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A sharper uncertainty principle ☆

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Abstract

The work strengthens the result established by L. Cohen on uncertainty principle involving phase derivative. We propose stronger uncertainty principles not only in the classical setting for Fourier transform, but also for self-adjoint operators. We also deduce the conditions that give rise to the equal relation of the uncertainty principle. Examples are provided to show that the new uncertainty principle is truly sharper than the existing ones in literature.

Keywords: Uncertainty principle; Phase derivative; Hardy spaces; Sobolev spaces; Self-adjoint operator

1. Introduction

The uncertainty principle is partly a description of a characteristic feature of quantum mechanical systems. It did not really sink into the minds of signal analysts until Gabor's fundamental work [12] in 1946 (see [11]):

$$\sigma_t \sigma_\omega \geqslant \frac{1}{2}, \tag{1.1}$$

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where σ_t and σ_ω are the duration and bandwidth of a signal $s(t) \in L^2(\mathbb{R})$ with $\|s\|_2 = 1$ defined by

$$\sigma_t^2 = \int_{-\infty}^{\infty} \bigl(t - \langle t\rangle\bigr)^2 \bigl|s(t)\bigr|^2\, dt \tag{1.2}$$

and

$$\sigma_\omega^2 = \int_{-\infty}^{\infty} \bigl(\omega - \langle \omega\rangle\bigr)^2 \bigl|\hat{s}(\omega)\bigr|^2\, d\omega, \tag{1.3}$$

where $\langle t\rangle$ and $\langle \omega\rangle$ are the means of time t and Fourier frequency ω, respectively, defined by

$$\langle t\rangle = \int_{-\infty}^{\infty} t\bigl|s(t)\bigr|^2\, dt$$

and

$$\langle \omega\rangle = \int_{-\infty}^{\infty} \omega\bigl|\hat{s}(\omega)\bigr|^2\, d\omega,$$

where $\hat{s}(\omega)$ is the Fourier transformation of $s(t)$.

The Fourier transform of $s \in L^1(\mathbb{R})$ is defined by

$$\hat{s}(\omega) = \mathbf{F}s(\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\omega} s(t)\, dt. \tag{1.4}$$

If $\hat{s}$ is also in $L^1(\mathbb{R})$, then the inversion Fourier transform formula holds, that is

$$s(t) = \mathbf{F}^{-1}\hat{s}(t) = (\hat{s})^\vee(t) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\omega} \hat{s}(\omega)\, d\omega, \quad \text{a.e.} \tag{1.5}$$

It is standard knowledge that through a density argument the restricted Plancherel Theorem

$$\|\hat{s}\|_2^2 = \|s\|_2^2, \quad s \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$$

may be extended to $L^2(\mathbb{R})$. Below, when we use the formulas (1.4) and (1.5) for $L^2(\mathbb{R})$ functions, we keep in mind that the convergence of the integrals is in the L^2 sense.

A number of different forms of the uncertainty principle arose through mathematical formulations since Gabor's work [1,4,8,9,11,15,16]. The inequality (1.1) is the most concise version but not the best one. In fact, a stronger result is available [3]:

$$\sigma_t\sigma_\omega \geqslant \frac{1}{2}\sqrt{1+4\mathrm{Cov}^2}, \tag{1.6}$$

where

$$\mathrm{Cov} = \langle t\varphi'(t)\rangle - \langle t\rangle\langle\omega\rangle = \int_{-\infty}^{\infty} t\varphi'(t)\left|s(t)\right|^2 dt - \langle t\rangle\langle\omega\rangle$$

is the covariance of the signal $s(t)$, defined in [3], where, as usual, $s(t) = \rho(t)e^{i\varphi(t)}$, $\rho(t) = |s(t)|$, $\varphi(t)$ is a real-valued function, and $\varphi'(t)$ is the classical derivative of $\varphi(t)$. The covariance has a symmetric representation

$$\mathrm{Cov} = \langle -\omega\psi(\omega)\rangle - \langle t\rangle\langle\omega\rangle = \int_{-\infty}^{\infty} -\omega\psi'(\omega)\left|\hat{s}(\omega)\right|^2 d\omega - \langle t\rangle\langle\omega\rangle,$$

where $\hat{s}(\omega) = |\hat{s}(\omega)|e^{i\psi(\omega)}$, $\psi(\omega)$ is a real-valued function, and $\psi'(\omega)$ is the classical derivative of $\psi(\omega)$. It was shown in [3] that the equality (1.6) holds if and only if $s(t) = e^{\alpha(t-\langle t\rangle)^2+\gamma}e^{-i[\beta(t-\langle t\rangle)^2+\langle\omega\rangle t]}$, where α, β and γ are arbitrary constants and $\alpha < 0$. However, both the statement and proof of (1.6) depend on the classical differentiability of $s(t) = \rho(t)e^{i\varphi(t)}$, $\rho(t)$ and $\varphi(t)$. General signals do not have such good smoothness properties. In order to establish those fundamental results for signals in general function spaces [7] defines derivatives of $s(t)$, $\rho(t)$ and $\varphi(t)$ through non-tangential boundary limits when $s(t), \omega\hat{s}(\omega) \in L^2(\mathbb{R})$ (i.e., s is in the Sobolev space) and offers a proof for the uncertainty principle (1.6) for $s(t)$ in the Sobolev space with the extra condition $ts(t) \in L^2(\mathbb{R})$ (i.e., $\hat{s}$ is also in the Sobolev space).

In this paper, we propose a form of uncertainty principle strictly stronger than (1.6), that is,

$$\sigma_t^2\sigma_\omega^2 \geqslant \frac{1}{4} + \left[\int \left|\big(t-\langle t\rangle\big)\big(\varphi'(t)-\langle\omega\rangle\big)\right|\left|s(t)\right|^2 dt\right]^2, \tag{1.7}$$

where $\varphi'(t)$ is suitably defined in our proofs.

It is the stronger uncertainty principle (1.7) that inspires us to study uncertainty principles for operators. There have been many studies on uncertainty principles for operators [2,3,11,14,16].

[10] gives an uncertainty principle for self-adjoint operators as follows:

$$\left\|(A-\alpha)s\right\|\left\|(B-\beta)s\right\| \geqslant \frac{1}{2}\left|\langle [A,B]s,s\rangle\right|, \quad s\in D(AB)\cap D(BA), \tag{1.8}$$

where A and B are self-adjoint operators, $\alpha,\beta\in\mathbb{C}$, and $[A,B] \triangleq AB-BA$.

[3] gives a stronger uncertainty principle for self-adjoint operators A, B:

$$\left\|(A-\alpha)s\right\|\left\|(B-\beta)s\right\| \geqslant \frac{1}{2}\sqrt{\left|\langle [A,B]s,s\rangle\right|^2 + \left|\langle [A-\alpha I, B-\beta I]_+ s,s\rangle\right|^2},$$
$$s\in D(AB)\cap D(BA), \tag{1.9}$$

where I is the identity operator and $[A-\alpha I, B-\beta I]_+ = (A-\alpha I)(B-\beta I)+(B-\beta I)(A-\alpha I)$.

Based on these results for self-adjoint operators, [16] derives the same inequalities for symmetric or normal operators in a Hilbert space $\mathcal{H}$.

If we assume that operators A and B are given by

$$As(t) = ts(t), \qquad B = \frac{1}{i}\frac{ds}{dt} \tag{1.10}$$

on the domains of all $s \in L^2(\mathbb{R})$ such that $ts(t) \in L^2(\mathbb{R})$ or $\omega\hat{s}(\omega) \in L^2(\mathbb{R})$, respectively, and $\alpha = \langle t\rangle$, $\beta = \langle\omega\rangle$, the formula (1.8) is reduced to (1.1) and the formula (1.9) is reduced to (1.6). In the present paper, we prove a corresponding uncertainty principle for self-adjoint operators that is stronger than (1.9). In fact, the obtained uncertainty principle reduces to (1.7) if the operators are taken to be those defined through (1.10).

The paper is organized as follows. In Section 2, we will prove the stronger form of uncertainty principle for a signal $s(t) = \rho(t)e^{i\varphi(t)}$ for which $s'(t)$, $\rho'(t)$ and $\varphi'(t)$ exist at all points in the classical derivative sense. Two examples are given to show that the lower bound estimate in our uncertainty principle is strictly sharper than those in the literature. In Section 3, we will use the Fourier transform derivative instead of the classical derivative. Once we use the Fourier transform derivative, we can prove the stronger form of uncertainty principle for signals $s(t)$ with the assumptions $s(t), \omega\hat{s}(\omega), ts(t) \in L^2(\mathbb{R})$. In Section 4, we prove the new form of uncertainty principle for signals $s(t)$ with the assumptions $s(t), \omega\hat{s}(\omega) \in L^2(\mathbb{R})$ and $zs_{\pm}(z) \in H^2(\mathbb{C}^{\pm})$, where $s_{\pm}$ are, respectively, the projections of s onto the Hardy $H^2(\mathbb{C}^{\pm})$ spaces (see Section 4). Sections 3 and 4 adopt different strategies. In Section 3 an absolutely continuous representative function of the Sobolev space function is used, and thus the proof requires delicate analysis. In Section 4 by using Hardy spaces decomposition everything can be done in the upper- and lower-half spaces and thus the proof becomes straightforward. The Hardy space decomposition strategy accordingly uses Hardy–Sobolev derivatives that were originally introduced in [6]. In Section 5, we derive our new stronger uncertainty principle for self-adjoint operators. We note that [7] contains a technical error (see Section 4) in the proof of the Cohen type uncertainty principle for non-smooth functions. In Sections 3 and 4 we offer two different strategies to treat the error, as well as replace the result with the stronger uncertainty principle.

We note that, with the classical L^2-setting, what we provide in this paper, as far as we are aware of, is the strongest uncertainty principle so far, but with the weakest assumptions. We also characterize, under certain necessary conditions, the forms of the signals that make the equal relation in the uncertainty principle to hold. The Sobolev space condition added is merely to guarantee existence of phase and amplitude derivatives in either the distribution or the Hardy space non-tangential boundary limit sense so as to accommodate the stronger forms of uncertainty principle.

In the sequel we denote by $\mathbb{R}$ the real axis, by $\mathbb{C}$ the complex plane, and by $\mathbb{C}^+$ and $\mathbb{C}^-$ the upper- and lower-half complex planes, respectively.

2. Classical derivative and uncertainty principle

The following result can be found in [3].

