Nonlinear Analysis 95 (2014) 436–449

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Unbounded holomorphic Fourier multipliers on starlike Lipschitz surfaces and applications to Sobolev spaces

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ARTICLE INFO

Article history:
Received 27 February 2013
Accepted 23 September 2013
Communicated by Enzo Mitidieri

MSC:
primary 35Q30
76D03
42B35
46E30

Keywords:
Quaternionic space
Fourier multiplier
Singular integral
Starlike Lipschitz surface
Hardy–Sobolev spaces

ABSTRACT

By a generalization of Fueter's result, we establish the correspondence between the convolution operators and the Fourier multipliers on starlike Lipschitz surfaces. As applications, we obtain the Sobolev-boundedness of the Fourier multipliers and prove the equivalence between two classes of Hardy–Sobolev spaces on starlike Lipschitz surfaces.

1. Introduction

Since 1970s, the singular integral operators on Lipschitz curves and surfaces have been studied extensively. In 1977, Calderón [1] first proved the L^2 boundedness of the singular Cauchy integral operators on a Lipschitz curve γ, where the Lipschitz constant is small. Later, Coifman–McIntosh–Meyer [2] eliminated this restriction. We refer the reader to Coifman–Jones–Semmes [3], Coifman–Meyer [4] and Jerison–Kenig [5] for further information.

In higher dimensional spaces, let Σ be a Lipschitz surface given by

$$\Sigma = \left\{ g(x)e_0 + x \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n \right\},$$

where g is a Lipschitz function such that $\|\nabla g\|_\infty \le \tan\omega$, $\omega \in [0, \frac{\pi}{2})$. Li–McIntosh–Semmes [6] embedded $\mathbb{R}^{n+1}$ in Clifford algebra $\mathbb{R}_{(n)}$ with identity e_0 and considered the right monogenic functions ϕ satisfying $|\phi(x)| \le C|x|^{-n}$ on a sector S^0_μ, $\mu > \omega$. C. Li, A. McIntosh and T. Qian proved that the convolution singular integral operator

$$T_{(\phi,\underline{\phi})}u(x) = \lim_{\varepsilon\to 0+} \left\{ \int_{y\in\Sigma, |y-x|\ge\varepsilon} \phi(x-y)n(y)u(y)dS_y + \underline{\phi}(\varepsilon n(x))u(x) \right\} \quad (1.1)$$

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0362-546X/$ – see front matter
http://dx.doi.org/10.1016/j.na.2013.09.026

is bounded on $L^p(\Sigma)$, $1 < p < \infty$. Gaudry–Long–Qian [7] gave another proof by a $T(b)$ type theorem and a martingale approach.

It is well-known that there exists a correspondence between the convolution singular integrals and the Fourier multipliers. See E. M. Stein's book [8] for details. On infinite Lipschitz curves γ, by the Fourier transform on the sectors and H^∞-functional calculus, McIntosh–Qian [9,10] established the theory of L^p-Fourier multipliers and proved that the convolution singular integral operators on γ are equivalent to Fourier multipliers associated with Dirac operators. See also Gaudry–Qian–Wang [11] and Qian [12] for the theory of L^p Fourier multipliers on the closed Lipschitz curves.

On a Lipschitz surface Σ, Li–McIntosh–Qian [13] generalized the Fourier transform holomorphically to a function of m complex variables. Let $-iD_\Sigma = \sum_{k=1}^m -ie_k D_{k,\Sigma}$, where $D_{k,\Sigma} = (\partial/\partial x_k)|_\Sigma u$, $k = 1, \ldots, m$. Li–McIntosh–Qian [13] proved $T_{(\phi,\underline{\phi})}$ defined by (1.1) can be written as $T_{(\phi,\underline{\phi})} = b(-iD_\Sigma) = b(-iD_{1,\Sigma}, -iD_{2,\Sigma}, \ldots, -iD_{m,\Sigma})$, where b is the Fourier transform of ϕ.

The above theory was further extended to the starlike Lipschitz surfaces by Qian [14,15]. Let $S^c_{\omega,\pm}$ and S^c_ω be the sectors defined in Definition 3.4. The results of T. Qian are restricted to the bounded holomorphic Fourier multipliers b belonging to the following classes

$$H^\infty(S^c_{\omega,\pm}) = \left\{ b : S^c_{\omega,\pm} \to \mathbb{C}, b \text{ is bounded holomorphic on } S^c_\omega \text{ and satisfies } |b(z)| \le C_\mu, z \in S^c_{\mu,\pm}, 0 < \mu < \omega \right\}.$$

A natural question is whether there exists a relation between the convolution operators on Σ and the multipliers b dominated by a polynomial, that is,

$$|b(z)| \le C_\mu |z|^s, \quad z \in S^c_{\mu,\pm},\ 0 < \mu < \omega.$$

If Σ is the Euclidean space or its unit sphere, such multipliers are the classical fractional differential and integral operators and are widely applied to the study of harmonic analysis and partial differential equations.

Li–Leong–Qian [16] obtained the following result. Let $\{P^{(k)}\}$ be the monogenic polynomials defined in Definition 3.1. If $b \in H^s(S^c_\omega)$, then the function

$$\phi(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} b(k) P^{(k)}(x) \in K^s(H_\omega).$$

See Section 3 for the definitions of $H^s(S^c_\omega)$ and $K^s(H_\omega)$. If $s = 0$, the multipliers b become the bounded Fourier multipliers. Hence the above-mentioned result is a generalization of those of McIntosh–Qian [9] and Qian [14,15].

