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Analytic Sampling Approximation by Projection Operator with Application in Decomposition of Instantaneous Frequency

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A sequence of special functions in Hardy space $\mathcal{H}^2(\mathbb{T}^s)$ are constructed from Cauchy kernel on unit disk $\mathbb{D}$. Applying projection operator of the sequence of functions leads to an analytic sampling approximation to f, any given function in $\mathcal{H}^2(\mathbb{T}^s)$. That is, f can be approximated by its analytic samples in $\mathbb{D}^s$. Under a mild condition, f is approximated exponentially by its analytic samples. By the analytic sampling approximation, a signal in $\mathcal{H}^2(\mathbb{T})$ can be approximately decomposed into components of positive instantaneous frequency. Using circular Hilbert transform, we apply the approximation scheme in $\mathcal{H}^s(\mathbb{T}^s)$ to $L^s(\mathbb{T}^2)$ such that a signal in $L^s(\mathbb{T}^2)$ can be approximated by its analytic samples on $\mathbb{C}^s$. A numerical experiment is carried out to illustrate our results.

Keywords: Hardy space; Cauchy kernel; analytic sampling approximation; instantaneous frequency; Hilbert transform.

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1. Introduction and motivation

Let $L^2(\mathbb{T}^s)$ be the space of $2\pi\mathbb{Z}^s$-periodic and square integrable functions, where $\mathbb{T}$ is the unit circle of complex plane $\mathbb{C}$ and $\mathbb{T}^s$ is the s-torus, $s \in \mathbb{N}$. The inner product $\langle \cdot, \cdot \rangle$ in $L^2(\mathbb{T}^s)$ is defined by

$$\langle f, g \rangle = \frac{1}{(2\pi)^s} \int_{[0,2\pi)^s} f(e^{it})\overline{g(e^{it})}dt, \quad \forall f, g \in L^2(\mathbb{T}^s),$$

with associated norm $||\cdot||_2$, where $e^{it} = (e^{it_1}, \ldots, e^{it_s})$, $t = (t_1, \ldots, t_s) \in [0, 2\pi)^s$, and $dt = dt_1 \ldots dt_s$. Any $f \in L^2(\mathbb{T}^s)$ can be expressed by $f(e^{it}) = \sum_{n\in\mathbb{Z}^s} \widehat{f}(n)e^{in\cdot t}$, where $n \cdot t$ is the Euclid inner product of $n \in \mathbb{Z}^s$ and $t \in [0, 2\pi)^s$, and

$\widehat{f}(n) = \frac{1}{(2\pi)^s}\int_{[0,2\pi)^s} f(t)e^{-in\cdot t}dt$. Plancherel's Theorem gives us that $||f||_2 = (\sum_{n\in\mathbb{Z}^s}|\widehat{f}(n)|^2)^{1/2}$. The truncated operator $+ : \mathbb{R} \longrightarrow \mathbb{R}^+ \cup \{0\}$ is defined by $x \mapsto 0$ if $x \leq 0$ and $x \mapsto x$ elsewhere.

In digital signal communication, a signal is usually reconstructed and approximated by its samples. Wavelet has many applications in sampling theory [2,11,12,13,14]. Sampling in wavelet subspaces of $L^2(\mathbb{R}^s)$ has been extensively studied by [16,6,11,12,13,14] and the numerous references therein. Goh, Han and Shen [9] recently constructed a special pair of tight frames for $L^2(\mathbb{T})$, and investigated their approximation. Motivated by [9], this paper aims at establishing an analytic sampling approximation to a signal in $L^2(\mathbb{T}^s)$ by its *analytic samples* on $\mathbb{C}^s$. That approximation originates from a sequence of nonstationary refinable functions in Hardy space $\mathcal{H}^2(\mathbb{T}^s)$, which is defined by

$$\mathcal{H}^2(\mathbb{T}^s) = \{f \in L^2(\mathbb{T}^s) : \widehat{f}(n) = 0, n \in \mathbb{Z}^s \setminus \mathbb{N}_0^s\},\ \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Recall that any f in $\mathcal{H}^2(\mathbb{T}^s)$ is the non-tangential boundary value of an analytic function on $\mathbb{D}^s$, where $\mathbb{D}$ is the unit disk of $\mathbb{C}$. Precisely, any function $f(r_1e^{it_1}, \ldots, r_se^{it_s})$ analytic on $\mathbb{D}^s$ can be identified with $f(e^{it}) \in \mathcal{H}^2(\mathbb{T}^s)$, namely,

$$f(e^{it}) = \lim_{(r_1,\ldots,r_s)\to(1,\ldots,1)-} f(r_1e^{it_1}, \ldots, r_se^{it_s}),$$

where $(r_1, \ldots, r_s) \to (1, \ldots, 1)-$ means $r_j \to 1-$, $j = 1, \ldots, s$.

Next we turn to our motivation. From the physics point of view, it makes a good sense to decompose a signal into intrinsic mode functions (IMFs) such as ones of nonnegative instantaneous frequency [22,24,8,27], of which the definition will be given in Section 3. Moreover, fast approximation is desired for practice. Therefore rational orthogonal systems in $\mathcal{H}^2(\mathbb{T})$, which may meet the two requirements mentioned above [22], attracted a lot of attention [27,20,25]. A popular rational orthogonal basis for $\mathcal{H}^2(\mathbb{T})$ is Takenaka-Malmquist (T-M) system $\{B_k : k \in \mathbb{N}\}$ [3,4,5,19,20], which is defined by

$$\begin{cases} B_1(e^{it}) = B_{\{a_1\}}(e^{it}) = \dfrac{1}{\sqrt{2\pi}}\dfrac{\sqrt{1-|a_1|^2}}{1-\overline{a}_1e^{it}}, \\ B_k(e^{it}) = B_{\{a_1,\ldots,a_k\}}(e^{it}) = \dfrac{1}{\sqrt{2\pi}}\dfrac{\sqrt{1-|a_k|^2}}{1-\overline{a}_ke^{it}}\displaystyle\prod_{j=1}^{k-1}\dfrac{e^{it}-a_j}{1-\overline{a}_je^{it}}, k \geq 2, \end{cases} , t \in [0, 2\pi), \tag{1.1}$$

with $a_j \in \mathbb{D}$ and $\sum_{j=1}^{\infty}(1 - |a_j|) = \infty$. When $a_1 = 0$, B_k in (1.1) has nonnegative instantaneous frequency [22]. Due to the orthogonality of $\{B_k : k \in \mathbb{N}\}$, any $f \in \mathcal{H}^2(\mathbb{T})$ can be reconstructed by

$$f(e^{it}) = \sum_{k=1}^{\infty}\langle f, B_{\{a_1,a_2,\ldots,a_k\}}\rangle B_{\{a_1,a_2,\ldots,a_k\}}(e^{it}), t \in [0, 2\pi). \tag{1.2}$$

It is necessary for practical application to estimate the converging rate of series in (1.2). However the truncated error

$$f(e^{it}) - \sum_{k=1}^{n} \langle f, B_{\{a_1,a_2,\ldots,a_k\}}\rangle B_{\{a_1,a_2,\ldots,a_k\}}(e^{it}) \tag{1.3}$$

is difficult to estimate in the sense of $\|\cdot\|_2$. Bultheel and Carrette [5] estimated (1.3) under some conditions but in the sense of $\|\cdot\|_1$. Apart from [5], to our best knowledge, there are few results on that problem in the literature. On the other hand, it is not easy to generalize some results on rational orthogonal basis from $\mathcal{H}^2(\mathbb{T})$ to $\mathcal{H}^2(\mathbb{T}^s)$, $s > 1$, about which readers may refer to [26] for adaptive Fourier decomposition (AFD) in higher dimensional Hardy spaces and in the Clifford algebra setting. Since the inner product $\langle f, B_{\{a_1,a_2,\ldots,a_k\}}\rangle$, $k \in \mathbb{N}$, is completely determined by analytic samples $f(a_j)$ [21], $j = 1, \ldots, k$, $P_n(f) := \sum_{k=1}^{n} \langle f, B_{\{a_1,a_2,\ldots,a_k\}}\rangle B_{\{a_1,a_2,\ldots,a_k\}}(e^{it})$ is actually an analytic sampling approximation to f. Therefore it is necessary to establish fast analytic sampling approximation of $\mathcal{H}^2(\mathbb{T}^s)$, and furthermore of $L^2(\mathbb{T}^s)$ from other perspective. Motivated by [9], we shall solve this problem by projection operator generating from a sequence of special nonstationary refinable functions. The following concepts and denotations are necessary for further proceeding.

