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Shannon-type sampling for multivariate non-bandlimited signals

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Abstract In this paper, starting from a function analytic in a neighborhood of the unit disk and based on Bessel functions, we construct a family of generalized multivariate sinc functions, which are radial and named radial Bessel-sinc (RBS) functions being time-frequency atoms with nonlinear phase. We obtain a recursive formula for the RBS functions in $\mathbb{R}^d$ with d being odd. Based on the RBS function, a corresponding sampling theorem for a class of non-bandlimited signals is established. We investigate a class of radial functions and prove that each of these functions can be extended to become a monogenic function between two parallel planes, where the monogencity is taken to be of the Clifford analysis sense.

Keywords analytic function, Fourier transform, radial Bessel-Sinc function, Shannon sampling

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1 Introduction

Sinc function $\mathrm{sinc}(t) = \frac{\sin t}{t}, t \in \mathbb{R}$, introduced by Woodward [14], is fundamental in digital signal processing and information theory due to the Shannon sampling theorem for reconstructing a bandlimited signal [5, 8]. The Fourier transform of sinc is the characteristic function of $[-1, 1]$, which is well-known as an ideal low-pass filter leading to the multi-resolution analysis of Haar and Shannon wavelets [7, 15, 16].

Shannon sampling holds only for bandlimited signals and as such ones naturally ask whether it is possible, and if yes, how to construct "sinc functions" for reconstructing non-bandlimited signals by their samples? This problem of one dimensional case has been completely solved by [2, 3]. In this paper, we focus on the higher dimensional case. Incidentally, our theory has the following three features.

(I) It presents a case where a Shannon-type sampling holds for non-bandlimited signals. Moreover, the classical Shannon sampling is imbedded in our theory, more details about which can be seen in Example 1 of Section 2.

(II) The common method to construct a sinc function is tensor product [9, 10]. However, the sinc function in this paper is constructed through time-frequency atoms with **nonlinear phases**, namely, it is radial.

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(III) Our theory is closely related to analytical signal theory. The notions of the instantaneous amplitude (IA), instantaneous phase (IP) and instantaneous frequency (IF) are used in many applications to measure and detect local details of a signal. A commonly acceptable approach to define them is through Hilbert transform, which leads to the theory of analytical signal. An important fact about IA and IP is as follows. When an IP is taken from the boundary value of a function analytic in a neighborhood of the closed unit disk, the corresponding IA is in the even-shift-invariant space generated by the integer-shifts of a generalized sinc function and the even-integer-shifts of the Hilbert transform of the generalized sinc function [1]. The fact inspires us to investigate the corresponding result in the multivariate case. The present paper is, to our best of knowledge, the first touch of higher dimensional analytic signals related to sinc functions. Our theory offers a potential in applications such as designing filters.

Our main idea is sketched as follows. We divide $\mathbb{R}^d$ into a cascade collection of the annulus $\{S_n : n\in\mathbb{Z}_+\}$ defined by

$$S_n := B_{(0,n+1)} \setminus B_{(0,n)} \tag{1.1}$$

with $B_{(0,0)} = \emptyset$ and $B(0,\sigma)$ being the ball in $\mathbb{R}^d$ centered at 0 with radius σ. We define even functions such that they are piecewise constants in $\{S_n : n \in \mathbb{Z}_+\}$. These constants correspond to the coefficients of power series expansion of a function analytic in a neighborhood of unit circle. Now the generalized sinc function is defined to be the Fourier transform of the piecewise constant function mentioned above, where the Fourier transform of any $f \in L^2(\mathbb{R}^d)$ is defined by

$$\hat{f}(\underline{\xi}) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(\underline{t}) \mathrm{e}^{-\mathrm{i}\langle \underline{\xi},\underline{t}\rangle} d\underline{t}, \quad \underline{\xi} \in \mathbb{R}^d$$

with

$$\langle \underline{\xi}, \underline{t}\rangle = \sum_{j=1}^{d} \xi_j t_j,$$

for $\underline{\xi} = (\xi_1, \ldots, \xi_d), \underline{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d$. We will see in Section 3 that the generalized sinc functions are closely related to Bessel function. For better understanding the generalized sinc function, we need more details about the Bessel function. Recall that the Bessel function $J_\alpha(z)$, $\alpha \in \mathbb{C}$, is defined to be

$$J_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\alpha+2k}}{k!\Gamma(k+\alpha+1)}, \quad z \in \mathbb{C} \tag{1.2}$$

with Γ being the Gamma function on $\mathbb{C}$. It is well known that $J_\alpha(z)$ satisfies Bessel's differential equation

$$z^2 \frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - \alpha^2)y = 0.$$

For a complex number α with $\mathrm{Re}(\alpha) > -1/2$, it holds that

$$J_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(1/2)\Gamma(\alpha+1/2)} \int_0^\pi \cos(z\cos\theta)\sin^{2\alpha}\theta d\theta.$$

In particular,

$$J_{n+\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}}(-1)^n z^{n+\frac{1}{2}} \mathcal{D}_z^n(\mathrm{sinc}(z)) \tag{1.3}$$

with the operator $\mathcal{D}$ defined by

$$\mathcal{D}_t = \frac{1}{t}\frac{d}{dt}.$$

Readers are referred to [13, 3.1.8, 3.1.1 and 3.3.1] for more materials on a Bessel function.

We need more notations for convenient narration. If $f \in L^2(\mathbb{R}^d)$ and supp $\hat{f} \subset B(0,\sigma)$, then f is called a *bandlimited function with bandwidth* σ. Denote the set of positive integers by $\mathbb{N}$ and the one of nonnegative integers by $\mathbb{Z}_+$. Let $\mathbb{N}_m = \{1, 2, \ldots, m\}$ and $\mathbb{Z}^d$ be the d-dimensional integer lattice. The unit sphere in $\mathbb{R}^d$ is denoted by $S^{d-1} = \{\underline{x} \in \mathbb{R}^d : |\underline{x}| = 1\}$, where $|\cdot| = \langle \cdot, \cdot \rangle^{1/2}$. A function $f(\underline{t}), \underline{t} \in \mathbb{R}^d$, is said to be radial if there exists a univariate function $F(r), r \in [0, \infty)$, such that $f(\underline{t}) = F(|\underline{t}|)$ for any $\underline{t} \in \mathbb{R}^d$.

The structure of this paper is organized as follows. Two approaches are given in Section 2 to construct generalized sinc functions in univariate case, one is Fourier transform approach, the other is boundary-value approach of analytic function. Section 3 aims at constructing radial-Bessel-sinc functions and addressing their properties. Since they are dimensionally dependent, we establish a recursive formula in odd dimensional case. In Section 4, generalized sinc functions are applied to the linear time-invariant system and sampling for multivariate non-bandlimited signals. A Clifford monogenic extension result for generalized sinc functions is established in Section 5.

2 Two approaches to construct univariate sinc functions

Definition 2.1. Let $\mathcal{R}(\Delta)$ be the set of functions analytic in some neighborhood of the closed unit disk $\overline{\Delta}$, real-valued on the real axis and normalized such that any $G \in \mathcal{R}(\Delta)$ satisfies $G(1) = 1$ and $G'(1) \neq 0$, where Δ is the open unit disk.

For any $G \in \mathcal{R}(\Delta)$, define a complex-valued function g by

$$g(z) = G(\mathrm{e}^{\mathrm{i}z}), \quad z \in \mathbb{C}, \tag{2.1}$$

then introduce functions u_G and v_G on $\mathbb{R}$ via the boundary value on $\mathbb{R}$ of $g(z)$ by

$$g(t) = u_G(t) + \mathrm{i}v_G(t), \quad t \in \mathbb{R}. \tag{2.2}$$

The fact that G being analytic in some neighborhood of $\overline{\Delta}$ leads to a power series expansion

$$G(z) = \sum_{k \in \mathbb{Z}_+} g_k z^k \tag{2.3}$$

with g_k real for $k \in \mathbb{Z}_+$. It is easy to check that there exists constants c (> 0) and $\lambda \in (0, 1)$ such that for any $k \in \mathbb{Z}_+$, it holds

$$|g_k| \leqslant c\lambda^k,$$

namely, the sequence $\{g_k : k \in \mathbb{Z}_+\}$ decays exponentially. Now u_G and v_G can be also expressed by

$$v_G(t) = \sum_{k \in \mathbb{N}} g_k \sin(kt), \quad u_G(t) = \sum_{k \in \mathbb{Z}_+} g_k \cos(kt), \quad t \in \mathbb{R},$$

which implies that both u_G and v_G are 2π-periodic and in $C^\infty(\mathbb{R})$. Direct observation on (2.2) gives us

$$1 = G(1) = g(2\pi n) = u_G(2\pi n) + \mathrm{i}v_G(2\pi n), \quad \forall\, n \in \mathbb{Z}. \tag{2.4}$$

Hence,

$$v_G(2\pi n) = 0, \quad u_G(2\pi n) = 1, \tag{2.5}$$

of which the last relation equals

$$\sum_{k=0}^{\infty} g_k = 1. \tag{2.6}$$

Next, we introduce the first approach to construct generalized sinc function.

Definition 2.2. For any $G \in \mathcal{R}(\Delta)$, we define the generalized sinc function $\mathrm{sinc}_G : \mathbb{R} \to \mathbb{R}$ and cosinc function $\mathrm{cosinc}_G : \mathbb{R} \to \mathbb{R}$ by

$$\mathrm{sinc}_G(t) = \frac{v_G(t)}{t}, \quad t \in \mathbb{R} \tag{2.7}$$

and

$$\mathrm{cosinc}_G(t) = \frac{1 - u_G(t)}{t}, \quad t \in \mathbb{R}. \tag{2.8}$$

By (2.5), it is easy to check that sinc_G and cosinc_G both belong to $L^2(\mathbb{R})$. Functions $\mathrm{sinc}_{\boldsymbol{a}}$ and $\mathrm{cosinc}_{\boldsymbol{a}}$ constructed in [2] correspond to $G = B_{\boldsymbol{a}}$, the Blaschke product defined by

$$B_{\boldsymbol{a}}(z) = \frac{z - a}{1 - \overline{a}z}, \quad a \in \Delta.$$

For better understanding sinc_G and cosinc_G, we explain more about v_G and u_G. For any $G \in \mathcal{R}(\Delta)$, define $F \in H^2(\Delta)$ by

$$F(z) = \frac{G(z) - 1}{z - 1}, \quad z \in \Delta, \tag{2.9}$$

which can be expressed as the power series

$$F(z) = \sum_{k \in \mathbb{Z}_+} c_k z^k, \quad z \in \mathbb{C}. \tag{2.10}$$

It is easy to see that

$$v_G(t) = \sum_{k \in \mathbb{N}} (c_{k-1} - c_k) \sin(kt) \tag{2.11}$$

and

$$u_G(t) = 1 - c_0 + \sum_{k \in \mathbb{N}} (c_{k-1} - c_k) \cos(kt). \tag{2.12}$$

Direct calculation gives us

$$c_n = 1 - (g_0 + \cdots + g_n) = \sum_{j=n+1}^{\infty} g_j$$

implying c_n satisfies the same exponential decay estimate as g_n, i.e.,

$$|c_k| \leqslant c\lambda^n \tag{2.13}$$

for some positive constants c and $\lambda \in (0, 1)$.