Lemma 2.1. *Let $s(t) = \rho(t)e^{i\varphi(t)} \in L^2(\mathbb{R})$ and $\|s\|_2 = 1$. Assume that the classical derivatives $\rho'(t)$, $\varphi'(t)$, $s'(t)$ exist at all points, and $s'(t)$ is in $L^2(\mathbb{R})$. Then there holds*

$$\sigma_\omega^2 = \int_{-\infty}^{\infty} \rho'^2(t)\,dt + \int_{-\infty}^{\infty} \left[\varphi'(t) - \langle\omega\rangle\right]^2 \rho^2(t)\,dt. \tag{2.11}$$

Theorem 2.2. *Assume $s(t)=\rho(t)e^{i\varphi(t)}$ and $\|s\|_2=1$ and the classical derivatives $\rho'(t)$, $\varphi'(t)$ and $s'(t)$ exist at all points, and $s'(t), ts(t)\in L^2(\mathbb{R})$. Then*

$$\sigma_t^2\sigma_\omega^2 \geqslant \frac{1}{4}+\left[\int_{-\infty}^{\infty}\left|\bigl(t-\langle t\rangle\bigr)\bigl(\varphi'(t)-\langle\omega\rangle\bigr)\right|\rho^2(t)\,dt\right]^2. \tag{2.12}$$

If $\varphi'(t)$ is continuous and ρ is non-zero almost everywhere, then the equality holds if and only if $s(t)$ is a chirp signal with the form

$$s(t)=e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_2]}, \tag{2.13}$$

$$s(t)=e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_3]}, \tag{2.14}$$

$$s(t)=\begin{cases} e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_4]} & \text{if } t\geqslant\langle t\rangle,\\ e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_5]} & \text{if } t<\langle t\rangle,\end{cases} \tag{2.15}$$

or

$$s(t)=\begin{cases} e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_6]} & \text{if } t\geqslant\langle t\rangle,\\ e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_7]} & \text{if } t<\langle t\rangle,\end{cases} \tag{2.16}$$

for some $d_1,d_2,d_3,d_4,d_5,d_6,d_7,\zeta,\varepsilon\in\mathbb{R}$, $\zeta,\varepsilon>0$, where $e^{2d_1}\sqrt{\frac{\zeta\pi}{2}}=1$.

Proof. To prove inequality (2.12), by taking into account Lemma 2.1, it suffices to prove two separated inequalities:

$$\int_{-\infty}^{\infty}\bigl(t-\langle t\rangle\bigr)^2\left|s(t)\right|^2dt\int_{-\infty}^{\infty}\rho'^2(t)\,dt\geqslant\frac{1}{4},$$

and

$$\int_{-\infty}^{\infty}\bigl(t-\langle t\rangle\bigr)^2\left|s(t)\right|^2dt\int_{-\infty}^{\infty}\bigl[\varphi'(t)-\langle\omega\rangle\bigr]^2\rho^2(t)\,dt\geqslant\left[\int_{-\infty}^{\infty}\left|\bigl(t-\langle t\rangle\bigr)\bigl[\varphi'(t)-\langle\omega\rangle\bigr]\right|\rho^2(t)\,dt\right]^2.$$

The proof of the first inequality is proceeded as follows. Due to the smoothness and integrability assumptions of $\rho^2(t)$ and $t\rho^2(t)$, for two particular sequences of numbers, M_n, N_n, tending to infinity as $n\to\infty$, we have

$$\frac{1}{4}=\left[\frac{1}{2}\int_{-\infty}^{\infty}\rho^2(t)\,dt\right]^2=\left[\lim_{n\to\infty}\frac{1}{2}\int_{-N_n}^{M_n}\rho^2(t)\,dt\right]^2$$

$$=\left\{\lim_{n\to\infty}\left[\frac{1}{2}\bigl(t-\langle t\rangle\bigr)\rho^2(t)\Big|_{-N_n}^{M_n}\right]-\lim_{n\to\infty}\left[\frac{1}{2}\int_{-N_n}^{M_n}\bigl(t-\langle t\rangle\bigr)2\rho(t)\rho'(t)\,dt\right]\right\}^2$$

$$= \left[\int_{-\infty}^{\infty} \big(t - \langle t\rangle\big)\rho(t)\rho'(t)\,dt \right]^2$$
$$\leqslant \left[\int_{-\infty}^{\infty} \big|\big(t - \langle t\rangle\big)\rho(t)\rho'(t)\big|\,dt \right]^2$$
$$\leqslant \int_{-\infty}^{\infty} \big|\big(t - \langle t\rangle\big)\rho(t)\big|^2\,dt \int_{-\infty}^{\infty} \big|\rho'(t)\big|^2\,dt. \tag{2.17}$$

To show the second inequality we have

$$\left\{ \int_{-\infty}^{\infty} \big|\big(t - \langle t\rangle\big)\big[\varphi'(t) - \langle\omega\rangle\big]\big|\rho^2(t)\,dt \right\}^2$$
$$= \left\{ \int_{-\infty}^{\infty} \big|\big(t - \langle t\rangle\big)\rho(t)\big[\varphi'(t) - \langle\omega\rangle\big]\rho(t)\big|\,dt \right\}^2$$
$$\leqslant \int_{-\infty}^{\infty} \big(t - \langle t\rangle\big)^2\rho^2(t)\,dt \int_{-\infty}^{\infty} \big[\varphi'(t) - \langle\omega\rangle\big]^2\rho^2(t)\,dt. \tag{2.18}$$

The inequality (2.12) follows from (2.17) and (2.18).

Next we deduce the conditions under which the equality holds in (2.12). The first equality in (2.17) holds if and only if $t - \langle t\rangle$ and $\rho'(t)$ have the same sign or opposite signs. The second equality in (2.17) is attained if and only if there exists a positive number ζ such that

$$\big|\big(t - \langle t\rangle\big)\rho(t)\big| = \zeta\big|\rho'(t)\big|.$$

If $t - \langle t\rangle$ and $\rho'(t)$ have the same sign, then $(t - \langle t\rangle)\rho(t) = \zeta\rho'(t)$, that is, $\frac{\rho'(t)}{\rho(t)} = \frac{1}{\zeta}(t - \langle t\rangle)$. Using indefinite integral, we have

$$\rho(t) = e^{\frac{1}{\zeta}(t-\langle t\rangle)^2+d}.$$

Obviously, the function $\rho(t) = e^{\frac{1}{\zeta}(t-\langle t\rangle)^2+d}$ such obtained cannot be in $L^2(\mathbb{R})$. Therefore, $t - \langle t\rangle$ and $\rho'(t)$ cannot have the same sign, but have to be of opposite signs. Then $-(t - \langle t\rangle)\rho(t) = \zeta\rho'(t)$, and, as consequence, $\frac{\rho'(t)}{\rho(t)} = -\frac{1}{\zeta}(t - \langle t\rangle)$. Using indefinite integral again we have

$$\rho(t) = e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}.$$

Since signals we discuss are of unit energy, we derive that ζ and d_1 should satisfy $e^{2d_1}\sqrt{\frac{\zeta\pi}{2}} = 1$.

The inequality (2.18) brings in conditions obeyed by the phase. The equality relation holds in (2.18) if and only if there exists a positive number ε such that

$$\big|\big(t - \langle t\rangle\big)\rho(t)\big| = \varepsilon\big|\big[\varphi'(t) - \langle\omega\rangle\big]\rho(t)\big|. \tag{2.19}$$

Deleting ρ that is a.e. non-zero, under the continuity assumption of φ', we have

$$\left|t-\langle t\rangle\right| = \varepsilon\left|\varphi'(t)-\langle\omega\rangle\right| \quad \text{for all } t.$$

Since the left-hand side is the absolute value of a linear function, there can be altogether four cases:

$$\varphi'(t) = \frac{1}{\varepsilon}\bigl(t-\langle t\rangle\bigr)+\langle\omega\rangle, \quad \text{or} \quad \varphi'(t) = -\frac{1}{\varepsilon}\bigl(t-\langle t\rangle\bigr)+\langle\omega\rangle,$$

or

$$\varphi'(t) = \begin{cases} \frac{1}{\varepsilon}(t-\langle t\rangle)+\langle\omega\rangle & \text{if } t \geqslant \langle t\rangle, \\ -\frac{1}{\varepsilon}(t-\langle t\rangle)+\langle\omega\rangle & \text{if } t < \langle t\rangle, \end{cases}$$

or

$$\varphi'(t) = \begin{cases} -\frac{1}{\varepsilon}(t-\langle t\rangle)+\langle\omega\rangle & \text{if } t \geqslant \langle t\rangle, \\ \frac{1}{\varepsilon}(t-\langle t\rangle)+\langle\omega\rangle & \text{if } t < \langle t\rangle. \end{cases}$$

When

$$\varphi'(t) = \frac{1}{\varepsilon}\bigl(t-\langle t\rangle\bigr)+\langle\omega\rangle,$$

then

$$\varphi(t) = \frac{1}{2\varepsilon}\bigl(t-\langle t\rangle\bigr)^2+\langle\omega\rangle t+d_2, \qquad s_1 = e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1} e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_2]}.$$

When

$$\varphi'(t) = -\frac{1}{\varepsilon}\bigl(t-\langle t\rangle\bigr)+\langle\omega\rangle,$$

then

$$\varphi(t) = -\frac{1}{2\varepsilon}\bigl(t-\langle t\rangle\bigr)^2+\langle\omega\rangle t+d_3, \qquad s_2 = e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1} e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_3]}.$$

When

$$\varphi'(t) = \begin{cases} \frac{1}{\varepsilon}(t-\langle t\rangle)+\langle\omega\rangle & \text{if } t \geqslant \langle t\rangle, \\ -\frac{1}{\varepsilon}(t-\langle t\rangle)+\langle\omega\rangle & \text{if } t < \langle t\rangle, \end{cases}$$

then

$$\varphi(t) = \begin{cases} \frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_4 & \text{if } t \geqslant \langle t\rangle, \\ -\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_5 & \text{if } t < \langle t\rangle, \end{cases}$$

and

$$s_3(t)=\begin{cases} e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_4]} & \text{if } t\geqslant\langle t\rangle,\\ e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_5]} & \text{if } t<\langle t\rangle.\end{cases}$$

Finally, when

$$\varphi'(t)=\begin{cases} -\frac{1}{\varepsilon}(t-\langle t\rangle)+\langle\omega\rangle & \text{if } t\geqslant\langle t\rangle,\\ \frac{1}{\varepsilon}(t-\langle t\rangle)+\langle\omega\rangle & \text{if } t<\langle t\rangle,\end{cases}$$

then

$$\varphi(t)=\begin{cases} -\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_6 & \text{if } t\geqslant\langle t\rangle,\\ \frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_7 & \text{if } t<\langle t\rangle,\end{cases}$$

and

$$s_4(t)=\begin{cases} e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_6]} & \text{if } t\geqslant\langle t\rangle,\\ e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_7]} & \text{if } t<\langle t\rangle.\end{cases}$$