For any $k \in \mathbb{N}_0$, let $\mathcal{L}_k = \{(n_1, n_2, \ldots, n_s) : n_j \in \{0, 1, \ldots, 2^k - 1\}, j = 1, \ldots, s\}$, and $d = \det(2I_s) = 2^s$ throughout this paper. For $\ell \in \mathbb{Z}^s$, define the $2\pi\ell/2^k$-shift operator $T_k^\ell : \mathcal{H}^2(\mathbb{T}^s) \longrightarrow \mathcal{H}^2(\mathbb{T}^s)$ by

$$T_k^\ell f = f(e^{i(\cdot - 2\pi\ell/2^k)}), \forall f \in \mathcal{H}^2(\mathbb{T}^s).$$

Since f is $2\pi\mathbb{Z}^s$-periodic, it is sufficient to consider $\ell \in \mathcal{L}_k$. Our projection operator is from a sequence of functions $\{\phi_k\}_{k=0}^{\infty} \subseteq \mathcal{H}^2(\mathbb{T}^s)$ satisfying the $2I_s$-refinement equation

$$\phi_{k+1} = \sum_{\ell \in \mathcal{L}_{k+1}} P_{k+1}(\ell) T_{k+1}^\ell \phi_0, k \geq 0, \tag{1.4}$$

where $P_{k+1} \triangleq \{P_{k+1}(\ell)\}_{\ell \in \mathcal{L}_{k+1}}$ belongs to $\mathcal{S}(2^{k+1}I_s)$. Here $\mathcal{S}(2^{k+1}I_s)$ is the space of $2^{k+1}\mathbb{N}_0^s$-periodic complex-valued sequences supported on $\mathbb{N}_0^s$, namely, for any $F = \{F(p)\}_{p \in \mathbb{N}_0^s} \in \mathcal{S}(2^{k+1}I_s)$, it holds that $F(p) = F(p + 2^{k+1}j)$, $\forall p, j \in \mathbb{N}_0^s$. Since $\{\phi_k\}_{k=0}^{\infty} \subset \mathcal{H}^2(\mathbb{T}^s) \subset L^2(\mathbb{T}^s)$, (1.4) is equivalent to

$$\widehat{\phi_{k+1}}(n) = \widehat{P_{k+1}}(n)\widehat{\phi_0}(n), \forall n \in \mathbb{N}_0^s, \tag{1.5}$$

where the discrete Fourier transform $\widehat{P_{k+1}} \triangleq \{\widehat{P_{k+1}}(j)\}$ of P_{k+1} is defined by $\widehat{P_{k+1}}(j) = \sum_{\ell \in \mathcal{L}_{k+1}} P_{k+1}(\ell) e^{-ij\cdot 2\pi\ell/2^{k+1}}$. Clearly $\widehat{P_{k+1}} \in \mathcal{S}(2^{k+1}I_s)$ if and only if $P_{k+1} \in \mathcal{S}(2^{k+1}I_s)$. Now it is ready to define the *projection operator* $\mathfrak{P}_k$ with respect to the refinable functions in (1.4) by

$$\mathfrak{P}_k(f) = \sum_{\ell \in \mathcal{L}_k} \langle f, T_k^\ell \phi_k\rangle T_k^\ell \phi_k, \tag{1.6}$$

for any $f \in \mathcal{H}^2(\mathbb{T}^s)$. In next section, we shall construct special $\{\phi_k\}_{k=0}^{\infty} \subseteq \mathcal{H}^2(\mathbb{T}^s)$ satisfying (1.4) such that the inner products in (1.6) can be calculated by analytic samples of f, and $||f - \mathfrak{P}_k(f)||_2$ can be estimated.

2. Analytic sampling approximation and its application in decomposition of instantaneous frequency

For any $x = (x_1, \ldots, x_s)$ and $y = (y_1, \ldots, y_s)$ in $\mathbb{C}^s$, the operator $\circ$ is defined by

$$x \circ y = (x_1 y_1, \ldots, x_s y_s), \tag{2.1}$$

and $||x|| = (\sum_{k=1}^{s} |x_k|^2)^{1/2}$ is the square norm of $\mathbb{C}^s$. A signal $f \in L^2(\mathbb{T}^s)$ has Sobolev smoothness ν if $\sum_{n\in\mathbb{Z}^s} |\widehat{f}(n)|^2(1 + ||n||^2)^{\nu} < \infty$. Define $\nu_2(f)$ to be $\sup\{\nu : f$ has Sobolev smoothness $\nu\}$. Clearly $\nu_2(f) \geq 0$ for any $f \in L^2(\mathbb{T}^s)$. Sobolev space $W_2^{\nu}(\mathbb{T}^s)$ is defined to be $\{f \in L^2(\mathbb{T}^s) : \nu_2(f) > \nu\}$ with the norm $||f||_{W_2^{\nu}(\mathbb{T}^s)} = (\sum_{n\in\mathbb{Z}^s} |\widehat{f}(n)|^2(1 + ||n||^2)^{\nu})^{1/2}$ and the seminorm $|f|_{W_2^{\nu}(\mathbb{T}^s)} = (\sum_{n\in\mathbb{Z}^s} |\widehat{f}(n)|^2||n||^{2\nu})^{1/2}$ for any $f \in W_2^{\nu}(\mathbb{T}^s)$. It is straightforward to see that $|f|_{W_2^{\nu}(\mathbb{T}^s)} \leq ||f||_{W_2^{\nu}(\mathbb{T}^s)}$.

In signal analysis, instantaneous frequency of a real-valued and 2π-periodic signal $s(t), t \in [0, 2\pi)$, is defined to be $\theta'(t)$, where $\theta(t)$ is defined through the analytic signal associated with s, viz. $s(t) + iHs(t) = \rho(t)e^{i\theta(t)}$. Here $\rho(t)(\geq 0)$ and $\theta(t)$ are regarded as instantaneous amplitude and instantaneous phase, respectively, and H is the circular Hilbert transform defined by

$$Hf(t) = \frac{1}{2\pi} p.v. \int_0^{2\pi} f(x) \cot \frac{t-x}{2} dx, \ \forall f \in L^2(\mathbb{T}). \tag{2.2}$$

If $\theta'(t) \geq 0$, then $s(t)$ and $s(t) + iHs(t)$ are regarded as a *real mono-component* and a *complex mono-component*, respectively. As mentioned in Section 1, it is of great physics importance to decompose a signal into mono-components [22,24,8,23]. More details on mono-component can be seen in [22,24,23], where we use AFD to decompose a signal in $L^2(\mathbb{T})$ into mono-components. We shall see that a signal in $W_2^{\nu}(\mathbb{T}) \cap \mathcal{H}^2(\mathbb{T})$, $\nu > 0$, can be decomposed into mono-components by its analytic samples, with the decomposing series converging exponentially.

Note 2.1. Since the inverse discrete Fourier transform(IDFT) is necessary for proving Theorem 2.1, we need to formulate it from the perspective of matrix for fast computation and better programming in Algorithm 2.2. For any $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_s) \in \mathbb{N}_0^s$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_s) \in \mathbb{N}_0^s$, we say that $\kappa \prec \mu$ in the lexicographic order if $\kappa_m = \mu_m$ for $m = 1, \ldots, i-1$ and $\kappa_i < \mu_i$. The Fourier transform $\widehat{P_{k+1}} = \{\widehat{P_{k+1}}(\nu_j)\}$ of $P_{k+1} = \{P_{k+1}(\nu_\ell)\} \in \mathcal{S}(2^{k+1}I_s)$ is given by

$$\widehat{P_{k+1}}(\nu_j) = \sum_{\nu_\ell \in \mathcal{L}_{k+1}} P_{k+1}(\nu_\ell) e^{-i\nu_j \cdot 2\pi\nu_\ell/2^{k+1}}, \tag{2.3}$$

where $\{\nu_j\}$ is the ordered set $\mathcal{L}_{k+1}$ under the lexicographic order, i.e., $\nu_1 \prec \nu_2 \prec$