Next, we introduce the second approach to construct generalized sinc function.

Definition 2.3. A *symmetric cascade filter* (SCF) $H_{\boldsymbol{c}}$ is a piecewise constant function on $\mathbb{R}$ defined by

$$H_{1,\boldsymbol{c}}(t) = c_n, \quad t \in (-n-1, -n] \cup [n, n+1) \tag{2.14}$$

with the sequence $\boldsymbol{c} := \{c_n : n \in \mathbb{Z}_+\}$. Here we use the indices pair to show that the filter is dependent on the dimension 1 as well as on the vector $\boldsymbol{c}$. Radial cascade filter associated with general dimension d will be seen in (3.1).

Note that $H_{1,\boldsymbol{c}}$ in (2.14) can be rewritten by

$$H_{1,\boldsymbol{c}} = \sum_{n \in \mathbb{Z}} c_n \chi(\cdot - n), \tag{2.15}$$

where the components of the *bi-infinite* sequence $\boldsymbol{c} := \{c_n : n \in \mathbb{Z}\}$ are extended by $c_n = c_{-n-1}, n \in \mathbb{Z}_+$, and χ is the *characteristic function* of the interval $[0, 1)$.

Corresponding to the filter $H_{1,\boldsymbol{c}}$ is the *impulse response* function ϕ_F defined by

$$\phi_F := \sqrt{\frac{\pi}{2}} \mathcal{F}^{-1} H_{1,\boldsymbol{c}}. \tag{2.16}$$

An explicit form of ϕ_F can be given by

$$\phi_F(t) = \mathrm{sinc}\left(\frac{t}{2}\right)\mathrm{Re}\{F(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{\frac{1}{2}\mathrm{i}t}\}, \quad t \in \mathbb{R}. \tag{2.17}$$

A direct computation confirms that

$$\phi_F = \mathrm{sinc}_G, \tag{2.18}$$

which will be proved in Appendix.

Example 1.1. Let $G = z$, i.e., $F = 1$. This case leads to $\mathrm{sinc}_G = \mathrm{sinc}$.

Example 1.2. Let G be the Blaschke product of order n, i.e.,

$$G(z) = B_{\boldsymbol{a}}(z) = \prod_{j=1}^{n} \frac{z - a_j}{1 - a_j z}$$

with $\boldsymbol{a} := (a_1, a_2, \dots, a_n) \in (-1, 1)^n$. Then by (2.17) and (2.18), we can check that

$$\mathrm{sinc}_G(t) = \frac{\sin \theta_{\boldsymbol{a}}(t)}{t},$$

where $\theta_{\boldsymbol{a}}$ is called a nonlinear phase and decided by the boundary value of Blaschke product by

$$\mathrm{e}^{\mathrm{i}\theta_{\boldsymbol{a}}(t)} = B_{\boldsymbol{a}}(\mathrm{e}^{\mathrm{i}t}), \quad t \in \mathbb{R}.$$

Correspondingly, $\mathrm{e}^{\mathrm{i}\theta_{\boldsymbol{a}}(\cdot)}$ is a time-frequency atom with nonlinear phase $\theta_{\boldsymbol{a}}$.

We remark that, from the above discussion, there are two equivalent ways to construct a generalize sinc function in one dimensional case. One is the boundary value approach of analytic functions in a neighborhood of the closed unit disk given by (2.7), and the other is the Fourier transform approach given by (2.17).

3 Radial Bessel-sinc functions

Let the vector $\boldsymbol{c} = (c_n : n \in \mathbb{Z}_+) \in l^2(\mathbb{Z}_+)$ be defined by (2.10). Extend $\boldsymbol{c}$ to be a symmetric *bi-infinite* vector $\boldsymbol{c} := (c_n : n \in \mathbb{Z})$ by $c_n = c_{-n-1}, n \in \mathbb{Z}_+$. We say that a *radial cascade filter* (RCF) $H_{d,\boldsymbol{c}}$ is a piecewise constant function if

$$H_{d,\boldsymbol{c}}(\underline{t}) = c_n, \quad \underline{t} \in S_n \tag{3.1}$$

with S_n defined in (1.1). The filter $H_{d,\boldsymbol{c}}$ in (3.1) can be rewritten as

$$H_{d,\boldsymbol{c}}(\underline{t}) = \sum_{n\in\mathbb{Z}} c_n \chi(|\underline{t}| - n) = H_{1,\boldsymbol{c}}(|\underline{t}|), \quad \underline{t} \in \mathbb{R}^d, \tag{3.2}$$

where χ is the characteristic function of the interval $[0, 1)$.

Definition 3.1. The Radial Bessel-sinc function $\mathrm{sinc}_{d,\boldsymbol{c}}$ in $\mathbb{R}^d$ associated with the vector $\boldsymbol{c}$ is defined to be

$$\mathrm{sinc}_{d,\boldsymbol{c}}(\underline{\xi}) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \mathcal{F}^{-1}(H_{d,\boldsymbol{c}})(\underline{\xi}). \tag{3.3}$$

For (3.3), we need to investigate the Fourier transform of $H_{d,\boldsymbol{c}}$.

Lemma 3.2. *The Fourier transform of* $H_{d,\boldsymbol{c}}, d \geqslant 1$, *is*

$$\mathrm{sinc}_{d,\boldsymbol{c}}(\underline{\xi}) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} |\underline{\xi}|^{-d} \sum_{k\in\mathbb{Z}_+} c_k \int_{k|\underline{\xi}|}^{(k+1)|\underline{\xi}|} r^{\frac{d}{2}} J_{\frac{d-2}{2}}(r) dr, \quad \underline{\xi} \in \mathbb{R}^d. \tag{3.4}$$

Proof. Using the computing technique of the Fourier transform of a radial function, we have

$$\begin{aligned}\operatorname{sinc}_{d,\boldsymbol{c}}(\underline{\xi}) &= (2\pi)^{-\frac{d}{2}}\left(\frac{\pi}{2}\right)^{\frac{d}{2}}\int_{\mathbb{R}^d} H_{1,\boldsymbol{c}}(|\underline{t}|)\mathrm{e}^{\mathrm{i}\langle\underline{\xi},\underline{t}\rangle}d\underline{t}\\ &= 2^{-d}\int_0^\infty H_{1,\boldsymbol{c}}(r)r^{d-1}dr\int_{S^{d-1}}\mathrm{e}^{\mathrm{i}r|\underline{\xi}|\langle\frac{\underline{\xi}}{|\underline{\xi}|},\underline{t}'\rangle}d\underline{t}', \quad \underline{\xi}\in\mathbb{R}^d.\end{aligned}$$

Recalling the formula

$$\int_{S^{d-1}}\mathrm{e}^{\mathrm{i}r\langle\underline{\omega},\underline{t}'\rangle}d\underline{t}' = (2\pi)^{\frac{d}{2}}r^{-\frac{d}{2}+1}J_{\frac{d-2}{2}}(r), \quad \underline{\omega}\in S^{d-1}, \quad r\in(0,\infty), \tag{3.5}$$

being independent of $\underline{\omega}\in S^{d-1}$, we obtain

$$\operatorname{sinc}_{d,\boldsymbol{c}}(\underline{\xi}) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}}|\underline{\xi}|^{-\frac{d}{2}+1}\int_0^\infty r^{\frac{d}{2}}H_{1,\boldsymbol{c}}(r)J_{\frac{d-2}{2}}(r|\underline{\xi}|)dr, \quad \underline{\xi}\in\mathbb{R}^d.$$

Noting the definition of $H_{1,\boldsymbol{c}}$, recalling the decaying rate of $\boldsymbol{c}$ and using Lebesgue dominated convergence theorem, we get

$$\begin{aligned}\operatorname{sinc}_{d,\boldsymbol{c}}(\underline{\xi}) &= \left(\frac{\pi}{2}\right)^{\frac{d}{2}}|\underline{\xi}|^{-\frac{d}{2}+1}\int_0^\infty r^{\frac{d}{2}}\sum_{k\in\mathbb{Z}}c_k\chi(r-k)J_{\frac{d-2}{2}}(r|\underline{\xi}|)dr\\ &= \left(\frac{\pi}{2}\right)^{\frac{d}{2}}|\underline{\xi}|^{-\frac{d}{2}+1}\sum_{k\in\mathbb{Z}_+}c_k\int_k^{k+1}r^{\frac{d}{2}}J_{\frac{d-2}{2}}(r|\underline{\xi}|)dr\\ &= \left(\frac{\pi}{2}\right)^{\frac{d}{2}}|\underline{\xi}|^{-d}\sum_{k\in\mathbb{Z}_+}c_k\int_{k|\underline{\xi}|}^{(k+1)|\underline{\xi}|}r^{\frac{d}{2}}J_{\frac{d-2}{2}}(r)dr.\end{aligned}$$

□

It is easy to see from Lemma 3.2 that $\operatorname{sinc}_{d,\boldsymbol{c}}$ is of dimensional dependence. In particular, when d is odd, due to the explicit representation (1.3) of Bessel function $J_{m+\frac{1}{2}}$, we can handle the right-sided integral in (3.4) in the rest of this section. We first set $d=2m+3$, $m\in\mathbb{Z}_+$, and define the function

$$\gamma_m(t) := t^{-2m-3}\sum_{k\in\mathbb{Z}_+}c_k\int_{kt}^{(k+1)t}r^{\frac{2m+3}{2}}J_{m+\frac{1}{2}}(r)dr, \quad t\in\mathbb{R}. \tag{3.6}$$

Consequently,

$$\operatorname{sinc}_{2m+3,\boldsymbol{c}}(\underline{\xi}) = \left(\frac{\pi}{2}\right)^{\frac{2m+3}{2}}\gamma_m(|\underline{\xi}|), \quad \underline{\xi}\in\mathbb{R}^{2m+3}. \tag{3.7}$$

Lemma 3.3. *The function $\gamma_m, m\geqslant 0$, defined in* (3.6) *satisfies*

$$\gamma_{m+1}(t) = Q_m(t) + (2m+3)t^{-2}\gamma_m(t), \quad t\in\mathbb{R}, \tag{3.8}$$

where

$$Q_m(t) := \sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2m-5}\sum_{k\in\mathbb{Z}_+}c_k r^{2m+3}\mathcal{D}_r^m(\operatorname{sinc}(r))|_{r=kt}^{(k+1)t}, \quad t\in\mathbb{R}, \tag{3.9}$$

with the operator $\mathcal{D}$ mentioned in (1.3).