In this article, we consider the converse of the result obtained in [16]. Further, we establish a correspondence between our Fourier multipliers and the kernels of the convolution operators on starlike Lipschitz surfaces. Precisely, we prove that the following are equivalent:

(i) $\phi(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} b_k P^{(k)}(x) \in K^s(H_{\omega,\pm})$;
(ii) There exists $b \in H^s(S^c_{\omega,\pm})$ such that $b_k = b(k)$, $k \in \mathbb{Z} \setminus \{0\}$.

For the Fourier multipliers on Lipschitz surfaces, there exists a substantial difficulty. On the unit sphere, the L^2-boundedness of Fourier multipliers can be obtained by the Plancherel theorem directly. In the new context, i.e. the starlike Lipschitz surfaces, the Plancherel theorem does not hold. For $s = 0$ and $n = 3$, Qian [14] obtained the equivalence of (i) and (ii). For the case $s \ne 0$, our method is similar to that of Qian [14,15] but with some necessary modification in technology.

(i)$\Longrightarrow$(ii). This part has been obtained by Li-Leong-Qian [16]. For the cases $s > 0$ and $s < 0$, Li-Leong-Qian [16] used the Kelvin inversion and Fueter's result, respectively. We point out that if the dimension n is odd, we could estimate the kernels of the Fourier multipliers for the cases $s > 0$ and $s < 0$ by the same method. We omit the details. See Theorem 3.9.

(ii)$\Longrightarrow$(i). Given ϕ as above. For $s = 0$, such a relation has been obtained by Qian [14]. However, for the case $s \ne 0$, if we apply the method of [14], we can only get $b \in H^{s+2}(S^c_{\omega,\pm})$ rather than $H^s(S^c_{\omega,\pm})$. See Theorem 3.10. Applying Proposition 3.3, we construct the function b by using z^k. This method also helps us avoid the difficulties occurring in the estimate of $|P^{(z)}(x)|$. See Theorem 3.12.

Remark 1.1. In this paper, we assume that the dimension n is odd. We point out that our method cannot be applied to the Clifford algebras with an even number of generators. In fact, our proof of Theorem 3.12 is based on a generalization of Fueter's result which holds when n is odd. See (6) of Proposition 3.3.

In Section 4, we give two applications of the theory of the Fourier multipliers obtained in Section 3. Let $s \in \mathbb{Z}_+ \cup \{0\}$. In Section 4.1, we introduce the Sobolev spaces $W^{2,s}_{\Gamma_\xi}(\Sigma)$ on the starlike Lipschitz surfaces. We prove that if $b \in H^s(S^c_\omega)$, $s > 0$, $r \in \mathbb{Z}_+ \cup \{0\}$, the multiplier operator M_b is bounded from $W^{2,r+s_1}_{\Gamma_\xi}(\Sigma)$ to $W^{2,r}_{\Gamma_\xi}(\Sigma)$. See Theorem 4.3. This result implies that our Fourier multipliers could exert an influence on the index of the Sobolev spaces associated with Γ_ξ.

In Section 4.2, we establish the equivalence between two classes of Hardy–Sobolev spaces on Σ. We can see that there exist two methods to define the Hardy–Sobolev spaces associated with Γ_ξ. By the Fourier multiplier theory, we prove the two classes of Hardy–Sobolev spaces are equivalent. See Theorem 4.4.

Our paper is organized as follows. In Section 2, we state notations, knowledge and terminology which will be used throughout the paper. In Section 3, we obtain the correspondence between the kernel functions ϕ and the multipliers b by use of Fueter's result. In Section 4, we give two applications of the Fourier multiplier theory on starlike Lipschitz surfaces.

2. Preliminaries

In this paper we work with the real Clifford algebra $\mathbb{R}^{(n)}$ generated by $e_1, e_2, \ldots, e_n$ as its basic vectors, over the real number field under the multiplication relations:

$$\begin{cases} e_0 = 1; \\ e_i^2 = -e_0 = -1, \quad 1 \le i \le n; \\ e_ie_j + e_je_i = 0, \quad i \neq j, 1 \le i, j \le n. \end{cases}$$

Denoted by $\mathbb{R}_1^n$ and $\mathbb{R}^n$ the linear subspaces of $\mathbb{R}^{(n)}$ generated by $\{e_0, e_1, e_2, \ldots, e_n\}$ and by $\{e_1, e_2, \ldots, e_n\}$, respectively. A vector in $\mathbb{R}_1^n$ is represented as $x = x_0e_0 + \underline{x}$, where $x_0 \in \mathbb{R}$ and $\underline{x} = \sum_{l=1}^n x_le_l \in \mathbb{R}^n$. We call x_0e_0 and $\underline{x}$ the real and imaginary parts of x, respectively. There are two basic operations on the basic elements:

$$\begin{cases} (e_{i_1} \cdots e_{i_l})^* = e_{i_l} \cdots e_{i_1}, \\ (e_{i_1} \cdots e_{i_l})' = (e_{i_1})' \cdots (e_{i_l})', \end{cases}$$

where $(e_0)' = e_0$, $(e_j)' = -e_j, j = 1, \ldots, n$. By linearity, they can be extended to $\mathbb{R}^{(n)}$, $\mathbb{R}_1^n$, $\mathbb{R}^n$. We define the operation "$-$" by $\bar{x} = (x^*)'$. If $x, y \in \mathbb{R}^{(n)}$, then $\overline{xy} = \bar{y}\,\bar{x}$. If $x = x_0 + \underline{x}$, then $\bar{x} = x_0 - \underline{x}$. If for a nonzero vector x, its inverse x^{-1} exists: $x^{-1} = \frac{\bar{x}}{|x|^2}$.