Hence the equality in (2.12) is attained if and only if the signal s is of one of the four forms given by s_1, s_2, s_3 and s_4. □

Remark 2.3. We will use the relation

$$\langle\omega\rangle=\int_{-\infty}^{\infty}\varphi'(t)\rho^2(t)\,dt \tag{2.20}$$

that holds in a wide sense. For the classical derivative case it is proved in [3]. For the Hardy–Sobolev derivative case and the Fourier transform derivative case it is proved in, respectively, [6] and Lemma 3.5 of the present paper. By using the relation (2.20) we have

$$\begin{aligned}&\left[\int_{-\infty}^{\infty}\big(t-\langle t\rangle\big)\big[\varphi'(t)-\langle\omega\rangle\big]\rho^2(t)\,dt\right]^2\\ &=\left[\int_{-\infty}^{\infty}t\varphi'(t)\rho^2(t)\,dt-\langle\omega\rangle\int_{-\infty}^{\infty}t\rho^2(t)\,dt-\langle t\rangle\int_{-\infty}^{\infty}\varphi'(t)\rho^2(t)\,dt+\langle t\rangle\langle\omega\rangle\int_{-\infty}^{\infty}\rho^2(t)\,dt\right]^2\\ &=\left[\int_{-\infty}^{\infty}t\varphi'(t)\rho^2(t)\,dt-\langle\omega\rangle\langle t\rangle-\langle t\rangle\langle\omega\rangle+\langle t\rangle\langle\omega\rangle\right]^2=\mathrm{Cov}^2,\end{aligned}$$

and

$$\left[\int_{-\infty}^{\infty}\big(t-\langle t\rangle\big)\big[\varphi'(t)-\langle\omega\rangle\big]\rho^2(t)\,dt\right]^2\leqslant\left[\int_{-\infty}^{\infty}\big|t-\langle t\rangle\big|\big|\varphi'(t)-\langle\omega\rangle\big|\rho^2(t)\,dt\right]^2.$$

Therefore

$$\frac{1}{4}+\mathrm{Cov}^2\leqslant\frac{1}{4}+\left[\int_{-\infty}^{\infty}\big|t-\langle t\rangle\big|\big|\varphi'(t)-\langle\omega\rangle\big|\rho^2(t)\,dt\right]^2.$$

This shows that (2.12) is stronger than (1.6).

Next we show that there exist signals for which the lower bound in (2.12) is indeed strictly sharper than that in (1.6).

Example 2.4. Let

$$s(t)=\begin{cases}(\frac{\alpha}{\pi})^{\frac{1}{4}}e^{-\frac{\alpha}{2}(t-\langle t\rangle)^2}e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+\beta_1]} & \text{if } t\geqslant\langle t\rangle,\\ (\frac{\alpha}{\pi})^{\frac{1}{4}}e^{-\frac{\alpha}{2}(t-\langle t\rangle)^2}e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+\beta_2]} & \text{if } t<\langle t\rangle,\end{cases}\tag{2.21}$$

that is a signal of the form (2.15), where $\varepsilon,\beta_1,\beta_2\in\mathbb{R}$, and $\alpha>0$. Then

$$\begin{aligned}\sigma_t^2&=\int_{-\infty}^{\infty}\big(t-\langle t\rangle\big)^2\big|s(t)\big|^2\,dt\\&=\int_{-\infty}^{\infty}\big(t-\langle t\rangle\big)^2\left|\left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}}e^{-\frac{\alpha}{2}(t-\langle t\rangle)^2}\right|^2dt\\&=\int_{-\infty}^{\infty}\big(t-\langle t\rangle\big)^2\left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}}e^{-\alpha(t-\langle t\rangle)^2}\,dt\\&=\left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}t^2e^{-\alpha t^2}\,dt=\frac{1}{2\alpha},\end{aligned}$$

$$\begin{aligned}\sigma_\omega^2&=\int_{-\infty}^{\infty}\rho'^2(t)\,dt+\int_{-\infty}^{\infty}\big[\varphi'(t)-\langle\omega\rangle\big]^2\big|s(t)\big|^2\,dt\\&=\int_{-\infty}^{\infty}\left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}}\alpha^2\big(t-\langle t\rangle\big)^2e^{-\alpha(t-\langle t\rangle)^2}\,dt+\int_{\langle t\rangle}^{\infty}\left[\frac{1}{\varepsilon}\big(t-\langle t\rangle\big)+\langle\omega\rangle-\langle\omega\rangle\right]^2\big|s(t)\big|^2\,dt\end{aligned}$$

$$
\begin{aligned}
&+\int_{-\infty}^{\langle t\rangle}\left[-\frac{1}{\varepsilon}\big(t-\langle t\rangle\big)+\langle\omega\rangle-\langle\omega\rangle\right]^2\big|s(t)\big|^2\,dt\\
&=\frac{\alpha}{2}+\frac{1}{\varepsilon^2}\int_{-\infty}^{\infty}\big(t-\langle t\rangle\big)^2\left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}}e^{-\alpha(t-\langle t\rangle)^2}\,dt\\
&=\frac{\alpha}{2}+\frac{1}{2\varepsilon^2\alpha}=\frac{\alpha^2\varepsilon^2+1}{2\varepsilon^2\alpha},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cov}&=\int_{-\infty}^{\infty}\big(t-\langle t\rangle\big)\big[\varphi'(t)-\langle\omega\rangle\big]\big|s(t)\big|^2\,dt\\
&=\int_{\langle t\rangle}^{\infty}\big(t-\langle t\rangle\big)\left[\frac{1}{\varepsilon}\big(t-\langle t\rangle\big)+\langle\omega\rangle-\langle\omega\rangle\right]\big|s(t)\big|^2\,dt\\
&\quad+\int_{-\infty}^{\langle t\rangle}\big(t-\langle t\rangle\big)\left[-\frac{1}{\varepsilon}\big(t-\langle t\rangle\big)+\langle\omega\rangle-\langle\omega\rangle\right]\big|s(t)\big|^2\,dt\\
&=\int_{\langle t\rangle}^{\infty}\big(t-\langle t\rangle\big)\left[\frac{1}{\varepsilon}\big(t-\langle t\rangle\big)\right]\big|s(t)\big|^2\,dt+\int_{-\infty}^{\langle t\rangle}\big(t-\langle t\rangle\big)\left[-\frac{1}{\varepsilon}\big(t-\langle t\rangle\big)\right]\big|s(t)\big|^2\,dt\\
&=\frac{1}{\varepsilon}\int_{\langle t\rangle}^{\infty}\big(t-\langle t\rangle\big)^2\left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}}e^{-\alpha(t-\langle t\rangle)^2}\,dt-\frac{1}{\varepsilon}\int_{-\infty}^{\langle t\rangle}\big(t-\langle t\rangle\big)^2\left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}}e^{-\alpha(t-\langle t\rangle)^2}\,dt\\
&=\frac{1}{\varepsilon}\int_{0}^{\infty}t^2\left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}}e^{-\alpha t^2}\,dt-\frac{1}{\varepsilon}\int_{-\infty}^{0}t^2\left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}}e^{-\alpha t^2}\,dt\\
&=0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\big|\big(t-\langle t\rangle\big)\big[\varphi'(t)-\langle\omega\rangle\big]\big|\big|s(t)\big|^2\,dt\\
&\quad=\int_{\langle t\rangle}^{\infty}\left|\big(t-\langle t\rangle\big)\left[\frac{1}{\varepsilon}\big(t-\langle t\rangle\big)+\langle\omega\rangle-\langle\omega\rangle\right]\right|\big|s(t)\big|^2\,dt\\
&\qquad+\int_{-\infty}^{\langle t\rangle}\left|\big(t-\langle t\rangle\big)\left[-\frac{1}{\varepsilon}\big(t-\langle t\rangle\big)+\langle\omega\rangle-\langle\omega\rangle\right]\right|\big|s(t)\big|^2\,dt
\end{aligned}
$$

$$= \int_{\langle t\rangle}^{\infty} \left|\big(t-\langle t\rangle\big)\left[\frac{1}{\varepsilon}\big(t-\langle t\rangle\big)\right]\right| \left|s(t)\right|^2 dt + \int_{-\infty}^{\langle t\rangle} \left|\big(t-\langle t\rangle\big)\left[-\frac{1}{\varepsilon}\big(t-\langle t\rangle\big)\right]\right| \left|s(t)\right|^2 dt$$

$$= \frac{1}{\varepsilon}\int_{\langle t\rangle}^{\infty} \big(t-\langle t\rangle\big)^2 \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} e^{-\alpha(t-\langle t\rangle)^2}\, dt + \frac{1}{\varepsilon}\int_{-\infty}^{\langle t\rangle} \big(t-\langle t\rangle\big)^2 \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} e^{-\alpha(t-\langle t\rangle)^2}\, dt$$

$$= \frac{1}{\varepsilon}\int_{0}^{\infty} t^2 \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} e^{-\alpha t^2}\, dt + \frac{1}{\varepsilon}\int_{-\infty}^{0} t^2 \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} e^{-\alpha t^2}\, dt$$

$$= \frac{1}{\varepsilon}\int_{-\infty}^{\infty} t^2 \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} e^{-\alpha t^2}\, dt$$

$$= \frac{1}{2\alpha\varepsilon}.$$

We therefore conclude that

$$\sigma_t^2\sigma_\omega^2 = \frac{1}{4} + \left\{ \int_{-\infty}^{\infty} \left|\big(t-\langle t\rangle\big)\big[\varphi'(t)-\langle\omega\rangle\big]\right| \left|s(t)\right|^2 dt \right\}^2 > \frac{1}{4} + \mathrm{Cov}^2.$$

The following example corresponds to the signal class (2.13) that gives equality in the old uncertainty principle. This then forces the equal sign to hold in the new uncertainty principle.

Example 2.5. Let

$$s(t) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\alpha}{2}(t-\langle t\rangle)^2} e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+\gamma]}, \tag{2.22}$$

where $\alpha > 0$, $\varepsilon, \gamma \in \mathbb{R}$. It is a signal of the form (2.13).