$\ldots \prec \nu_{d^{k+1}}$. Now (2.3) can be rewritten as

$$\begin{bmatrix} e^{-i\nu_1\cdot 2\pi\nu_1/2^{k+1}} & \cdots & e^{-i\nu_1\cdot 2\pi\nu_{d^{k+1}}/2^{k+1}} \\ e^{-i\nu_2\cdot 2\pi\nu_1/2^{k+1}} & \cdots & e^{-i\nu_2\cdot 2\pi\nu_{d^{k+1}}/2^{k+1}} \\ \vdots & \ddots & \vdots \\ e^{-i\nu_{d^{k+1}}\cdot 2\pi\nu_1/2^{k+1}} & \cdots & e^{-i\nu_{d^{k+1}}\cdot 2\pi\nu_{d^{k+1}}/2^{k+1}} \end{bmatrix} \begin{bmatrix} P_{k+1}(\nu_1) \\ P_{k+1}(\nu_2) \\ \vdots \\ P_{k+1}(\nu_{d^{k+1}}) \end{bmatrix} = \begin{bmatrix} \widehat{P_{k+1}}(\nu_1) \\ \widehat{P_{k+1}}(\nu_2) \\ \vdots \\ \widehat{P_{k+1}}(\nu_{d^{k+1}}) \end{bmatrix},$$

from which we deduce the IDFT

$$\begin{aligned} \begin{bmatrix} P_{k+1}(\nu_1) \\ P_{k+1}(\nu_2) \\ \vdots \\ P_{k+1}(\nu_{d^{k+1}}) \end{bmatrix} &= \tfrac{1}{d^{k+1}} \begin{bmatrix} e^{-i\nu_1\cdot 2\pi\nu_1/2^{k+1}} & \cdots & e^{-i\nu_1\cdot 2\pi\nu_{d^{k+1}}/2^{k+1}} \\ e^{-i\nu_2\cdot 2\pi\nu_1/2^{k+1}} & \cdots & e^{-i\nu_2\cdot 2\pi\nu_{d^{k+1}}/2^{k+1}} \\ \vdots & \ddots & \vdots \\ e^{-i\nu_{d^{k+1}}\cdot 2\pi\nu_1/2^{k+1}} & \cdots & e^{-i\nu_{d^{k+1}}\cdot 2\pi\nu_{d^{k+1}}/2^{k+1}} \end{bmatrix}^* \begin{bmatrix} \widehat{P_{k+1}}(\nu_1) \\ \widehat{P_{k+1}}(\nu_2) \\ \vdots \\ \widehat{P_{k+1}}(\nu_{d^{k+1}}) \end{bmatrix} \\ &= \tfrac{1}{d^{k+1}} \begin{bmatrix} e^{i\nu_1\cdot 2\pi\nu_1/2^{k+1}} & \cdots & e^{i\nu_{d^{k+1}}\cdot 2\pi\nu_1/2^{k+1}} \\ e^{i\nu_1\cdot 2\pi\nu_2/2^{k+1}} & \cdots & e^{i\nu_{d^{k+1}}\cdot 2\pi\nu_2/2^{k+1}} \\ \vdots & \ddots & \vdots \\ e^{i\nu_1\cdot 2\pi\nu_{d^{k+1}}/2^{k+1}} & \cdots & e^{i\nu_{d^{k+1}}\cdot 2\pi\nu_{d^{k+1}}/2^{k+1}} \end{bmatrix} \begin{bmatrix} \widehat{P_{k+1}}(\nu_1) \\ \widehat{P_{k+1}}(\nu_2) \\ \vdots \\ \widehat{P_{k+1}}(\nu_{d^{k+1}}) \end{bmatrix}, \end{aligned} \tag{2.4}$$

where we use the relation $\sum_{m=1}^{d^{k+1}} e^{-i(\nu_j-\nu_k)\cdot 2\pi\nu_m/2^{k+1}} = d^{k+1}\delta_{j-k}$ for any $j,k \in \{1,2,\ldots,d^{k+1}\}$.

2.1. *Analytic sampling approximation*

Theorem 2.1. *Let* $\phi_0(e^{it_1},\ldots,e^{it_s}) = \prod_{j=1}^{s} \frac{1}{(1-\overline{a}_j e^{it_j})}, t = (t_1,t_2,\ldots,t_s) \in [0,2\pi)^s$. *Construct* $\phi_k, k \geq 1$, *by*

$$\widehat{\phi_k}(n) = \widehat{P_k}(n)\widehat{\phi_0}(n), \forall n \in \mathbb{N}_0^s, \tag{2.5}$$

where $\{\widehat{P_k}(n)\} \in \mathcal{S}(2^k I_s)$ *such that*

$$\widehat{\phi_k}(m) = d^{-k/2} \tag{2.6}$$

for any $m \in \mathcal{L}_k$. *Then* $\mathfrak{P}_k(f)$ *defined in (1.6) can be expressed by*

$$\mathfrak{P}_k(f) = \sum_{\ell\in\mathcal{L}_k}\sum_{\ell''\in\mathcal{L}_k}\sum_{\ell'\in\mathcal{L}_k} \overline{P_k(\ell'')}P_k(\ell')f(\boldsymbol{a}\circ e^{i2\pi 2^{-k}(\ell''+\ell'+\ell)})\phi_0(e^{i(\cdot-2\pi 2^{-k}(\ell+\ell'))}), \tag{2.7}$$

where $\boldsymbol{a} := (a_1,\ldots,a_s) \in \mathbb{D}^s$, *and* $\{P_k(n)\}$ *is the DFT of* $\{\widehat{P_k}(n)\}$. *Moreover, for any* $f \in \mathcal{H}^2(\mathbb{T}^s) \cap W_2^\nu(\mathbb{T}^s)$, $\nu > 0$, *there holds*

$$\begin{aligned} ||f-\mathfrak{P}_k(f)||_2^2 \leq |f|^2_{W_2^\nu(\mathbb{T}^s)} 2^{-2k\nu+1} + \tfrac{4(|\widehat{f}(0)|^2+2||f||_2^2)}{\prod_{j=1}^{s}(1-|a_j^{2^k}|^2)} \textstyle\sum_{\ell=1}^{s} |a_\ell^{2^k}|^2 \\ + \tfrac{||f||_2^2}{\prod_{j=1}^{s}(1-|a_j^{2^k}|^2)} \textstyle\sum_{\ell=1}^{s} |a_\ell^{2^k}|^2. \end{aligned} \tag{2.8}$$

Proof. Direct calculation together with the Cauchy integral formula gives us

$$\begin{aligned}
\langle f, \phi_0 \rangle &= \int_{\mathbb{T}^s} f(e^{it_1}, \ldots, e^{it_s}) \overline{\phi_0(e^{it_1}, \ldots, e^{it_s})} dt_1 \ldots dt_s \\
&= \int_{\mathbb{T}^s} f(e^{it_1}, \ldots, e^{it_s}) \prod_{p=1}^{s} \frac{1}{(1 - a_p e^{-it_p})} dt_1 \ldots dt_s \\
&= 2\pi \int_{\mathbb{T}^{s-1}} f(a_1, e^{it_2}, \ldots, e^{it_s}) \prod_{p=2}^{s} \frac{1}{(1 - a_p e^{-it_p})} dt_2 \ldots dt_s \\
&\vdots \\
&= (2\pi)^s f(a_1, a_2, \ldots, a_s) \\
&= (2\pi)^s f(\mathbf{a}).
\end{aligned} \tag{2.9}$$

It follows from (2.5) that

$$\begin{aligned}
\mathfrak{P}_k(f) &= \sum_{\ell \in \mathcal{L}_k} \langle f, T_k^{\ell} \phi_k \rangle T_k^{\ell} \phi_k \\
&= \sum_{\ell \in \mathcal{L}_k} \sum_{\ell'' \in \mathcal{L}_k} \sum_{\ell' \in \mathcal{L}_k} \overline{P_k(\ell'')} P_k(\ell') f(\mathbf{a} \circ e^{i2\pi 2^{-k}(\ell'' + \ell' + \ell)}) T_k^{\ell} T_k^{\ell'} \phi_0 \\
&= \sum_{\ell \in \mathcal{L}_k} \sum_{\ell'' \in \mathcal{L}_k} \sum_{\ell' \in \mathcal{L}_k} \overline{P_k(\ell'')} P_k(\ell') f(\mathbf{a} \circ e^{i2\pi 2^{-k}(\ell'' + \ell' + \ell)}) \phi_0(e^{i(\cdot - 2\pi 2^{-k}(\ell + \ell'))}).
\end{aligned} \tag{2.10}$$

For any $f \in \mathcal{H}^2(\mathbb{T}^s)$, it can be proved as [9] that

$$\widehat{\mathfrak{P}_k(f)}(n) = d^k \widehat{\phi_k}(n) \sum_{p \in \mathbb{Z}^s} \widehat{f}(n + 2^k p) \overline{\widehat{\phi_k}(n + 2^k p)}, \forall n \in \mathbb{N}_0^s. \tag{2.11}$$

By Plancherel's Theorem, we have

$$||f - \mathfrak{P}_k(f)||_2^2 = \sum_{j \in \mathcal{L}_k} |\widehat{f}(j) - \widehat{\mathfrak{P}_k(f)}(j)|^2 + \sum_{j \in \mathcal{L}_k} \sum_{q \in \mathbb{N}_0^s \setminus \{0\}} |\widehat{f}(j + 2^k q) - \widehat{\mathfrak{P}_k(f)}(j + 2^k q)|^2. \tag{2.12}$$