We also use the complex Clifford algebra $\mathbb{C}^{(n)}$ generated by $e_1, \ldots, e_n$ over the complex number field, whose elements are also denoted by $x, y, \ldots$. The complex imaginary element i commutes with all the $e_j, j = 0, 1, \ldots, n$ and $i' = -i$. Therefore we can extend the definitions of $*$, $'$ and $-$ to $\mathbb{C}^{(n)}$, respectively. The natural inner product $\langle x, y\rangle$ between x and y in $\mathbb{C}^{(n)}$ is the complex number $\Sigma_S x_S \bar{y}_S$, where $x = \Sigma_S x_S e_S$, $y = \Sigma_S y_S e_S$ and S runs over all the ordered subsets $(i_1, \ldots, i_l)$ with $i_1 < i_2 < \cdots < i_l$ of the set $\{1, 2, \ldots, n\}$ and $e_S = e_{i_1} \cdots e_{i_l}$. Hence the norm associated with $\langle\cdot, \cdot\rangle$ is defined as $|x| = \langle x, x\rangle^{1/2} = (\sum_S |x_S|^2)^{1/2}$. It is easy to see that $\langle x, y\rangle = \frac{1}{4}(|x + y|^2 - |x - y|^2)$. The angle between two vectors x and y is defined as $\arg(x, y) = \arccos\langle x, y\rangle/(|x||y|)$, where the function arccos takes values in $[0, \pi)$. We denote the unit sphere $\{x \in \mathbb{R}_1^n : |x| = 1\}$ by $S_{\mathbb{R}_1^n}$ and the unit sphere $\{\underline{x} \in \mathbb{R}^n : |\underline{x}| = 1\}$ by $S_{\mathbb{R}^n}$.

Let $f = \sum_S f_S e_S$ be a $\mathbb{C}^1$-function and $\underline{D} = \sum_{k=1}^n \frac{\partial}{\partial x_k} e_k$. The differential operator $D = \frac{\partial}{\partial x_0} + \underline{D}$ applies to f and gives

$$\begin{cases} Df = \displaystyle\sum_{k=0}^n \frac{\partial f_S}{\partial x_k} e_k e_S, \\ fD = \displaystyle\sum_{k=0}^n \frac{\partial f_S}{\partial x_k} e_S e_k. \end{cases}$$

In the polar coordinate system, the operator D can be decomposed into

$$D = \xi\,\partial_r - \frac{1}{r}\partial_\xi = \xi\left(\partial_r - \frac{1}{r}\Gamma_\xi\right),$$

where Γ_ξ is a first order differential operator depending only on the angular coordinates known as the spherical Dirac operator. See Delanghe–Sommen–Souček [17].

A $\mathbb{C}^1$-function defined on an open subset of $\mathbb{R}^{n+1}$ with values in $\mathbb{R}_{(n)}$ or $\mathbb{C}_{(n)}$ is called left monogenic if $Df = 0$ and right monogenic if $fD = 0$. All real analytic functions f in domains in $\mathbb{R}^n$ have both left- and right-monogenic extensions to domains in $\mathbb{R}^{n+1}$. These two extensions coincide if and only if $Df = fD$, so they coincide if f is scalar valued particularly.

One basic example of both left- and right-monogenic functions is the Cauchy kernel $E(x) = \bar{x}/|x|^{n+1}$. By this kernel, we can define the Kelvin inversion $I(f)(x) = E(x)f(x^{-1})$. This operation preserves the monogenicity of the functions. Take the unit sphere $S_{\mathbb{R}_1^n}$ for example, for f is a monogenic function defined on the interior of $S_{\mathbb{R}_1^n}$, $I(f)$ is also a monogenic one defined on the exterior of $S_{\mathbb{R}_1^n}$. It is easy to see that $|I(f)(x)| \le |f(x)|/|x|^n$.

3. The correspondence between the kernel functions and the multipliers

3.1. The kernel functions

In this subsection, we estimate the kernels of the multipliers generated by a class of monogenic polynomials.

Definition 3.1. Let $\mathbb{Z}_+$ be the set of positive integers. For $k \in \mathbb{Z}_+$, we define the monogenic polynomials in $\mathbb{R}_1^n$ as

$$\begin{cases} P^{(-k)}(x) = \dfrac{(-1)^{k-1}}{(k-1)!}\left(\dfrac{\partial}{\partial x_0}\right)^{k-1} E(x), \\ P^{(k)}(x) = I(P^{(-k)})(x). \end{cases}$$

Given a set O in the complex plane, we could define a set $\overrightarrow{O}$ in $\mathbb{R}^n_1$. Then for any function f defined on $\overrightarrow{O}$, we establish a relation between f and f^0, where f^0 is a function defined on O. Therefore, we can use f^0 to estimate f. Now we give the definition of the induced functions and the induced sets.