Then

$$\sigma_t^2 = \frac{1}{2\alpha}, \qquad \sigma_\omega^2 = \frac{\alpha^2\varepsilon^2+1}{2\varepsilon^2\alpha},$$

and

$$\mathrm{Cov} = \int_{-\infty}^{\infty} \big(t-\langle t\rangle\big)\big[\varphi'(t)-\langle\omega\rangle\big]\left|s(t)\right|^2 dt$$

$$= \int_{-\infty}^{\infty} \big(t-\langle t\rangle\big)\left[\frac{1}{\varepsilon}\big(t-\langle t\rangle\big)+\langle\omega\rangle-\langle\omega\rangle\right]\left|s(t)\right|^2 dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{\varepsilon}\big(t - \langle t\rangle\big)^2 \big|s(t)\big|^2\,dt$$

$$= \frac{1}{2\alpha\varepsilon}$$

$$= \int_{-\infty}^{\infty} 2\big|\big(t - \langle t\rangle\big)\big[\varphi'(t) - \langle\omega\rangle\big]\big|\big|s(t)\big|^2\,dt.$$

We therefore conclude that

$$\sigma_t^2\sigma_\omega^2 = \frac{1}{4} + \left\{\int_{-\infty}^{\infty} \big|\big(t - \langle t\rangle\big)\big[\varphi'(t) - \langle\omega\rangle\big]\big|\big|s(t)\big|^2\,dt\right\}^2$$
$$= \frac{1}{4} + \mathrm{Cov}^2.$$

The proof of Theorem 2.2 depends on integrability of the involved functions and existence of various derivatives. Those conditions cannot be satisfied by general signals of finite energy. The next two sections will treat the problem and develop the theory of uncertainty principle for signals in more general classes.

3. Fourier transform derivative and uncertainty principle

In the following sections we basically deal with signals in the Sobolev spaces. We adopt the notation $L_n^2(\mathbb{R})$ for the Sobolev spaces [17], that is

$$L_n^2(\mathbb{R}) = \left\{s(t) \in L^2(\mathbb{R}):\ \left(\frac{d^*}{dt}\right)^n s(t) \in L^2(\mathbb{R})\right\}$$

with the norm defined by

$$\sqrt{\|s\|_2^2 + \left\|\left(\frac{d^*}{dt}\right)^n s\right\|_2^2},$$

where $(\frac{d^*}{dt})^n s(t)$ stands for the n-th distributional derivative of s. This paper concerns signals in $L_1^2(\mathbb{R})$ or subspaces of it.

Lemma 3.1. *Assume $s(t) \in L^2(\mathbb{R})$, $ts(t) \in L^2(\mathbb{R})$, then $s(t) \in L^1(\mathbb{R})$ and the Fourier transform $\hat{s}(\omega) \in C_0(\mathbb{R})$.*

Proof. Let $E_0 = [-1, 1]$, $E_1 = \{t:\ |t| > 1,\ |s(t)| > 1\}$, $E_2 = \{t:\ |t| > 1,\ |s(t)| \leqslant 1\}$. Then

$$\int_{E_0} \big|s(t)\big|\,dt \leqslant \left(\int_{E_0} \big|s(t)\big|^2\,dt\right)^{\frac{1}{2}} 2^{\frac{1}{2}} < \infty,$$

$$\int_{E_1} |s(t)|\,dt \leqslant \int_{E_1} |s(t)|^2\,dt < \infty,$$

$$\int_{E_2} |s(t)|\,dt \leqslant \left(\int_{E_2} |ts(t)|^2\,dt\right)^{\frac{1}{2}} \left(\int_{E_2} |t|^{-2}\,dt\right)^{\frac{1}{2}} < \infty.$$

So $s(t) \in L^1(\mathbb{R})$, and its Fourier transform $\hat{s}(\omega) \in C_0(\mathbb{R})$. $\square$

Lemma 3.2. *Assume that $s(t)$, $ts(t)$ and $\omega\hat{s}(\omega) \in L^2(\mathbb{R})$. Then $\hat{s} \in L^1(\mathbb{R})$, and $s(t)$ is almost everywhere equal to a function in $C_0(\mathbb{R})$. Moreover, there exists the Fourier transform derivative $(Ds)(t) \in L^2(\mathbb{R})$ of s such that $(Ds)^\wedge(\omega) = i\omega\hat{s}(\omega) \in L^2(\mathbb{R})$ and*

$$\lim_{a\to 0} \int_{-\infty}^{+\infty} \left|a^{-1}\big(s(t+a) - s(t)\big) - (Ds)(t)\right|^2 dt = 0. \tag{3.23}$$

Therefore,

$$\liminf_{a\to 0} \left|a^{-1}\big(s(t+a) - s(t)\big) - (Ds)(t)\right| = 0$$

holds almost everywhere on $\mathbb{R}$. If, in particular, $s(t)$ has classical derivatives $s'(t)$ almost everywhere on $\mathbb{R}$, then $(Ds)(t) = s'(t)$ almost everywhere on $\mathbb{R}$.

Proof. Since

$$\|s\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{+\infty} |s(t)|^2\,dt = \int_{-\infty}^{+\infty} |\hat{s}(\omega)|^2\,d\omega = \|\hat{s}\|^2_{L^2(\mathbb{R})}$$

and $\omega\hat{s}(\omega) \in L^2(\mathbb{R})$, by invoking Lemma 3.1, we have $\hat{s} \in L^1(\mathbb{R})$, $s \in L^1(\mathbb{R})$ and the inverse Fourier transform $\mathbf{F}^{-1}\hat{s}(t) = \mathbf{F}\hat{s}(-t) \in C_0(\mathbb{R})$, which is almost everywhere equal to $s(t)$. Let $s_a(t) = s(t+a)$, then

$$\mathbf{F}\big(a^{-1}(s_a - s)\big)(\omega) = a^{-1}\big(e^{ia\omega} - 1\big)\mathbf{F}s(\omega),$$
$$\left|a^{-1}\big(e^{ia\omega} - 1\big)\right| \leqslant |\omega|,$$

and

$$a^{-1}\big(e^{ia\omega} - 1\big) \to i\omega \quad \text{as } a \to 0.$$

By Lebesgue's Dominated Convergence Theorem, there holds

$$\mathbf{F}\big(a^{-1}(s_a - s)\big)(\omega) \to i\omega\mathbf{F}s(\omega) = g_1(\omega)$$

in $L^2(\mathbb{R})$. Let $\mathbf{F}^{-1}(g_1)(t) = (Ds)(t)$. The Plancherel Theorem implies

$$a^{-1}\big(s_a(t)-s(t)\big)\to(Ds)(t)$$

in $L^2(\mathbb{R})$. So, (3.23) holds. □

In the following lemma, we prove that $s(t)$ is identical with an absolutely continuous function almost everywhere. Although this is a known result [20], we prove it in a concise way.

Lemma 3.3. *Assume that $1\leqslant p_1\leqslant 2$, $1\leqslant p_2\leqslant 2$, $s(t)\in L^{p_1}(\mathbb{R})$, $h(\omega)=i\omega\hat{s}(\omega)\in L^{p_2}(\mathbb{R})$. Let*

$$g(t)=\int_a^t (Ds)(u)\,du+s(a),$$

where a is a Lebesgue point of s. Then $s(t)$ is identical almost everywhere with the absolutely continuous function $g(t)$, and

$$(Ds)(t)=g'(t)\quad \textit{for almost all } t\in\mathbb{R}. \tag{3.24}$$

Proof. Since $h(\omega)$ is in $L^{p_2}(\mathbb{R})$, by Hausdorff–Young's inequality, $(Ds)(t)$ exists as an $L^{q_2}(\mathbb{R})$ function, where $\frac{1}{q_2}+\frac{1}{p_2}=1$.

Let $W(t)=e^{-\frac{t^2}{2}}$, $W_\varepsilon(t)=\frac{1}{\varepsilon}W(\frac{t}{\varepsilon})$, and

$$s_\varepsilon(t)=(s*W_\varepsilon)(t)=\int_{-\infty}^{\infty} s(t-x)W_\varepsilon(x)\,dx,\qquad \varepsilon>0.$$

Then $s_\varepsilon(t)\in C^\infty(\mathbb{R})$. By calculation, we have $\hat{W}(\omega)=W(\omega)\in C^\infty(\mathbb{R})$, $W(0)=1$, $\widehat{W_\varepsilon}(\omega)=W(\varepsilon\omega)$, and $\widehat{s_\varepsilon}(\omega)=\hat{s}(\omega)W(\varepsilon\omega)$. Therefore, by Lebesgue's Dominated Convergence Theorem,

$$\int_{-\infty}^{+\infty}|\omega|^{p_2}\big|\widehat{s_\varepsilon}(\omega)-\hat{s}(\omega)\big|^{p_2}\,d\omega=\int_{-\infty}^{+\infty}|\omega|^{p_2}\big|\hat{s}(\omega)\big|^{p_2}\big|1-W(\varepsilon\omega)\big|^{p_2}\,d\omega\to 0,$$

as $\varepsilon\to 0$. Since $s'_\varepsilon(t)=(\mathbf{F}^{-1}\widehat{s'_\varepsilon})(t)=(\mathbf{F}\widehat{s'_\varepsilon})(-t)$, by Hausdorff–Young's inequality,

$$\left(\int_{-\infty}^{+\infty}\big|s'_\varepsilon(-t)-(Ds)(-t)\big|^{q_2}\,dt\right)^{\frac{1}{q_2}}\leqslant\left(\int_{-\infty}^{+\infty}|\omega|^{p_2}\big|\widehat{s_\varepsilon}(\omega)-\hat{s}(\omega)\big|^{p_2}\,d\omega\right)^{\frac{1}{p_2}}\to 0, \tag{3.25}$$

as $\varepsilon\to 0$. The relation (3.25) implies that

$$\lim_{\varepsilon\to 0}\int_{[a,b]} s'_\varepsilon(t)\,dt=\int_{[a,b]}(Ds)(t)\,dt \tag{3.26}$$

over any finite interval $[a.b]$. But if a and b are Lebesgue points of s, the left-hand side of (3.26) is equal to

$$\lim_{\varepsilon\to 0}\big(s_\varepsilon(b)-s_\varepsilon(a)\big)=s(b)-s(a). \tag{3.27}$$

Since the points in $\mathbb{R}$ are almost everywhere Lebesgue points of s, we see, from (3.26) and (3.27), that $s(t)$ is equal almost everywhere to an absolutely continuous function $g(t)$, and (3.24) holds. □

Definition 3.4. Assume $s(t)=\rho(t)e^{i\varphi(t)}\in L^2(\mathbb{R})$ and $\omega\hat{s}(\omega)\in L^2(\mathbb{R})$. We can define the Fourier transform derivative for $\rho(t)$ and $\varphi(t)$ as follows:

$$(D\rho)(t)=\begin{cases}\rho(t)\operatorname{Re}\frac{(Ds)(t)}{s(t)}, & \text{if } s(t)\neq 0,\\ 0, & \text{if } s(t)=0,\end{cases}$$

and

$$(D\varphi)(t)=\begin{cases}\operatorname{Im}\frac{(Ds)(t)}{s(t)}, & \text{if } s(t)\neq 0,\\ 0, & \text{if } s(t)=0.\end{cases}$$

We note that the L^2 space consists of equivalent classes of almost everywhere equal functions and the definition is based on a representation of the equivalent class of s. If we take an alternative representation of the same equivalent class, then the defined amplitude and phase derivative are in the respective but the same equivalent classes of almost everywhere equal measurable functions.