By (2.11), we get

$$
\begin{aligned}
&\sum_{j\in\mathcal{L}_k} |\widehat{f}(j) - \widehat{\mathfrak{P}_k(f)}(j)|^2 \\
&= \sum_{j\in\mathcal{L}_k} |\widehat{f}(j) - d^k\widehat{\phi_k}(j) \sum_{p\in\mathbb{N}_0^s} \widehat{f}(j+2^k p)\overline{\widehat{\phi}_k(j+2^k p)}|^2 \\
&= \sum_{j\in\mathcal{L}_k} |d^k\widehat{\phi_k}(j) \sum_{p\in\mathbb{N}_0^s\setminus\{0\}} \widehat{f}(j+2^k p)\overline{\widehat{\phi}_k(j+2^k p)}|^2 \\
&= \sum_{j\in\mathcal{L}_k} | \sum_{p\in\mathbb{N}_0^s\setminus\{0\}} \widehat{f}(j+2^k p) a_1^{2^k p_1} a_2^{2^k p_2} \dots a_s^{2^k p_s}|^2 \\
&\le ||f||_2^2 \sum_{p\in\mathbb{N}_0^s\setminus\{0\}} |a_1^{2^k p_1} a_2^{2^k p_2} \dots a_s^{2^k p_s}|^2 \\
&\le ||f||_2^2 \sum_{\ell=1}^{s} \sum_{\widetilde{q}_\ell=1}^{\infty} |a_\ell^{2^k \widetilde{q}_\ell}|^2 \sum_{\widetilde{q}_j\in\mathbb{N}_0, j\neq\ell} |\prod_{j\neq\ell} a_j^{2^k\widetilde{q}_j}|^2 \\
&\le ||f||_2^2 \sum_{\ell=1}^{s} \sum_{\widetilde{q}_\ell=1}^{\infty} |a_\ell^{2^k \widetilde{q}_\ell}|^2 \frac{1}{\prod_{j\neq\ell}(1-|a_j^{2^k}|^2)} \\
&\le ||f||_2^2 \frac{1}{\prod_{j=1}^{s}(1-|a_j^{2^k}|^2)} \sum_{\ell=1}^{s} |a_\ell^{2^k}|^2,
\end{aligned}
\tag{2.13}
$$

where the third identity derives from (2.6) and the period of $\widehat{P_{k+1}}$, the first inequality from Cauchy-Schwarz inequality, and the second one from

$$
\mathbb{N}_0^s \setminus \{0\} \subseteq \bigcup_{\ell=1}^{s} \{(q_1,\dots,q_s) : q_\ell \in \mathbb{N}, q_j \in \mathbb{N}_0, j \neq \ell\}.
$$

By (2.11) and (2.6), it can be proved as [9] that

$$
\begin{aligned}
&\sum_{j\in\mathcal{L}_k} \sum_{q\in\mathbb{N}_0^s\setminus\{0\}} |\widehat{\mathfrak{P}_k(f)}(j+2^k q) - \widehat{f}(j+2^k q)|^2 \\
&\le 2 \sum_{j\in\mathcal{L}_k} \sum_{q\in\mathbb{N}_0^s\setminus\{0\}} |\widehat{f}(j+2^k q)|^2 + 4|\widehat{f}(0)|^2|\widehat{\phi_k}(0)|^2 \sum_{q\in\mathbb{N}_0^s\setminus\{0\}} d^{2k}|\widehat{\phi_k}(2^k q)|^2 \\
&\quad +4 \sum_{j\in\mathcal{L}_k\setminus\{0\}} |\widehat{f}(j)|^2|\widehat{\phi_k}(j)|^2 \sum_{q\in\mathbb{N}_0^s\setminus\{0\}} d^{2k}|\widehat{\phi_k}(j+2^k q)|^2 \\
&\quad +4 \sum_{j\in\mathcal{L}_k} \sum_{p\in\mathbb{N}_0^s\setminus\{0\}} |\widehat{f}(j+2^k p)\overline{\widehat{\phi}_k(j+2^k p)}|^2 \sum_{q\in\mathbb{N}_0^s\setminus\{0\}} d^{2k}|\widehat{\phi_k}(j+2^k q)|^2.
\end{aligned}
\tag{2.14}
$$

Next we continue to estimate $\sum_{j\in\mathcal{L}_k} \sum_{q\in\mathbb{N}_0^s\setminus\{0\}} |\widehat{\mathfrak{P}_k(f)}(j+2^k q) - \widehat{f}(j+2^k q)|^2$ in

(2.14). We firstly estimate that

$$
\begin{aligned}
&2\sum_{j\in\mathcal{L}_k}\sum_{q\in\mathbb{N}_0^s\backslash\{0\}}|\widehat{f}(j+2^kq)|^2\\
&=2\sum_{j\in\mathcal{L}_k}\sum_{q\in\mathbb{N}_0^s\backslash\{0\}}||j+2^kq||^{-2\nu}||j+2^kq||^{2\nu}|\widehat{f}(j+2^kq)|^2\\
&=2^{-2k\nu+1}\sum_{j\in\mathcal{L}_k}\sum_{q\in\mathbb{N}_0^s\backslash\{0\}}||2^{-k}j+q||^{-2\nu}||j+2^kq||^{2\nu}|\widehat{f}(j+2^kq)|^2\\
&\le 2^{-2k\nu+1}\sum_{j\in\mathcal{L}_k}\sum_{q\in\mathbb{N}_0^s\backslash\{0\}}||j+2^kq||^{2\nu}|\widehat{f}(j+2^kq)|^2\\
&\le 2^{-2k\nu+1}|f|^2_{W_2^\nu(\mathbb{T}^s)}.
\end{aligned}
\tag{2.15}
$$

By

$$\widehat{\phi}_k(2^kq)=\widehat{P}_k(2^kq)\widehat{\phi}_0(2^kq)=\widehat{P}_k(0)\widehat{\phi}_0(2^kq),\forall q=(q_1,\ldots,q_s)\in\mathbb{N}_0^s,$$

we get

$$
\begin{aligned}
&4|\widehat{f}(0)|^2|\widehat{\phi_k}(0)|^2\sum_{q\in\mathbb{N}_0^s\backslash\{0\}}d^{2k}|\widehat{\phi_k}(2^kq)|^2\\
&=4|\widehat{f}(0)|^2|\widehat{\phi_k}(0)|^2|\widehat{P}_k(0)|^2d^{2k}\sum_{q\in\mathbb{N}_0^s\backslash\{0\}}|\widehat{\phi}_0(2^kq)|^2\\
&=4|\widehat{f}(0)|^2\sum_{q\in\mathbb{N}_0^s\backslash\{0\}}|\widehat{\phi_0}(2^kq)|^2\\
&=4|\widehat{f}(0)|^2\sum_{q\in\mathbb{N}_0^s\backslash\{0\}}|a_1^{2^kq_1}\cdot a_2^{2^kq_2}\ldots a_s^{2^kq_s}|^2\\
&\le 4|\widehat{f}(0)|^2\sum_{\ell=1}^{s}\sum_{\widetilde{q}_\ell=1}^{\infty}|a_\ell^{2^k\widetilde{q}_\ell}|^2\sum_{\widetilde{q}_j\in\mathbb{N}_0,j\neq\ell}|\prod_{j\neq\ell}a_j^{2^k\widetilde{q}_j}|^2\\
&\le 4|\widehat{f}(0)|^2\sum_{\ell=1}^{s}\sum_{\widetilde{q}_\ell=1}^{\infty}|a_\ell^{2^k\widetilde{q}_\ell}|^2\frac{1}{\prod_{j\neq\ell}(1-|a_j^{2^k}|^2)}\\
&\le 4|\widehat{f}(0)|^2\frac{1}{\prod_{j=1}^s(1-|a_j^{2^k}|^2)}\sum_{\ell=1}^{s}|a_\ell^{2^k}|^2,
\end{aligned}
\tag{2.16}
$$

of which the second identity derives from

$$|\widehat{\phi_k}(0)|=|\widehat{P}_k(0)||\widehat{\phi_0}(0)|=|\widehat{P}_k(0)|$$

and (2.6). The sum $4\sum_{j\in\mathcal{L}_k\backslash\{0\}}|\widehat{f}(j)|^2|\widehat{\phi_k}(j)|^2\sum_{q\in\mathbb{N}_0^s\backslash\{0\}}d^{2k}|\widehat{\phi_k}(j+2^kq)|^2$ in (2.14) is estimated by

$$
\begin{aligned}
&4\sum_{j\in\mathcal{L}_k\backslash\{0\}}|\widehat{f}(j)|^2|\widehat{\phi_k}(j)|^2\sum_{q\in\mathbb{N}_0^s\backslash\{0\}}d^{2k}|\widehat{\phi_k}(j+2^kq)|^2\\
&=4\sum_{j\in\mathcal{L}_k\backslash\{0\}}|\widehat{f}(j)|^2d^{-k}\sum_{q\in\mathbb{N}_0^s\backslash\{0\}}d^{2k}d^{-k}|a_1^{2^{k+1}q_1}a_2^{2^{k+1}q_2}\ldots a_s^{2^{k+1}q_s}|\\
&\le 4||f||_2^2\frac{1}{\prod_{j=1}^s(1-|a_j^{2^k}|^2)}\sum_{\ell=1}^{s}|a_\ell^{2^k}|^2.
\end{aligned}
\tag{2.17}
$$