Definition 3.2. Let f^0 be holomorphic, defined in an open set O in the upper half plane of $\mathbb{C}$ and $f^0(z) = u(x, y) + iv(x, y)$, $z = x + iy$, where u and v are real valued functions. Then

(1) we call $\overrightarrow{f^0}$ the "induced function" from f^0, which is defined by

$$\overrightarrow{f^0}(x_0 + \underline{x}) = u(x_0, |\underline{x}|) + \frac{\bar{\underline{x}}}{|\underline{x}|} v(x_0, |\underline{x}|);$$

(2) we call $\overrightarrow{O}$ the "induced set" from O, which is defined by

$$\overrightarrow{O} = \left\{(x_0, \underline{x}) \in \mathbb{R}^n_1 : (x_0, |\underline{x}|) \in O\right\}.$$

Denote by τ^0 the mapping

$$\tau^0 : f^0 \longrightarrow k_n^{-1} \Delta^{(n-1)/2} \overrightarrow{f^0}.$$

It is noted that τ^0 is linear with respect to the addition and real scalar multiplication. The main result in this subsection is based on a generalization of Fueter's result. We state it as the following lemma.

Proposition 3.3 *([15, Proposition 1]). Let $k \in \mathbb{Z}_+$. Then (1) $P^{(-1)}(x) = E(x)$; (2) $P^{(-k)}(x) = \frac{(-1)^{k-1}}{(k-1)!}(\frac{\partial}{\partial x_0})^{k-1}E(x)$; (3) $P^{(-k)}$ and $P^{(k-1)}$ are monogenic; (4) $P^{(-k)}$ is homogeneous of degree $-n+1-k$ and $P^{(k-1)}$ homogeneous of degree $k-1$; (5) $P^{(-k)} = I(P^{(k-1)})$; (6) if n is odd, then $P^{(k-1)} = \tau((\cdot)^{n+k-2})$.*

We define our Fourier multipliers in the following sets in the complex plane.

Definition 3.4. For $\omega \in (0, \frac{\pi}{2})$, let

$$\begin{cases} S^c_{\omega,\pm} = \left\{z \in \mathbb{C} : |\arg(\pm z)| < \omega\right\}, \\ S^c_{\omega} = S^c_{\omega,+} \cup S^c_{\omega,-}, \end{cases}$$

where $\arg(z)$ is the angle of z and takes value in $(-\pi, \pi]$. Let

$$\begin{cases} S^c_{\omega,\pm}(\pi) = \left\{z \in \mathbb{C} : |\mathrm{Re}(z)| \le \pi,\ z \in S^c_{\omega,\pm}\right\}, \\ S^c_{\omega}(\pi) = S^c_{\omega,+}(\pi) \cup S^c_{\omega,-}(\pi); \end{cases}$$

$$W^c_{\omega,\pm}(\pi) = \left\{z \in \mathbb{C} : |\mathrm{Re}(z)| \le \pi \text{ and } \mathrm{Im}(\pm z) > 0\right\} \cup S^c_{\omega}(\pi);$$

and

$$\begin{cases} H^c_{\omega,\pm} = \left\{z = \exp(i\eta) \in \mathbb{C} :\ \eta \in W^c_{\omega,\pm}(\pi)\right\}, \\ H^c_{\omega} = H^c_{\omega,+} \cap H^c_{\omega,-}. \end{cases}$$

Remark 3.5. In the complex plane, a starlike Lipschitz curve has the parameterization $\gamma = \gamma(\theta) = \exp i(\theta + iA(\theta))$, where A is a 2π-periodic Lipschitz function. Let $\|A'\|_\infty = \tan(\omega_0)$, $\omega_0 \in (0, \frac{\pi}{2})$. If $z, \eta \in \gamma$, we can write $z = \exp i(\theta_1 + iA(\theta_1))$ and $\eta = \exp i(\theta_2 + iA(\theta_2))$. Then $z\eta^{-1} = \exp(iu)$, where $u = (\eta_1 - \eta_2) + i(A(\theta_1) - A(\theta_2))$. Hence we get $|\tan(\arg u)| \le \|A'\|_\infty$ and $\arg u \le \omega$ for $\omega \in (\omega_0, \frac{\pi}{2})$. This implies $u \in S^c_{\omega,\pm}$ and $z\eta^{-1} = \exp(iu) \in H^c_\omega$.

Remark 3.6. We call $H^c_{\omega,+}$ and the complement of $H^c_{\omega,-}$ heart-shaped regions. Please see [14, Figure 1] for the shapes of $H^c_{\omega,\pm}$.

In [11,12], the authors studied a class of bounded Fourier multipliers in the complex plane. They proved that the kernels of the Fourier multipliers belong to some function space in the heart-shaped region. To deal with our Fourier multipliers on starlike Lipschitz surfaces, we define the following spaces.

For $-\infty < s < \infty$, we define the kernel function spaces and the multiplier spaces, respectively, as follows.

$$K^s(H^c_{\omega,\pm}) = \left\{\phi^0 : H^c_{\omega,\pm} \longrightarrow \mathbb{C},\ \phi^0 \text{ is holomorphic and satisfies } |\phi^0(z)| \le \frac{C_\nu}{|1-z|^{1+s}} \text{ in every } H^c_{\nu,\pm},\ 0 < \nu < \omega\right\}$$

and

$$K^s(H^c_{\omega}) = \left\{\phi^0 : H^c_{\omega} \longrightarrow \mathbb{C} : \phi^0 = \phi^{0,+} + \phi^{0,-},\ \phi^{0,\pm} \in K^s(H^c_{\omega,\pm})\right\}.$$

The corresponding multiplier spaces $H^s(S^c_{\omega,\pm})$ and $H^s(S^c_\omega)$ are defined by

$$H^s(S^c_{\omega,\pm}) = \left\{ b : S^c_{\omega,\pm} \longrightarrow \mathbb{C},\ b \text{ is holomorphic and satisfies } |b(z)| \le C_\nu |z|^s \text{ in every } S^c_{\nu,\pm},\ 0 < \nu < \omega \right\}$$

and

$$H^s(S^c_\omega) = \left\{ b : S^c_\omega \longrightarrow \mathbb{C},\ b_\pm = b\chi_{\{z\in\mathbb{C};\pm Rez>0\}} \in H^s(S^c_{\omega,\pm}) \right\}.$$

Qian [18] got the following holomorphic extension result on the complex plane.