Proof. It follows from (1.3) that

$$\gamma_m(t) = t^{-2m-3}\sum_{k\in\mathbb{Z}_+}c_k\int_{kt}^{(k+1)t}r^{\frac{2m+3}{2}}\sqrt{\frac{2}{\pi}}(-1)^m r^{m+\frac{1}{2}}\mathcal{D}_r^m(\operatorname{sinc}(r))dr,$$

which can be simplified as

$$\gamma_m(t) = \sqrt{\frac{2}{\pi}}(-1)^m t^{-2m-3}\sum_{k\in\mathbb{Z}_+}c_k\int_{kt}^{(k+1)t}r^{2m+2}\mathcal{D}_r^m(\operatorname{sinc}(r))dr. \tag{3.10}$$

Now direct computation gives us

$$\begin{aligned}\gamma_{m+1}(t) &= \sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2m-5}\sum_{k\in\mathbb{Z}_+}c_k\int_{kt}^{(k+1)t}r^{2m+4}\mathcal{D}_r^{m+1}(\mathrm{sinc}(r))dr\\ &= \sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2m-5}\sum_{k\in\mathbb{Z}_+}c_k\int_{kt}^{(k+1)t}r^{2m+3}d[\mathcal{D}_r^{m}(\mathrm{sinc}(r))]\\ &= \sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2m-5}\sum_{k\in\mathbb{Z}_+}c_k r^{2m+3}\mathcal{D}_r^{m}(\mathrm{sinc}(r))|_{r=kt}^{(k+1)t}\\ &\quad -(2m+3)\sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2m-5}\sum_{k\in\mathbb{Z}_+}c_k\int_{kt}^{(k+1)t}r^{2m+2}\mathcal{D}_r^{m}(\mathrm{sinc}(r))dr.\end{aligned}$$

Recalling the definition Q_m in (3.9) and noting (3.10), we conclude the proof. □

Below we investigate the function Q_m.

Lemma 3.4. *The function Q_m defined in* (3.9) *satisfies*

$$Q_m(t) = (-1)^m\sqrt{\frac{2}{\pi}}t^{-2}\mathcal{D}_t^m\left(\frac{v_G''(t)}{t}\right), \quad t\in\mathbb{R}, \tag{3.11}$$

where v_G is defined in (2.2).

Proof. We prove this lemma by induction method on m. Firstly, we show that

$$Q_0(t) = \sqrt{\frac{2}{\pi}}t^{-2}\frac{v_G''(t)}{t}, \quad t\in\mathbb{R}.$$

To this end, setting $m=0$ in (3.9), we have

$$\begin{aligned}Q_0(t) &= -\sqrt{\frac{2}{\pi}}t^{-5}\sum_{k\in\mathbb{Z}_+}c_k r^3\mathrm{sinc}(r)|_{r=kt}^{(k+1)t}\\ &= -\sqrt{\frac{2}{\pi}}t^{-3}\sum_{k\in\mathbb{Z}_+}c_k[(k+1)^2\sin(k+1)t - k^2\sin(kt)]\\ &= -\sqrt{\frac{2}{\pi}}t^{-3}\sum_{k\in\mathbb{N}}(c_{k-1}-c_k)k^2\sin(kt).\end{aligned}$$

Combining this with the identity (2.11), that is,

$$v_G(t) = \sum_{k\in\mathbb{N}}(c_{k-1}-c_k)\sin(kt),$$

we conclude (3.11) for $m=0$.

Secondly, we need to show that Q_m satisfies the differential equation

$$Q_{m+1}(t) = -t^{-1}\frac{d}{dt}Q_m(t) - 2t^{-2}Q_m(t), \quad t\in\mathbb{R}. \tag{3.12}$$

Direct calculation leads to an equality chain

$$\begin{aligned}Q_m(t) &= \sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2m-5}\sum_{k\in\mathbb{Z}_+}c_k r^{2m+3}\mathcal{D}_r^m(\mathrm{sinc}(r))|_{r=kt}^{(k+1)t}\\ &= \sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2}\sum_{k\in\mathbb{Z}_+}c_k(k+1)^{2m+3}\mathcal{D}_r^m(\mathrm{sinc}(r))|_{r=(k+1)t}\end{aligned}$$

$$- \sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2}\sum_{k\in\mathbb{Z}_+} c_k k^{2m+3}\mathcal{D}_r^m(\mathrm{sinc}(r))|_{r=kt},$$

which implies that

$$Q_m(t) = \sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2}\sum_{k\in\mathbb{N}}(c_{k-1}-c_k)k^{2m+3}\mathcal{D}_r^m(\mathrm{sinc}(r))|_{r=kt}. \tag{3.13}$$

Differentiating on both sides of (3.13) gives us

$$\frac{d}{dt}Q_m(t) = -2\sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-3}\sum_{k\in\mathbb{N}}(c_{k-1}-c_k)k^{2m+3}\mathcal{D}_r^m(\mathrm{sinc}(r))|_{r=kt}$$
$$+\sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2}\sum_{k\in\mathbb{N}}(c_{k-1}-c_k)k^{2m+3}\frac{d}{dt}\{\mathcal{D}_r^m(\mathrm{sinc}(r))|_{r=kt}\}.$$

Invoking the above equality chain, we have

$$\begin{aligned}\frac{d}{dt}Q_m(t) &= -2t^{-1}Q_m(t) + \sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2}\sum_{k\in\mathbb{N}}(c_{k-1}-c_k)k^{2m+3}\{r\mathcal{D}_r^{m+1}(\mathrm{sinc}(r))|_{r=kt}\}\frac{dr}{dt}\\ &= -2t^{-1}Q_m(t) + \sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-1}\sum_{k\in\mathbb{N}}(c_{k-1}-c_k)k^{2m+5}\{\mathcal{D}_r^{m+1}(\mathrm{sinc}(r))|_{r=kt}\}\\ &= -2t^{-1}Q_m(t) - tQ_{m+1}(t),\end{aligned}$$

from which, we get (3.12).

Now we continue with the induction. Assuming that (3.11) is true for m, we proceed to show that it also true for $m+1$. Applying the formula (3.12), we have

$$\begin{aligned}Q_{m+1}(t) &= -t^{-1}\frac{d}{dt}\left\{(-1)^m\sqrt{\frac{2}{\pi}}t^{-2}\mathcal{D}_t^m\left(\frac{v_G''(t)}{t}\right)\right\} - 2t^{-2}Q_m(t)\\ &= 2t^{-2}(-1)^m\sqrt{\frac{2}{\pi}}t^{-2}\mathcal{D}_t^m\left(\frac{v_G''(t)}{t}\right) + (-1)^{m+1}\sqrt{\frac{2}{\pi}}t^{-3}\frac{d}{dt}\left\{\mathcal{D}_t^m\left(\frac{v_G''(t)}{t}\right)\right\} - 2t^{-2}Q_m(t).\end{aligned}$$

By the induction hypothesis, we obtain

$$\begin{aligned}Q_{m+1}(t) &= (-1)^{m+1}\sqrt{\frac{2}{\pi}}t^{-3}\frac{d}{dt}\left\{\mathcal{D}_t^m\left(\frac{v_G''(t)}{t}\right)\right\}\\ &= (-1)^{m+1}\sqrt{\frac{2}{\pi}}t^{-2}\mathcal{D}_t\left\{\mathcal{D}_t^m\left(\frac{v_G''(t)}{t}\right)\right\}\\ &= (-1)^{m+1}\sqrt{\frac{2}{\pi}}t^{-2}\left\{\mathcal{D}_t^{m+1}\left(\frac{v_G''(t)}{t}\right)\right\}.\end{aligned}$$

The proof is completed. □

Now, (3.7) and the following recursive formulas A and B lead to our main result in this section.

Recursive formula A: Functions $\{\gamma_j(t) : j\in\mathbb{Z}_+\}$ can be recursively calculated by

$$\gamma_{m+1}(t) = (2m+3)t^{-2}\gamma_m(t) + (-1)^m\sqrt{\frac{2}{\pi}}t^{-2}\mathcal{D}_t^m\left(\frac{[v_G(t)]''}{t}\right), \quad t\in\mathbb{R}, \tag{3.14}$$

with the initial function

$$\gamma_0(t) = -\sqrt{\frac{2}{\pi}}\mathcal{D}_t\left(\frac{v_G(t)}{t}\right)$$

and v_G defined in (2.2).

Recursive formula B: Construct $\{g_j : j \in \mathbb{Z}_+\}$ by

$$g_{m+1}(t) = -\mathcal{D}_t g_m(t), \quad t \in \mathbb{R} \tag{3.15}$$

with the initial function

$$g_0(t) = \sqrt{\frac{2}{\pi}} t^{-3} \sum_{k\in\mathbb{Z}_+} c_k \int_{kt}^{(k+1)t} r \sin r dr. \tag{3.16}$$

Theorem 3.5. *The radial Bessel-sinc function* $\mathrm{sinc}_{d,\boldsymbol{c}}$ *satisfies the following identity for odd* $d = 3, 5, 7, \dots,$

$$\mathrm{sinc}_{d,\boldsymbol{c}}(\underline{\xi}) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \gamma_{\frac{d-3}{2}}(|\underline{\xi}|), \quad \xi \in \mathbb{R}^d, \tag{3.17}$$

where γ_m *is determined by the recursive formulas A and B.*

Proof. The recursive relation (3.14) can be concluded from Lemmas 3.3 and 3.4. As such, we just need to prove the explicit expression of γ_0. To this end, it follows from (3.10) that

$$\begin{aligned}
\gamma_0(t) &= \sqrt{\frac{2}{\pi}} t^{-3} \sum_{k\in\mathbb{Z}_+} c_k \int_{kt}^{(k+1)t} r \sin r dr = \sqrt{\frac{2}{\pi}} t^{-3} \sum_{k\in\mathbb{Z}_+} c_k \int_{kt}^{(k+1)t} r d(-\cos r) \\
&= \sqrt{\frac{2}{\pi}} t^{-2} \sum_{k\in\mathbb{Z}_+} c_k (k\cos(kt) - (k+1)\cos(k+1)t) + \sqrt{\frac{2}{\pi}} t^{-3} \sum_{k\in\mathbb{Z}_+} c_k \int_{kt}^{(k+1)t} \cos r dr \\
&= \sqrt{\frac{2}{\pi}} t^{-2} \sum_{k\in\mathbb{N}} (c_k - c_{k-1}) k \cos(kt) + \sqrt{\frac{2}{\pi}} t^{-3} \sum_{k\in\mathbb{N}} (c_{k-1} - c_k) \sin(kt) \\
&= \sqrt{\frac{2}{\pi}} [-t^{-2} v_G'(t) + t^{-3} v_G(t)],
\end{aligned}$$

where we used the formula (2.11) twice in the last equality. Now we get

$$\gamma_0(t) = -\sqrt{\frac{2}{\pi}} t^{-1} \frac{v_G'(t) t - v_G(t)}{t^2} = -\sqrt{\frac{2}{\pi}} \mathcal{D}_t(\mathrm{sinc}_G(t))$$

to complete the proof. □

We know from Theorem 3.5 that the recursive formula (3.14) is crucial for constructing a radial sinc function. For convenient discussion, we need a further investigation into (3.14) and as such we present an alternative expression of (3.14), precisely, we offer a differential-operator-based characterization of (3.14).

