L^p POLYHARMONIC DIRICHLET PROBLEMS IN REGULAR DOMAINS III: THE UNIT BALL

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ABSTRACT. In this article, we consider a class of Dirichlet problems with L^p boundary data for polyharmonic functions in the unit ball. By introducing a sequence of new kernel functions for the unit ball, which are called higher order Poisson kernels, we give the integral representation solutions of the problems.

1. INTRODUCTION

In recent years, there has been a great deal of investigation on various boundary value problems (simply, BVPs) for polyanalytic, polyharmonic, metaanalytic and metaharmonic functions etc. in some plane domains. Those include Riemann, Hilbert, Dirichlet, Neumann, Schwarz and Robin problems [2–5,8–10,14]. The main objective is to obtain integral representation solutions of BVPs in various settings such as Hölder continuity, continuity, Sobolev boundary data and so on. All of those works are to generalize the classical integral representation theory for analytic and harmonic functions in planar domains. Among other things, Dirichlet problems for polyharmonic functions (for short, PHD problems) attract considerable interest. In the higher dimensional cases, there were also a lot of studies published on Dirichlet, Neumann, and mixed problems etc., with different boundary data including L^p, Hölder, Hardy, Sobolev, Besov and so forth, in various types of domains of $\mathbb{R}^n$ and Riemann manifolds, such as regular, polyhedral, convex, semiconvex, C^1, Lipschitz domains and so on [6, 7, 17, 20–24, 29–31]. However, there have been little research in PHD problems of higher dimension ([20, 25, 30] and references therein) to get the estimates, existence, and uniqueness under appropriate assumptions, of the solutions for the problems. In [18], Kalmenov, Koshanov and Nemchenko have obtained a Green function representation of solutions to a class of Dirichlet problems of the inhomogeneous polyharmonic equation in a ball. By the method of layer potentials, Verchota studied the representation of solutions of a biharmonic Neumann problem in Lipschitz domains in [31].

The main purpose of this article is to solve the following PHD problems with L^p boundary data in the unit ball, i.e.,

$$\begin{cases} \Delta^m u = 0 \text{ in } B_n, \\ \Delta^j u = f_j \text{ on } S^{n-1}, \end{cases} \tag{1.1}$$

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where $n \geq 3$, $\Delta \equiv \Delta_n := \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$ for any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, B_n is the unit ball, S^{n-1} is the unit sphere in $\mathbb{R}^n$, $f_j \in L^p(S^{n-1})$, $m \in \mathbb{N}$ consisting of all positive integers, $0 \leq j < m$, and $1 \leq p \leq \infty$. In [2], Begehr, Du and Wang studied the same PHD problems with Hölder continuous boundary data in two-dimensional plane $\mathbb{R}^2$, that is, there all f_j were Hölder continuous. The main technique of [2] was transformation of the PHD problems to the classical Riemann problems for analytic functions on the unit circle. That was done by means of certain decompositions of polyanalytic and polyharmonic functions originated by the authors, as well as Schwarz reflection principle. By introducing a class of kernel functions, integral representation solutions of the problems were given. Those kernel functions are higher order analogs of the Poisson kernel for the unit disc (So they are called higher order Poisson kernels). Explicit formulas for the kernels were not available due to the complexity until the presence of [10] in which they were given for the continuous boundary data setting. By establishing a new decomposition theorem for polyharmonic functions, Du et al gave explicit representations of higher order Poisson kernels, and further extended the integral representation results in [2] to the continuous boundary data setting [10] and L^p data setting [11, 12]. In the present paper, we introduce a sequence of new kernel functions, which are higher dimensional analogues to the higher order Poisson kernels in the unit disc. Using these kernels, we shall give the integral representation solutions of PHD problems (1.1) in L^p boundary data setting.

2. Higher order Poisson kernels

Definition 2.1. Let D be a simply connected (bounded or boundless) domain in $\mathbb{R}^n$ with smooth boundary ∂D and $k \in \mathbb{N} \cup \{\infty\}$, $C^k(D)$ denotes the set of all functions that have continuous partial derivatives of k order in D. If f is a continuous function defined on $D \times \partial D$ satisfying $f(\cdot, v) \in C^k(D)$ for any fixed $v \in \partial D$ and $f(x, \cdot) \in C(\partial D)$ for any fixed $x \in D$, then f is said to be of $C^k \times C$ smoothness on $D \times \partial D$ and written as $f \in (C^k \times C)(D \times \partial D)$.

Definition 2.2. A sequence $\{\, g_m(x, v) \,\}_{m=1}^{\infty}$ of real-valued functions defined on $B_n \times S^{n-1}$ is called a sequence of higher order Poisson kernels, or more precisely, $g_m(x, v)$ is an mth order Poisson kernel, if it satisfies the following conditions.

1. For any $m \in \mathbb{N}$, $g_m \in (C^\infty \times C)(B_n \times S^{n-1})$, $\frac{\partial g_m}{\partial x_j}$ and $\frac{\partial^2 g_m}{\partial x_j^2}$ belong to $C(B_n \times S^{n-1})$, $1 \leq j \leq n$; the non-tangential boundary value

$$\lim_{\substack{x \to u \\ x \in B_n,\, u \in S^{n-1}}} g_m(x, v) = g_m(u, v)$$

exists for all v but $v \neq u$, where u is any fixed unit vector belonging to S^{n-1}. Moreover, $g_m(\cdot, u)$ can be continuously extended to $\overline{B}_n \setminus \{u\}$ for all $u \in S^{n-1}$;

2. $\Delta g_1(x, v) = 0$ and $\Delta g_m(x, v) = g_{m-1}(x, v)$ for $m > 1$;

3. $\lim_{x \to u,\, x \in B_n,\, u \in S^{n-1}} \int_{S^{n-1}} g_1(x, v)\gamma(v)dv = \gamma(u)$ a.e. for any $\gamma \in L^p(S^{n-1})$, $p \geq 1$;

4. $\lim_{x \to u,\, x \in B_n,\, u \in S^{n-1}} \int_{S^{n-1}} g_m(x, v)\gamma(v)dv = 0$ for any $2 \leq m \leq n-1$ and $\gamma \in L^p(S^{n-1})$, $p \geq 1$; and

5. $\lim_{x \to u,\, x \in B_n,\, u \in S^{n-1}} g_m(x, v) = 0$ uniformly on $v \in S^{n-1}$ for any fixed $u \in S^{n-1}$, $m \geq n$,

where all the limits are non-tangential [27].