Theorem 3.7. *Let $-\infty < s < \infty, s \neq -1, -2, \ldots$. If $b \in H^s(S^c_{\omega,\pm})$ and $\phi(z) = \sum_{k\in\mathbb{Z}\setminus\{0\}} b(k)z^k$, then $\phi \in K^s(H^c_{\omega,\pm})$.*

Define

$$\begin{cases} H_{\omega,\pm} = \left\{ x \in \mathbb{R}^n_1 :\ \dfrac{(\pm \ln |x|)}{\arg(x, e_0)} < \tan\omega \right\} = \overrightarrow{H^c_{\omega,\pm}}, \\ H_\omega = H_{\omega,+} \cap H_{\omega,-} = \left\{ x \in \mathbb{R}^n_1 :\ \dfrac{|\ln |x||}{\arg(x, e_0)} < \tan\omega \right\}. \end{cases}$$

Similar to Qian [18], we consider the Fourier multiplier operators defined on regions $H_{\omega,\pm}$. Set

$$K^s(H_{\omega,\pm}) = \left\{ \phi : H_{\omega,\pm} \longrightarrow \mathbb{C}^{(n)}, \phi = \sum_{k\in\mathbb{Z}\setminus\{0\}} c_k P^{(k)}, c_k \in \mathbb{C}, \text{ is monogenic} \right.$$
$$\left. \text{and satisfies } |\phi(x)| \le \frac{C_\nu}{|1-x|^{n+s}}, x \in H_{\nu,\pm},\ 0 < \nu < \omega \right\}$$

and

$$K^s(H_\omega) = \left\{ \phi : H_\omega \longrightarrow \mathbb{R}^{(n)}, \phi = \phi^+ + \phi^-, \phi^\pm \in K^s(H_{\omega,\pm}) \right\}.$$

As a preparatory of the main result, we state an estimate of the derivation of $\phi^0 \in K^s(H^c_{\omega,\pm})$.

Lemma 3.8 *([16, Theorem 3.5]). For $\phi^0 \in K^s(H^c_{\omega,\pm})$ and any $0 < \upsilon < \omega$, we have*

$$|(\phi^0)^{(j)}(z)| \le \frac{2j!C_\nu}{\delta^j(\nu)} \frac{1}{|1-z|^{1+j+s}},$$

where C_ν is a constant and $\delta(\nu) = \min\left\{\frac{1}{2}, \tan(\omega - \nu)\right\}$.

Based on the above preliminary definitions and lemmas, we could estimate the kernel functions of the multipliers. In the complex plane $\mathbb{C}$, the corresponding result has been obtained by Qian [18]. Hence our result generalizes that of Qian [18] to the general Clifford algebras with an odd number of generators.

Theorem 3.9. *Let n be odd and $b \in H^s(S^c_{\omega,\pm})$ with $-\infty < s < \infty, s \neq -1, -2, \ldots$. If $\phi(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} b(k)P^{(k)}(x)$, we have $\phi \in K^s(H_{\omega,\pm})$.*

Proof. The proof is similar to that of [16, Theorems 3.7 and 3.9]. We point out that when the dimension is odd, we can deal with the cases $s > 0$ and $s < 0$ by the same method. For simplicity, we omit the details. □

3.2. The converse result

On $\mathbb{R}^n$, the Fourier theory justifies that there exists a correspondence between the kernels of the convolution singular integral operators and the symbols of the multipliers. By Theorem 3.9, for $b \in H^s(S^c_\omega)$, there exists a function $\phi \in K^s(H_\omega)$. Now we consider the converse of Theorem 3.9. For $\phi \in K^s(H_{\omega,\pm})$, we prove that there exists a function $b^\nu(z) \in H^s(S^c_{\nu,\pm})$ such that $b_k = b^\nu(k), 0 < \nu < \omega$.

Let $n = 3$. Such b^ν has been obtained by Qian [14] for the case $s = 0$. The main tool is the following polynomials $P^{(k)}$. For any $z \in S^c_\omega$, let

$$\begin{cases} P^{(z)}_- = \tau^0((\cdot)^z),\ z \in S^c_{\omega,-}, \\ P^{(z)}_+ = \tau^0((\cdot)^{z+2}),\ z \in S^c_{\omega,+}, \end{cases}$$

where $(\cdot)^z = \exp(z \ln(\cdot))$, where in the first case the ln function is defined by cutting the positive x-axis, and in the second case, defined by cutting the negative x-axis.

By use of the new functions $P^{(z)}_-$ and $P^{(z)}_+$, we can get the following result. For the ease of computation, we still assume $n = 3$.

