rangle_s \triangleq \sigma^2_{t,s}.\end{aligned}$$

$$\begin{aligned}\sigma^2_{\xi,P} \triangleq \langle (\xi-\langle \xi\rangle_P)^2\rangle_P &= \int\int (\xi-\langle \xi\rangle_P)^2 P(t,\xi)dtd\xi \\ &= \int\int \xi^2 P(t,\xi)dtd\xi - 2\langle \xi\rangle_P \int\int \xi P(t,\xi)dtd\xi + \langle \xi\rangle_P^2 \int\int P(t,\xi)dtd\xi \\ &= \langle {\varphi'}^2(t)\rangle_s - 2\langle \xi\rangle_P\langle \varphi'(t)\rangle_s + \langle \xi\rangle_P^2\int |s|^2 dt + \frac{1}{12M^2}\int |s(t)|^2 dt \\ &= \langle \omega^2\rangle_s - \int {\rho'}^2(t)dt - 2\langle \omega\rangle_s\langle \omega\rangle_s + \langle \omega\rangle_s^2 \int |s|^2 dt + \frac{1}{12M^2}\int |s(t)|^2 dt \\ &= \langle (\omega - \langle \omega\rangle_s)^2\rangle_s - \int {\rho'}^2(t)dt + \frac{1}{12M^2}\int |s(t)|^2 dt \\ &= \int (\varphi'(t) - \langle \varphi'(t)\rangle_s)^2|s(t)|^2 dt + \frac{1}{12M^2}\int |s(t)|^2 dt,\end{aligned}$$

The last equality is due to the relation $\sigma^2_{\omega,s} = \int(\varphi'(t) - \langle\omega\rangle)^2|s(t)|^2 dt + \int {\rho'}^2(t)dt$ (see page 16, Ref. 5). The last quantity in the equality chain tends to

$$\langle (\varphi' - \langle \varphi'\rangle_s)^2\rangle_s$$

as $M \to \infty$.

For multi-components, by adopting the notation $\langle\ \rangle_{\tilde{s}}$ as the average with respect to the density $|\tilde{s}|^2$ as introduced in (3.15) and used in (3.17), as well as $\langle\ \rangle_{\tilde{\tau}}$ as the average with respect to the density $|\tilde{\tau}|^2$, we have

$$\sigma_{t,P}^2 \triangleq \langle (t - \langle t \rangle_P)^2 \rangle_P = \langle (t - \langle t \rangle_{\tilde{s}})^2 \rangle_{\tilde{s}} \triangleq \sigma_{t,\tilde{s}}^2$$

and

$$\begin{aligned}\sigma_{\xi,P}^2 &\triangleq \langle (\xi - \langle \xi \rangle_P)^2 \rangle_P = \langle (\xi - \langle \xi \rangle_{\tilde{\tau}})^2 \rangle_{\tilde{\tau}} \\ &= \int \langle (\widetilde{\varphi' - \langle \xi \rangle_P})^2(t), \tilde{s}(t) \bigotimes \bar{\tilde{s}}(t) \rangle dt + \frac{1}{12M^2} \int |s(t)|^2 dy,\end{aligned}$$

where $(\widetilde{\varphi' - \langle \xi \rangle_P})^2 = ((\varphi_1' - \langle \xi \rangle_P)^2, ..., (\varphi_n' - \langle \xi \rangle_P)^2, ...)$.

Covariance

The covariances with respect to the TTFD of a mono-component $s(t) = \rho(t)e^{i\varphi(t)}$ and to itself are respectively given by

$$\mathrm{Cov}_P = \langle t\xi \rangle_P - \langle t \rangle_P \langle \xi \rangle_P, \quad \text{and} \quad \mathrm{Cov}_s = \langle t\varphi'(t) \rangle_s - \langle t \rangle_s \langle \varphi' \rangle_s.$$

For mono-components $\langle t \rangle_P = \langle t \rangle_s, \langle \xi \rangle_P = \langle \varphi' \rangle_s$. The average of $t\xi$ with respect to P is computed as

$$\langle t\xi \rangle_P = \int\int t\xi P(t,\xi) dt d\xi = \int\int t\xi \rho^2(t)\delta(\xi - \varphi'(t)) dt d\xi = \int t\varphi'(t)\rho^2(t) dt = \langle t\varphi'(t) \rangle_s.$$