Higher order Poisson kernels are the key in our approach to solve the PHD problems (1.1). In what follows, we shall show their explicit expressions of power series of $|x|$ with coefficients in terms of ultraspherical (or Gegenbauer) polynomials, $P_l^{(\lambda)}$. The latter can be defined through a generating function [1,27]. Let

$$(1-2r\xi+r^2)^{-\lambda}=\sum_{l=0}^{\infty}P_l^{(\lambda)}(\xi)r^l, \tag{2.1}$$

where $0\le |r|<1$, $|\xi|\le 1$ and $\lambda>-\frac{1}{2}$, then $P_l^{(\lambda)}$ is called the ultraspherical polynomials of degree l associated with λ. $P_l^{(\lambda)}$ is a polynomial of precise degree l, and has the following explicit expression (see [16,28]):

$$P_l^{(\lambda)}(\xi)=\sum_{j=0}^{[\frac{l}{2}]}(-1)^j\frac{\Gamma(l-j+\lambda)}{\Gamma(\lambda)j!(l-2j)!}(2\xi)^{l-2j}. \tag{2.2}$$

Introduce the spherical coordinate

$$\begin{cases} x_1=r\cos\theta_1,\\ x_2=r\sin\theta_1\cos\theta_2,\\ x_3=r\sin\theta_1\sin\theta_2\cos\theta_3,\\ \vdots\\ x_{n-1}=r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1},\\ x_n=r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1}, \end{cases} \tag{2.3}$$

and set

$$u=(\cos\theta_1,\sin\theta_1\cos\theta_2,\sin\theta_1\sin\theta_2\cos\theta_3,\dots,\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1}), \tag{2.4}$$

then the polar coordinate is

$$x=ru, \tag{2.5}$$

where $r=(x_1^2+x_2^2+\cdots+x_n^2)^{1/2}$, $0\le\theta_1,\theta_2,\dots,\theta_{n-2}\le\pi$ and $0\le\theta_{n-1}\le 2\pi$. By a straightforward calculation, the polar coordinate form of the Laplacian is

$$\Delta=\frac{1}{\rho^{n-1}}\frac{\partial}{\partial\rho}(\rho^{n-1}\frac{\partial}{\partial\rho})+\frac{1}{\rho^2}\Delta_{S^{n-1}}, \tag{2.6}$$

where the Laplace-Beltrami operator

$$\begin{aligned}\Delta_{S^{n-1}}=&\frac{\partial^2}{\partial\theta_1^2}+\frac{1}{\sin^2\theta_1}\frac{\partial^2}{\partial\theta_2^2}+\cdots+\frac{1}{\sin^2\theta_1\cdots\sin^2\theta_{n-2}}\frac{\partial^2}{\partial\theta_{n-1}^2}\\ &+(n-2)\cot\theta_1\frac{\partial}{\partial\theta_1}+(n-3)\frac{\cot\theta_2}{\sin^2\theta_1}\frac{\partial}{\partial\theta_2}\\ &+(n-4)\frac{\cot\theta_3}{\sin^2\theta_1\sin^2\theta_2}\frac{\partial}{\partial\theta_3}+\cdots\\ &+\frac{\cot\theta_{n-2}}{\sin^2\theta_1\cdots\sin^2\theta_{n-3}}\frac{\partial}{\partial\theta_{n-2}}.\end{aligned} \tag{2.7}$$

Consider the partial differential equation on the unit sphere,

$$\Delta_{S^{n-1}}\Phi=\lambda\Phi, \tag{2.8}$$

where Φ is a function defined on the unit sphere, λ is a constant. If the above partial differential equation has a nonzero solution for some λ, then λ is called an eigenvalue of $\Delta_{S^{n-1}}$ and the nonzero solutions Φ are called eigenfunctions corresponding to such λ.

In [16, 17], Hua established some excellent results, one of which is the following

Lemma 2.3 ([17]). *All eigenvalues of the Laplace-Beltrami operator $\Delta_{S^{n-1}}$ given by (2.7) are*

$$\lambda_l = -l(l+n-2), \tag{2.9}$$

where $l = 0, 1, 2, \ldots$. The eigenfunctions corresponding to any such λ_l are $P_l^{(\frac{n}{2}-1)}(u \cdot v)$ on $u \in S^{n-1}$ for any fixed $v \in S^{n-1}$, where $u \cdot v = \sum_{k=1}^n u_k v_k$ is the euclidian inner product of the unit vectors $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ in S^{n-1}.

Remark 2.4. If we define

$$Z_v^{(l)}(u) = c_{l,n} P_l^{(\frac{n}{2}-1)}(u \cdot v), \tag{2.10}$$

where the constant (see [16])

$$c_{l,n} = \frac{1}{2\pi^{\frac{n}{2}}} \left(l + \frac{n}{2} - 1\right) \Gamma\left(\frac{n}{2} - 1\right), \tag{2.11}$$

$Z_v^{(l)}$ is called the zonal harmonic of degree l with pole v [27]; the space $\mathscr{H}_l$ consisting of all zonal harmonics is called the space of spherical harmonics of degree l; the dimension of $\mathscr{H}_l$ is finite, more precisely,

$$\dim \mathscr{H}_l = \binom{n+l-1}{l} - \binom{n+l-3}{l-2} \triangleq a_l. \tag{2.12}$$

So the dimension of the eigenspace corresponding to eigenvalue λ_l is finite and equals to $\dim \mathscr{H}_l$. Such eigenspace consists of all $P_l^{(\frac{n}{2}-1)}(u \cdot v)$. From [27],

$$|Z_v^{(l)}(u)| \le |Z_v^{(l)}(v)| = a_l \omega_{n-1}^{-1} \tag{2.13}$$

for any $u, v \in S^{n-1}$, where ω_{n-1} is the surface area of the unit sphere S^{n-1}, $\omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$. Therefore, by (2.10), (2.11) and (2.13),

$$|P^{(l)}(u \cdot v)| \le |P^{(l)}(v \cdot v)| = \frac{n-2}{2l+n-2} a_l \tag{2.14}$$

for any $u, v \in S^{n-1}$.

Set $x = ru$, $u \in S^{n-1}$. From the generating function (2.1), by a direct calculus, the Poisson kernel for the unit ball in $\mathbb{R}^n$ can be expanded as (see [17])

$$\begin{aligned} P_n(x, v) &= \frac{1}{\omega_{n-1}} \frac{1-|x|^2}{|x-v|^n} \\ &= \frac{1}{\omega_{n-1}} \frac{1-r^2}{(1-2ru\cdot v + r^2)^{\frac{n}{2}}} \\ &= \frac{1}{\omega_{n-1}} \sum_{l=0}^{\infty} \frac{2l+n-2}{n-2} r^l P_l^{(\frac{n}{2}-1)}(u \cdot v), \end{aligned} \tag{2.15}$$

where $r = |x| < 1$ and $v \in S^{n-1}$.

Lemma 2.5.

$$\Delta\left(r^s P_l^{(\frac{1}{2}n-1)}(u\cdot v)\right) = (\lambda_l-\lambda_s)r^{s-2}P_l^{(\frac{1}{2}n-1)}(u\cdot v), \tag{2.16}$$

for any nonzero $s\in\mathbb{R}$ and $l\in\mathbb{N}\cup\{0\}$.