The relations (2.20) and (2.11) for the classical derivative case are generalized in the following

Lemma 3.5. *Assume $s(t)=\rho(t)e^{i\varphi(t)}\in L^2(\mathbb{R})$, $\|s\|_2=1$ and $\omega\hat{s}(\omega)\in L^2(\mathbb{R})$. Then the mean of the Fourier frequency and the bandwidth of s are, respectively, given by*

$$\langle\omega\rangle=\int_{-\infty}^{\infty}(D\varphi)(t)\rho^2(t)\,dt,$$

and

$$\sigma_\omega^2=\int_{-\infty}^{\infty}(D\rho)^2(t)\,dt+\int_{-\infty}^{\infty}\big[(D\varphi)(t)-\langle\omega\rangle\big]^2\rho^2(t)\,dt. \tag{3.28}$$

Proof. Since $s(t)=\rho(t)e^{i\varphi(t)}\in L^2(\mathbb{R})$ and $\omega\hat{s}(\omega)\in L^2(\mathbb{R})$, $\langle\omega\rangle$ and σ_ω^2 are well defined.

$$\begin{aligned}\langle\omega\rangle&=\int_{-\infty}^{\infty}\omega\big|\hat{s}(\omega)\big|^2\,d\omega=\int_{-\infty}^{\infty}\omega\hat{s}(\omega)\overline{\hat{s}(\omega)}\,d\omega\\&=-i\int_{-\infty}^{\infty}(Ds)(t)\overline{s(t)}\,dt=\int_{\mathbb{R}\setminus E}\operatorname{Im}\frac{(Ds)(t)}{s(t)}\big|s(t)\big|^2\,dt=\int_{-\infty}^{\infty}(D\varphi)(t)\rho^2(t)\,dt,\end{aligned}$$

where $E = \{t \in \mathbb{R} \mid s(t) = 0\}$.

$$\begin{aligned}
\sigma_\omega^2 &= \int_{-\infty}^{\infty} \big(\omega - \langle\omega\rangle\big)^2 \big|\hat{s}(\omega)\big|^2 \, d\omega \\
&= \int_{-\infty}^{\infty} \big(\omega - \langle\omega\rangle\big)\hat{s}(\omega)\overline{\big(\omega - \langle\omega\rangle\big)\hat{s}(\omega)} \, d\omega \\
&= \int_{-\infty}^{\infty} \big[-i(Ds)(t) - \langle\omega\rangle s(t)\big]\big[\overline{-i(Ds)(t) - \langle\omega\rangle s(t)}\big] \, dt \\
&= \int_{-\infty}^{\infty} (Ds)(t)\overline{(Ds)(t)} \, dt + \int_{-\infty}^{\infty} i\langle\omega\rangle(Ds)(t)\overline{s(t)} \, dt \\
&\quad - \int_{-\infty}^{\infty} i\langle\omega\rangle s(t)\overline{(Ds)(t)} \, dt + \int_{-\infty}^{\infty} \langle\omega\rangle^2 \big|s(t)\big|^2 \, dt \\
&= \int_{\mathbb{R}\setminus E} \left|\frac{(Ds)(t)}{s(t)}\right|^2 \big|s(t)\big|^2 \, dt - 2\langle\omega\rangle \int_{-\infty}^{\infty} \mathrm{Im}\big[(Ds)(t)\overline{s(t)}\big] \, dt + \int_{-\infty}^{\infty} \langle\omega\rangle^2 \big|s(t)\big|^2 \, dt \\
&= \int_{\mathbb{R}\setminus E} \mathrm{Re}^2\left[\frac{(Ds)(t)}{s(t)}\right]\big|s(t)\big|^2 \, dt + \int_{\mathbb{R}\setminus E} \mathrm{Im}^2\left[\frac{(Ds)(t)}{s(t)}\right]\big|s(t)\big|^2 \, dt \\
&\quad - 2\langle\omega\rangle \int_{-\infty}^{\infty} (D\varphi)(t)\big|s(t)\big|^2 \, dt + \int_{-\infty}^{\infty} \langle\omega\rangle^2 \big|s(t)\big|^2 \, dt \\
&= \int_{-\infty}^{\infty} (D\rho)^2(t)\big|s(t)\big|^2 \, dt + \int_{-\infty}^{\infty} (D\varphi)^2(t)\big|s(t)\big|^2 \, dt \\
&\quad - 2\langle\omega\rangle \int_{-\infty}^{\infty} (D\varphi)(t)\big|s(t)\big|^2 \, dt + \int_{-\infty}^{\infty} \langle\omega\rangle^2 \big|s(t)\big|^2 \, dt \\
&= \int_{-\infty}^{\infty} (D\rho)^2(t)\big|s(t)\big|^2 \, dt + \int_{-\infty}^{\infty} \big[(D\varphi)(t) - \langle\omega\rangle\big]^2 \big|s(t)\big|^2 \, dt. \qquad \square
\end{aligned}$$

Theorem 3.6. *Let $s(t) = \rho(t)e^{i\varphi(t)}$, $ts(t)$ and $\omega\hat{s}(\omega) \in L^2(\mathbb{R})$, $\|s\|_2 = 1$. Then*

$$\sigma_t^2\sigma_\omega^2 \geqslant \frac{1}{4} + \left[\int_{-\infty}^{+\infty} \big|t - \langle t\rangle\big|\big|(D\varphi)(t) - \langle\omega\rangle\big|\big|s(t)\big|^2 \, dt\right]^2. \tag{3.29}$$

Under the extra assumptions that $s(t)=\rho(t)e^{i\varphi(t)}$ has the classical derivatives $s'(t)$, $\varphi'(t)$, $\rho'(t)$, where $\varphi'(t)$ is continuous and ρ is almost everywhere non-zero, then the equality is attained if and only if $s(t)$ has one of the following four forms

$$s(t)=e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_2]},$$
$$s(t)=e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_3]},$$
$$s(t)=\begin{cases} e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_4]} & \text{if } t\geqslant\langle t\rangle,\\ e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_5]} & \text{if } t<\langle t\rangle,\end{cases}$$

and

$$s(t)=\begin{cases} e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_6]} & \text{if } t\geqslant\langle t\rangle,\\ e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_7]} & \text{if } t<\langle t\rangle,\end{cases}$$

for some $d_1,d_2,d_3,d_4,d_5,d_6,d_7,\zeta,\varepsilon\in\mathbb{R}$, $\zeta,\varepsilon>0$, where $e^{2d_1}\sqrt{\frac{\zeta\pi}{2}}=1$.

Note that this theorem is of the same type as Theorem 2.2. But there are two main differences between them. The first is that Theorem 3.6 extends the inequality part of Theorem 2.2 to generalized signals $s(t)\in L^2$ with $ts(t),\omega\hat{s}(\omega)\in L^2(\mathbb{R})$. The second is that we, essentially, can only verify some sufficient conditions giving equality in (3.29). We, however, cannot show the same conditions to be necessary. This is caused by the distribution nature of the Fourier derivative. To prove the necessity of the conditions we have to assume the same smoothness conditions on s and the related objects as we do in Theorem 2.2.

Proof. By recalling Lemma 3.5, we have

$$\sigma_\omega^2=\int_{-\infty}^{\infty}(D\rho)^2(t)\,dt+\int_{-\infty}^{\infty}\big[(D\varphi)(t)-\langle\omega\rangle\big]^2\rho^2(t)\,dt.$$

As in the proof of Theorem 2.2, to prove the inequality (3.29) it suffices to prove

$$\int_{-\infty}^{+\infty}\big(t-\langle t\rangle\big)^2\big|s(t)\big|^2\,dt\int_{-\infty}^{\infty}(D\rho)^2(t)\,dt\geqslant\frac{1}{4} \tag{3.30}$$

and

$$\int_{-\infty}^{+\infty}\big(t-\langle t\rangle\big)^2\big|s(t)\big|^2\,dt\int_{-\infty}^{\infty}\big[(D\varphi)(t)-\langle\omega\rangle\big]^2\rho^2(t)\,dt$$
$$\geqslant\left[\int_{-\infty}^{+\infty}\big|t-\langle t\rangle\big|\big|(D\varphi)(t)-\langle\omega\rangle\big|\big|s(t)\big|^2\,dt\right]^2. \tag{3.31}$$

We first prove (3.30).

By Lemma 3.3, we may assume that $s_0(t)$ is an absolutely continuous function that is equal to $s(t)$ almost everywhere. Then, for two particular sequences of numbers, M_n, N_n, tending to infinity as $n \to \infty$,

$$\begin{aligned}
\frac{1}{4} &= \left[\frac{1}{2}\int_{-\infty}^{\infty} \left|s(t)\right|^2 dt\right]^2 \\
&= \left[\frac{1}{2}\int_{-\infty}^{\infty} \left|s_0(t)\right|^2 dt\right]^2 \\
&= \left[\lim_{n\to\infty} \frac{1}{2}\int_{-N_n}^{M_n} \left|s_0(t)\right|^2 dt\right]^2 \\
&= \left\{\lim_{n\to\infty}\left[\frac{1}{2}\big(t-\langle t\rangle\big)\left|s_0(t)\right|^2\Big|_{-N_n}^{M_n}\right] - \lim_{n\to\infty}\left[\frac{1}{2}\int_{-N_n}^{M_n}\big(t-\langle t\rangle\big)\left[s_0'(t)\overline{s_0(t)} + s_0(t)\overline{s_0'(t)}\right] dt\right]\right\}^2 \\
&= \left\{\frac{1}{2}\int_{-\infty}^{\infty}\big(t-\langle t\rangle\big)\left[s_0'(t)\overline{s_0(t)} + s_0(t)\overline{s_0'(t)}\right] dt\right\}^2 \\
&= \left\{\frac{1}{2}\int_{-\infty}^{\infty}\big(t-\langle t\rangle\big)\left[(Ds)(t)\overline{s(t)} + s(t)\overline{(Ds)(t)}\right] dt\right\}^2 \\
&= \left\{\frac{1}{2}\int_{\mathbb{R}\setminus E}\big(t-\langle t\rangle\big)\left|s(t)\right|^2\left[\frac{(Ds)(t)}{s(t)} + \frac{\overline{(Ds)(t)}}{\overline{s(t)}}\right] dt\right\}^2 \\
&= \left\{\int_{\mathbb{R}\setminus E}\big(t-\langle t\rangle\big)\left|s(t)\right|\left|s(t)\right| \operatorname{Re}\frac{(Ds)(t)}{s(t)}\, dt\right\}^2 \\
&\leqslant \left[\int_{-\infty}^{\infty}\left|\big(t-\langle t\rangle\big)\rho(t)(D\rho)(t)\right| dt\right]^2 \\
&\leqslant \int_{-\infty}^{\infty}\left|\big(t-\langle t\rangle\big)\rho(t)\right|^2 dt \int_{-\infty}^{\infty}\left|(D\rho)(t)\right|^2 dt,
\end{aligned}$$

where $E = \{t \in \mathbb{R} \mid s(t) \neq 0\}$.