Therefore, the covariance of TTFD is identical with the covariance of the signal

$$\mathrm{Cov}_P = \langle t\xi \rangle_P - \langle t \rangle_P \langle \xi \rangle_P = \langle t\varphi'(t) \rangle_s - \langle t \rangle_s \langle \varphi' \rangle_s = \mathrm{Cov}_s.$$

Higher Moments

For mono-components and their TTFDs we have, for $l = 2, 3, ...$,

$$\langle t^l \rangle_P = \langle t^l \rangle_s$$

and

$$\langle \xi^l \rangle_P = \langle \xi^l \rangle_\tau = \langle \varphi'^l \rangle_s + O(\frac{1}{M^2}).$$

For multi-components and their CTTFDs we have

$$\langle t^l \rangle_P = \langle t^l \rangle_{\tilde{s}}$$

and

$$\langle \xi^l \rangle_P = \langle \xi^l \rangle_{\tilde{\tau}} = \int \langle \widetilde{\varphi'^l}(t), \tilde{s}(t) \bigotimes \bar{\tilde{s}}(t) \rangle dt + O\left(\frac{1}{M^2}\right),$$

where $\widetilde{\varphi'^l} = (\varphi_1'^{\,l}, ..., \varphi_n'^{\,l}, ...)$.

3.7. *Local average*

Treating TTFD and CTTFD as general densities, we can immediately argue that the density of frequency at a given time and the density of time at a given frequency are respectively given by

$$P(\xi|t) = \frac{P(t,\xi)}{P_1(t)} \quad \text{and} \quad P(t|\xi) = \frac{P(t,\xi)}{P_2(\xi)}. \tag{3.20}$$

where for mono-component $P_1(t)$ is time marginal distribution given in (3.9) and $P_2(\xi)$ is frequency marginal distribution given in (3.10). For multi-components and the related CTTFD $P_1(t)$ and $P_2(\xi)$ are defined through the corresponding mono-component decomposition.

The conditional average value of a function at a given time or frequency with respect to the TTFD or CTTFD is

$$\langle g(\xi)\rangle_{t,P} = \int g(\xi)\frac{P(t,\xi)}{P_1(t)}d\xi = \frac{1}{P_1(t)}\int g(\xi)P(t,\xi)d\xi,$$

$$\langle g(t)\rangle_{\xi,P} = \int g(t)\frac{P(t,\xi)}{P_2(\xi)}dt = \frac{1}{P_2(\xi)}\int g(t)P(t,\xi)dt.$$

For mono-components the mean frequency at a given time is computed as follows:

$$\langle \xi\rangle_{t,P} = \int \xi\frac{P(t,\xi)}{P_1(t)}d\xi = \frac{\int \xi\rho^2(t)\delta_M(\xi-\varphi'(t))d\xi}{P_1(t)} = \varphi'(t).$$

As expected the mean frequency of a mono-component at a given time is just the instantaneous frequency at the time.

The following are second conditional moments of frequency and standard deviation. .

$$\langle \xi^2\rangle_{t,P} = \int \xi^2\frac{P(t,\xi)}{P_1(t)}d\xi = \frac{\int \xi^2\rho^2(t)\delta_M(\xi-\varphi'(t))d\xi}{P_1(t)} = \varphi'^2(t) + \frac{1}{12M^2}.$$

$$\begin{aligned}\sigma^2_{\xi|t,P} &= \langle(\xi-\langle\xi\rangle_{t,P})^2\rangle_{t,P} = \int(\xi-\langle\xi\rangle_{t,P})^2\frac{P(t,\xi)}{P_1(t)}d\xi \\ &= \frac{\int\xi^2P(t,\xi)d\xi - 2\langle\xi\rangle_{t,P}\int\xi P(t,\xi)d\xi + {\langle\xi\rangle_{t,P}}^2\int P(t,\xi)d\xi}{P_1(t)} \\ &= \frac{|s(t)|^2[\frac{1}{12M^2}+\varphi'^2(t)] - 2\langle\xi\rangle_{t,P}|s(t)|^2\varphi'(t) + {\langle\xi\rangle_{t,P}}^2|s(t)|^2}{P_1(t)} \\ &= \frac{1}{12M^2}.\end{aligned}$$

The conditional standard deviation denotes the deviations of frequency about the instantaneous frequency at a given time. As M is close to infinity, we have $\sigma^2_{\xi|t,P}$ close to zero, which implies that TTFD for mono-component signals is infinitely concentrated along the instantaneous frequency graph.