Proof. As $l=0$, for any nonzero $s\in\mathbb{R}$, the LHS of (2.16) $=\Delta r^s=\left(\frac{\partial^2}{\partial r^2}+\frac{n-1}{r}\frac{\partial}{\partial r}\right)r^s=s(s+n-2)r^{s-2}=-\lambda_s$; For any nonzero $s\in\mathbb{R}$ and positive integer l, by Lemma 2.3,

$$\begin{aligned}\Delta\left(r^s P_l^{(\frac{1}{2}n-1)}(u\cdot v)\right) &= \left(\frac{\partial^2}{\partial r^2}+\frac{n-1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\Delta_{S^{n-1}}\right)r^sP_l^{(\frac{1}{2}n-1)}(u\cdot v)\\ &= (\lambda_l-\lambda_s)r^{s-2}P_l^{(\frac{1}{2}n-1)}(u\cdot v).\end{aligned}$$

Thus (2.16) is established. □

In what follows, for any positive integer l, we denote

$$\Lambda_k^{(l)}=\lambda_l-\lambda_{l+2k}, \tag{2.17}$$

$$\left(\frac{1}{\mathbf{\Lambda}^{(l)}}\right)_0=\{1\}, \tag{2.18}$$

$$\left(\frac{1}{\mathbf{\Lambda}^{(l)}}\right)_k=\left\{\frac{1}{\Lambda_1^{(l)}},\frac{1}{\Lambda_2^{(l)}},\ldots,\frac{1}{\Lambda_k^{(l)}}\right\} \tag{2.19}$$

for any positive integer k, and

$$\left(\frac{1}{\mathbf{\Lambda}^{(l)}}\right)_\infty=\left\{\frac{1}{\Lambda_1^{(l)}},\frac{1}{\Lambda_2^{(l)}},\ldots,\frac{1}{\Lambda_k^{(l)}},\ldots\right\}. \tag{2.20}$$

Introduce a class of index transformation operator, which is defined as follows:

$$\begin{aligned} A_j^{(l)}:\ \prod\left(\frac{1}{\mathbf{\Lambda}^{(l)}}\right)_\infty &\longrightarrow \prod\left(\frac{1}{\mathbf{\Lambda}^{(l)}}\right)_\infty\\ \prod_{p=1}^m\left(\frac{1}{\Lambda_p^{(l)}}\right)^{k_p} &\longmapsto \frac{1}{\Lambda_j^{(l)}}\times\prod_{p=1}^m\left(\frac{1}{\Lambda_p^{(l)}}\right)^{k_p},\end{aligned} \tag{2.21}$$

where $1\le j<\infty$, and the set $\prod\left(\frac{1}{\mathbf{\Lambda}^{(l)}}\right)_\infty=\left\{\prod_{p=1}^m\left(\frac{1}{\Lambda_p^{(l)}}\right)^{k_p}: m\in\mathbb{N}, k_p\in\mathbb{N}\cup\{0\}\right\}$. Namely, $\prod\left(\frac{1}{\mathbf{\Lambda}^{(l)}}\right)_\infty$ is composed of the products of all the members in the set $\left(\frac{1}{\mathbf{\Lambda}^{(l)}}\right)_m$ $(\subset\left(\frac{1}{\mathbf{\Lambda}^{(l)}}\right)_\infty)$ with non-negative integer indices for any $m\in\mathbb{N}$; the operator $A_j^{(l)}$ makes the index of the factor $\frac{1}{\Lambda_j^{(l)}}$ in any such product increase by 1. (It is noteworthy that $\prod_{p=1}^m\frac{1}{\left(\Lambda_p^{(l)}\right)^{k_p}}=\prod_{p=1}^j\frac{1}{\left(\Lambda_p^{(l)}\right)^{k_p}}$ with $k_p=0$, $m+1\le p\le j$ as $m\le j$. Of course, in the following, we only consider the products as some part of the members of the set $\prod\left(\frac{1}{\mathbf{\Lambda}^{(l)}}\right)_\infty$.)

Moreover, define a formal operation of vector operators:

$$(T_1,T_2,\ldots,T_q)\circ(h_1,h_2,\ldots,h_q)=(T_1h_1,T_2h_2,\ldots,T_qh_q), \tag{2.22}$$

where $q\in\mathbb{N}$, T_q is an operator on some appropriate linear space H_q and $h_q\in H_q$.

Set

$$\mathcal{A}_m^{(l)} = \Big(A_{m-1}^{(l)}, \underbrace{A_{m-2}^{(l)}}_{2^0 \text{ term}}, \underbrace{A_{m-3}^{(l)}, A_{m-3}^{(l)}}_{2^1 \text{ terms}}, \ldots, \underbrace{A_k^{(l)}, \ldots, A_k^{(l)}}_{2^{m-2-k} \text{ terms}}, \ldots, \underbrace{A_3^{(l)}, \ldots, A_3^{(l)}}_{2^{m-5} \text{ terms}}, \underbrace{A_2^{(l)}, \ldots, A_2^{(l)}}_{2^{m-4} \text{ terms}}, \underbrace{A_1^{(l)}, \ldots, A_1^{(l)}}_{2^{m-3} \text{ terms}}\Big), \tag{2.23}$$

and

$$X_m = \Big(|x|^{2(m-1)} - 1, \underbrace{|x|^{2(m-2)} - 1}_{2^0 \text{ term}}, \ldots, \underbrace{|x|^{2k} - 1, \ldots, |x|^{2k} - 1}_{2^{m-2-k} \text{ terms}}, \ldots, \underbrace{|x|^{2\times 2} - 1, \ldots, |x|^{2\times 2} - 1}_{2^{m-4} \text{ terms}}, \underbrace{|x|^{2\times 1} - 1, \ldots, |x|^{2\times 1} - 1}_{2^{m-3} \text{ terms}}\Big)^T \tag{2.24}$$

for any $m \in \mathbb{N}$ and $m \geq 3$, where and in what follows, either $(\cdots)^T$ or $[\cdots]^T$ always denotes the transpose operation. With the above preliminaries, we can give the following definitions:

$$c_1^{(l)} := 1, \tag{2.25}$$

$$c_2^{(l)} := A_1^{(l)} c_1^{(l)} = \frac{1}{\Lambda_1^{(l)}}, \tag{2.26}$$

and for any $m \geq 3$,

$$c_m^{(l)} := \mathcal{A}_m^{(l)} \circ \begin{pmatrix} c_{m-1}^{(l)} \\ -c_{m-1}^{(l)} \end{pmatrix}. \tag{2.27}$$