Proposition 3.6. *For* $m \in \mathbb{Z}_+$*, it holds that* $g_m = \gamma_m$*, which implies that* (3.14) *is equivalent to* (3.15)*.*

Proof. We prove this proposition by induction method on m. It follows from (3.14) and (3.16) that $g_0 = \gamma_0$. Now we just need to show that if $g_m = \gamma_m$, then $g_{m+1} = \gamma_{m+1}$. Recursive formula A leads to the sequence of functions γ_m defined by (3.10), that is,

$$\gamma_m(t) = \sqrt{\frac{2}{\pi}} (-1)^m t^{-2m-3} \sum_{k\in\mathbb{Z}_+} c_k \int_{kt}^{(k+1)t} r^{2m+2} \mathcal{D}_r^m(\mathrm{sinc}(r)) dr.$$

Now it suffices to show that

$$\frac{d}{dt}\gamma_m(t) = -(2m+3)t^{-1}\gamma_m(t) + (-1)^{m+1}\sqrt{\frac{2}{\pi}} t^{-1} \mathcal{D}_t^m\left(\frac{[v_G(t)]''}{t}\right), \quad t \in \mathbb{R}. \tag{3.18}$$

Differentiating on both sides of (3.10) gives us that

$$\frac{d}{dt}\gamma_m(t) = \frac{d}{dt}\left(\sqrt{\frac{2}{\pi}} (-1)^m t^{-2m-3} \sum_{k\in\mathbb{Z}_+} c_k \int_{kt}^{(k+1)t} r^{2m+2} \mathcal{D}_r^m(\mathrm{sinc}(r)) dr\right)$$

$$= -(2m+3)\frac{1}{t}\gamma_m(t) + \sqrt{\frac{2}{\pi}}(-1)^m t^{-1} \sum_{k\in\mathbb{Z}_+} c_k (k+1)^{2m+3}\mathcal{D}_r^m(\mathrm{sinc}(r))|_{r=(k+1)t}$$

$$- \sqrt{\frac{2}{\pi}}(-1)^m t^{-1} \sum_{k\in\mathbb{Z}_+} c_k k^{2m+3}\mathcal{D}_r^m(\mathrm{sinc}(r))|_{r=kt}$$

$$= -(2m+3)\frac{1}{t}\gamma_m(t) + \sqrt{\frac{2}{\pi}}(-1)^m t^{-1} \sum_{k\in\mathbb{N}} (c_{k-1}-c_k) k^{2m+3}\mathcal{D}_r^m(\mathrm{sinc}(r))|_{r=kt}.$$

Now by comparing (3.13) and (3.11), we obtain (3.18). The proof is completed. □

As a consequence of Proposition 3.6, we have the following corollary.

Corollary 3.7. *Suppose that the two sequences* $\{\gamma_m : m \in \mathbb{Z}_+\}$ *and* $\{g_m : m \in \mathbb{Z}_+\}$ *of functions are defined by* (3.14) *and* (3.15)*, respectively. Then we have*

$$\gamma_m(t) = g_m(t) = \sqrt{\frac{2}{\pi}}(-1)^{m+1}\mathcal{D}_t^{m+1}(\mathrm{sinc}_G(t)), \quad t \in \mathbb{R}, \tag{3.19}$$

where v_G *is defined by* (2.2).

Proof. By comparing recursive formula A with formula B, the required result follows immediately. □

Finally, we offer an estimation for decaying rate of the generalized sinc function.

Proposition 3.8. *The following inequality holds*

$$|\mathrm{sinc}_{d,\boldsymbol{c}}(\underline{\xi})| \leqslant \frac{\mathrm{const}}{|\underline{\xi}|^{\frac{d+1}{2}}}, \quad \underline{\xi} \in \mathbb{R}^d, \quad |\underline{\xi}| \geqslant 1, \tag{3.20}$$

where const *is some constant independent of* $\underline{\xi}$.

Proof. Recalling (3.17), it suffices to show that

$$|\gamma_m(t)| \leqslant \frac{\mathrm{const}}{|t|^{m+2}}, \quad t \in \mathbb{R}, \quad |t| \geqslant 1,$$

where const is some constant independent of t. By using (3.19), we need to prove

$$|\mathcal{D}_t^m(\mathrm{sinc}_G(t))| \leqslant \frac{\mathrm{const}}{|t|^{m+1}}, \quad t \in \mathbb{R}, \quad |t| \geqslant 1.$$

This inequality can be verified by adopting induction method on m, by using the formula

$$\mathcal{D}_t^m = \frac{1}{t^m}\sum_{k=0}^{m-1} b_k^{(m)}\left(-\frac{1}{t}\right)^k \frac{d^{m-k}}{dt^{m-k}}$$

with some constants $b_k^{(m)}, k = 0, \ldots, m-1$, independent of t, and by noting that both of sinc_G and v_G are infinitely differential and their derivatives are bounded. □

4 Applications in LTI system and sampling

Let T be the transform function of a continuous linear time-invariant (LTI) system. When the input signal is f_{in}, then the output signal f_{out} is the convolution of T and f_{in}, namely,

$$f_{\mathrm{out}}(\underline{t}) = \int_{\mathbb{R}^d} f_{\mathrm{in}}(\underline{t} - \underline{x})T(\underline{x})d\underline{x}, \quad \underline{t} \in \mathbb{R}^d.$$

There is also an equivalent representation of the output signal in the frequency domain

$$\hat{f}_{\mathrm{out}}(\underline{\xi}) = (2\pi)^{-\frac{d}{2}}\hat{f}_{\mathrm{in}}(\underline{\xi})\hat{T}(\underline{\xi}), \quad \underline{\xi} \in \mathbb{R}^d.$$

If, in particular, in univariate case, we choose $\hat{T}$ to be the indicator function $\chi_{(-1,1)}$, then the response function is just the usual sinc function up to a multiplicative constant. In higher dimensional cases, one adopts the tensor product approach, that is, $\hat{T}$ is the indicator function of the cube $(-1,1)^d$ and $T(\underline{t}) = \prod_{k=1}^{d} \frac{\sin t_k}{t_k}$ (see [10]). To apply RBS functions to LTI system, we propose the following filtering process: for input signal $f_{\rm in}$, the output signal $f_{\rm out}$ keeps the frequency information of $f_{\rm in}$ with different scales in the different frequency bands, i.e., in the annulau $S_n = B_{(0,n+1)} \setminus B_{(0,n)}$, namely,

$$\begin{aligned}
\hat{f}_{\rm out}(\underline{\xi}) &= c_0 \hat{f}_{\rm in}(\underline{\xi}), \quad \underline{\xi} \in S_0 = B_{(0,1)},\\
\hat{f}_{\rm out}(\underline{\xi}) &= c_1 \hat{f}_{\rm in}(\underline{\xi}), \quad \underline{\xi} \in S_1,\\
&\vdots\\
\hat{f}_{\rm out}(\underline{\xi}) &= c_n \hat{f}_{\rm in}(\underline{\xi}), \quad \underline{\xi} \in S_n, \quad n = 1, 2, \dots,\\
&\vdots
\end{aligned}$$

where $\boldsymbol{c} = \{c_j : j \in \mathbb{Z}_+\}$ is the vector defined in (2.10). Corresponding to this LTI system, we know from (3.3) that the transform function in frequency domain is the cascade radial filter $H_{d,\boldsymbol{c}}$ and the impulse response is just the radial Bessel sinc function $\mathrm{sinc}_{d,\boldsymbol{c}}$.

In order to introduce a sampling space in $L^2(\mathbb{R}^d)$, we need to recall the classical Whittaker-Kotelnikov-Shannon sampling theorem by the tensor product approach, which states that any bandlimited signal f with $\mathrm{supp}\hat{f} \subset [-\sigma, \sigma]^d$ can be reconstructed from its sampling sequence $\{f(\underline{n}\frac{\pi}{\sigma}) : \underline{n} \in \mathbb{Z}^d\}$, that is,

$$f(\underline{t}) = \sum_{\underline{n} \in \mathbb{Z}^d} f\left(\underline{n}\frac{\pi}{\sigma}\right) \prod_{j=1}^{d} \frac{\sin(\sigma t_j - n_j \pi)}{\sigma t_j - n_j \pi}, \quad \underline{t} \in \mathbb{R}^d. \tag{4.1}$$

Any bandlimited signal f with $\mathrm{supp}\hat{f} \subset [-\sigma, \sigma]^d$ corresponds to a multivariate integrable complex valued function in the d-dimensional cube of width $2\frac{\pi}{\sigma}$, i.e.,

$$M_{f,\sigma}(\underline{t}) = \left(\frac{\sqrt{2\pi}}{2\sigma}\right)^d \sum_{\underline{n} \in \mathbb{Z}^d} f\left(\underline{n}\frac{\pi}{\sigma}\right) \mathrm{e}^{-\mathrm{i}\frac{\pi}{\sigma}\langle \underline{n}, \underline{t}\rangle}, \quad \underline{t} \in \mathbb{R}^d. \tag{4.2}$$

Taking Fourier transform to both sides of (4.1), we can define the space of bandlimited signals in frequency domain to be

$$B_\sigma = \{f \in L^2(\mathbb{R}^d) : \hat{f}(\underline{\xi}) = M_{f,\sigma}(\underline{\xi}) \chi_{(-\sigma,\sigma)^d}(\underline{\xi}), \underline{\xi} \in \mathbb{R}^d\}.$$

In order to derive a sampling space of non-bandlimited signals, motivated by (4.2), we define function $G_{f,\boldsymbol{c},\sigma}$ by

$$G_{f,\boldsymbol{c},\sigma}(\underline{t}) = \begin{cases}
c_0 M_{f,\sigma}(\underline{t}), & \underline{t} \in B_{(0,\sigma)},\\
c_1 M_{f,\sigma}(\underline{t}), & \underline{t} \in B_{(0,2\sigma)} \setminus B_{(0,\sigma)},\\
\quad\vdots & \\
c_n M_{f,\sigma}(\underline{t}), & \underline{t} \in B_{(0,(n+1)\sigma)} \setminus B_{(0,n\sigma)}, \quad n = 1, 2, \dots,\\
\quad\vdots &
\end{cases} \tag{4.3}$$

Now, we define the space of non-bandlimited signals by

$$B_{\boldsymbol{c},\sigma} = \{f \in L^2(\mathbb{R}^d) : \hat{f}(\underline{\xi}) = G_{f,\boldsymbol{c},\sigma}(\underline{\xi}), \underline{\xi} \in \mathbb{R}^d\}. \tag{4.4}$$