Similarly,

$$\begin{aligned}
&4\sum_{j\in\mathcal{L}_k}\sum_{p\in\mathbb{N}_0^s\setminus\{0\}}|\widehat{f}(j+2^kp)\overline{\widehat{\phi}_k(j+2^kp)}|^2\sum_{q\in\mathbb{N}_0^s\setminus\{0\}}d^{2k}|\widehat{\phi_k}(j+2^kq)|^2\\
&=4\sum_{j\in\mathcal{L}_k}\sum_{p\in\mathbb{N}_0^s\setminus\{0\}}\sum_{q\in\mathbb{N}_0^s\setminus\{0\}}|\widehat{f}(j+2^kp)|^2||a_1^{2^kp_1}a_2^{2^kp_2}\dots a_s^{2^kp_s}|^2|a_1^{2^kq_1}a_2^{2^kq_2}\dots a_s^{2^kq_s}|^2\\
&\le 4||f||_2^2\frac{1}{\prod_{j=1}^s(1-|a_j^{2^k}|^2)}\sum_{\ell=1}^s|a_\ell^{2^k}|^2.
\end{aligned}\tag{2.18}$$

Now we deduce from(2.12)-(2.18) that (2.8) holds. □

Algorithm 2.2.

Step 1: Let $\phi_0(e^{it})=\prod_{j=1}^s\frac{1}{(1-\overline{a}_je^{it_j})}$ with $a_j\in\mathbb{D}\backslash\{0\}$. Construct $\widehat{P_{k+1}}=\{\widehat{P_{k+1}}(n)\}_{n\in\mathcal{L}_{k+1}}\in\mathcal{S}(2^{k+1}I_s)$, $k\in\mathbb{N}_0$, such that $|\widehat{P_{k+1}}(n)|=\frac{d^{-(k+1)/2}}{|\widehat{\phi}_0(n)|}=\frac{d^{-(k+1)/2}}{|a_1^{n_1}\cdot a_2^{n_2}\cdots a_s^{n_s}|}$, where $n=(n_1,n_2,\dots,n_s)$.

Step 2: Implementing inverse discrete Fourier transform on $\widehat{P_{k+1}}$ by (2.4), we get $P_{k+1}=\{P_{k+1}(n)\}_{n\in\mathcal{L}_{k+1}}$.

Step 3: For any $f\in\mathcal{H}^2(\mathbb{T}^s)\cap W_2^\nu(\mathbb{T}^s)$, $\nu>0$, it can be approximated by (2.7), where $\mathbf{a}=(a_1,\dots,a_s)$ and $\circ$ is defined in (2.1). The approximating error $||f-\mathfrak{P}_k(f)||_2$ is given by (2.8).

Numerical Experiment 2.3.

In this experiment, we use Algorithm 2.2 to approximate

$$f(e^{it})=\frac{1+g_1e^{it_1}}{1.5-be^{i(t_1+t_2)}+g_2e^{i(t_1+2t_2)}},t=(t_1,t_2)\in[0,2\pi)^2,g_1\in\mathbb{C},g_2,b\in\mathbb{D},$$

by the following procedures holding for approximation to other signals in $\mathcal{H}^2(\mathbb{T}^2)$.

Step 1: Select

$$\phi_0(e^{it})=\frac{1}{(1-\overline{a}_1e^{it_1})(1-\overline{a}_2e^{it_2})},$$

where $a_1,a_2\in\mathbb{D}\setminus\{0\}$. Obviously, $\mathcal{L}_k=\{(n_1,n_2):n_1,n_2\in\{0,1,\dots,2^k-1\}\}$. Construct $\widehat{P_k}=\{\widehat{P_k}(n)\}_{n\in\mathcal{L}_k}\in\mathcal{S}(2^kI_2)$, $k\in\mathbb{N}_0$, by

$$\widehat{P_k}(n)=\frac{4^{-k/2}}{\overline{a}_1^{n_1}\overline{a}_2^{n_2}}.\tag{2.19}$$

Step 2: Implementing IDFT on $\widehat{P_k}$ by (2.4), we get $P_k=\{P_k(n)\}_{n\in\mathcal{L}_k}$.

Step 3: For sufficiently large k, f can be approximated by

$$\mathfrak{P}_k(f)=\sum_{\ell\in\mathcal{L}_k}\sum_{\ell'\in\mathcal{L}_k}\sum_{\ell''\in\mathcal{L}_k}\overline{P_k(\ell'')}P_k(\ell')f(\mathbf{a}\circ e^{i2\pi2^{-k}(\ell''+\ell'+\ell)})\phi_0(e^{i(\cdot-2\pi2^{-k}(\ell+\ell'))}),\tag{2.20}$$

where $\mathbf{a}=(a_1,a_2)$.

Table 1 shows the approximating error ratios

$$ratio1 = \frac{||f - \mathfrak{P}_k(f)||_2}{||f||_2},\ ratio2 = \frac{||\mathrm{Re}(f - \mathfrak{P}_k(f))||_2}{||\mathrm{Re}(f)||_2},\ ratio3 = \frac{||\mathrm{Im}(f - \mathfrak{P}_k(f))||_2}{||\mathrm{Im}(f)||_2}$$

corresponding to different choices of g_1, g_2, b, k, a_1 and a_2, where $||\cdot||_2$ is approximatively computed by

$$||F||_2 \approx \frac{2\pi}{80}\Big(\sum_{j=0}^{79}\sum_{\ell=0}^{79}|F(e^{i\frac{2\pi}{80}j}, e^{i\frac{2\pi}{80}\ell})|^2\Big)^{1/2}, F \in \mathcal{H}^2(\mathbb{T}^2),$$

and Re and Im are the real and imaginary parts, respectively. See Figure 1 and Figure 2 for the graphs of $\mathrm{Re}(f)$, $\mathrm{Im}(f)$, $\mathrm{Re}(\mathfrak{P}_k(f))$ and $\mathrm{Im}(\mathfrak{P}_k(f))$ corresponding to the third and last rows of Table 1.

g_1	g_2	b	ratio1	ratio2	ratio3	k	a_1	a_2
1	0.3	0.2778	0.0577	0.0391	0.1103	1	0.0672	0.0110
1	0.3	0.2778	0.0020	0.0014	0.0038	2	0.0272	0.0110
0	0	0.6667	0.1975	0.1270	0.4444	1	0.0056	0.0053
0	0	0.6667	0.0390	0.0251	0.0878	2	0.0056	0.0053
0	0	0.6667	0.0015	0.0010	0.0035	3	0.0056	0.0053

Table 1. Every row of this table shows the approximating error ratios corresponding to different choices of g_1, g_2, b, k, a_1, and a_2.

2.2. *Application: decomposing a signal into mono-component by analytic samples*

Next based on Theorem 2.1, we shall approximately decompose a signal in $\mathcal{H}^2(\mathbb{T})$ into mono-components. We first investigate the decomposition of the Cauchy kernel on $\mathbb{D}$. For any $a \in \mathbb{D} \setminus \{0\}$, the function $\frac{1}{1-ae^{it}}, t \in [0, 2\pi)$, can be decomposed by

$$\frac{1}{1-ae^{it}} = \frac{1}{\overline{a}}\Big(-\frac{e^{it}-\overline{a}}{1-ae^{it}} + \frac{e^{it}}{1-ae^{it}}\Big). \tag{2.21}$$

Recall that $\frac{e^{it}-\overline{a}}{1-ae^{it}}$ is the boundary value on unit circle of Möbius transform, and is a mono-component [20,22]. Precisely, $\frac{e^{it}-\overline{a}}{1-ae^{it}} = e^{i\theta_a(t)}$ with

$$\theta_a(t) = t + 2\mathrm{arctan}\frac{|a|\sin(t-\gamma_a)}{1-|a|\cos(t-\gamma_a)}, \theta_a'(t) = \frac{1-|a|^2}{1-2|a|\cos(t-\gamma_a)+|a|^2},$$

where $t \in [0, 2\pi)$ and $a = |a|e^{i\gamma_a}$. On the other hand, $\frac{e^{it}}{1-ae^{it}}$ is a starlike function, in particular, a convex function, and a mono-component as well [22]. Calculate