Therefore, we have

Lemma 2.6. *For any $x \in \mathbb{R}^n$ and fixed $v \in S^{n-1}$, let*

$$\left(H_v^{(l)}\right)_1 (x) = |x|^l P_l^{(\frac{n}{2}-1)} \left(\frac{x}{|x|} \cdot v\right), \tag{2.28}$$

$$\left(H_v^{(l)}\right)_2 (x) = c_2^{(l)} \left(|x|^2 - 1\right) \left(H_v^{(l)}\right)_1 (x) \tag{2.29}$$

and

$$\left(H_v^{(l)}\right)_m (x) = \left(H_v^{(l)}\right)_1 (x) \left[\left(c_m^{(l)}\right)^T X_m\right], \tag{2.30}$$

where $m = 3, 4, \ldots$, $l = 0, 1, 2, \ldots$, and λ_l given by (2.9) are the eigenvalues of the Laplace-Beltrami operator $\Delta_{S^{n-1}}$ defined in (2.7). Then

$$\Delta \left(H_v^{(l)}\right)_1 = 0 \tag{2.31}$$

and

$$\Delta \left(H_v^{(l)}\right)_m = \left(H_v^{(l)}\right)_{m-1}, \ m \geq 2. \tag{2.32}$$

Proof. It is immediat from Lemma 2.5 that $\Delta\left(H_v^{(l)}\right)_1(x)=\Delta\left[|x|^l P_l^{(\frac{n}{2}-1)}\left(\frac{x}{|x|}\cdot v\right)\right]=0$. Similarly, as $m=2$, by Lemma 2.5,

$$\begin{aligned}\Delta\left(H_v^{(l)}\right)_2(x)&=\Delta\left[\frac{1}{\Lambda_1^{(l)}}\left(|x|^{l+2}-|x|^l\right)P_l^{(\frac{n}{2}-1)}\left(\frac{x}{|x|}\cdot v\right)\right]\\&=|x|^l P_l^{(\frac{n}{2}-1)}\left(\frac{x}{|x|}\cdot v\right)\\&=\left(H_v^{(l)}\right)_1(x).\end{aligned}\tag{2.33}$$

That is to say that (2.32) holds for $m=2$. For any $m\geq 3$, noting the definitions (2.23)-(2.27) and (2.30), we can write that

$$\begin{aligned}\left(H_v^{(l)}\right)_{m-1}(x)=&\Bigg\{\left[d_{0,m-2}[|x|^{2(m-2)}-1]-\frac{\Lambda_{m-2}^{(l)}}{\Lambda_1^{(l)}}d_{0,m-2}\left[|x|^2-1\right]\right]\\&-\sum_{j=2}^{m-3}\sum_{p=1}^{2^{m-3-j}}\left[d_{p,j}[|x|^{2j}-1]-\frac{\Lambda_j^{(l)}}{\Lambda_1^{(l)}}d_{p,j}\left[|x|^2-1\right]\right]\Bigg\}\\&\times\left(H_v^{(l)}\right)_1(x),\end{aligned}\tag{2.34}$$

where the coefficients $d_{0,m-2}$ and $d_{p,j}$ depend only on $\Lambda_s^{(l)}$, $s=1,2,\dots,m-2$. Therefore, by the definitions (2.27) and (2.30),

(2.35)

$$\begin{aligned}\left(H_v^{(l)}\right)_m(x)&=\left[c_m^{(l)}\right]^T\left[\left(\left(H_v^{(l)}\right)_1(x)\right)X_m\right]\\&=\Bigg\{\frac{1}{\Lambda_{m-1}^{(l)}}\left[d_{0,m-2}(|x|^{2(m-1)}-1)-\frac{\Lambda_{m-2}^{(l)}}{\Lambda_1^{(l)}}d_{0,m-2}\left(|x|^2-1\right)\right]\\&\quad-\sum_{j=3}^{m-2}\sum_{p=1}^{2^{m-2-j}}\frac{1}{\Lambda_j^{(l)}}\left[d_{p,j-1}(|x|^{2j}-1)-\frac{\Lambda_{j-1}^{(l)}}{\Lambda_1^{(l)}}d_{p,j-1}\left(|x|^2-1\right)\right]\\&\quad-\frac{1}{\Lambda_1^{(l)}}\left[\left(d_{0,m-2}-\frac{\Lambda_{m-2}^{(l)}}{\Lambda_1^{(l)}}d_{0,m-2}\right)-\sum_{j=3}^{m-2}\sum_{p=1}^{2^{m-2-j}}\left(d_{p,j-1}-\frac{\Lambda_{j-1}^{(l)}}{\Lambda_1^{(l)}}d_{p,j-1}\right)\right]\\&\quad\times\left(|x|^2-1\right)\Bigg\}\left(H_v^{(l)}\right)_1(x).\end{aligned}$$

So, by Lemma 2.5 and a straightforward calculation,

$$\Delta\left(H_v^{(l)}\right)_m=\left(H_v^{(l)}\right)_{m-1}.$$

□

Remark 2.7. Define an ordering relation $\preceq$ in $\mathbb{R}^n$: For any $\alpha=(a_1,a_2,\dots,a_n)$ and $\beta=(b_1,b_2,\dots,b_n)$, comparing the corresponding components in the order of 1 to n, if there exists the first $j\in\{1,2,\dots,n\}$ such that $a_j<b_j$, then $\alpha\prec\beta$; otherwise, $\alpha=\beta$. In (2.34), the coefficients $d_{0,m-2}=\prod_{s=1}^{m-2}\frac{1}{\Lambda_s^{(l)}}$ and $d_{p,j}=(-1)^{k_{1,p,j}}\prod_{s=1}^{m-3}\frac{1}{\left(\Lambda_s^{(l)}\right)^{k_{s,p,j}}}$, with the indices $(k_{1,p,j},k_{2,p,j},\dots,k_{m-3,p,j})$ satisfying

the following properties:

$$k_{1,p,j} + k_{2,p,j} + \cdots + k_{m-3,p,j} = m - 2, \tag{2.36}$$

$$k_{1,p,j} \geq k_{2,p,j} \geq \cdots \geq k_{m-3,p,j}, \tag{2.37}$$

$$(k_{1,s,j}, k_{2,s,j}, \ldots, k_{m-3,s,j}) \preceq (k_{1,s+1,j}, k_{2,s+1,j}, \ldots, k_{m-3,s+1,j}) \tag{2.38}$$

and

$$(k_{1,1,t}, k_{2,1,t}, \ldots, k_{m-3,1,t}) \preceq (k_{1,1,t+1}, k_{2,1,t+1}, \ldots, k_{m-3,1,t+1}), \tag{2.39}$$

where $2 \leq j, t, t+1 \leq m-3$ and $1 \leq p, s, s+1 \leq 2^{m-2-j}$.