Theorem 3.10. *Let $n = 3$ and $-\infty < s < -2$. If $\phi(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} b_k P^{(k)}(x) \in K^s(H_{\omega,\pm})$, then for every $\nu \in (0, \omega)$ there exists a function $b^\nu \in H^{s+2}(S^c_{\nu,\pm})$ such that $b_i = b^\nu(i)$, $i = \pm 1, \pm 2, \ldots$. Moreover,*

$$b^\nu(z) = \lim_{r\to 1-} \frac{1}{2\pi^2} \int_{L^\pm(\nu)} P^{(z)}(y^{-1})E(y)n(y)\phi(r^{\pm 1}y)d\sigma(y),$$

where $L^\pm(\nu) = \overrightarrow{\exp(il^\pm(\nu))}$ and the path $l^\pm(\nu)$ is defined as

$$l^\pm(\nu) = \Big\{z \in \mathbb{C} : z = r\exp(i(\pi \pm \nu)), r \text{ is from } \pi\sec(\nu) \text{ to } 0;\ \text{and then } z = r\exp(-(\pm i\nu)), r \text{ is from } 0 \text{ to } \pi\sec(\nu)\Big\}.$$

Proof. Because $\tau^0 : f^0 \longrightarrow \frac{1}{4}\Delta\overrightarrow{f^0}$, write $f^0 = \eta^z$, where $\eta = x + iy$. For $x = (x_0, |\underline{x}|) \in L^\pm(\nu)$, there exists $\eta \in \exp(il^\pm(\nu))$ such that $\eta = (x_0, |\underline{x}|)$. Set $\mathbf{e} = \underline{x}/|\underline{x}|$. We have known that

$$\Delta\overrightarrow{f^0} = \Delta\overrightarrow{((\cdot)^z)} = \frac{2}{|\underline{x}|}\frac{\partial u}{\partial y}(x_0, |\underline{x}|) + 2\mathbf{e}\left(\frac{1}{|\underline{x}|}\frac{\partial v}{\partial y}(x_0, |\underline{x}|) - \frac{1}{|\underline{x}|^2}v(x_0, |\underline{x}|)\right).$$

Now $f^0 = e^{i\eta z}$, where $\eta \in l^\pm(\nu)$. Then $f = u + iv$, where u and v are the real and imaginary parts of f, respectively. We have $\frac{\partial}{\partial\eta}(e^{i\eta z}) = ize^{i\eta z}$. Set $\eta = re^{-i\mu}$ and $z = |z|e^{i\theta}$. We get

$$e^{-i\eta z} = \exp(-ir|z|e^{i(\theta-\mu)}) = \exp(r|z|\sin(\theta-\mu))\exp(-ir|z|\cos(\theta-\mu)).$$

Because $\phi \in K^s(S_\omega)$, we have

$$|\phi(x)| \le \frac{C}{|1-x|^{s+3}}, \quad \text{where } x = x_0 + \underline{x} \in L^\pm(\nu).$$

For such an x, there exists an $z = x+iy \in \exp(il^\pm(\nu))$ such that $z = e^{i\eta} = \exp(r\sin\mu + ir\cos\mu)$ and $|\underline{x}| = e^{r\sin\mu}\sin(r\cos\mu)$. Then we have

$$|b^\mu(z)| \le C\int_0^{\pi\sec\mu} |z|e^{-r|z|\sin(\mu-\theta)}\frac{1}{|1-e^{i\eta}|^{s+3}}\frac{1}{|x|}\frac{1}{|\underline{x}|}r^2 dr.$$

For the factor $1/|1-e^{i\eta}|^{s+3}$, we have

$$|1-e^{i\eta}|^2 = 1 + e^{2r\sin\mu} - 2e^{r\sin\mu}\cos(r\cos\mu).$$

Let $f(r) = r^2$ and $g(r) = 1 + e^{2r\sin\mu} - 2e^{r\sin\mu}\cos(r\cos\mu)$, we have $\lim_{r\to 0}\frac{f(r)}{g(r)} = 1$. Hence we can find a constant C such that

$$\frac{r}{|1-e^{r\sin\mu}e^{ir\cos\mu}|} \le C, \quad r \in (0, \pi\sec\mu),$$

that is, $1/|1-e^{r\sin\mu}e^{ir\cos\mu}|^{s+3} \sim r^{s+3}$. Finally we have

$$\begin{aligned}|b^\mu(z)| &\le C\int_0^{\pi\sin\mu} |z|e^{-r|z|\sin(\mu-\theta)}\frac{1}{r^{s+3}}\frac{1}{e^{3r\sin\mu}}\frac{r^2}{e^{r\sin\mu}\sin(r\cos\mu)}dr \\ &\le C|z|\int_0^{\pi\sin\mu} e^{-r|z|\sin(\mu-\theta)}\frac{r^2}{r^{s+4}}e^{-4r\sin\mu}dr \\ &\le C|z|^{s+2},\end{aligned}$$

where in the second inequality we have used the fact that $s < -2$. □

Theorem 3.10 implies that by the method of Qian [14], we can only get $b \in H^{s+2}(S^c_{\omega,\pm})$ rather than $b \in H^s(S^c_{\omega,\pm})$ for the case $s \neq 0$. In order to get a better result, we apply a new method. Our idea is based on two observations. On one hand, the function b which we want should be defined on $S^c_{\omega,\pm} \subset \mathbb{C}$. On the other, by Proposition 3.3, we know that when the dimension n is odd, the polynomials $P^{(-k)}$ and $P^{(k-1)}$, $k \in \mathbb{Z}_+$, satisfy the following relations:

$$P^{(-k)} = \tau((\cdot)^{-k}), \qquad P^{(k-1)} = \tau((\cdot)^{k+n-2}).$$

Hence our method is to construct a function $\phi^0 \in K^s(H^c_{\omega,\pm})$ by use of $\phi \in K^s(H_{\omega,\pm})$. Then we could represent the function b via ϕ^0 and the technique on the complex plane. At first we introduce a lemma about the relationship between $H^c_{\omega,\pm}$ and $H_{\omega,\pm}$.