We accordingly can also have the following results for multi-component signals.

$$\langle\xi\rangle_{t,P} = \int \xi \frac{P(t,\xi)}{P_1(t)} d\xi = \frac{\sum_{k=1}^{\infty} \varphi'_k(t)\rho_k^2(t)}{\sum_{k=1}^{\infty} \rho_k^2(t)}.$$

$$\langle\xi^2\rangle_{t,P} = \int \xi^2 \frac{P(t,\xi)}{P_1(t)} d\xi = \frac{\sum_{k=1}^{\infty} [{\varphi'}_k^2(t) + \frac{1}{12M^2}]\rho_k^2(t)}{\sum_{k=1}^{\infty} \rho_k^2(t)}.$$

$$\begin{aligned}\sigma^2_{\xi|t,P} &= \langle(\xi - \langle\xi\rangle_{t,P})^2\rangle_{t,P} = \int (\xi - \langle\xi\rangle_{t,P})^2 \frac{P(t,\xi)}{P_1(t)} d\xi \\ &= \frac{\sum_{k=1}^{\infty} \{\rho_k^2(t)[\frac{1}{12M^2} + {\varphi'}_k^2(t)] - 2\langle\xi\rangle_{t,P}\rho_k^2(t){\varphi'}_k(t) + {\langle\xi\rangle_{t,P}}^2\rho_k^2(t)\}}{\sum_{k=1}^{\infty} \rho_k^2(t)}.\end{aligned}$$

3.8. *Global average for Period signal*

Treating period signals $f(e^{it}) = \rho(t)e^{i\varphi(t)}$ as defined on $[0, 2\pi)$, the global averages of time and duration for a TTFD or a CTTFD are defined differently from the non-periodic case. In the following, we give the properties for the period case for mono-components.

Since t in the periodic signal $f(e^{it})$ is in $[0, 2\pi)$, we can regard t as an angular random variable (see Ref. 14), then the global average of time with respect to a TTFD or a CTTFD is given by

$$\langle t\rangle_P = \arg \int_0^{2\pi} \int_{-\infty}^{\infty} e^{it} P(t,\xi) dt d\xi,$$

and the duration is given by

$$\sigma^2_{t,P} = 1 - \langle\cos(t - \langle t\rangle_P)\rangle_P = 1 - \int_0^{2\pi} \int_{-\infty}^{\infty} \cos(t - \langle t\rangle_P) P(t,\xi) dt d\xi.$$

If $f(e^{it}) = \rho(t)e^{i\varphi(t)}$ is a mono-component, then we have

$$\langle t\rangle_P = \arg \int_0^{2\pi} \int_{-\infty}^{\infty} e^{it} P(t,\xi) dt d\xi = \arg \int_0^{2\pi} \int_{-\infty}^{\infty} e^{it} \rho^2(t) \delta_M(\xi - \varphi'(t)) dt d\xi = \arg \int_0^{2\pi} e^{it}\rho^2(t) dt = \langle t\rangle_f$$

and

$$\begin{aligned}\sigma^2_{t,P} &= 1 - \langle\cos(t - \langle t\rangle_P)\rangle_P \\ &= 1 - \int_0^{2\pi} \int_{-\infty}^{\infty} \cos(t - \langle t\rangle_P) P(t,\xi) dt d\xi \\ &= 1 - \int_0^{2\pi} \cos(t - \langle t\rangle_f)\rho^2(t) dt = \sigma^2_{t,f}.\end{aligned}$$

4. Examples of TTFD and CTTFD

This section provides examples of TTFD and CTTFD for periodic mono-component signals.

Example 1 Let $f(e^{it}) = \frac{1}{e^{i2t}+2}$, $t \in [0, 2\pi)$, With the threshold $\varepsilon = 0.000905$ for the energy, we have an AFD decomposition

$$f(e^{it}) \approx \sum_{k=1}^{5} f_k(e^{it}),$$

where

$$f_k(e^{it}) = \langle f, B_k(e^{it})\rangle B_k(e^{it})$$

and

$$B_k(e^{it}) = \frac{\sqrt{1-|a_k|^2}}{1-\bar{a}_k e^{it}} \prod_{l=1}^{k-1} \frac{e^{it}-a_l}{1-\bar{a}_l e^{it}}, \tag{4.1}$$