If we define a vertical sum as in [10]:

$$\sum \begin{cases} a_1 \\ a_2 \\ \vdots \\ a_n \end{cases} =: a_1 + a_2 + \cdots + a_n, \tag{2.40}$$

then, for example, we have the following expressions

$$\left(H_v^{(l)}\right)_4(x) = \left(H_v^{(l)}\right)_1(x) \times \sum \begin{cases} \frac{1}{\Lambda_1\Lambda_2\Lambda_3}(|x|^6 - 1) \\ -\frac{1}{\Lambda_1^2\Lambda_2}(|x|^4 - 1) \\ -\sum \begin{cases} \frac{1}{\Lambda_1^2\Lambda_2}(|x|^2 - 1) \\ -\frac{1}{\Lambda_1^3}(|x|^2 - 1) \end{cases} \end{cases}, \tag{2.41}$$

$$\left(H_v^{(l)}\right)_5(x) = \left(H_v^{(l)}\right)_1(x) \times \sum \begin{cases} \frac{1}{\Lambda_1\Lambda_2\Lambda_3\Lambda_4}(|x|^8 - 1) \\ -\frac{1}{\Lambda_1^2\Lambda_2\Lambda_3}(|x|^6 - 1) \\ -\sum \begin{cases} \frac{1}{\Lambda_1^2\Lambda_2^2}(|x|^4 - 1) \\ -\frac{1}{\Lambda_1^3\Lambda_2}(|x|^4 - 1) \end{cases} \\ -\sum \begin{cases} \frac{1}{\Lambda_1^2\Lambda_2\Lambda_3}(|x|^2 - 1) \\ -\frac{1}{\Lambda_1^3\Lambda_2}(|x|^2 - 1) \\ -\sum \begin{cases} \frac{1}{\Lambda_1^3\Lambda_2}(|x|^2 - 1) \\ -\frac{1}{\Lambda_1^4}(|x|^2 - 1) \end{cases} \end{cases} \end{cases} \tag{2.42}$$

and
(2.43)

$$\left(H_v^{(l)}\right)_6(x)=\left(H_v^{(l)}\right)_1(x)\times\sum\begin{cases}\frac{1}{\Lambda_1\Lambda_2\Lambda_3\Lambda_4\Lambda_5}(|x|^{10}-1)\\ -\frac{1}{\Lambda_1^2\Lambda_2\Lambda_3\Lambda_4}(|x|^{8}-1)\\ -\sum\begin{cases}\frac{1}{\Lambda_1^2\Lambda_2^2\Lambda_3}(|x|^{6}-1)\\ -\frac{1}{\Lambda_1^3\Lambda_2\Lambda_3}(|x|^{6}-1)\end{cases}\\ -\sum\begin{cases}\frac{1}{\Lambda_1^2\Lambda_2^2\Lambda_3}(|x|^{4}-1)\\ -\frac{1}{\Lambda_1^3\Lambda_2^2}(|x|^{4}-1)\\ -\sum\begin{cases}\frac{1}{\Lambda_1^3\Lambda_2^2}(|x|^{4}-1)\\ -\frac{1}{\Lambda_1^4\Lambda_2}(|x|^{4}-1)\end{cases}\end{cases}\\ -\sum\begin{cases}\frac{1}{\Lambda_1^2\Lambda_2\Lambda_3\Lambda_4}(|x|^{2}-1)\\ -\frac{1}{\Lambda_1^3\Lambda_2\Lambda_3}(|x|^{2}-1)\\ -\sum\begin{cases}\frac{1}{\Lambda_1^3\Lambda_2^2}(|x|^{2}-1)\\ -\frac{1}{\Lambda_1^4\Lambda_2}(|x|^{2}-1)\end{cases}\\ -\sum\begin{cases}\frac{1}{\Lambda_1^3\Lambda_2\Lambda_3}(|x|^{2}-1)\\ -\frac{1}{\Lambda_1^4\Lambda_2}(|x|^{2}-1)\\ -\sum\begin{cases}\frac{1}{\Lambda_1^4\Lambda_2}(|x|^{2}-1)\\ -\frac{1}{\Lambda_1^5}(|x|^{2}-1)\end{cases}\end{cases}\end{cases}\end{cases}.$$

Lemma 2.8. *Let K is a compact subset of $\mathbb{R}^n$, R is a positive real number. Assume that the variable coefficient power series $\sum_{l=0}^{\infty}a_l(u)r^l$ with $a_l\in C(K)$, $l=0,1,2,\ldots$, is uniformly convergent on any compact set $K\times[0,R']$, where $0<R'<R$, and the limit*

$$\lim_{\substack{r\to R-\\ K\ni u'\to u\in K}}\sum_{l=0}^{\infty}a_l(u')r^l$$

exists for some $u\in K$, then for such $u\in K$, the limit

$$\lim_{\substack{r\to R-\\ K\ni u'\to u\in K}}\sum_{l=0}^{\infty}\frac{1}{Al+B}a_l(u')r^l$$

also exists for any nonzero constants $A,B\in\mathbb{R}$ with $\frac{B}{A}\geq 0$.

Proof. It is enough to deal with the special case $A=1$ and $B>0$. To do so, set

$$F(r,u)=\sum_{l=0}^{\infty}a_l(u)r^l \tag{2.44}$$

and

$$G(r,u)=\sum_{l=0}^{\infty}\frac{1}{l+B}a_l(u)r^l, \tag{2.45}$$

where $0\le r<R$ and $u\in K$. Then, as $0<r<R$, by termwise integrations,

$$\begin{aligned} G(r,u)&=\sum_{l=0}^{\infty}\frac{1}{l+B}a_l(u)r^l \\ &=r^{-B}\sum_{l=0}^{\infty}\frac{1}{l+B}a_l(u)r^{l+B} \\ &=r^{-B}\sum_{l=0}^{\infty}\int_0^r a_l(u)s^{l+B-1}ds \\ &=r^{-B}\int_0^r\sum_{l=0}^{\infty}a_l(u)s^{l+B-1}ds \\ &=r^{-B}\int_0^r s^{B-1}F(s,u)ds. \end{aligned} \tag{2.46}$$

By the assumption, $\lim_{r\to R-,\,K\ni u'\to u\in K}F(r,u')$ exists. If $F(R,u)$ is defined by this limit, we get that $F\in C([0,R]\times K)$. Thus, from (2.46), by elementary calculus, the $\lim_{r\to R-,\,K\ni u'\to u\in K}G(r,u')$ also exists.

□

As a simple consequent, we have

Corollary 2.9. *The same conditions as in the above lemma are provided, then for any $m\in\mathbb{N}$,*

$$\lim_{\substack{r\to R-\\ K\ni u'\to u\in K}}\sum_{l=0}^{\infty}\prod_{p=1}^{m}\frac{1}{(A_pl+B_p)^{k_p}}a_l(u')r^l \tag{2.47}$$

also exists for any nonzero constants $A_p,B_p\in\mathbb{R}$ with $\frac{B_p}{A_p}\ge 0$, and $k_p\in\mathbb{N}$, $1\le p\le m$.