Theorem 4.1. *A signal $f \in B_{\boldsymbol{c},\sigma}$ if and only if*

$$f(\underline{t}) = \sum_{\underline{k} \in \mathbb{Z}^d} f\left(\frac{\pi}{\sigma}\underline{k}\right) \mathrm{sinc}_{d,c}(\sigma \underline{t} - \underline{k}\pi), \quad \underline{t} \in \mathbb{R}^d. \tag{4.5}$$

Proof. We first prove the necessity. By the definition of the space $B_{\boldsymbol{c},\sigma}$, we know that for any $f \in B_{\boldsymbol{c},\sigma}$, it has the following representation in the frequency domain

$$\begin{aligned}\hat{f}(\underline{\xi}) &= G_{f,\boldsymbol{c},\sigma}(\underline{\xi}) = \sum_{n=1}^{\infty} M_{f,\sigma}(\underline{\xi}) c_{n-1} \chi_{B(0,n\sigma)\backslash B(0,(n-1)\sigma)}(\underline{\xi}) \\ &= \left(\frac{\sqrt{2\pi}}{2\sigma}\right)^d \sum_{\underline{k}\in\mathbb{Z}^d} f\left(\underline{k}\frac{\pi}{\sigma}\right) \mathrm{e}^{-\mathrm{i}\frac{\pi}{\sigma}\langle \underline{k},\underline{\xi}\rangle} \sum_{n=1}^{\infty} c_{n-1} \chi_{B(0,n\sigma)\backslash B(0,(n-1)\sigma)}(\underline{\xi}).\end{aligned}$$

Applying the inverse Fourier transform to both sides of the above equation leads to

$$\begin{aligned}f(\underline{t}) &= \left(\frac{1}{2\sigma}\right)^d \sum_{\underline{k}\in\mathbb{Z}^d} f\left(\underline{k}\frac{\pi}{\sigma}\right) \sum_{n=1}^{\infty} c_{n-1} \int_{B(0,n\sigma)\backslash B(0,(n-1)\sigma)} \mathrm{e}^{-\mathrm{i}\frac{\pi}{\sigma}\langle \underline{k},\underline{\xi}\rangle} \mathrm{e}^{\mathrm{i}\langle \underline{t},\underline{\xi}\rangle} d\underline{\xi} \\ &= \left(\frac{1}{2\sigma}\right)^d \sum_{\underline{k}\in\mathbb{Z}^d} f\left(\underline{k}\frac{\pi}{\sigma}\right) \sum_{n=1}^{\infty} c_{n-1} \int_{B(0,n\sigma)\backslash B(0,(n-1)\sigma)} \mathrm{e}^{\mathrm{i}\langle \underline{t}-\frac{\pi}{\sigma}\underline{k},\underline{\xi}\rangle} d\underline{\xi} \\ &= 2^{-d} \sum_{\underline{k}\in\mathbb{Z}^d} f\left(\underline{k}\frac{\pi}{\sigma}\right) \sum_{n=1}^{\infty} c_{n-1} \int_{B(0,n)/B(0,n-1)} \mathrm{e}^{\mathrm{i}\langle \sigma\underline{t}-\pi\underline{k},\underline{\xi}\rangle} d\underline{\xi} \\ &= 2^{-d} \sum_{\underline{k}\in\mathbb{Z}^d} f\left(\underline{k}\frac{\pi}{\sigma}\right) \int_{\mathbb{R}^d} H_{d,\boldsymbol{c}}(\underline{\xi}) \mathrm{e}^{\mathrm{i}\langle \sigma\underline{t}-\pi\underline{k},\underline{\xi}\rangle} d\underline{\xi} \\ &= \sum_{\underline{k}\in\mathbb{Z}^d} f\left(\underline{k}\frac{\pi}{\sigma}\right) \mathrm{sinc}_{d,\boldsymbol{c}}(\sigma\underline{t}-\pi\underline{k}).\end{aligned}$$

Reversing the process above, we can prove the sufficiency. □

The Shannon type sampling formula (4.5) involves with infinite sums. From the practical point of view, we need to use finite sums to approximate the original signal f. The approximating error is measured in terms of the energy norm of $L^2(\mathbb{R}^d)$ by an adaptive truncated sum. That is, we use

$$S_n(\underline{t}) = \sum_{|\underline{t}-\frac{\pi}{\sigma}\underline{k}|\leqslant n} f\left(\frac{\pi}{\sigma}\underline{k}\right) \mathrm{sinc}_{d,c}(\sigma\underline{t}-\underline{k}\pi), \quad \underline{t}\in\mathbb{R}^d \tag{4.6}$$

to approximate f.

Theorem 4.2. *Suppose that* $f \in B_{\boldsymbol{c},\sigma}$ *and* $\boldsymbol{c}$ *satisfies* (2.13)*. Then*

$$\|f - S_n\|_\infty = \mathcal{O}(n^{-1/2}). \tag{4.7}$$

Proof. Using the Cauchy-Schwartz inequality, we get

$$\begin{aligned}|f(\underline{t}) - S_n(\underline{t})| &= \left| \sum_{|\underline{t}-\frac{\pi}{\sigma}\underline{k}|>n} f\left(\frac{\pi}{\sigma}\underline{k}\right) \mathrm{sinc}_{d,c}(\sigma\underline{t}-\underline{k}\pi) \right| \\ &\leqslant \left(\sum_{|\underline{t}-\frac{\pi}{\sigma}\underline{k}|>n} f^2\left(\frac{\pi}{\sigma}\underline{k}\right)\right)^{\frac{1}{2}} \left(\sum_{|\underline{t}-\frac{\pi}{\sigma}\underline{k}|>n} \mathrm{sinc}^2_{d,c}(\sigma\underline{t}-\underline{k}\pi)\right)^{\frac{1}{2}} \\ &\leqslant \left(\sum_{\underline{k}\in\mathbb{Z}^d} f^2\left(\frac{\pi}{\sigma}\underline{k}\right)\right)^{\frac{1}{2}} \left(\sum_{|\underline{t}-\frac{\pi}{\sigma}\underline{k}|>n} \mathrm{sinc}^2_{d,c}(\sigma\underline{t}-\underline{k}\pi)\right)^{\frac{1}{2}}\end{aligned}$$

with $\underline{t} \in [0, \frac{\pi}{\sigma})^d$. That the system $\{\mathrm{sinc}_{d,c}(\cdot - \underline{k}\frac{\pi}{\sigma}) : \underline{k} \in \mathbb{Z}^d\}$ is stable (a frame) in $B_{\boldsymbol{c},\sigma}$ leads to

$$\sum_{\underline{k}\in\mathbb{Z}^d} f^2\left(\frac{\pi}{\sigma}\underline{k}\right) \leqslant \mathrm{const}\|f\|^2_{L^2(\mathbb{R}^d)}$$

and correspondingly

$$|f(t)-S_n(t)| \leqslant \text{const}\Bigg(\sum_{|\underline{t}-\frac{\pi}{\sigma}\underline{k}|>n} \text{sinc}^2_{d,c}(\sigma\underline{t}-\underline{k}\pi)\Bigg)^{\frac{1}{2}}.$$

Substituting (3.20) into above inequality leads to

$$|f(t)-S_n(t)| \leqslant \text{const}\Bigg(\sum_{|\underline{t}-\frac{\pi}{\sigma}\underline{k}|>n} \frac{1}{|\sigma\underline{t}-\underline{k}\pi|^{d+1}}\Bigg)^{\frac{1}{2}}.$$

Note that

$$\left\{\underline{k}:\left|\underline{t}-\frac{\pi}{\sigma}\underline{k}\right|>n\right\} \subseteq \bigcup_{j=1}^{d}\left\{(k_1,k_2,\dots,k_d):\left|t_j-\frac{\pi}{\sigma}k_j\right| \geqslant \frac{n}{\sqrt{d}}, k_\ell\in\mathbb{Z}, \ell\neq j\right\}$$

with $\underline{k}=(k_1,k_2,\dots,k_d)$ and $\underline{t}=(t_1,t_2,\dots,t_d)$. Consequently,

$$\begin{aligned}\sum_{|\underline{t}-\frac{\pi}{\sigma}\underline{k}|>n} \frac{1}{|\sigma\underline{t}-\underline{k}\pi|^{d+1}} &\asymp \sum_{|\underline{t}-\frac{\pi}{\sigma}\underline{k}|>n} \frac{1}{(\sum_{j=1}^{d}|\sigma t_j-k_j\pi|)^{d+1}} \\ &\leqslant d \sum_{(k_1,\dots,k_{d-1})\in\mathbb{Z}^{d-1}} \sum_{|\sigma t_d-\pi k_d|\geqslant\frac{n\sigma}{\sqrt{d}}} \frac{1}{(\sum_{j=1}^{d}|\sigma t_j-k_j\pi|)^{d+1}},\end{aligned}$$

where the estimation $\asymp$ is guaranteed by the equivalence of norms in $\mathbb{R}^d$. Using the similar techniques in [4,6], we can first estimate that

$$\begin{aligned}&\sum_{|\sigma t_d-\pi k_d|>\frac{n\sigma}{\sqrt{d}}} \frac{1}{(\sum_{j=1}^{d}|\sigma t_j-k_j\pi|)^{d+1}} \\ &\quad\leqslant \text{const}\frac{1}{(\sum_{j=1}^{d-1}|\sigma t_j-k_j\pi|+\frac{n\sigma}{\pi\sqrt{d}})^{d}} + \text{const}\frac{1}{(\sum_{j=1}^{d-1}|\sigma t_j-k_j\pi|+\frac{n\sigma}{\pi\sqrt{d}})^{d+1}}.\end{aligned} \tag{4.8}$$

Furthermore, by the same estimation method in (4.8), we have

$$\begin{aligned}&\sum_{(k_1,\dots,k_{d-1})\in\mathbb{Z}^{d-1}} \frac{1}{(\sum_{j=1}^{d-1}|\sigma t_j-k_j\pi|+\frac{n\sigma}{\pi\sqrt{d}})^{d}} \\ &\quad\leqslant \text{const}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\frac{dx_1\dots dx_{d-1}}{(\sum_{j=1}^{d-1}|\sigma t_j-x_j\pi|+\frac{n\sigma}{\pi\sqrt{d}})^{d}} \leqslant \text{const}\frac{1}{n}.\end{aligned} \tag{4.9}$$

Similarly,

$$\sum_{(k_1,\dots,k_{d-1})\in\mathbb{Z}^{d-1}} \frac{1}{(\sum_{j=1}^{d-1}|\sigma t_j-k_j\pi|+\frac{n\sigma}{\pi\sqrt{d}})^{d+1}} \leqslant \text{const}\frac{1}{n^2}. \tag{4.10}$$

It follows from (4.9) and (4.10) that $\|f-S_n\|_\infty=\mathcal{O}(n^{-1/2})$. □