For each element $\mathbf{e}$ in the vector space $\mathcal{Q}$, the linear span of 1 and $\mathbf{e}$ over $\mathbb{R}$ is called the complex plane in $\mathbb{R}^n_1$ induced by $\mathbf{e}$, denoted by $\mathbb{C}^{\mathbf{e}}$. Denote by $H^{\mathbf{e}}_{\omega,\pm}$ and $H^{\mathbf{e}}_\omega$ the images on $\mathbb{C}^{\mathbf{e}} \subset \mathbb{R}^{(n)}_1$ of the sets $H^c_{\omega,\pm}$ and H^c_ω in $\mathbb{C}$, respectively, under the mapping $i_{\mathbf{e}} : a + bi \longrightarrow a + b\mathbf{e}$. By the same method of [14, Lemma 4], we can prove the following lemma.

Lemma 3.11.

$$H_{\omega,\pm}=\bigcup_{\mathbf{e}\in J}H^{\mathbf{e}}_{\omega,\pm}\quad\textit{and}\quad H_{\omega,\pm}=\bigcup_{\mathbf{e}\in J}H^{\mathbf{e}}_{\omega,\pm},$$

where the index set J is the set of all the unit vectors.

By Lemma 3.11, we could establish the correspondence between monogenic kernel functions and the holomorphic multipliers stated in Section 1.

Theorem 3.12. *Let n be odd and* $\phi(x)=\sum_{k\in\mathbb{Z}\setminus\{0\}}b_kP^{(k)}(x)\in K^s(H_{\omega,\pm})$. *If the series* $\sum_{k\in\mathbb{Z}\setminus\{0\}}b_kz^k$ *converges in* $H^c_{\omega,\pm}$, *then for every* $\nu\in(0,\omega)$ *there exists a function* $b^\nu\in H^s(S^c_{\nu,\pm})$ *such that* $b_k=b^\nu(k)$, $k\in\mathbb{Z}\setminus\{0\}$.

Proof. By Proposition 3.3, we know that if n is odd, for $k\in\mathbb{Z}_+$,

$$P^{(-k)}=\tau^0((\cdot)^{-k})\quad\text{and}\quad P^{(k-1)}=\tau^0((\cdot)^{n+k-1}).$$

For $\phi(x)=\sum_{k\in\mathbb{Z}\setminus\{0\}}b_kP^{(k)}(x)$ on $H_{\omega,\pm}$, we define the following function ϕ^0 on $H^c_{\omega,\pm}$ as $\phi^0(z)=\sum_{k\in\mathbb{Z}\setminus\{0\}}b_kz^k$, where $z\in H^c_{\omega,\pm}$. For simplicity, we only estimate ϕ^0 in the region $H^c_{\omega,+}$. Let $\mathbf{e}=\frac{\underline{x}}{|\underline{x}|}$. For any $z=u+iv\in H^c_{\omega,+}$, we have $x=u+v\mathbf{e}=(x_0,\underline{x})\in H^{\mathbf{e}}_{\omega,+}\subset H_{\omega,+}$ by Lemma 3.11. We have proved that for $z\in H^c_{\omega,+}$ there exists a constant $\delta(\nu)=\min\left\{\frac{1}{2},\tan(\omega-\nu)\right\}$ such that the ball $S_r(z)$ is contained in $H^c_{\omega,\pm}$, where z is the center and the radius r equals $\delta(\nu)|1-z|$. We denote by $B(x,r)$ the ball $\left\{y\in\mathbb{R}^{(n)}_1,\ |x-y|<\delta(\nu)|1-x|\right\}$ and have $B(x,r)\subset H^{\mathbf{e}}_{\omega,+}\subset H_{\omega,+}$.

Assume that f and g are the real and imaginary parts of $\phi^0(z)$, respectively. The induced function is defined by

$$\overrightarrow{\phi^0}(x)=f(x_0,|\underline{x}|)+\mathbf{e}g(x_0,|\underline{x}|)$$

and satisfies $\Delta^{(n-1)/2}\overrightarrow{\phi^0}(x)=\phi(x)$, where $x=(x_0,\underline{x})=u+v\mathbf{e}$. We can see that

$$|\overrightarrow{\phi^0}(x)|\le\int_{B(x,r)}\frac{c}{|x-y|^2}\frac{C_\nu}{|1-y|^{n+s}}dy.$$

For any $q\in B(x,\delta(\nu)|1-x|)$,

$$|1-y|\ge|1-x|-|x-y|>(1-\delta(\nu))|1-x|.$$

We can get

$$\begin{aligned}|\overrightarrow{\phi^0}(x)|&\le\frac{C_\nu}{|1-x|^{n+s}}\int_0^{\delta(\nu)|1-x|}\frac{1}{|x-y|^2}|x-y|^{n-1}d(|x-y|)\\&\le\frac{C_\nu}{|1-x|^{1+s}}.\end{aligned}$$

By the definition of $|\overrightarrow{\phi^0}|$, we have

$$|\phi^0(z)|=|\overrightarrow{\phi^0}(x)|\le\frac{C_\nu}{|1-x|^{1+s}}=\frac{C_\nu}{|1-z|^{1+s}}.$$

By the use of the above estimate, we could construct the function $b\in H^s(S^\omega_{\omega,\pm})$ as follows.