Denoting $f_k(e^{it}) = \rho_k(t)e^{i\varphi_k(t)}$, then

$$\rho_k(t) = |\langle f, B_k(e^{it})\rangle| \frac{\sqrt{1-|a_k|^2}}{1-|a_k|\cos(t-\theta_{a_k})+|a_k|^2},$$

$$\varphi_k'(t) = \frac{|a_k|\cos(t-\theta_{a_k})-|a_k|^2}{1-2|a_k|\cos(t-\theta_{a_k})+|a_k|^2} + \sum_{l=1}^{k-1} \frac{1-|a_l|^2}{1-2|a_l|\cos(t-\theta_{a_l})+|a_l|^2},$$

where $a_k = |a_k|e^{i\theta_{a_k}}$.

The selected points in the decomposition process (see Appendix) are

$a_1 = 0$, $a_2 = 0.0000+0.8150i$, $a_3 = -0.0000-0.7450i$, $a_4 = 0.0000+0.7500i$, $a_5 = -0.0000-0.7500i$.

We study the CTTFD of $\sum_{k=1}^{5} f_k(e^{it})$ given by

$$P(t,\xi) = \sum_{k=1}^{5} P_k(t,\xi) = \sum_{k=1}^{5} \rho_k^2(t)\delta(\xi-\varphi_k'(t)).$$

The graph of the CTTFD is given by Figure 4.2(a). The color index represents the logarithm of the normalized energy density.

Example 2 Let $f(e^{it}) = \cos t^2$. Its Hardy space projection is $f^+(e^{it}) \triangleq g(e^{it}) = \frac{1}{2}(\cos t^2 + i\widetilde{H}\cos t^2)$, where $\widetilde{H}$ is the circular Hilbert transformation. Up to a small threshold, we have an AFD decomposition

$$g(e^{it}) \approx \sum_{k=1}^{11} g_k(e^{it}) = \sum_{k=1}^{11} \rho_k(t)e^{i\varphi_k(t)}.$$

Then

$$f(e^{it}) \approx 2\sum_{k=1}^{11} \rho_k(t)\cos\varphi_k(t) - c_0,$$

where $c_0 = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(e^{it})dt$. We regard the time-frequency distribution of $f^+(e^{it})$ as being that of $f(e^{it})$. The selected points are

$$\begin{aligned} &a_1 = 0, && a_2 = 0.3580 - 0.5920i, && a_3 = -0.6460 - 0.5340i, \\ &a_4 = -0.8660 + 0.2120i, && a_5 = -0.5580 + 0.7200i, && a_6 = 0.8840 + 0.2640i, \\ &a_7 = -0.0160 + 0.9180i, && a_8 = 0.5280 + 0.7700i, && a_9 = 0.7520 + 0.5060i, \\ &a_{10} = 0.0840 - 0.7280i, && a_{11} = -0.8300 - 0.2600i. && \end{aligned}$$

The graph of the CTTFD is given by Figure 4.2(b).

Appendix A. Adaptive Fourier Decompositions

Fourier expansion may converge slowly for the entries that contribute the essential part of the total energy may come late. In a Hilbert space when expending a function in a given basis the same problem occurs. To treat the problem greedy algorithm comes into play. The proposal that adaptive mono-component decomposition is a fundamental method in time-frequency analysis is based on the fact that it gives rise to intrinsic building blocks of the signal. Indeed, an adaptive Fourier decomposition (AFD) gives a fast decomposition in norm into frequency-increasing mono-components. Below we give an introduction to AFD. An AFD is based on a Takenaka-Malmquist (TM) system,[23,21,25,24] that, however, is not a traditional one. The spirit of general greedy algorithm in general is fast convergence achieved through selection of the best possible or close-to-best parameters defining the system. In AFD under the frame of a TM system the best parameter can always be obtained at every step, and the recursive process accumulating factors with positive instantaneous frequency in the Hardy space formulation is applicable. For the basic theory and further development we refer to Ref. 18, Ref. 23, Ref. 20, Ref. 21, and Ref. 24. Below we provide a description of the AFD algorithm. We concentrate in the unit disc case. The AFD theory and algorithm for a half plane is similar.