We remark that, compared with the sampling class defined by the usually approach

$$S_{\boldsymbol{c},\sigma}=\left\{f: f(\underline{t})=\sum_{\underline{k}\in\mathbb{Z}^d} r_{\underline{k}}\text{sinc}_{d,c}(\sigma\underline{t}-\underline{k}\pi), t\in\mathbb{R}^d, r_{\underline{k}}\in l^2(\mathbb{Z}^d)\right\},$$

the sampling class $B_{\boldsymbol{c},\sigma}$ defined in (4.4) may look unnatural for the function $G_{f,\boldsymbol{c},\sigma}$ and $M_{f,\sigma}$ defined in (4.3) are both dependent of f. In fact, when $d=1$, the two function classes $B_{\boldsymbol{c},\sigma}$ and $S_{\boldsymbol{c},\sigma}$ coincide. This is because that the corresponding sinc function $\text{sinc}_{1,c}(\pi(\cdot))$ is of cardinality. It is easy to show that the two classes coincide if and only the involved sinc function has cardinality. Recall that a function ϕ is of cardinality if $\phi(0)=1$ and $\phi(\underline{k})=0$ for $0\neq\underline{k}\in\mathbb{Z}^d$. The cardinality of $\text{sinc}_{1,c}(\pi(\cdot))$ is proved as follows. We first note that $\mathcal{A} = G'(1) = \sum_{k\in\mathbb{Z}_+} c_k \neq 0$. Then (2.11) implies that $\mathrm{sinc}_{1,c}(k\pi) = 0$ for $0 \neq k \in \mathbb{Z}$. For $k = 0$, we have

$$\begin{aligned}\frac{1}{\mathcal{A}}\mathrm{sinc}_{1,c}(0) &= \frac{1}{\mathcal{A}}\lim_{t\to 0}\sum_{k\in\mathbb{N}}(c_{k-1}-c_k)\frac{\sin(kt)}{t} = \frac{1}{\mathcal{A}}\sum_{k\in\mathbb{N}}k(c_{k-1}-c_k)\\ &= \frac{1}{\mathcal{A}}\sum_{k\in\mathbb{Z}_+}(k+1)c_k - \sum_{k\in\mathbb{N}}kc_k = \frac{1}{\mathcal{A}}\sum_{k\in\mathbb{Z}_+}c_k = 1.\end{aligned}$$

Therefore, we have that $B_{\boldsymbol{c},\sigma} = S_{\boldsymbol{c},\sigma}$ in the one dimensional case (see [11]).

For $d > 1$, it is easily seen that the corresponding sinc function $\mathrm{sinc}_{d,c}(\pi(\cdot))$ does not have the cardinality. Therefore, the two classes $B_{\boldsymbol{c},\sigma}$ and $S_{\boldsymbol{c},\sigma}$ are not identical.

5 A Paley-Wiener type extension theorem for a class of radial functions

We proved in [2] that, in the case of $d = 1$, all functions in $B_{\boldsymbol{c},\sigma}$ $(= S_{\boldsymbol{c},\sigma})$ can be extended to be analytic in a strip symmetric about the real line. In higher dimensional case, we shall investigate the monogenic extension of the following functions related to $H_{d,\mathbf{c}}(\xi)$ in (3.2).

Definition 5.1. Let $S_n := B_{(0,n+1)} \setminus B_{(0,n)}$. Suppose that $R(r)$ is defined in $[0,\infty)$ and 1-periodic, and a sequence $\mathbf{c} = \{c_n\}$ decays exponentially, i.e., (2.13) holds. Define

$$\mathbb{E} = \left\{ f \in L^2(\mathbb{R}^d) : \hat{f}(\underline{\xi}) = R(|\underline{\xi}|)H_{d,\mathbf{c}}(\xi) = R(|\underline{\xi}|)\sum_{k=0}^{\infty} c_k\chi_{S_k}(\underline{\xi}), \xi \in \mathbb{R}^d \right\}. \tag{5.1}$$

It is not difficult to check that $\mathbb{E} = B_{\boldsymbol{c},\sigma} = S_{\boldsymbol{c},\sigma}$ when $d = 1$, but when $d > 1$, $\mathbb{E} \neq B_{\boldsymbol{c},\sigma} \neq S_{\boldsymbol{c},\sigma}$. Next, we show that any $f \in \mathbb{E}$ can be extended to be monogenic between two parallel planes, where the monogencity is in the sense of Clifford analysis. This result can be regarded as the counterpart of [9] concerning Paley-Wiener theorem in the Clifford analysis setting. For basic knowledge of Clifford analysis, we refer the reader to [9].

Theorem 5.2. *Suppose that $f \in \mathbb{E}$. Then f can be monogenically extended to the region between two planes given by*

$$\left\{ x = y + \underline{x} \;\middle|\; \frac{\log\lambda}{\sigma} < y < -\frac{\log\lambda}{\sigma} \right\},$$

where λ is given in (2.13). *Moreover the extended function in the region above can be estimated by*

$$|f(y+\underline{x})| \leqslant \frac{C_{\boldsymbol{c},\sigma}}{1-\mathrm{e}^{-\sigma|y|+\log\lambda}},$$

where $C_{\boldsymbol{c},\sigma}$ is a constant depending on σ and $\mathbf{c}$.

Proof. Consider a possible Clifford vector $x = y + \underline{x} = y + x_1e_1 + \cdots + x_de_d$ that makes both of the following two integrals well defined,

$$\begin{aligned} f^+(x) &= (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}\langle\underline{\xi},\underline{x}\rangle}\mathrm{e}^{-y|\underline{\xi}|}\chi_+(\underline{\xi})\hat{f}(\underline{\xi})d\underline{\xi},\\ f^-(x) &= (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}\langle\underline{\xi},\underline{x}\rangle}\mathrm{e}^{y|\underline{\xi}|}\chi_-(\underline{\xi})\hat{f}(\underline{\xi})d\underline{\xi}, \end{aligned}$$

where

$$\chi_\pm(\underline{\xi}) = \frac{1}{2}\left(1 \pm \mathrm{i}\frac{\underline{\xi}}{|\underline{\xi}|}\right)$$

are the Fourier multipliers corresponding to the Cauchy kernels in, respectively, the upper and the lower spaces [12]. Recalling the partition of the integral domain $\mathbb{R}^d = \bigcup_{n=0}^{\infty} S_n$, we have

$$f^+(x) = (2\pi)^{-\frac{d}{2}}\sum_{n=0}^{\infty}\int_{S_n}\mathrm{e}^{\mathrm{i}\langle\underline{\xi},\underline{x}\rangle}\mathrm{e}^{-y|\underline{\xi}|}\chi_+(\underline{\xi}))c_nR(|\underline{\xi}|)d\underline{\xi}$$

$$= (2\pi)^{-\frac{d}{2}} \sum_{n=0}^{\infty} \int_0^1 \int_{S^{d-1}} \mathrm{e}^{\mathrm{i}\langle \sigma(n+r)\underline{\omega}, \underline{x}\rangle} \mathrm{e}^{-y\sigma(r+n)} \chi_+(\underline{\omega}) c_n R(\sigma(r+n))(\sigma r)^{d-1} dr d\underline{\omega}$$

$$= (2\pi)^{-\frac{d}{2}} \int_0^1 \int_{S^{d-1}} \mathrm{e}^{\mathrm{i}\langle \sigma r\underline{\omega}, \underline{x}\rangle} \mathrm{e}^{-\sigma y r} \chi_+(\underline{\omega}) R(\sigma r)(\sigma r)^{d-1} \left[\sum_{n=0}^{\infty} \mathrm{e}^{\mathrm{i}\langle \sigma n\underline{\omega}, \underline{x}\rangle} \mathrm{e}^{-\sigma y n} c_n\right] dr d\underline{\omega}. \tag{5.2}$$

Using the estimate (2.13), when

$$y > \frac{\log \lambda}{\sigma},$$

the infinite series in (5.2) is dominated by a geometric series, and the exchange of the infinite summation with the integration can be justified. Thus, we obtain a convergent integral. Replacing the infinite series with the dominating geometric series, we obtain the desired estimate of the integral, i.e.,

$$|f^+(x)| \leqslant \frac{C_{\mathbf{c},\sigma}}{1 - \mathrm{e}^{-\sigma y + \log \lambda}}.$$

Similar to [2], we can exchange the differentiation with the integration and verify the monogenicity of f^+ based on the monogenicity of $\mathrm{e}^{\mathrm{i}\langle \underline{\xi}, \underline{x}\rangle} \mathrm{e}^{-\mathrm{y}|\underline{\xi}|} \chi_+(\underline{\xi})$. The same properties for f^- with $y < -\frac{\log \lambda}{\sigma}$ can be proved similarly. □

Now we investigate the properties of the function $f \in \mathbb{E}$.

Lemma 5.3. *Any function $f \in \mathbb{E}$ takes the form*

$$f(\underline{t}) = |\underline{t}|^{-d} \sum_{k \in \mathbb{Z}_+} c_k \int_{k|\underline{t}|}^{(k+1)|\underline{t}|} r^{\frac{d}{2}} R\left(\frac{r}{|\underline{t}|}\right) J_{\frac{d-2}{2}}(r) dr, \quad \underline{t} \in \mathbb{R}^d. \tag{5.3}$$

Proof. Using the computing technique of the Fourier transform of radial functions, we have

$$f(\underline{t}) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \sum_{k=0} c_k \chi_{S_k}(\underline{\xi}) R(|\underline{\xi}|) \mathrm{e}^{\mathrm{i}\langle \underline{t}, \underline{\xi}\rangle} d\underline{\xi}$$

$$= (2\pi)^{-\frac{d}{2}} \int_0^{\infty} H_{1,\boldsymbol{c}}(r) r^{d-1} R(r) dr \int_{S^{d-1}} \mathrm{e}^{\mathrm{i}r|\underline{t}|\langle \frac{\underline{t}}{|\underline{t}|}, \underline{t}'\rangle} d\underline{t}', \quad \underline{\xi} \in \mathbb{R}^d.$$

Recalling the formula

$$\int_{S^{d-1}} \mathrm{e}^{\mathrm{i}r\langle \underline{\omega}, \underline{t}'\rangle} d\underline{t}' = (2\pi)^{\frac{d}{2}} r^{-\frac{d}{2}+1} J_{\frac{d-2}{2}}(r), \quad \underline{\omega} \in S^{d-1}, \quad r \in (0, \infty), \tag{5.4}$$

being independent of $\underline{\omega} \in S^{d-1}$, we obtain that

$$f(\underline{t}) = |\underline{t}|^{-\frac{d}{2}+1} \int_0^{\infty} r^{\frac{d}{2}} H_{1,\boldsymbol{c}}(r) R(r) J_{\frac{d-2}{2}}(r|\underline{t}|) dr, \quad \underline{t} \in \mathbb{R}^d.$$