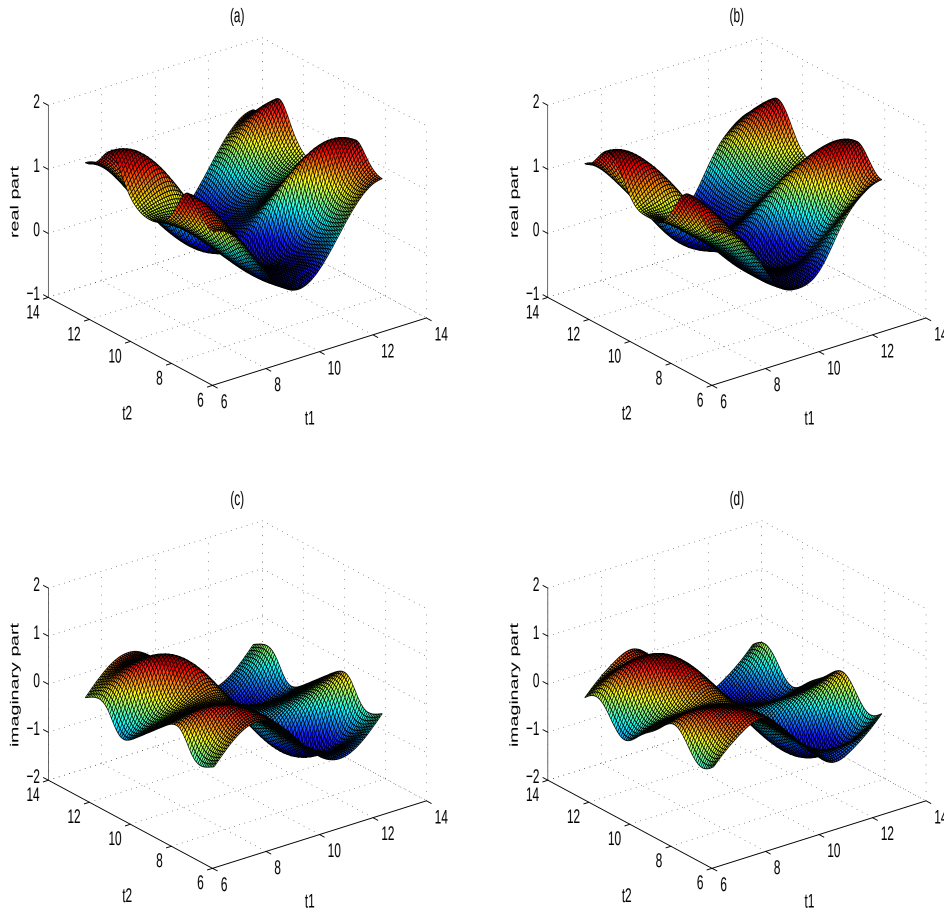

Fig. 1. The graphs of $\mathrm{Re}(f)$, $\mathrm{Re}(\mathfrak{P}_k(f))$, $\mathrm{Im}(f)$, and $\mathrm{Im}(\mathfrak{P}_k(f))$ corresponding to the third row of Table 1 are shown on (a)-(d), respectively.

$$
\begin{aligned}
\frac{e^{it}}{1-ae^{it}} &= \frac{e^{it}-\overline{a}}{|1-ae^{it}|^2} \\
&= \frac{e^{it}-|a|e^{-i\gamma_a}}{|1-ae^{it}|^2} \\
&= \frac{|e^{it}-\overline{a}|e^{i\Theta_a(t)}}{|1-ae^{it}|^2} \\
&= \frac{e^{i\Theta_a(t)}}{|1-ae^{it}|},
\end{aligned}
$$

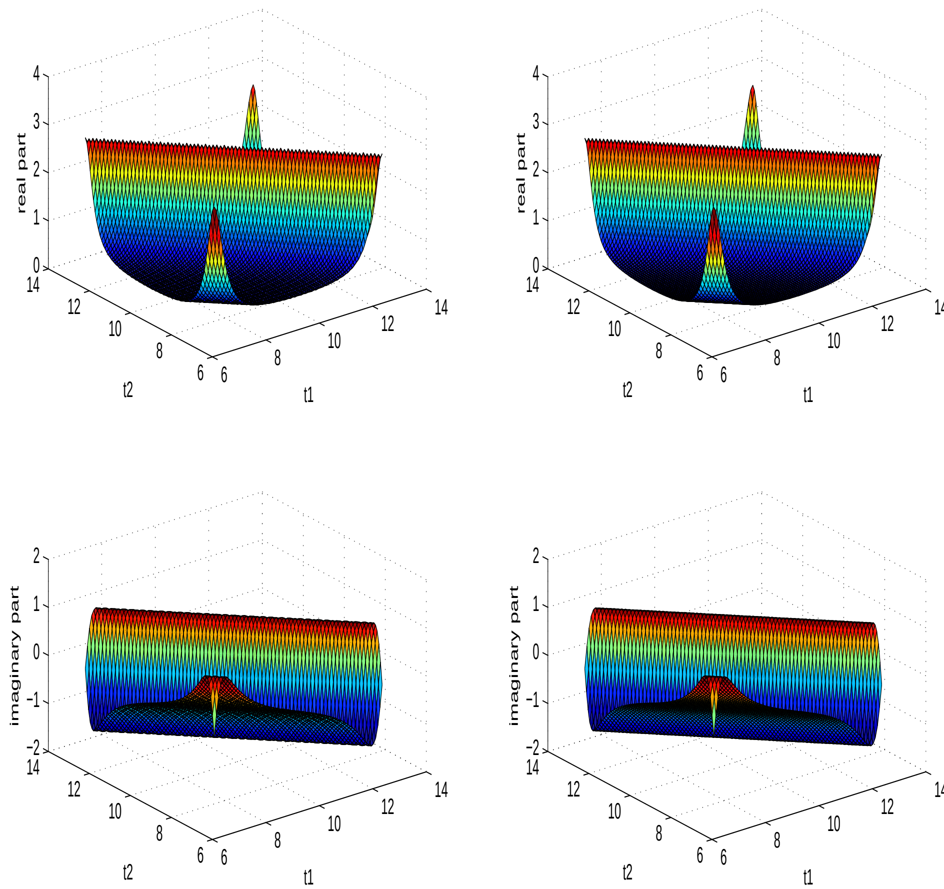

Fig. 2. The graphs of Re(f), Re$(\mathfrak{P}_k(f))$, Im(f), and Im$(\mathfrak{P}_k(f))$ corresponding to the last row of Table 1 are shown on (a)-(d), respectively.

where $\Theta_a(t) = \arctan \frac{\sin t + |a| \sin \gamma_a}{\cos t - |a| \cos \gamma_a}$, and $\Theta'_a(t) = \frac{1-|a|\cos(t+\gamma_a)}{1-2|a|\cos(t+\gamma_a)+|a|^2}$, $t \in [0, 2\pi)$. See Figures 3 and 4 for the graphs of $\theta'_{e^{i2\pi 2^{-j}/2^3}}(t)$ and $\Theta'_{e^{i2\pi 2^{-j}/2^3}}(t)$, $j = 0, 1, \ldots, 7$. In a word, the Cauchy kernel on $\mathbb{D}$ can be decomposed into two mono-components by (2.21).

Denote

$$\frac{1}{(1-\overline{a}e^{it})} = \phi_a^1(e^{it}) + \phi_a^2(e^{it})$$

with $\phi_a^1(e^{it}) = -\frac{e^{it}-a}{a(1-\overline{a}e^{it})}$ and $\phi_a^2(e^{it}) = \frac{e^{it}}{a(1-\overline{a}e^{it})}$. Since $\mathfrak{P}_k(f)$ in (2.10) tends to f

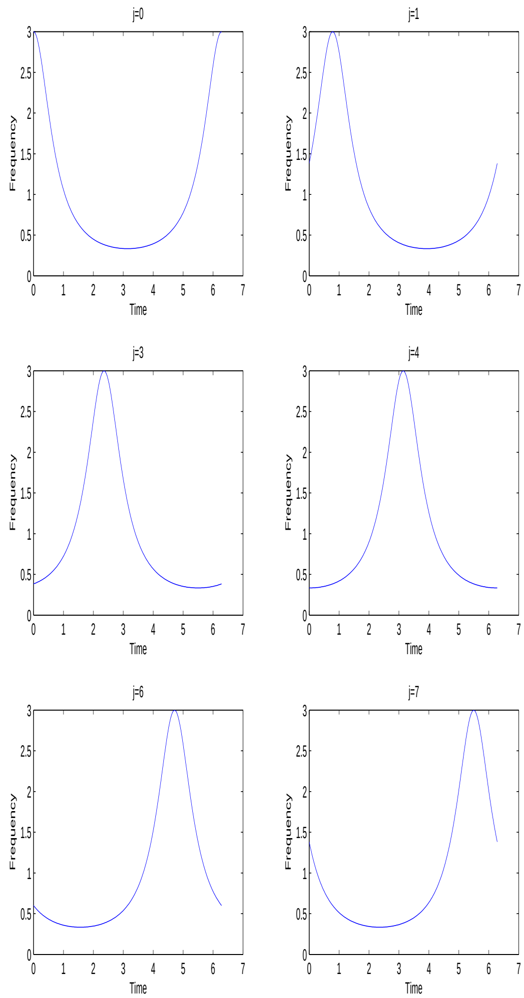

Fig. 3. The graphs of $\theta'_{e^{i2\pi 2^{-j}/2^3}}(t)$.