Below we write $H^2(\mathbb{D})$ as H^2. In AFD we have a "dictionary" consisting of the elementary functions

$$e_{\{a\}}(z) := B_{\{a\}}(z) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}, \quad a \in \mathbb{D}.$$

The function $e_{\{a\}}$ is called *the evaluator at a* (also reproducing kernel and shifted-Cauchy kernel). Each evaluator gives rise to, essentially, the evaluating functional. In fact, for any $F \in H^2$, by using the Cauchy Formula, we have

$$\langle F, e_{\{a\}} \rangle = \sqrt{2\pi}\sqrt{1-|a|^2} F(a).$$

Since for a real-valued signal $\tilde{G}$ there holds the relation

$$\tilde{G} = 2\text{Re}G^+ - c_0, \tag{A.1}$$

we can concentrate in decomposing $G = G^+ \in H^2$. Setting $G_1 = G = G^+$, the first step is to maximize the projection $|\langle G_1, e_{\{a\}} \rangle|^2 = 2\pi(1-|a|^2)|G_1(a)|^2$ among

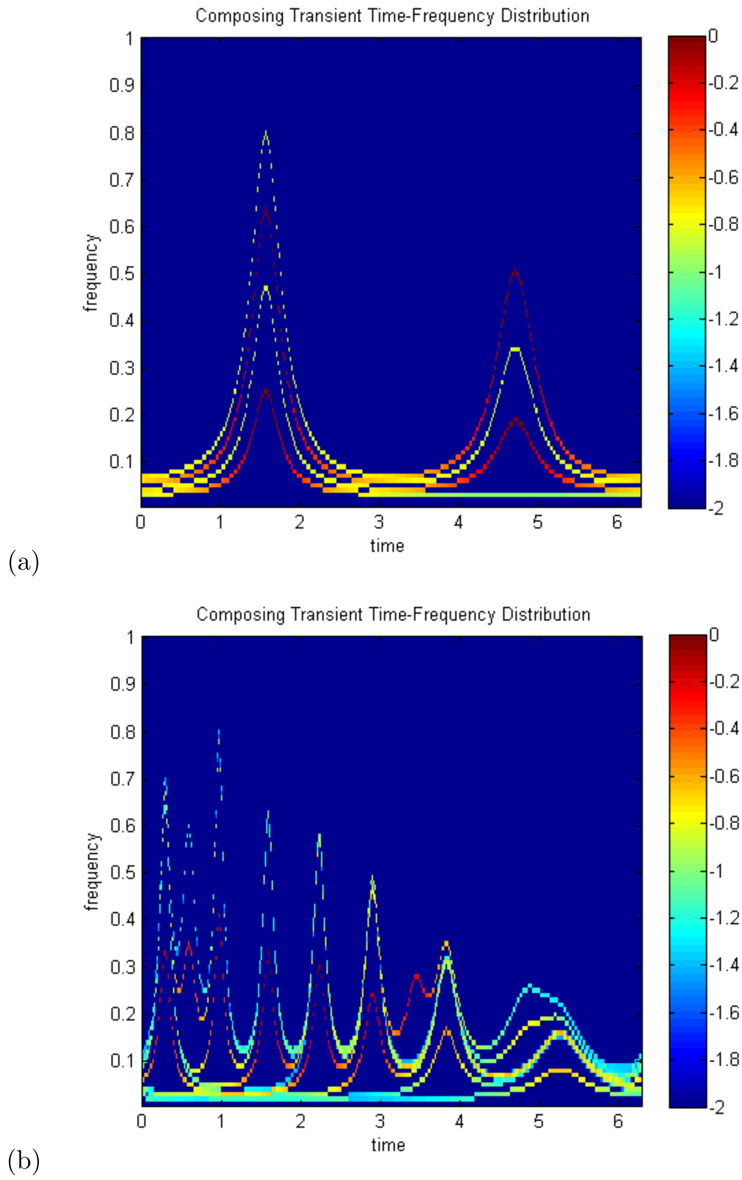

Fig. 4.2. Example of TTFD. (a)The TTFD of $f(e^{it}) = \frac{1}{e^{i2t}+2}$. (b) The TTFD of $f(e^{it}) = \cos t^2$.

all possible selections of $a \in \mathbb{D}$. What is crucial is the ***Maximal Projection Principle*** asserting that there exists $a_1 \in \mathbb{D}$ such that

$$|\langle G_1, e_{\{a_1\}}\rangle|^2 = \max\{|\langle G_1, e_{\{a\}}\rangle|^2 \ : \ a \in \mathbb{D}\}.$$