Noting the definition of $H_{1,\boldsymbol{c}}$, recalling the decaying rate of $\boldsymbol{c}$ and using Lebesgue dominated convergence theorem, we get

$$f(\underline{t}) = |\underline{t}|^{-\frac{d}{2}+1} \int_0^{\infty} r^{\frac{d}{2}} \sum_{k \in \mathbb{Z}} c_k \chi(r-k) R(r) J_{\frac{d-2}{2}}(r|\underline{t}|) dr$$

$$= |\underline{t}|^{-\frac{d}{2}+1} \sum_{k \in \mathbb{Z}_+} c_k \int_k^{k+1} r^{\frac{d}{2}} R(r) J_{\frac{d-2}{2}}(r|\underline{t}|) dr$$

$$= |\underline{t}|^{-d} \sum_{k \in \mathbb{Z}_+} c_k \int_{k|\underline{t}|}^{(k+1)|\underline{t}|} r^{\frac{d}{2}} R\left(\frac{r}{|\underline{t}|}\right) J_{\frac{d-2}{2}}(r) dr.$$

□

We shall see from Theorem 5.7 that any $f \in \mathbb{E}$ can be radially expressed as g_m defined by

$$g_m(t) = t^{-2m-3} \sum_{k \in \mathbb{Z}_+} c_k \int_{kt}^{(k+1)t} r^{\frac{2m+3}{2}} R\left(\frac{r}{t}\right) J_{m+\frac{1}{2}}(r) dr. \tag{5.5}$$

Therefore, to prove Theorem 5.7, we need to investigate g_m. Using the identity

$$J_{m+\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi}}(-1)^m z^{m+\frac{1}{2}}\mathcal{D}_z^m(\mathrm{sinc}(z))$$

and the Fourier expansion of

$$R(r)=\sum_{j\in\mathbb{Z}}R_j\mathrm{e}^{-\mathrm{i}2\pi jr},$$

we get

$$\begin{aligned}g_m(t)&=t^{-2m-3}\sum_{k\in\mathbb{Z}_+}c_k\int_{kt}^{(k+1)t}r^{\frac{2m+3}{2}}R\left(\frac{r}{t}\right)J_{m+\frac{1}{2}}(r)dr\\&=t^{-2m-3}\sum_{k\in\mathbb{Z}_+}c_k\int_{kt}^{(k+1)t}r^{\frac{2m+3}{2}}\sum_{j\in\mathbb{Z}}R_j\mathrm{e}^{-\mathrm{i}2\pi j\frac{r}{t}}\sqrt{\frac{2}{\pi}}(-1)^m r^{m+\frac{1}{2}}\mathcal{D}_r^m(\mathrm{sinc}(r))dr,\end{aligned}$$

which can be simplified as

$$g_m(t)=\sqrt{\frac{2}{\pi}}(-1)^m t^{-2m-3}\sum_{k\in\mathbb{Z}_+}c_k\sum_{j\in\mathbb{Z}}R_j\int_{kt}^{(k+1)t}r^{2m+2}\mathrm{e}^{-\mathrm{i}2\pi j\frac{r}{t}}\mathcal{D}_r^m(\mathrm{sinc}(r))dr. \tag{5.6}$$

Then

$$\begin{aligned}g_{m+1}(t)&=\sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2m-5}\sum_{k\in\mathbb{Z}_+}c_k\sum_{j\in\mathbb{Z}}R_j\int_{kt}^{(k+1)t}r^{2m+4}\mathrm{e}^{-\mathrm{i}2\pi j\frac{r}{t}}\mathcal{D}_r^{m+1}(\mathrm{sinc}(r))dr\\&=\sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2m-5}\sum_{k\in\mathbb{Z}_+}c_k\sum_{j\in\mathbb{Z}}R_j\int_{kt}^{(k+1)t}r^{2m+3}\mathrm{e}^{-\mathrm{i}2\pi j\frac{r}{t}}d[\mathcal{D}_r^m(\mathrm{sinc}(r))]\\&=\sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2m-5}\sum_{k\in\mathbb{Z}_+}c_k\sum_{j\in\mathbb{Z}}R_j r^{2m+3}\mathrm{e}^{-\mathrm{i}2\pi j\frac{r}{t}}\mathcal{D}_r^m(\mathrm{sinc}(r))|_{r=kt}^{(k+1)t}\\&\quad-\sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2m-5}\sum_{k\in\mathbb{Z}_+}c_k\sum_{j\in\mathbb{Z}}R_j\int_{kt}^{(k+1)t}\mathcal{D}_r^m(\mathrm{sinc}(r))d[r^{2m+3}\mathrm{e}^{-\mathrm{i}2\pi j\frac{r}{t}}]\\&=\sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2m-5}\sum_{k\in\mathbb{Z}_+}c_k\sum_{j\in\mathbb{Z}}R_j r^{2m+3}\mathrm{e}^{-\mathrm{i}2\pi j\frac{r}{t}}\mathcal{D}_r^m(\mathrm{sinc}(r))|_{r=kt}^{(k+1)t}\\&\quad-(2m+3)\sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2m-5}\sum_{k\in\mathbb{Z}_+}c_k\sum_{j\in\mathbb{Z}}R_j\int_{kt}^{(k+1)t}\mathcal{D}_r^m(\mathrm{sinc}(r))r^{2m+2}\mathrm{e}^{-\mathrm{i}2\pi j\frac{r}{t}}dr\\&\quad+\sqrt{\frac{2}{\pi}}\mathrm{i}2\pi(-1)^{m+1}t^{-2m-6}\sum_{k\in\mathbb{Z}_+}c_k\sum_{j\in\mathbb{Z}}jR_j\int_{kt}^{(k+1)t}\mathcal{D}_r^m(\mathrm{sinc}(r))r^{2m+3}\mathrm{e}^{-\mathrm{i}2\pi j\frac{r}{t}}dr.\end{aligned}$$

For simplicity, we denote it by

$$g_{m+1}(t)=J_1+(2m+3)t^{-2}g_m(t)+J_2, \tag{5.7}$$

where

$$J_1=\sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2m-5}\sum_{k\in\mathbb{Z}_+}c_k\sum_{j\in\mathbb{Z}}R_j r^{2m+3}\mathrm{e}^{-\mathrm{i}2\pi j\frac{r}{t}}\mathcal{D}_r^m(\mathrm{sinc}(r))|_{r=kt}^{(k+1)t}$$

and

$$J_2=\sqrt{\frac{2}{\pi}}\mathrm{i}2\pi(-1)^{m+1}t^{-2m-6}\sum_{k\in\mathbb{Z}_+}c_k\sum_{j\in\mathbb{Z}}jR_j\int_{kt}^{(k+1)t}\mathcal{D}_r^m(\mathrm{sinc}(r))r^{2m+3}\mathrm{e}^{-\mathrm{i}2\pi j\frac{r}{t}}dr.$$

We first investigate J_1 as follows.

Lemma 5.4.

$$J_1 = \left(\sum_{j\in\mathbb{Z}} R_j\right)(-1)^m\sqrt{\frac{2}{\pi}}t^{-2}\mathcal{D}_t^m\left(\frac{v_G''(t)}{t}\right). \tag{5.8}$$

Proof.

$$\begin{aligned} J_1 &= \sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2m-5}\sum_{k\in\mathbb{Z}_+} c_k \sum_{j\in\mathbb{Z}} R_j r^{2m+3}\mathrm{e}^{-\mathrm{i}2\pi j\frac{r}{t}}\mathcal{D}_r^m(\operatorname{sinc}(r))|_{r=kt}^{(k+1)t} \\ &= \sum_{j\in\mathbb{Z}} R_j\sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2}\sum_{k\in\mathbb{Z}_+} c_k(k+1)^{2m+3}\mathcal{D}_r^m(\operatorname{sinc}(r))|_{r=(k+1)t} \\ &\quad - \sum_{j\in\mathbb{Z}} R_j\sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2}\sum_{k\in\mathbb{Z}_+} c_k k^{2m+3}\mathcal{D}_r^m(\operatorname{sinc}(r))|_{r=kt}. \end{aligned}$$

A further calculation leads to

$$J_1 = \left(\sum_{j\in\mathbb{Z}} R_j\right)\sqrt{\frac{2}{\pi}}(-1)^{m+1}t^{-2}\sum_{k\in\mathbb{N}}(c_{k-1}-c_k)k^{2m+3}\mathcal{D}_r^m(\operatorname{sinc}(r))|_{r=kt}. \tag{5.9}$$

By comparing (3.13) and (3.11) in Lemma 3.4, we conclude the proof of (5.8). □

Lemma 5.5.

$$J_2 = -\frac{1}{t}\frac{d}{dt}g_m(t) - (2m+3)\frac{1}{t^2}g_m(t) - J_1, \quad t\in\mathbb{R}. \tag{5.10}$$

Proof. Differentiating to both sides of (5.6)

$$g_m(t) = \sqrt{\frac{2}{\pi}}(-1)^m t^{-2m-3}\sum_{k\in\mathbb{Z}_+} c_k\sum_{j\in\mathbb{Z}} R_j\int_{kt}^{(k+1)t} r^{2m+2}\mathrm{e}^{-\mathrm{i}2\pi j\frac{r}{t}}\mathcal{D}_r^m(\operatorname{sinc}(r))dr,$$

we have

$$\begin{aligned} \frac{d}{dt}g_m(t) &= -(2m+3)\sqrt{\frac{2}{\pi}}(-1)^m t^{-2m-4}\sum_{k\in\mathbb{Z}_+} c_k\sum_{j\in\mathbb{Z}} R_j\int_{kt}^{(k+1)t} r^{2m+2}\mathrm{e}^{-\mathrm{i}2\pi j\frac{r}{t}}\mathcal{D}_r^m(\operatorname{sinc}(r))dr \\ &\quad + \sqrt{\frac{2}{\pi}}(-1)^m t^{-2m-3}\frac{d}{dt}\left(\sum_{k\in\mathbb{Z}_+} c_k\sum_{j\in\mathbb{Z}} R_j\int_{kt}^{(k+1)t} r^{2m+2}\mathrm{e}^{-\mathrm{i}2\pi j\frac{r}{t}}\mathcal{D}_r^m(\operatorname{sinc}(r))dr\right) \\ &= -(2m+3)\frac{1}{t}g_m(t) \\ &\quad + \sqrt{\frac{2}{\pi}}(-1)^m t^{-2m-3}\sum_{k\in\mathbb{Z}_+} c_k\sum_{j\in\mathbb{Z}} R_j\int_{kt}^{(k+1)t} r^{2m+3}\frac{2\pi \mathrm{i}j}{t^2}\mathrm{e}^{-\mathrm{i}2\pi j\frac{r}{t}}\mathcal{D}_r^m(\operatorname{sinc}(r))dr \\ &\quad + \sqrt{\frac{2}{\pi}}(-1)^m t^{-2m-3}\sum_{k\in\mathbb{Z}_+}(k+1)c_k\sum_{j\in\mathbb{Z}} R_j r^{2m+2}\mathrm{e}^{-\mathrm{i}2\pi j\frac{r}{t}}\mathcal{D}_r^m(\operatorname{sinc}(r))|_{r=(k+1)t} \\ &\quad - \sqrt{\frac{2}{\pi}}(-1)^m t^{-2m-3}\sum_{k\in\mathbb{Z}_+} kc_k\sum_{j\in\mathbb{Z}} R_j r^{2m+2}\mathrm{e}^{-\mathrm{i}2\pi j\frac{r}{t}}\mathcal{D}_r^m(\operatorname{sinc}(r))|_{r=kt} \\ &= -(2m+3)\frac{1}{t}g_m(t) - tJ_2 \\ &\quad + \sqrt{\frac{2}{\pi}}(-1)^m t^{-1}\sum_{k\in\mathbb{Z}_+}(k+1)^{2m+3}c_k\sum_{j\in\mathbb{Z}} R_j\mathcal{D}_r^m(\operatorname{sinc}(r))|_{r=(k+1)t} \\ &\quad - \sqrt{\frac{2}{\pi}}(-1)^m t^{-1}\sum_{k\in\mathbb{Z}_+} k^{2m+3}c_k\sum_{j\in\mathbb{Z}} R_j\mathcal{D}_r^m(\operatorname{sinc}(r))|_{r=kt} \end{aligned}$$