Theorem 2.10. *Let*

$$g_m(x,v)=\frac{1}{\omega_{n-1}}\sum_{l=0}^{\infty}\frac{2l+n-2}{n-2}\left(H_v^{(l)}\right)_m(x), \tag{2.48}$$

where $m\in\mathbb{N}$, $x\in B_n$, $v\in S^{n-1}$, and $\left(H_v^{(l)}\right)_m(x)$ given as in Lemma 2.6. Then $\{\,g_m(x,v)\,\}_{m=1}^{\infty}$ is a sequence of higher order Poisson kernels defined as in Definition 2.2.

Proof. Since $\left(H_v^{(l)}\right)_1(x)=|x|^lP_l^{(\frac{n}{2}-1)}(x\cdot v/|x|)$ is a homogeneous polynonials of degree l with respect to $x\in\mathbb{R}^n$ [27], g_m given by (2.48) is a power series in n variables [19].

Owing to (2.15) and (2.28), g_1 is just the Poisson kernel for unit ball in $\mathbb{R}^n$. So g_1 satisfies the related properties stated in Definition 2.2. By (2.30),

$$g_m(x,v)=\frac{1}{\omega_{n-1}}\sum_{l=0}^{\infty}\frac{2l+n-2}{n-2}\left(H_v^{(l)}\right)_1(x)\left[\left(c_m^{(l)}\right)^TX_m\right].$$

By (2.14),

$$\left|\left(H_v^{(l)}\right)_1(x)\right| \leq \frac{n-2}{2l+n-2} a_l |x|^l, \tag{2.49}$$

where a_l is given by (2.12). Since from (2.12) and (2.17),

$$\Lambda_k^{(l)} = 2k(2l+2k+n-2) = O(l+1) \ (l \to +\infty),$$

by (2.24) and (2.28), for any $x \in B_n$,

$$\left|\left(c_m^{(l)}\right)^T X_m\right| \leq C \frac{1}{(l+1)^{m-1}} \left(1-|x|^2\right), \tag{2.50}$$

where C is a positive constant, and in which we use the basic facts as

$$|x|^{2p} - 1 = (|x|^2-1)(1+|x|^2+\cdots+|x|^{2p-2}) \tag{2.51}$$

and

$$(m-1) + (m-2)\times 2^0 + (m-3)\times 2^1 + \cdots + 1 \times 2^{m-3} = 2^{m-1}. \tag{2.52}$$

From (2.30), (2.49) and (2.50), due to the fact

$$a_l = \frac{2l+n-2}{l}\binom{n+l-3}{l} = O\left((l+1)^{n-2}\right) \ (l \to \infty), \tag{2.53}$$

$$\left|\frac{2l+n-2}{n-2}\left(H_v^{(l)}\right)_m(x)\right| \leq C(1-|x|^2)^{\min\{m-1,1\}} \frac{a_l}{(l+1)^{m-1}} |x|^l \tag{2.54}$$

for any $m \in \mathbb{N}$, $x \in B_n$ and $v \in S^{n-1}$. Especially, when $m \geq n$,

$$\begin{aligned} \left|\frac{2l+n-2}{n-2}\left(H_v^{(l)}\right)_m(x)\right| &\leq C(1-|x|^2)\frac{a_l}{(l+1)^{m-1}}|x|^l \\ &\leq C(1-|x|^2)\frac{1}{(l+1)^{m+1-n}}|x|^l. \end{aligned} \tag{2.55}$$

Therefore, The power seriers of RHS of (2.48) is uniformly convergent on $K \times S^{n-1}$ for any $m \in \mathbb{N}$, where K is any closed set of B_n. So g_m is real analytic in $B_n \times S^{n-1}$ (see [19]). Then, the following facts are obtained:

(1) By Lemma 2.6 and termwise differentiations (also see [19]),

$$\Delta g_1(x,v) = 0 \text{ and } \Delta g_m(x,v) = g_{m-1}(x,v), \ m \geq 2;$$

(2)

$$\lim_{\substack{x \to u, \\ x \in B_n,\, u \in S^{n-1}}} g_m(x,v) = 0$$

uniformly on $v \in S^{n-1}$ for any fixed $u \in S^{n-1}$, $m \geq n$.

Thus the properties 2 and 5 in Definition 2.2 are established.

Denote

$$[p_l(\gamma)](u) = \int_{S^{n-1}} P_l^{(\frac{n}{2}-1)}(u\cdot v)\gamma(v)dv \tag{2.56}$$

for any $\gamma \in L^p\left(S^{n-1}\right)$ and fixed $u \in S^{n-1}$. By Hölder inequality,

$$|[p_l(\gamma)](u)| \leq \frac{n-2}{2l+n-2} a_l \omega_{n-1}^{\frac{p-1}{p}} ||\gamma||_p < +\infty, \tag{2.57}$$

where a_l is given by (2.12), and ω_{n-1} is the surface area of the unit sphere S^{n-1}. Then, by termwise integrations, for any $\gamma \in L^p\left(S^{n-1}\right)$,

$$
\begin{aligned}
F_1^\gamma(ru) :&= \int_{S^{n-1}} g_1(ru, v)\gamma(v)dv \\
&= \frac{1}{\omega_{n-1}} \sum_{l=0}^{\infty} \frac{2l+n-2}{n-2}[p_l(\gamma)](u)r^l,
\end{aligned} \tag{2.58}
$$

$$
\begin{aligned}
F_2^\gamma(ru) :&= \int_{S^{n-1}} g_m(ru, v)\gamma(v)dv \\
&= \omega_{n-1}^{-1}\left(r^2-1\right) \sum_{l=0}^{\infty} \frac{1}{\Lambda_1^{(l)}} \frac{2l+n-2}{n-2}[p_l(\gamma)](u)r^l
\end{aligned} \tag{2.59}
$$

and

$$
\begin{aligned}
F_m^\gamma(ru) :&= \int_{S^{n-1}} g_m(ru, v)\gamma(v)dv \\
&= \frac{1}{\omega_{n-1}} \sum_{l=0}^{\infty} \left[\left(c_m^{(l)}\right)^T X_m\right] \frac{2l+n-2}{n-2}[p_l(\gamma)](u)r^l \\
&= \omega_{n-1}^{-1}\left(r^2-1\right) \sum_{l=0}^{\infty} \left[\left(c_m^{(l)}\right)^T \widetilde{X}_m\right] \frac{2l+n-2}{n-2}[p_l(\gamma)](u)r^l,
\end{aligned} \tag{2.60}
$$

where $x = ru$, $r = |x|$, $m \geq 3$, $\widetilde{X}_m = \left(|x|^2-1\right)^{-1} X_m$, X_m is given by (2.24). By the definitions of $c_m^{(l)}$ and $\widetilde{X}_m$, similar to (2.34), we can write that