as $k \to \infty$, we regard $\mathfrak{P}_k(f)$ as the approximate decomposition of f for sufficiently large k. That is, f is approximately decomposed into mono-components as follow

$$f(e^{it}) \approx \sum_{\ell \in \mathcal{L}_k} \sum_{\ell'' \in \mathcal{L}_k} \sum_{\ell' \in \mathcal{L}_k} \sum_{n=1}^{2} \overline{P_k(\ell'')} P_k(\ell') f(ae^{i2\pi 2^{-k}(\ell''+\ell'+\ell)}) T_k^{\ell} T_k^{\ell'} \left(\phi_a^n(e^{it}) \right) \tag{2.22}$$

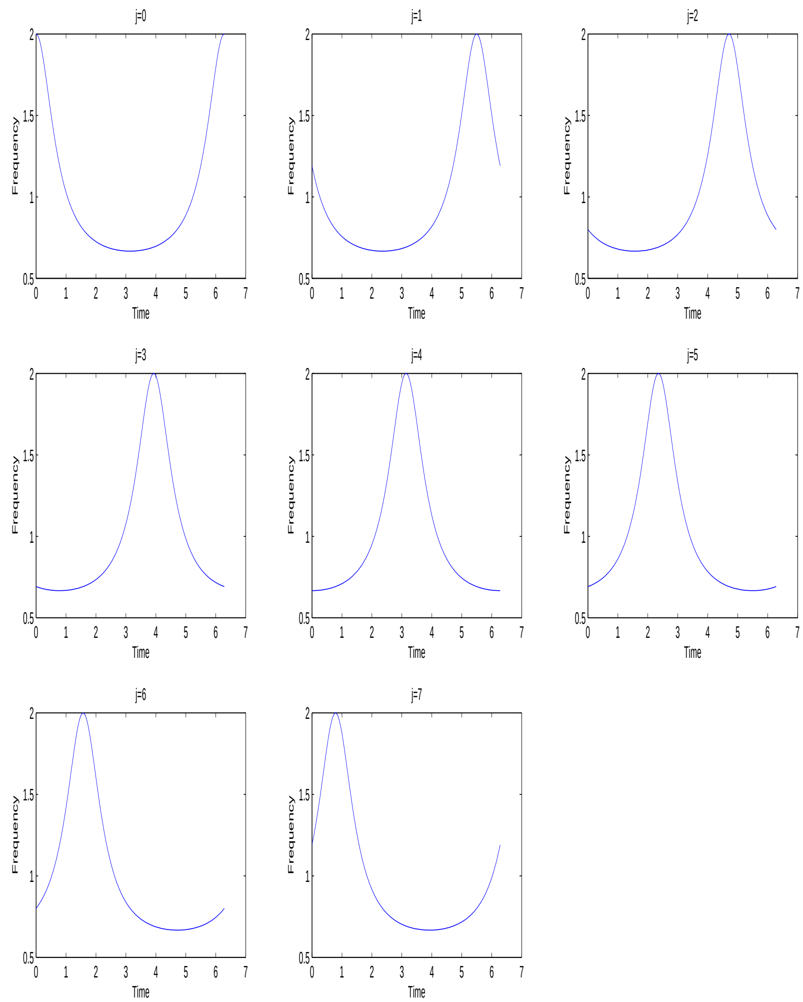

Fig. 4. The graphs of $\Theta'_{e^{i2\pi 2^{-j}/2^3}}(t)$.

for sufficiently large k. The approximating error of (2.22) is estimated by (2.8).

3. Analytic sampling approximation to a signal in $L^2(\mathbb{T}^s)$

Making use of Hilbert transform H defined in (2.2), we shall apply the approximation scheme in Theorem 2.1 to the setting of $L^2(\mathbb{T}^s)$. Recall that Hilbert transform has the alternative Fourier series expression

$$Hf(t) = \sum_{k\in\mathbb{Z}} -i\mathrm{sgn}(k)\widehat{f}(k)e^{ikt},\ t \in [0, 2\pi), \forall f \in L^2(\mathbb{T}). \tag{3.1}$$

Since we treat s real variables, we denote by $H^{(j)}$ the Hilbert transform with respect to the jth variable t_j, namely, for $t = (t_1, t_2, \dots, t_s) \in [0, 2\pi)^s$,

$$H^{(j)}f(t) = \frac{1}{2\pi} p.v. \int_0^{2\pi} f(t_1, \dots, t_{j-1}, x, t_{j+1}, \dots, t_s) \cot \frac{t_j - x}{2} dx,\ \forall f \in L^2(\mathbb{T}^s),$$

$j = 1, \dots, s$. Now we define the projection operators

$$\begin{aligned} P_1^{(j)} &= \tfrac{1}{2} f + i\tfrac{1}{2} H^{(j)} f(t) = \tfrac{1}{2}(I + iH^{(j)}), \\ P_{-1}^{(j)} &= \tfrac{1}{2} f - i\tfrac{1}{2} H^{(j)} f(t) = \tfrac{1}{2}(I - iH^{(j)}), \end{aligned}$$

where I is the identity projection. It is straightforward to see from (3.1) that the Fourier series of $P_1^{(j)} f$ and $P_{-1}^{(j)} f$ take the form

$$\left.\begin{aligned} P_1^{(j)} f(e^{it}) &= \sum_{n_j=0}^{\infty} \sum_{n_1\in\mathbb{Z}} \cdots \sum_{n_{j-1}\in\mathbb{Z}} \sum_{n_{j+1}\in\mathbb{Z}} \cdots \sum_{n_s\in\mathbb{Z}} c_n e^{in\cdot t}, \\ P_{-1}^{(j)} f(e^{it}) &= \sum_{n_j=-\infty}^{0} \sum_{n_1\in\mathbb{Z}} \cdots \sum_{n_{j-1}\in\mathbb{Z}} \sum_{n_{j+1}\in\mathbb{Z}} \cdots \sum_{n_s\in\mathbb{Z}} d_n e^{in\cdot t}, \end{aligned}\right. \quad n = (n_1, n_2, \dots, n_s).$$

Now we accordingly have the following expression

$$\begin{aligned} f &= \prod_{j=1}^{s} (P_1^{(j)} + P_{-1}^{(j)}) f \\ &= \sum_{\epsilon_j\in\{1,-1\}, j=1,\dots,s} P_{\epsilon_1}^{(1)} P_{\epsilon_2}^{(2)} \dots P_{\epsilon_s}^{(s)} f. \end{aligned} \tag{3.2}$$

Theorem 3.1. *For any* $f \in W_2^\nu(\mathbb{T}^s)$, $\nu > 0$, $\boldsymbol{a} = (a_1, a_2, \dots, a_s) \in \mathbb{D}^s$ *with* $a_j \neq 0, j = 1, 2, \dots, s$, *we can construct rational polynomials* $\mathbf{P}_k(f), k \in \mathbb{N}_0$, *such that* $\mathbf{P}_k(f)$ *tends to* f *as* $k \to \infty$, *and the approximating error is estimated by*

$$\begin{aligned} ||\mathbf{P}_k(f) - f||_2^2 \leq 2^{s+1}|f|^2_{W_2^\nu(\mathbb{T}^s)} 2^{-2k\nu} &+ \frac{12|\widehat{f}(0)|^2 + 2^{s+3}||f||_2^2}{\prod_{j=1}^s (1 - |a_j^{2^k}|^2)} \sum_{\ell=1}^{s} |a_\ell^{2^k}|^2 \\ &+ \frac{|\widehat{f}(0)|^2 + 2^s||f||_2^2}{\prod_{j=1}^s (1 - |a_j^{2^k}|^2)} \sum_{\ell=1}^{s} |a_\ell^{2^k}|^2. \end{aligned}$$

Moreover, $\mathbf{P}_k(f)$ *can be expressed by the analytic samples of* $P_{\epsilon_1}^{(1)} P_{\epsilon_2}^{(2)} \dots P_{\epsilon_s}^{(s)} f$ *defined in* (3.2), *where* $\epsilon_j \in \{1, -1\}, j = 1, \dots, s$.