The proof of (3.31) is the same as that for Theorem 2.2, except that we need to replace $\varphi'(t)$ with $(D\varphi)(t)$.

It is clear that the four types of signals in the statement of the theorem give equality in (3.29). When consider necessity of the four types, the assumptions that $s'(t)$, $\rho'(t)$ and $\varphi'(t)$ all exist in the classical derivative sense, and $\varphi'(t)$ is continuous, and ρ is almost everywhere non-zero are needed. In the case

$$(D\rho)(t) = \rho'(t) \quad \text{and} \quad (D\varphi)(t) = \varphi'(t).$$

Based on these assumptions, the same proof as in that of Theorem 2.2 is valid. □

4. Hardy–Sobolev derivative and uncertainty principle

In this section we will be working in Hardy spaces. For an introduction to these spaces see [13]. We adopt the Hardy–Sobolev spaces decomposition technique developed in [7] and [6]. Some closely related work may be found in [5].

If $s \in L_1^2(\mathbb{R})$, then we have Hardy–Sobolev decomposition (see [6]) $s = s_+ + s_-$, where

$$s_\pm(t) = \frac{\pm 1}{\sqrt{2\pi}} \int_0^{\pm\infty} e^{it\omega}\hat{s}(\omega)\, d\omega$$

and

$$s_\pm(z) = \frac{\pm 1}{2\pi i} \int_{-\infty}^{\infty} \frac{s(u)}{u - z}\, du, \quad z \in \mathbb{C}^\pm.$$

The decomposition is orthogonal and unique. In [7] we show that $s_\pm(z), s'_\pm(z) \in H^2(\mathbb{C}^\pm)$, where $H^2(\mathbb{C}^+)$ and $H^2(\mathbb{C}^-)$ denote the H^2 Hardy spaces in the upper-half and lower-half complex planes.

Below we define a new type of derivatives, called Hardy–Sobolev derivatives [6].

Definition 4.1. If $s(t) = \rho(t)e^{i\varphi(t)} \in L_1^2(\mathbb{R})$, then the Hardy–Sobolev derivatives of $s(t)$, $\rho(t)$ and $\varphi(t)$ are defined to be

$$s^*(t) = \begin{cases} s'_+(t) + s'_-(t) & \text{if } s'_+(t) \text{ and } s'_-(t) \text{ are defined,} \\ 0 & \text{if } s'_+(t) \text{ or } s'_-(t) \text{ is not defined,} \end{cases}$$

$$\rho^*(t) = \begin{cases} \rho(t)\,\mathrm{Re}[\frac{s'_+(t)+s'_-(t)}{s_+(t)+s_-(t)}] & \text{if } t \in \mathbb{R} \setminus L, \\ 0 & \text{if } t \in L, \end{cases}$$

and

$$\varphi^*(t) = \begin{cases} \mathrm{Im}[\frac{s'_+(t)+s'_-(t)}{s_+(t)+s_-(t)}] & \text{if } t \in \mathbb{R} \setminus L, \\ 0 & \text{if } t \in L, \end{cases}$$

where $L = \{t \in \mathbb{R} \mid \text{at least one of } s_+(t),\ s_-(t),\ s'_+(t), \text{ or } s'_-(t) \text{ is not defined; or } s_+(t) + s_-(t) = 0\}$.

Note that the above definition is based on non-tangential boundary limits of functions in the Hardy spaces and hence the respective Hardy–Sobolev derivatives are uniquely defined without using any representative of the L^2 functions. We also note that if a function $s(t)=\rho(t)e^{i\varphi(t)}\in L_1^2(\mathbb{R})$ has the classical derivatives $s'(t)$, $\rho'(t)$ and $\varphi'(t)$, then the Hardy–Sobolev derivatives of $s(t)$, $\rho(t)$ and $\varphi(t)$ coincide with the classical ones [6].

In [6], there are corresponding results about the mean of Fourier frequency and bandwidth.

Lemma 4.2. *Assume $s(t)=\rho(t)e^{i\varphi(t)}\in L_1^2(\mathbb{R})$ and $\|s\|_2=1$. The mean Fourier frequency is identical with*

$$\langle\omega\rangle=\int_{-\infty}^{\infty}\varphi^*(t)\rho^2(t)\,dt, \tag{4.32}$$

and the bandwidth

$$\sigma_\omega^2=\int_{-\infty}^{\infty}\rho^{*2}(t)\,dt+\int_{-\infty}^{\infty}\big[\varphi^*(t)-\langle\omega\rangle\big]^2\rho^2(t)\,dt. \tag{4.33}$$

Theorem 4.3. *Let $s(t)=\rho(t)e^{i\varphi(t)}\in L_1^2(\mathbb{R})$, $zs_\pm(z)\in H^2(\mathbb{C}^\pm)$, and $\|s\|_2=1$. Then,*

$$\sigma_t^2\sigma_\omega^2\geqslant\frac{1}{4}+\left[\int_{-\infty}^{\infty}\big|\big(t-\langle t\rangle\big)\big(\varphi^*(t)-\langle\omega\rangle\big)\big|\rho^2(t)\,dt\right]^2. \tag{4.34}$$

If $s(t)=\rho(t)e^{i\varphi(t)}$ has the classical derivatives $s'(t)$, $\varphi'(t)$, $\rho'(t)$, $\varphi'(t)$ is continuous, $\rho(t)$ is a.e. non-zero, then the equality holds if and only if $s(t)$ has one of the following four forms, namely,

$$s(t)=e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_2]},$$
$$s(t)=e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_3]},$$
$$s(t)=\begin{cases}e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_4]} & \text{if } t\geqslant\langle t\rangle,\\ e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_5]} & \text{if } t<\langle t\rangle,\end{cases}$$

or

$$s(t)=\begin{cases}e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[-\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_6]} & \text{if } t\geqslant\langle t\rangle,\\ e^{-\frac{1}{\zeta}(t-\langle t\rangle)^2+d_1}e^{i[\frac{1}{2\varepsilon}(t-\langle t\rangle)^2+\langle\omega\rangle t+d_7]} & \text{if } t<\langle t\rangle,\end{cases}$$

for some $d_1,d_2,d_3,d_4,d_5,d_6,d_7,\zeta,\varepsilon\in\mathbb{R}$, $\zeta,\varepsilon>0$, where $e^{2d_1}\sqrt{\frac{\zeta\pi}{2}}=1$.

Proof. Besides replacing $(D\varphi)(t)$ and $(D\rho)(t)$ with $\varphi^*(t)$ and $\rho^*(t)$ in the proof of Theorem 3.6 the only essential difference is the proof of the inequality

$$\int_{-\infty}^{\infty}\big(t-\langle t\rangle\big)^2\big|s(t)\big|^2\,dt\int_{-\infty}^{\infty}\rho^{*2}(t)\,dt\geqslant\frac{1}{4}.$$

We proceed as follows:

$$\begin{aligned}
\frac{1}{4} &= \left[\frac{1}{2}\int_{-\infty}^{\infty}\left|s(t)\right|^2 dt\right]^2 \\
&= \left[\frac{1}{2}\int_{-\infty}^{\infty}\left[s_+(t)+s_-(t)\right]\overline{s_+(t)+s_-(t)}\,dt\right]^2 \\
&= \lim_{y\to 0}\lim_{M,N\to\infty}\left\{\frac{1}{2}\int_{-N}^{M}\left[s_{+,y}(t)+s_{-,-y}(t)\right]\overline{s_{+,y}(t)+s_{-,-y}(t)}\,dt\right\}^2 \\
&= \lim_{y\to 0}\lim_{M,N\to\infty}\left\{\frac{1}{2}\big(t-\langle t\rangle\big)\left[s_{+,y}(t)+s_{-,-y}(t)\right]\overline{s_{+,y}(t)+s_{-,-y}(t)}\Big|_{-N}^{M}\right. \\
&\quad - \frac{1}{2}\int_{-N}^{M}\big(t-\langle t\rangle\big)\Big[\big(s'_{+,y}(t)+s'_{-,-y}(t)\big)\overline{s_{+,y}(t)+s_{-,-y}(t)} \\
&\quad \left. + \big(s_{+,y}(t)+s_{-,-y}(t)\big)\overline{s'_{+,y}(t)+s'_{-,-y}(t)}\Big]\,dt\right\}^2 \\
&= \left\{\frac{1}{2}\int_{-\infty}^{\infty}\big(t-\langle t\rangle\big)\Big[\big(s'_+(t)+s'_-(t)\big)\overline{s_+(t)+s_-(t)}+\big(s_+(t)+s_-(t)\big)\overline{s'_+(t)+s'_-(t)}\Big]\,dt\right\}^2 \\
&= \left\{\frac{1}{2}\int_{-\infty}^{\infty}\big(t-\langle t\rangle\big)\left[s_+(t)+s_-(t)\right]\overline{s_+(t)+s_-(t)}\left[\frac{s'_+(t)+s'_-(t)}{s_+(t)+s_-(t)}+\overline{\frac{s'_+(t)+s'_-(t)}{s_+(t)+s_-(t)}}\right]dt\right\}^2 \\
&= \left[\int_{-\infty}^{\infty}\big(t-\langle t\rangle\big)\left|s_+(t)+s_-(t)\right|^2\operatorname{Re}\left\{\frac{s'_+(t)+s'_-(t)}{s_+(t)+s_-(t)}\right\}dt\right]^2 \\
&\leqslant \left[\int_{-\infty}^{\infty}\left|t-\langle t\rangle\right|\left|s_+(t)+s_-(t)\right|\left|s_+(t)+s_-(t)\right|\left|\operatorname{Re}\left\{\frac{s'_+(t)+s'_-(t)}{s_+(t)+s_-(t)}\right\}\right|dt\right]^2 \\
&\leqslant \int_{-\infty}^{\infty}\big(t-\langle t\rangle\big)^2\left|s_+(t)+s_-(t)\right|^2 dt\int_{-\infty}^{\infty}\left|s_+(t)+s_-(t)\right|^2\operatorname{Re}^2\left\{\frac{s'_+(t)+s'_-(t)}{s_+(t)+s_-(t)}\right\}dt \\
&= \int_{-\infty}^{\infty}\big(t-\langle t\rangle\big)^2\left|s(t)\right|^2 dt\int_{-\infty}^{\infty}\rho^{*2}(t)\,dt. \qquad \square
\end{aligned}$$

Remark 4.4. There is an error in the proof of uncertainty principle in [7]. In [7] we thought that the condition $s(t), \hat{s} \in L_1^2(\mathbb{R})$ would imply $zs_\pm(z) \in H^2(\mathbb{C}^\pm)$, the latter being necessary in the proof of uncertainty principle in [7], and we gave a wrong proof. A counter example is as follows. Let $s(t) = \frac{2}{t^2+1}$. Then $s(t), ts(t), \hat{s}(\omega), \omega\hat{s}(\omega)$ are all in $L^2(\mathbb{R})$. We have $s_\pm(z) = \frac{1}{1 \mp iz} \in H^2(\mathbb{C}^\pm)$, but $zs_+(z)$, for instance, does not belong to $H^2(\mathbb{C}^+)$.