$$
\left(c_m^{(l)}\right)^T \widetilde{X}_m = \widetilde{d}_{0,m-1} \sum_{q=0}^{m-2} |x|^{2q} + \sum_{j=1}^{m-2} \sum_{p=1}^{2^{m-2-j}} \widetilde{d}_{p,j} \sum_{q=0}^{j-1} |x|^{2q} \tag{2.61}
$$

where the coeficients $\widetilde{d}_{0,m-1} = \prod_{s=1}^{m-1} \frac{1}{\Lambda_s^{(l)}}$ and $d_{p,j} = (-1)^{k_{1,p,j}} \prod_{s=1}^{m-2} \frac{1}{\left(\Lambda_s^{(l)}\right)^{k_{s,p,j}}}$, in which the indices $(k_{1,p,j}, k_{2,p,j}, \ldots, k_{m-2,p,j})$ have the properties stated in Remark 2.7. Since $\lim_{r\to 1-,\, S^{n-1}\ni u'\to u\in S^{n-1}} F_1^\gamma(ru') = \gamma(u)$ a.e. on $u \in S^{n-1}$ [27], by Corollary 2.9 and (2.58)-(2.61), we obtain

$$
\lim_{\substack{r\to 1- \\ S^{n-1}\ni u'\to u\in S^{n-1}}} F_m^\gamma(ru') = 0, \text{ a.e. on } u \in S^{n-1} \tag{2.62}
$$

for any $m \geq 2$. That is, the property 4 in Definition 2.2 is verified.

Finally, it is noted that the non-tangential boundary value

$$
\lim_{\substack{x\to u \\ x\in B_n,\, u\in S^{n-1}}} g_1(x, v) = g_1(u, v)
$$

exists for all v but $v \neq u$, where u is any fixed unit vector belonging to S^{n-1}. Moreover, $g_1(\cdot, u)$ can be continuously extended to $\overline{B}_n \setminus \{u\}$ for all $u \in S^{n-1}$. By an argument similar to (2.62), for $m \geq 2$, the non-tangential boundary value

$$
\lim_{\substack{x\to u \\ x\in B_n,\, u\in S^{n-1}}} g_m(x, v) = g_m(u, v)
$$

exists for all v but $v \neq u$, where u is any fixed unit vector belonging to S^{n-1}; and $g_m(\cdot, u)$ can be continuously extended to $\overline{B}_n \setminus \{u\}$ for all $u \in S^{n-1}$. The property 1 in Definition 2.2 is also established. Thus we complete this theorem. □

Remark 2.11. As $n = 2$, set $x = (x_1, x_2) = r\vartheta$, where $\vartheta = (\cos\theta, \sin\theta) \in S^1$, $\theta \in [0, 2\pi)$, S^1 is the unit circle. Let $\mathbb{T} = S^1$ and $\mathbb{D}$ denotes the unit disc, and introduce the Chebyshev polynomials of the first kind $T_l(x) = \cos(l \arccos x)$, $x \in [-1, 1]$ and $l = 0, 1, 2, \ldots$ (see [28]). Then we have the the following expression of the Poisson kernel for the unit disc in $\mathbb{R}^2$;

$$P_2(r\vartheta, \varphi) = \frac{1}{2\pi} \frac{1 - r^2}{|r\vartheta - \varphi|^2} = \frac{1}{2\pi}\left[1 + 2\sum_{l=1}^{\infty} r^l \cos(l(\theta - \phi))\right] = \frac{1}{2\pi}\sum_{l=0}^{\infty}\left(1 - \frac{\delta_{l0}}{2}\right) T_l\left(\cos(l(\theta - \phi))\right) r^l, \tag{2.63}$$

where $\varphi = (\cos\phi, \sin\phi)$, $\phi \in [0, 2\pi)$, and δ_{l0} is the Kronecker's symble. Denote $P_l^{[0]}(x) = |x|^l T_l\left(\frac{x}{|x|}\right)$, where $x \in \mathbb{R}^2$, then, by the properties of Chebyshev polynomials of the first kind and some direct calculations, $P_l^{[0]}$ have the following nice properties:

A: $P_l^{[0]}$ is a homogenous polynomials of degree l;
B: $\Delta P_l^{[0]}(x) = 0$, where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2}$, which is the Laplacian in $\mathbb{R}^2$;
C: $\frac{\partial^2}{\partial \theta^2} P_l^{[0]}(\cos\theta) = -l^2 P_l^{[0]}(\cos\theta)$.

It must be pointed out that the above $P_l^{[0]}$ is different from the Gegenbauer polynomials $P_l^{(0)}$ since the latter vanishs identically for $l \geq 1$ [28]. Based on the above facts, by a similar argument, we can give alternative explicit expressions (similar to (2.48)) of the higher order Possion kernels for the unit disc, which are expressed by some vertical sums detailed in [10].

3. Polyharmonic Dirichlet problems in the unit ball

In this section, we solve the PHD problem (1.1), i.e.,

$$\begin{cases} \Delta^m u = 0 \text{ in } B_n, \\ \Delta^j u = f_j \text{ on } S^{n-1}, \end{cases}$$

where B_n is the unit ball, S^{n-1} is the unit sphere in $\mathbb{R}^n$, $f_j \in L^p(S^{n-1})$, $m \in \mathbb{N}$, $0 \leq j < m$, and $p \geq 1$.

To do so, at first, as a special case extension of Theorem 2.27 in [13], we establish

Lemma 3.1. *Let D be a simply connected bounded domain in $\mathbb{R}^n$ with smooth boundary ∂D. If $f \in (C^1 \times C)(D \times \partial D)$ and $\frac{\partial f}{\partial x_j} \in C(D \times \partial D)$, $1 \leq j \leq n$, then*

$$\frac{\partial}{\partial x_j}\left(\int_{\partial D} f(x, v)dv\right) = \int_{\partial D} \frac{\partial f}{\partial x_j}(x, v)dv \tag{3.1}$$

for any $x = (x_1, x_2, \ldots, x_n) \in D$ and $1 \leq j \leq n$.