$$= -(2m+3)\frac{1}{t}g_m(t) - tJ_2$$
$$+ \left(\sum_{j\in\mathbb{Z}} R_j\right)\sqrt{\frac{2}{\pi}}(-1)^m t^{-1}\sum_{k\in\mathbb{N}} k^{2m+3}(c_{k-1}-c_k)\mathcal{D}_r^m(\mathrm{sinc}(r))|_{r=kt}.$$

By noting (5.9), we get

$$\frac{d}{dt}g_m(t) = -(2m+3)\frac{1}{t}g_m(t) - tJ_2 - tJ_1,$$

from which we conclude that

$$J_2 = -\frac{1}{t}\frac{d}{dt}g_m(t) - (2m+3)\frac{1}{t^2}g_m(t) - J_1. \qquad \square$$

Corollary 5.6. *Suppose that the sequence of functions* $\{g_m : m \in \mathbb{Z}_+\}$ *is defined in* (5.5). *Then for any* $m \in \mathbb{Z}_+$, g_m *satisfies the following recursive equation*:

$$g_{m+1}(t) = -\mathcal{D}_t g_m(t), \quad t \in \mathbb{R}. \tag{5.11}$$

Proof. This is a direct consequence of (5.7) and (5.10). $\square$

Theorem 5.7. *Suppose that* $f \in \mathbb{E}$. *Then for odd* $d = 3, 5, \ldots,$ *the following identity holds,*

$$f(\underline{t}) = g_{\frac{d-3}{2}}(|\underline{t}|), \quad \underline{t} \in \mathbb{R}^d,$$

where g_m *is determined by the recursive equation* $g_{m+1}(t) = -\frac{1}{t}\frac{d}{dt}g_m(t), t \in \mathbb{R}$ *with the initial function*

$$g_0(t) = \sqrt{\frac{2}{\pi}}t^{-3}\sum_{k\in\mathbb{Z}_+} c_k \sum_{j\in\mathbb{Z}} R_j \int_{kt}^{(k+1)t} r\sin r e^{-\mathrm{i}2\pi j\frac{r}{t}}dr.$$

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References

1 Chen Q, Li L, Malonek H. The construction of generalized B-spline low-pass filters related to possibility density. Sci China Math, 2012, 55, 12: 2481–2491

2 Chen Q, Qian T. Sampling theorem and multiscale specturm based on nonlinear Foureir atoms. Appl Anal, 2009, 88: 903–919

3 Chen Q, Wang Y B, Wang Y. A sampling theorem for non-bandlimited signals using generalized sinc functions. Comput Math Appl, 2008, 56: 1650–1661

4 Feichtinger H G, Gröchenig K. Error analysis in regular and irregular sampling theory. Appl Anal, 1993, 50: 167–189

5 Gröchenig K, Heil C, Walnut D. Nonperiodic sampling and the local three squares theorem. Ark Mat, 2000, 38: 77–92

6 García A G, Villalón P G. On the truncation error of generalized sampling expansions in shift-invariant spaces. Sampl Theory Signal Image Process, 2007, 6: 53–69

7 Heil C, Colella D. Matrix refinement equations: Existence and uniqueness. J Fourier Anal Appl, 1996, 2: 363–377

8 Heil C, Koo Y Y, Lim J K. Duals of frame sequences. Acta Appl Math, 2009, 107: 75–90

9 Kou K I, Qian T. The Paley-Wiener theorem in R_n with the Clifford analysis setting. J Funct Anal, 2002, 189: 227–241

10 Kou K I, Qian T. Shannon sampling in the Clifford analysis setting. Z Anal Anwendungen, 2005, 24: 825–842

11 Li Y, Chen Q, Qian T, et al. Sampling error analysis and some properties of non-bandlimited signals that are reconstructed by generalized sinc functions. Appl Anal, doi: 10.1080/00036811.2013.769135

12 Li C, McIntosh A, Qian T. Clifford algebras, Fourier transforms, and singular convolution operators on Lipschitz surfaces. Rev Mat Iberoamericana, 1994, 10: 665–695

13 Watson G N. A Treatise on the Theory of Bessel Functions, 2nd Ed. Cambridge: Cambridge University Press, 1966

14 Woodward P M. Probability and Information Theory with Applications to Radar. London: Pergamon Press, 1953

15 Yang S, Peng L. Raising approximation order of refinable vector by increasing multiplicity. Sci China Ser A, 2006, 49: 86–97

16 Yang S, Peng L. Construction of high order balanced multiscaling functions via PTST. Sci China Ser F, 2006, 49: 504–515

Appendix: Proof of (2.18)

Below we prove $\phi_F = \mathrm{sinc}_G$. It follows directly from (2.16) and (3.2) that

$$\begin{aligned}\phi_F(t) &= \sqrt{\frac{\pi}{2}}\mathcal{F}^{-1}H_{1,\boldsymbol{c}}(t) = \sqrt{\frac{\pi}{2}}\mathcal{F}^{-1}\bigg(\sum_{n\in\mathbb{Z}} c_n\chi(\cdot - n)\bigg)(t)\\ &= \sqrt{\frac{\pi}{2}}\sum_{n\in\mathbb{Z}} c_n(\mathcal{F}^{-1}\chi(\cdot - n))(t)\\ &= \sqrt{\frac{\pi}{2}}\sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{\mathrm{i}nt}\mathcal{F}^{-1}\chi(t).\end{aligned}$$

Using

$$\mathcal{F}^{-1}\chi(t) = \frac{1}{\sqrt{2\pi}}\int_0^1 \mathrm{e}^{\mathrm{i}\xi t}d\xi = \frac{1}{\sqrt{2\pi}}\frac{\mathrm{e}^{\mathrm{i}t}-1}{\mathrm{i}t}$$

gives us

$$\begin{aligned}\phi_F(t) &= \sqrt{\frac{\pi}{2}}\sum_{n\in\mathbb{Z}} c_n\mathrm{e}^{\mathrm{i}nt}\frac{1}{\sqrt{2\pi}}\frac{\mathrm{e}^{\mathrm{i}t}-1}{\mathrm{i}t}\\ &= \frac{1}{2}\sum_{n\in\mathbb{Z}} c_n\frac{\mathrm{e}^{\mathrm{i}(n+1)t}-\mathrm{e}^{\mathrm{i}nt}}{\mathrm{i}t}\\ &= \frac{1}{2}\sum_{n\in\mathbb{Z}} c_n\frac{\sin(n+1)t - \sin nt}{t} - \mathrm{i}\frac{1}{2}\sum_{n\in\mathbb{Z}} c_n\frac{\cos(n+1)t-\cos nt}{t}\\ &:= \mathrm{Re} + \mathrm{Im}.\end{aligned}$$

Using the symmetry, i.e., $c_n = c_{-n-1}$ and in particular $c_0 = c_{-1}$, we have

$$\begin{aligned}\mathrm{Re} &= \frac{1}{2}\sum_{n\in\mathbb{Z}} c_n\frac{\sin(n+1)t-\sin nt}{t}\\ &= \frac{1}{2}\sum_{n\in\mathbb{Z}\setminus\{0\}}(c_{n-1}-c_n)\frac{\sin nt}{t}\\ &= \frac{1}{2}\sum_{n=1}^{\infty}(c_{n-1}-c_n)\frac{\sin nt}{t} + \frac{1}{2}\sum_{n=1}^{\infty}(c_{-n-1}-c_{-n})\frac{\sin(-n)t}{t}\\ &= \frac{1}{2}\sum_{n=1}^{\infty}(c_{n-1}-c_n)\frac{\sin nt}{t} + \frac{1}{2}\sum_{n=1}^{\infty}(c_n-c_{n-1})\frac{\sin(-n)t}{t}\\ &= \sum_{n=1}^{\infty}(c_{n-1}-c_n)\frac{\sin nt}{t}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Im} &= \frac{1}{2}\sum_{n\in\mathbb{Z}} c_n\frac{\cos(n+1)t-\cos nt}{t}\\ &= \frac{1}{2}\sum_{n\in\mathbb{Z}\setminus\{0\}}(c_{n-1}-c_n)\frac{\cos nt}{t} \quad (\text{here } \ c_0 = c_{-1})\\ &= \frac{1}{2}\sum_{n=1}^{\infty}(c_{n-1}-c_n)\frac{\cos nt}{t} + \frac{1}{2}\sum_{n=1}^{\infty}(c_{-n-1}-c_{-n})\frac{\cos(-n)t}{t}\\ &= \frac{1}{2}\sum_{n=1}^{\infty}(c_{n-1}-c_n)\frac{\cos nt}{t} + \frac{1}{2}\sum_{n=1}^{\infty}(c_n-c_{n-1})\frac{\cos(-n)t}{t}\end{aligned}$$

$$= 0.$$

Therefore,

$$\phi_F(t) = \sum_{n=1}^{\infty}(c_{n-1} - c_n)\frac{\sin nt}{t}.$$

On the other hand, by $F(z) = \frac{G(z)-1}{z-1}$, we get

$$G(z) = 1 + (z-1)F(z) = 1 + \sum_{n=0}^{\infty} c_n z^{n+1} - \sum_{n=0}^{\infty} c_n z^n$$
$$= 1 - c_0 + \sum_{n=1}^{\infty}(c_{n-1} - c_n)z^n.$$

The formula $G(\mathrm{e}^{\mathrm{i}t}) = u_G(t) + \mathrm{i}v_G(t)$ leads to

$$\mathrm{sinc}_G(t) = v_G(t)/t = \sum_{n=1}^{\infty}(c_{n-1} - c_n)\sin(nt)/t.$$

Now we conclude the proof of (2.18).