Remark 4.5. To avoid using $zs_\pm(z) \in H^2(\mathbb{C}^\pm)$ in the proof of the uncertainty principle, Theorem 3.6 uses, instead, the Fourier transform derivative, and obtains a stronger uncertainty principle than that in [7]. In Theorem 4.3, by assuming $zs_\pm(z) \in H^2(\mathbb{C}^\pm)$ we obtain the same stronger form of uncertainty principle for the Hardy–Sobolev derivative.

Remark 4.6. Hardy–Sobolev derivatives are concrete representatives in the precise point-wise value sense of the corresponding distributional derivatives. The respective phase derivatives are just formal formulations $\mathrm{Im}[\chi_{\mathbb{R}\setminus L} \frac{s^*(t)}{s_+(t)+s_-(t)}]$ and $\mathrm{Im}[\chi_{s\neq 0} \frac{\frac{d^* s}{dt}}{s(t)}]$. With Theorem 4.3, the cost of the precise point-wise expression is the condition $zs_\pm(z) \in H^2(\mathbb{C})$.

5. Uncertainty principle for self-adjoint operator

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and with norm $\|\cdot\| \triangleq \langle \cdot,\cdot \rangle^{\frac{1}{2}}$. Let A, B be two self-adjoint operators on $\mathcal{H}$, with domains $D(A)$ and $D(B)$. Then the domain of the product AB is

$$D(AB) = \big\{ s \in D(B)\colon\ Bs \in D(A) \big\},$$

and likewise for $D(BA)$. The commutator and the anticommutator are, respectively, defined as

$$\begin{aligned} [A,B] &\triangleq AB - BA \quad \text{on } D\big([A,B]\big) = D(AB) \cap D(BA), \\ [A,B]_+ &\triangleq AB + BA \quad \text{on } D\big([A,B]_+\big) = D(AB) \cap D(BA). \end{aligned}$$

Definition 5.1. Let $s(t) \in L^2$ and A, B be two self-adjoint operators on L^2, with domains $D(A)$ and $D(B)$. Then the mean of the operator is defined by

$$\langle A \rangle_s \triangleq \langle As, s \rangle = \int \bar{s} As\, dt, \tag{5.35}$$

and the variance is

$$\sigma_A^2(s) \triangleq \big\langle \big(A - \langle A \rangle_s I\big)^2 \big\rangle = \int \bar{s} \big(A - \langle A \rangle_s I\big)^2 s\, dt, \tag{5.36}$$

and the covariance of operators A and B is defined by

$$\begin{aligned} \mathrm{Cov}_{A,B}(s) &\triangleq \frac{1}{2} \langle AB + BA \rangle_s - \langle A \rangle_s \langle B \rangle_s \\ &= \frac{1}{2} \big\langle [A,B]_+ \big\rangle_s - \langle A \rangle_s \langle B \rangle_s \\ &= \frac{1}{2} \big\langle \big[A - \langle A \rangle_s I, B - \langle B \rangle_s I\big]_+ \big\rangle, \end{aligned} \tag{5.37}$$

where I is the identity operator.

Lemma 5.2. *(See [3].) Let $s(t) \in L^2$ and B be a self-adjoint operator on L^2 with domain $D(B)$. Then*

$$\sigma_B^2(s) = \int \left[\operatorname{Im}\left(\frac{Bs}{s}\right)\right]^2 \left|s(t)\right|^2 dt + \int \left[\operatorname{Re}\left(\frac{Bs}{s}\right) - \langle B\rangle_s\right]^2 \left|s(t)\right|^2 dt,$$

where $\operatorname{Im}(\frac{Bs}{s})$ *and* $\operatorname{Re}(\frac{Bs}{s})$ *denote, respectively, the imaginary part and real part of* $\frac{Bs}{s}$.

Proof.

$$\begin{aligned}
\sigma_B^2(s) &= \int \overline{s(t)}\big(B - \langle B\rangle_s\big)^2 s(t)\, dt \\
&= \int \left|\big(B - \langle B\rangle_s\big)s(t)\right|^2 dt \\
&= \int \left|Bs(t) - \langle B\rangle_s s(t)\right|^2 dt \\
&= \int \left|\left(\frac{Bs(t)}{s(t)} - \langle B\rangle_s\right)s(t)\right|^2 dt \\
&= \int \left|\frac{Bs(t)}{s(t)} - \langle B\rangle_s\right|^2 \left|s(t)\right|^2 dt \\
&= \int \left[\operatorname{Im}\left(\frac{Bs}{s}\right)\right]^2 \left|s(t)\right|^2 dt + \int \left[\operatorname{Re}\left(\frac{Bs}{s}\right) - \langle B\rangle_s\right]^2 \left|s(t)\right|^2 dt. \qquad \square
\end{aligned}$$

Theorem 5.3. *Let A, B be two self-adjoint operators on L^2, with domains $D(A)$ and $D(B)$. Assume that $s \in D(AB) \cap D(BA)$ and $(As)(Bs) = sABs$. Then*

$$\sigma_A^2(s)\sigma_B^2(s) \geqslant \frac{1}{4}\left|\big\langle [A,B]\big\rangle_s\right|^2 + \left[\int \left|As(t) - \langle A\rangle_s s(t)\right| \left|\operatorname{Re}\left(\frac{Bs}{s}\right) - \langle B\rangle_s\right| \left|s(t)\right| dt\right]^2. \tag{5.38}$$

The equality is attained if and only if there exist positive numbers ζ, ε such that

$$\left|\big(A - \langle A\rangle_s\big)s\right| = \zeta \left|\operatorname{Im}\left(\frac{Bs}{s}\right)s\right| = \varepsilon \left|\left[\operatorname{Re}\left(\frac{Bs}{s}\right) - \langle B\rangle_s\right]s\right|. \tag{5.39}$$

Proof. Since

$$\sigma_B^2(s) = \int \left[\operatorname{Im}\left(\frac{Bs}{s}\right)\right]^2 \left|s(t)\right|^2 dt + \int \left[\operatorname{Re}\left(\frac{Bs}{s}\right) - \langle B\rangle_s\right]^2 \left|s(t)\right|^2 dt,$$

we need to prove the following two inequalities:

$$\int \left[\operatorname{Im}\left(\frac{Bs}{s}\right)\right]^2 \left|s(t)\right|^2 dt\, \sigma_A^2 \geqslant \frac{1}{4}\left|\big\langle [A,B]\big\rangle_s\right|^2, \tag{5.40}$$

and

$$\int\left[\mathrm{Re}\left(\frac{Bs}{s}\right)-\langle B\rangle_s\right]^2\left|s(t)\right|^2 dt\,\sigma_A^2(s)$$
$$\geqslant\left[\int\left|As(t)-\langle A\rangle_s s(t)\right|\left|\mathrm{Re}\left(\frac{Bs}{s}\right)-\langle B\rangle_s\right|\left|s(t)\right|dt\right]^2. \tag{5.41}$$

Now we prove the inequality (5.40).

$$\begin{aligned}
\int\left[\mathrm{Im}\left(\frac{Bs}{s}\right)\right]^2\left|s(t)\right|^2 dt\,\sigma_A^2(s) &= \int\left[\mathrm{Im}\left(\frac{Bs}{s}\right)\right]^2\left|s(t)\right|^2 dt\int\left|\big(A-\langle A\rangle_s\big)s(t)\right|^2 dt\\
&\geqslant\left|\int \mathrm{Im}\left(\frac{Bs}{s}\right)\overline{s(t)}\big(A-\langle A\rangle_s\big)s(t)\,dt\right|^2\\
&=\left|\int \mathrm{Im}\left(\frac{Bs}{s}\right)\overline{s(t)}\left(\frac{As}{s}-\langle A\rangle_s\right)s(t)\,dt\right|^2\\
&=\left|\int \mathrm{Im}\left(\frac{Bs}{s}\right)\left|s(t)\right|^2\left(\frac{As}{s}-\langle A\rangle_s\right)dt\right|^2\\
&=\left|\int \mathrm{Im}\left(\frac{Bs}{s}\left|s(t)\right|^2\right)\left(\frac{As}{s}-\langle A\rangle_s\right)dt\right|^2\\
&=\left|\int \mathrm{Im}\big[\overline{s(t)}Bs\big]\left(\frac{As}{s}-\langle A\rangle_s\right)dt\right|^2\\
&=\left|\int \frac{1}{2i}\big[\overline{s(t)}Bs-s(t)\overline{Bs}\big]\left(\frac{As}{s}-\langle A\rangle_s\right)dt\right|^2\\
&=\frac{1}{4}\left|\int \big[\overline{s(t)}Bs-s(t)\overline{Bs}\big]\frac{(A-\langle A\rangle_s)s}{s}\,dt\right|^2\\
&=\frac{1}{4}\left|\int \left[\overline{s(t)}Bs\frac{(A-\langle A\rangle_s)s}{s}-\overline{Bs}\big(A-\langle A\rangle_s\big)s\right]dt\right|^2\\
&=\frac{1}{4}\left|\int \big[\overline{s(t)}\big(A-\langle A\rangle_s\big)Bs-\bar{s}B\big(A-\langle A\rangle_s\big)s\big]\,dt\right|^2\\
&=\frac{1}{4}\left|\int \overline{s(t)}\big[\big(A-\langle A\rangle_s\big)B-B\big(A-\langle A\rangle_s\big)\big]s\,dt\right|^2\\
&=\frac{1}{4}\left|\big\langle[A-\langle A\rangle_s,B]\big\rangle_s\right|^2\\
&=\frac{1}{4}\left|\big\langle[A,B]\big\rangle_s\right|^2.
\end{aligned}\tag{5.42}$$