Proof. Fix $X = (x_1, x_2, \dots, x_n) \in D$ and $j \in \{1, 2, \dots, n\}$, take $X_l = X + t_l e_j$ with $\lim_{l\to+\infty} t_l = 0$, and $e_j = (0, \dots, 1, \dots, 0) \in \mathbb{R}^n$ whose the jth element is 1 and the other ones are zero. Denote

$$D_l(X, v) = \frac{f(X_l, v) - f(X, v)}{t_l} = \frac{\partial}{\partial x_j} f(X + \theta t_l e_j, v), \tag{3.2}$$

where $0 < \theta < 1$, then from $f \in (C^1 \times C)(D \times \partial D)$,

$$\lim_{l\to+\infty} D_l(X, v) = \frac{\partial f}{\partial x_j}(X, v),\ \ v \in \partial D \tag{3.3}$$

and by $\frac{\partial f}{\partial x_j} \in C(D \times \partial D)$,

$$\left|\frac{\partial f}{\partial x_j}(X, v)\right| \le M \tag{3.4}$$

holds on the compact set $D_c\times\partial D$ for any $1 \le j \le n$, where D_c is any compact subset of D including X, M is a positive constant depending on D_c and n. Therefore, as $X_l \in D_c$, by (3.3), (3.4) and Lebesgue's dominated convergence theorem,

$$\lim_{l\to+\infty} \int_{\partial D} D_l(X, v)dv = \int_{\partial D} \frac{\partial f}{\partial x_j}(X, v)dv,$$

i.e.,

$$\lim_{l\to+\infty} \frac{\int_{\partial D} f(X_l, v)dv - \int_{\partial D} f(X, v)dv}{t_l} = \int_{\partial D} \frac{\partial f}{\partial x_j}(X, v)dv. \tag{3.5}$$

Since X and the sequence X_l are arbitrarily chosen, then

$$\frac{\partial}{\partial x_j}\left(\int_{\partial D} f(X, v)dv\right) = \int_{\partial D} \frac{\partial f}{\partial x_j}(X, v)dv$$

for any $1 \le j \le n$ and $X \in D$. □

A natural consequent is obtained as the following

Corollary 3.2. *Let D be a simply connected bounded domain in $\mathbb{R}^n$ with smooth boundary ∂D. If $f \in (C^2 \times C)(D \times \partial D)$, $\frac{\partial f}{\partial x_j}$ and $\frac{\partial^2 f}{\partial x_j^2}$ belong to $C(D \times \partial D)$, $1 \le j \le n$, then*

$$\frac{\partial^2}{\partial x_j^2}\left(\int_{\partial D} f(x, v)dv\right) = \int_{\partial D} \frac{\partial^2}{\partial x_j^2} f(x, v)dv \tag{3.6}$$

for any $x = (x_1, x_2, \dots, x_n) \in D$ and $1 \le j \le n$. Immediately, it follows that

$$\Delta\left(\int_{\partial D} f(x, v)dv\right) = \int_{\partial D} \Delta f(x, v)dv. \tag{3.7}$$

Secondly, we establish an important theorem concerning the differentiability of integrals of higher order Poisson kernels as follows.

Theorem 3.3. *Let $\{ g_m(x,v) \}_{m=1}^{\infty}$ be the sequence of higher order Poisson kernels defined in Theorem 2.10, then for any $m > 1$ and $\gamma \in L^p(S^{n-1})$, $p \geq 1$,*

$$\Delta \left(\int_{S^{n-1}} g_m(x,v)\gamma(v)dv \right) = \int_{S^{n-1}} g_{m-1}(x,v)\gamma(v)dv. \tag{3.8}$$

Proof. Set $x = ru$ and $r = |x|$. Applying (2.58)-(2.60), similar to (2.54), we have

$$\left| \frac{2l+n-2}{n-2} \left(H_v^{(l)} \right)_m (x) \right| \leq C(1-|x|^2)^{\min\{m-1,1\}} \frac{a_l}{(l+1)^{m-1}} \omega_{n-1}^{\frac{p-1}{p}} ||\gamma||_p r^l \tag{3.9}$$

for any $m \in \mathbb{N}$, where C is a positive constant and a_l is given by (2.12). Note (2.53), $\lim_{l\to\infty} \sqrt[l]{\frac{a_l}{(l+1)^{m-1}}} = 1$. Therefore, the power series $\sum_{l=0}^{\infty} \frac{1}{(l+1)^{m-1}} a_l r^l$ convergents uniformly on the compact sets of $[-1,1]$. Then the power series in n variables given in (2.58)-(2.60) are uniformly convergent on $K \times S^{n-1}$ for any $m \in \mathbb{N}$, where K is any closed subset of B_n. So $\int_{S^{n-1}} g_m(x,v)\gamma(v)dv$ is real analytic in $B_n \times S^{n-1}$ (see [19]). Certainly, it belongs to $(C^2 \times C)(B_n \times S^{n-1})$ and satisfies the assumptions as in Corollary 3.2. Therefore, (3.8) immediately follows from Corollary 3.2. □

Now we can give the main result for polyharmonic Dirichlet problems in the unit ball as follows.

Theorem 3.4. *Let $\{ g_m(x,v) \}_{m=1}^{\infty}$ be the sequence of higher order Poisson kernels defined on $B_n \times S^{n-1}$, given by (2.48), then for any $m > 1$, the PHD problem (1.1) is solvable and its general solution is given by*

$$u(x) = \sum_{j=1}^{m} \int_{S^{n-1}} g_j(x,v) f_{j-1}(v)dv + u_h(x), \; x \in B_n, \tag{3.10}$$

where $u_h(x)$ denotes the general solution of the accompanying homogeneous PHD problem

$$\begin{cases} \Delta^m u = 0 \text{ in } B_n, \\ \Delta^j u = 0 \text{ on } S^{n-1} \end{cases} \tag{3.11}$$

where $0 \leq j \leq m-1$.

Proof. Note the inductive property of higher order Poisson kernels stated as in Definition 2.2, and let the polyharmonic operators Δ^l, $1 \leq l \leq n-1$, act on two sides of (3.10), by Theorem 3.3, we have

$$\Delta^l u(x) = \sum_{j=l+1}^{n} \int_{S^{n-1}} g_{j-l}(x,v) f_{j-1}(v)dv + \Delta^l u_h(x). \tag{3.12}$$

Thus, since $\Delta^l u_h = 0$ on S^{n-1}, the non-tangential boundary value

$$\Delta^l u(s) = f_l(s), \; s \in S^{n-1}, \; 0 \leq l \leq m-1 \tag{3.13}$$

follows from (2.62) and the nice property of g_1, i.e.,

$$\lim_{\substack{x \to s \\ x\in B_n, s\in S^{n-1}}} \int_{S^{n-1}} g_1(x,v)\gamma(v)dv = \gamma(s) \tag{3.14}$$

for any $\gamma \in L^p(S^{n-1})$, $p \geq 1$. Similarly, letting the polyharmonic operators Δ^n act on two sides of (3.10), we have $\Delta^n u(x) = 0$ for any $x \in B_n$. Thus (3.10) is a solution of the PHD problem (1.1).

Denote

$$u^*(x) = \sum_{j=1}^{m} \int_{S^{n-1}} g_j(x,v) f_{j-1}(v) dv, \tag{3.15}$$

the above argument show that u^* is a special solution of the PHD problem (1.1). Since u_h is the general solution of the accompanying homogenous PHD problem (3.11), then it is immediate from linear algebra that (3.10) is the general solution of the PHD problem (1.1). □

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