For a proof of this fact we refer to Ref. 23 or Ref. 19. Now write

$$\begin{aligned} G(z) = G_1(z) &= \langle G_1, e_{\{a_1\}}\rangle e_{\{a_1\}} + \big(G_1(z) - \langle G_1, e_{\{a_1\}}\rangle e_{\{a_1\}}\big) \\ &= \langle G_1, e_{\{a_1\}}\rangle e_{\{a_1\}} + R_1(z). \end{aligned}$$

The energy of the standard remainder $R_1(z)$ is thus minimized, where we have the factorization

$$R_1(z) = G_2(z)\frac{z - a_1}{1 - \overline{a}_1 z},$$

where

$$G_2(z) = \big(G_1(z) - \langle G_1, e_{\{a_1\}}\rangle e_{\{a_1\}}(z)\big)\,\frac{1 - \overline{a}_1 z}{z - a_1}.$$

Note that $G_2(z)$ is still in H^2, as the difference $G_1(z) - \langle G_1, e_{\{a_1\}}\rangle e_{\{a_1\}}(z)$ has zero at $z = a_1$.

We call the process getting G_2 from G_1 a *maximal sifting process*. Applying the process to G_2 we obtain G_3, and so on. Repeating such process to the nth step, we obtain

$$\begin{aligned} G(z) &= \sum_{k=1}^{n}\langle G_k, e_{\{a_k\}}\rangle B_{\{a_1,\ldots,a_k\}}(z) + R_n(z) \\ &= \sum_{k=1}^{n}\langle G_k, e_{\{a_k\}}\rangle B_{\{a_1,\ldots,a_k\}}(z) + G_{n+1}(z)\prod_{k=1}^{n}\frac{z - a_k}{1 - \overline{a}_k z}, \end{aligned}$$

where

$$G_{k+1}(z) = \big(G_k(z) - \langle G_k, e_{\{a_k\}}\rangle e_{\{a_k\}}\big)\,\frac{1 - \overline{a}_k z}{z - a_k},$$

and a_k is selected under the Maximal Projection Principle. The kth standard remainder has the expression

$$R_k(z) = G_{k+1}(z)\prod_{l=1}^{k}\frac{z - a_l}{1 - \overline{a}_l z}.$$

The orthogonality properties imply

$$\|G\|^2 = 2\pi\sum_{k=1}^{n}(1 - |a_k|^2)|G_k(a_k)|^2 + \|R_n\|^2.$$

It is further proved that there holds the Plancherel Theorem

$$\|G\|^2 = 2\pi\sum_{k=1}^{\infty}(1 - |a_k|^2)|G_k(a_k)|^2 = \sum_{k=1}^{\infty}|\langle G^+, B_k\rangle|^2,$$

and therefore the partial sums converge to the function itself in the Hardy space, as well in the L^2 space on the boundary. For a given threshold $\epsilon > 0$, one sets to have the consecutive maximal sifting processes cease at the first N that

$$\|R_N\|^2 = \|G\|^2 - 2\pi\sum_{k=1}^{N}(1 - |a_k|^2)|G_k(a_k)|^2 \le \epsilon.$$

This completes the algorithm. To summarize, the kth maximal sifting process produces the output

$$\langle G_k, e_{\{a_k\}}\rangle B_k(z) = \frac{(1-|a_k|^2)G_k(a_k)}{1-\overline{a}_k z}\prod_{l=1}^{k-1}\frac{z-a_l}{1-\overline{a}_l z}.$$

Due to the energy principle, this may be called *the* kth *intrinsic complex-mono-component* (*IMC*) of G^+. The approximation by the Nth partial sum of the IMCs is

$$G^+(z) \approx \sum_{k=1}^{N}\frac{(1-|a_k|^2)G_k(a_k)}{1-\overline{a}_k z}\prod_{l=1}^{k-1}\frac{z-a_l}{1-\overline{a}_l z}.$$

With an error less than 2ϵ the relation (A.1) gives

$$\tilde{G}(e^{it}) \approx -c_0 + 2\mathrm{Re}\sum_{k=1}^{N}\frac{(1-|a_k|^2)G_k(a_k)}{1-\overline{a}_k e^{it}}\prod_{l=1}^{k-1}\frac{e^{it}-a_l}{1-\overline{a}_l e^{it}}.$$

We remark that the selection $a_1 = 0$ makes all B_n' be mono-components.

For variations of AFD we refer to Ref. 21 and Ref. 22.

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