To prove the inequality (5.41), we have

$$\int\left|As(t)-\langle A\rangle_s s(t)\right|\left|\mathrm{Re}\left(\frac{Bs}{s}\right)-\langle B\rangle_s\right|\left|s(t)\right|dt$$

$$\leqslant \int \left|As(t) - \langle A\rangle_s s(t)\right|^2 dt \int \left[\mathrm{Re}\left(\frac{Bs}{s}\right) - \langle B\rangle_s\right]^2 \left|s(t)\right|^2 dt$$

$$= \sigma_A^2(s) \int \left[\mathrm{Re}\left(\frac{Bs}{s}\right) - \langle B\rangle_s\right]^2 \left|s(t)\right|^2 dt. \tag{5.43}$$

Owing to the Hölder inequality, the equality in (5.42) holds if and only if there exists a positive number ζ such that

$$\left|\left(A - \langle A\rangle_s\right)s\right| = \zeta \left|\mathrm{Im}\left(\frac{Bs}{s}\right)s\right|,$$

and the equality in (5.43) is attained if and only if there exists a positive number ε such that

$$\left|As(t) - \langle A\rangle_s s\right| = \varepsilon \left|\left[\mathrm{Re}\left(\frac{Bs}{s}\right) - \langle B\rangle_s\right]s\right|. \qquad \square$$

Remark 5.4. It is of interest to consider the following question. Let A, B be two self-adjoint operators on $L^2(\mathbb{R})$, with domains $D(A)$ and $D(B)$, and $(As)(Bs) = sABs$. Assume that $[A, B]$ is closable. Whether we have, for any $s \in D(\overline{[A, B]}) \cap D(A) \cap D(B)$,

$$\sigma_A^2(s)\sigma_B^2(s) \geqslant \left|\left\langle\overline{[A, B]}\right\rangle_s\right|^2/4 + \left[\int \left|(As)(t) - \langle A\rangle_s s(t)\right| \left|\mathrm{Re}\left(\frac{Bs}{s}\right) - \langle B\rangle_s\right| \left|s(t)\right| dt\right]^2. \tag{5.44}$$

Pointed by [11] that a weaker form of the above relation is generally false. In fact, the counter example constructed in [11] serves also as a counter example for the above relation. The problem is that we are not sure whether for any $s \in D(\overline{[A, B]}) \cap D(A) \cap D(B)$ there exists a sequence $s_n \in D(AB) \cap D(BA)$ such that $s_n \to s$, $As_n \to As$, and $Bs_n \to Bs$. We, however, can show that in case

$$As(t) = ts(t), \qquad Bs(t) = -is'(t),$$

then the uncertainty principle can be extended to the closure of $[A, B]$, which, in fact, is $i\mathrm{I}$. The latter is, in fact, an alternative proof of Theorem 3.6. We first note that $[A, B]$ is densely defined. Secondly, on a dense subset of L^2 we have $[A, B] = i\mathrm{I}$ that is bounded. Therefore, $D\overline{[A, B]} = L^2$, and $\overline{[A, B]} = i\mathrm{I}$. The operators A, B themselves are self-adjoint on $L^2(\mathbb{R})$, with domains $D(A)$ and $D(B)$. Since self-adjoint operators are dense and then closable (see [19]), we are with the understanding that $A = \overline{A}$, $B = \overline{B}$. The conditions $s \in D\overline{[A, B]}$, $s \in D(A)$ and $s \in D(B)$ amount $s \in L^2$, $ts(t) \in L^2$, $Ds \in L^2$. Now we show that for $s \in D\overline{[A, B]} \cap D(A) \cap D(B)$ we can find $s_n \in D([A, B])$ satisfying $s_n \to s$, $s_n' \to Ds$, $ts_n(t) \to ts(t)$. We take, for instance, h be the heat kernel, and $\epsilon_n \to 0+$. Then $h_n(t) = \frac{1}{\epsilon_n}h(\frac{t}{\epsilon_n})$ is an approximation to identity. Let $s_n = h_n * s$. Then $s_n \to s$ in L^2. $(s_n')\hat{}(\omega) = (i\omega)\widehat{h_n}(\omega)\hat{s}(\omega)$ and $\widehat{h_n}$ being bounded and tending to 1 imply that $s_n' \to Ds$ (the Dominated Convergence Theorem). To show $ts_n(t) \to ts(t)$ in L^2, we first have the decomposition

$$ts_n(t) = \int (t-u)h_n(t-u)s(u)\,du + \int h_n(t-u)us(u)\,du = I_n^1(t) + I_n^2(t).$$

Then, in the L^2 sense,

$$\lim_{n\to\infty} I_n^1(t) = \int uh_n(u)\,du\,s(t) = 0,$$

and

$$\lim_{n\to\infty} I_n^2(t) = ts(t)$$

(see [18]). The above proof is valid for any even kernel h with the extra integrability condition $th(t) \in L^1$.

In a similar way, from Theorem 5.3, we can also obtain Theorem 4.3. In the case we take $y_n \to 0+$, $s_n = s_{+,y_n} + s_{-,-y_n}$. Note that the s_n are essentially Poisson integrals, and hence $s_n \to s$. In the proof of $s_n' \to s_+' + s_-' = s^*$, along with an application of the Dominated Convergence Theorem, we use the Sobolev space assumption. The proof of $ts_n(t) \to ts(t)$ uses the condition $zs_\pm(z) \in H^2(\mathbb{C}^\pm)$, and, in applying the Dominate Convergence Theorem, uses the property that any maximal function of a Hardy H^p space function belongs to the L^p space on the boundary, $1 < p < \infty$.

Essentially, the above extensions of Theorem 5.3 for the case $As(t) = ts(t)$ and $Bs(t) = -is'(t)$ are the extensions of Theorem 2.2 to Theorem 3.6 and Theorem 4.3.

Example 5.5. We take

$$As(t) = ts(t), \qquad B = \frac{1}{i}s'(t)$$

for all $s(t) = \rho(t)e^{i\varphi(t)} \in L^2(\mathbb{R})$ with $ts(t), s'(t) \in L^2(\mathbb{R})$, where $s'(t)$, $\varphi'(t)$ exist as classical derivative, and $\|s\|_2 = 1$. Then

$$\begin{aligned}
\sigma_A^2(s) &= \big\langle \big(A - \langle A\rangle_s I\big)^2\big\rangle_s = \int \bar{s}\big(A - \langle A\rangle_s I\big)^2 s\,dt = \int \big(t - \langle A\rangle_s\big)^2 |s|^2\,dt,\\
\sigma_B^2(s) &= \int \bar{s}\big(B - \langle B\rangle_s I\big)^2 s\,dt\\
&= \int \left(\frac{1}{i}s'(t) - \langle B\rangle_s s\right)\overline{\frac{1}{i}s'(t) - \langle B\rangle_s s}\,dt\\
&= \int \big(\omega\hat{s}(\omega) - \langle B\rangle_s \hat{s}\big)\overline{\omega\hat{s}(\omega) - \langle B\rangle_s \hat{s}}\,d\omega\\
&= \int \big(\omega - \langle B\rangle_s\big)^2 |\hat{s}|^2\,d\omega,\\
\big\langle [A,B]\big\rangle_s &= \big\langle (AB - BA)s, s\big\rangle\\
&= \int \left(t\cdot\frac{1}{i}s'(t) - \frac{1}{i}\cdot s - \frac{1}{i}\cdot ts'(t)\right)\bar{s}\,dt\\
&= \int \left(-\frac{1}{i}\right)|s|^2\,dt = -\frac{1}{i}\|s\|^2,
\end{aligned}$$

and

$$\begin{aligned}\int \big|As - \langle A\rangle_s s\big| \left|\mathrm{Re}\left(\frac{Bs}{s}\right) - \langle B\rangle_s\right| |s|\,dt &= \int \big|ts - \langle t\rangle s\big| \big|\varphi'(t) - \langle B\rangle_s\big| |s|\,dt \\ &= \int \big|\big(t - \langle A\rangle_s\big)\big(\varphi'(t) - \langle B\rangle_s\big)\big| |s|^2\,dt.\end{aligned}$$

Applying Theorem 5.3, we have

$$\begin{aligned}&\int \big(t - \langle A\rangle_s\big)^2 \big|s(t)\big|^2\,dt \int \big(\omega - \langle B\rangle_s\big)\big|\hat{s}(\omega)\big|^2\,d\omega \\ &\quad\geqslant \frac{1}{4} + \left[\int \big|\big(t - \langle A\rangle_s\big)\big(\varphi'(t) - \langle B\rangle_s\big)\big| \big|s(t)\big|^2\,dt\right]^2.\end{aligned} \tag{5.45}$$

The formula (5.45) coincides with the result of Theorem 2.2.

Example 5.6. Now we consider the operators A_k and B_l on $L^2(\mathbb{R})$, defined by

$$A_k s(t) = t^k s(t) \quad \text{and} \quad B_l s(t) = \frac{1}{i^l} s^{(l)}(t)$$

for arbitrary but fixed $k, l \in \mathbb{N}$, with $D(A_k) = \{s \in L^2(\mathbb{R})\colon\ x^k s(t) \in L^2(\mathbb{R})\}$ and $D(B_l) = \{s \in L^2(\mathbb{R})\colon\ s^{(l)}(t) \in L^2(\mathbb{R}),\ \|s\|_2 = 1\}$. Then we have

$$\begin{aligned}\big\langle [A_k, B_l]\big\rangle_s &= \int \bar{s}(A_k B_l - B_l A_k)s\,dt \\ &= -(-i)^l \int \bar{s} \sum_{j=1}^{\min\{k,l\}} C_l^j \cdot \frac{k!}{(k-n)!} \cdot t^{k-j} s^{(l-j)}\,dt \\ &= -(-i)^l \sum_{j=1}^{\min\{k,l\}} C_l^j \cdot \frac{k!}{(k-n)!} \int \bar{s} t^{k-j} s^{(l-j)}\,dt.\end{aligned}$$

Applying Theorem 5.3, we obtain

$$\begin{aligned}&\int \big(t^k - \langle A_k\rangle_s\big)^2 |s|^2\,dt \int \left|\frac{1}{i^l} s^{(l)}(t) - \langle B_l\rangle_s s\right|^2 dt \\ &\quad\geqslant \frac{1}{4}\left|\sum_{j=1}^{\min\{k,l\}} C_l^j \cdot \frac{k!}{(k-n)!} \int \bar{s} t^{k-j} s^{(l-j)}\,dt\right|^2 \\ &\qquad + \left[\int \left|\big(t^k - \langle A_k\rangle_s\big)\left[\mathrm{Re}\left(\frac{B_l s}{s}\right) - \langle B_l\rangle_s\right]\right| \big|s(t)\big|^2\,dt\right]^2,\end{aligned}$$

where $s \in D(A_k B_l) \cap D(B_l A_k)$.

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