Proof. Let $f_{\epsilon_1,\dots,\epsilon_s}(t_1, \dots, t_s) \triangleq P_{\epsilon_1}^{(1)} P_{\epsilon_2}^{(2)} \dots P_{\epsilon_s}^{(s)} f(\epsilon_1 t_1, \dots, \epsilon_s t_s)$. Obviously, $f_{\epsilon_1,\dots,\epsilon_s}(t_1, \dots, t_s) \in \mathcal{H}^2(\mathbb{T}^s)$. By Algorithm 2.2, we can construct $\mathfrak{P}_k(f_{\epsilon_1,\dots,\epsilon_s}) \in \mathcal{H}^2(\mathbb{T}^s)$ such that

$$\mathfrak{P}_k(f_{\epsilon_1,\dots,\epsilon_s})[t_1, \dots, t_s] \to f_{\epsilon_1,\dots,\epsilon_s}[t_1, \dots, t_s] \text{ as } k \to \infty, \tag{3.3}$$

and the approximating error is given by (2.8) with f being replaced by $f_{\epsilon_1,\ldots,\epsilon_s}$. Precisely,

$$||f_{\epsilon_1,\ldots,\epsilon_s} - \mathfrak{P}_k(f_{\epsilon_1,\ldots,\epsilon_s})||_2^2 \le 2|f_{\epsilon_1,\ldots,\epsilon_s}|^2_{W_2^\nu(\mathbb{T}^s)} 2^{-2k\nu} + \frac{||f_{\epsilon_1,\ldots,\epsilon_s}||_2^2}{\prod_{j=1}^s (1-|a_j^{2^k}|^2)} \sum_{\ell=1}^s |a_\ell^{2^k}|^2 + \frac{4(|\widehat{f_{\epsilon_1,\ldots,\epsilon_s}}(0)|^2 + 2||f_{\epsilon_1,\ldots,\epsilon_s}||_2^2)}{\prod_{j=1}^s (1-|a_j^{2^k}|^2)} \sum_{\ell=1}^s |a_\ell^{2^k}|^2. \tag{3.4}$$

Replacing t_j with $\epsilon_j t_j, j = 1, \ldots, s$, gives us that

$$\mathfrak{P}_k(f_{\epsilon_1,\ldots,\epsilon_s})[\epsilon_1 t_1, \ldots, \epsilon_s t_s] \to P_{\epsilon_1}^{(1)} P_{\epsilon_2}^{(2)} \ldots P_{\epsilon_s}^{(s)} f(t_1, \ldots, t_s) \text{ when } k \to \infty \tag{3.5}$$

with the error estimation being the same as in (3.4). Denote

$$\mathbf{P}_k(f) = \sum_{\epsilon_j \in \{1,-1\}, j=1,\ldots,s} \mathfrak{P}_k(f_{\epsilon_1,\ldots,\epsilon_s})[t_1, \ldots, t_s].$$

It follows from (3.2), (3.5), and (3.4) that $\mathbf{P}_k(f) \to f$ when $k \to \infty$, and

$$\begin{aligned}
&||f - \mathbf{P}_k(f)||_2^2 \\
&\le 2^s \sum_{\epsilon_j \in \{1,-1\}, j=1,\ldots,s} ||f_{\epsilon_1,\ldots,\epsilon_s} - \mathfrak{P}_k(f_{\epsilon_1,\ldots,\epsilon_s})||_2^2 \\
&\le 2^{s+1} |f|^2_{W_2^\nu(\mathbb{T}^s)} 2^{-2k\nu} + \frac{4|\widehat{f}(0)|^2 + 8(|\widehat{f}(0)|^2 + 2^s||f||_2^2)}{\prod_{j=1}^s (1-|a_j^{2^k}|^2)} \sum_{\ell=1}^s |a_\ell^{2^k}|^2 \\
&\quad + \frac{(|\widehat{f}(0)|^2 + 2^s||f||_2^2)}{\prod_{j=1}^s (1-|a_j^{2^k}|^2)} \sum_{\ell=1}^s |a_\ell^{2^k}|^2 \\
&= 2^{s+1} |f|^2_{W_2^\nu(\mathbb{T}^s)} 2^{-2k\nu} + \frac{12|\widehat{f}(0)|^2 + 2^{s+3}||f||_2^2}{\prod_{j=1}^s (1-|a_j^{2^k}|^2)} \sum_{\ell=1}^s |a_\ell^{2^k}|^2 \\
&\quad + \frac{|\widehat{f}(0)|^2 + 2^s||f||_2^2}{\prod_{j=1}^s (1-|a_j^{2^k}|^2)} \sum_{\ell=1}^s |a_\ell^{2^k}|^2,
\end{aligned}$$

where we use $\widehat{f_{\epsilon_1,\ldots,\epsilon_s}}(0) = \widehat{f}(0)/2^s$ and $\sum_{\epsilon_j \in \{1,-1\}, j=1,\ldots,s} ||f_{\epsilon_1,\ldots,\epsilon_s}||_2^2 \le |\widehat{f}(0)|^2/2^s + ||f||_2^2$ in the second inequality.

Recall that $P_{\epsilon_1}^{(1)} P_{\epsilon_2}^{(2)} \ldots P_{\epsilon_s}^{(s)} f$ is analytic on $\bigotimes_{j=1}^s \mathfrak{P}(\epsilon_j)$, where the map $\mathfrak{P}$ is defined by $\mathfrak{P}(1) = \mathbb{D}$ and $\mathfrak{P}(-1) = \mathbb{C} \setminus (\mathbb{D} \cup \{0\})$. It follows from (2.20) that $\mathfrak{P}_k(f_{\epsilon_1,\ldots,\epsilon_s})[t_1, \ldots, t_s]$ in (3.3) can be expressed by its analytic samples on $\bigotimes_{j=1}^s \mathfrak{P}(\epsilon_j)$. Precisely,

$$\begin{aligned}
&\mathfrak{P}_k(f_{\epsilon_1,\ldots,\epsilon_s}) \\
&= \sum_{\ell \in \mathcal{L}_k} \sum_{\ell' \in \mathcal{L}_k} \sum_{\ell'' \in \mathcal{L}_k} \Big[\overline{P_k^{(\epsilon_1,\ldots,\epsilon_s)}(\ell'')} P_k^{(\epsilon_1,\ldots,\epsilon_s)}(\ell') P_{\epsilon_1}^{(1)} \ldots P_{\epsilon_s}^{(s)} f((\mathbf{a} \circ e^{i2\pi 2^{-k}(\ell''+\ell'+\ell)})^{(\epsilon_1,\ldots,\epsilon_s)}) \\
&\qquad \times \phi_0((e^{i(\cdot - 2\pi 2^{-k}(\ell+\ell'))})^{(\epsilon_1,\ldots,\epsilon_s)}) \Big],
\end{aligned}$$

where $\mathbf{g}^{(\epsilon_1,\ldots,\epsilon_s)} = (g_1^{\epsilon_1},\ldots,g_s^{\epsilon_s})$ for any $\mathbf{g} = (g_1,\ldots,g_s) \in \mathbb{C}^s$, $P_k^{(\epsilon_1,\ldots,\epsilon_s)}$ and $P_k^{(\epsilon_1,\ldots,\epsilon_s)}$ are some sequences in $\mathcal{S}(2^{k+1}I_s)$ constructed by Algorithm 2.2. Therefore,

$$\mathbf{P}_k(f) = \sum_{\epsilon_j\in\{1,-1\},j=1,\ldots,s}\sum_{\ell\in\mathcal{L}_k}\sum_{\ell''\in\mathcal{L}_k}\sum_{\ell'\in\mathcal{L}_k}\Big[\overline{P_k^{(\epsilon_1,\ldots,\epsilon_s)}(\ell'')}P_k^{(\epsilon_1,\ldots,\epsilon_s)}(\ell') \times P_{\epsilon_1}^{(1)}\ldots P_{\epsilon_s}^{(s)}f\big((\mathbf{a}\circ e^{i2\pi 2^{-k}(\ell''+\ell'+\ell)})^{(\epsilon_1,\ldots,\epsilon_s)}\big)\phi_0\big((e^{i(\cdot-2\pi 2^{-k}(\ell+\ell'))})^{(\epsilon_1,\ldots,\epsilon_s)}\big)\Big],$$

namely, $\mathbf{P}_k(f)$ can be expressed by the analytic samples of $P_{\epsilon_1}^{(1)}P_{\epsilon_2}^{(2)}\ldots P_{\epsilon_s}^{(s)}f$. □

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