For $s<0$ and $z\in S^c_{\mu,\pm}$,

$$b^\mu(z)=\frac{1}{2\pi}\int_{\lambda_\pm(\mu)}\exp(-i\eta z)\phi^0(\exp(i\eta))d\eta,$$

where

$$\lambda_\pm(\mu)=\left\{\eta\in H^c_{\omega,\pm}\mid\eta=r\exp(i(\pi\pm\mu)),\ r\text{ is from }\pi\sec\mu\text{ to; and then }\eta=r\exp(\mp i\mu),\ r\text{ is from 0 to }\pi\sec\mu\right\}$$

and for $s\ge 0$, $z\in S^c_{\mu,\pm}$,

$$b^\mu(z)=\frac{1}{2\pi}\lim_{\varepsilon\to0}\left(\int_{l(\varepsilon,|z|^{-1})\cup c_\pm(|z|^{-1},\mu)\cup\Lambda_\pm(|z^{-1}|,\mu)}\exp(-i\eta z)\phi^0(\exp(i\eta))d\eta+\phi^{|s|}_{\varepsilon,\pm}(z)\right),$$

where if $r\le\pi$

$$l(\varepsilon,r)=\left\{\eta=x+iy\mid y=0,x\text{ is from }-r\text{ to }-\varepsilon,\text{ and then from }\varepsilon\text{ to }r\right\},$$

$$c_\pm(r,\mu)=\left\{\eta=r\exp(i\alpha)\mid\alpha\text{ is from }\pi\pm\mu\text{ to }\pi,\text{ then from 0 to }\mp\mu\right\},$$

and

$$\Lambda_{\pm}(r,\mu) = \Big\{\eta \in W_{\omega,\pm} \mid \eta = \rho \exp(i(\pi \pm \mu)),\ \rho \text{ is from } \pi \sec \mu \text{ to } r; \\ \text{and then } \eta = \rho \exp(\mp i\mu),\ \rho \text{ is from } r \text{ to } \pi \sec \mu\Big\}$$

and if $r > \pi$,

$$l(\varepsilon, r) = l(\varepsilon, \pi), \qquad c_{\pm}(r,\mu) = c_{\pm}(\pi,\mu), \qquad \Lambda_{\pm}(r,\mu) = \Lambda_{\pm}(\pi,\mu)$$

and in any case,

$$\phi^{[s]}_{\varepsilon,\pm}(z) = \int_{L_{\pm}(\varepsilon)} \phi^0(\exp(i\eta)) \left(1 + (-i\eta z) + \cdots + \frac{(-i\eta z)^{[s]}}{[s]!}\right) d\eta,$$

where $L_{\pm}(\varepsilon)$ is any contour from $-\varepsilon$ to ε lying in $C_{\omega,\pm}$.

By Cauchy's theorem and Taylor series expansion, we could prove that $b^{\nu} \in H^s(S^c_{\omega})$ and $b_i = b^{\nu}(i)$, $i = \pm 1, \pm 2, \ldots$ by the estimate of the function ϕ^0 obtained above. We omit the details here and refer the reader to Qian [18]. This completes the proof. □

3.3. Unbounded holomorphic Fourier multipliers

Definition 3.13. A closed surface Σ is said to be a starlike Lipschitz surface, if it is star-shaped about the origin and there exists a constant $M < \infty$ such that for any $x,\ x' \in \Sigma$,

$$\frac{\left|\ln |x^{-1}x'|\right|}{\arg(x, x')} \le M.$$

The minimum value of M is called the Lipschitz constant of Σ, denoted by $N = \mathrm{Lip}(\Sigma)$.

In the sequel, we will be working on a fixed starlike Lipschitz surface Σ and assume that $\omega \in (\arctan(N), \frac{\pi}{2})$. Denote

$$\rho = \min\Big\{|x| : x \in \Sigma\Big\} \quad \text{and} \quad l = \max\Big\{|x| : x \in \Sigma\Big\}.$$

Our Fourier multiplier operators are defined on the following dense subclass of $L^2(\Sigma)$ (see [14]).

$$\mathcal{A} = \{f : f(x) \text{ is left monogenic in an annual } \rho - s < |x| < l + s \text{ for some } s > 0\}.$$

Let M_k be the finite dimensional right module of k homogeneous left monogenic functions in $\mathbb{R}^n_1$, and let $M_{-(k+n)}$ be the finite dimensional right module of $-(k+n)$ homogeneous left monogenic functions in $\mathbb{R}^n_1 \setminus \{0\}$. The spaces M_k and $M_{-(k+n)}$ are eigenspaces of the left spherical Dirac operator. We define the projection operators on M_k and $M_{-(k+n)}$, respectively, as

$$P_k : f \to P_k(f) \quad \text{and} \quad Q_k : f \to Q_k(f).$$

For $f \in \mathcal{A}$, in the annuals where f is defined, we have the Laurent series expansion

$$f(x) = \sum_{k=0}^{\infty} P_k(f)(x) + \sum_{k=0}^{\infty} Q_k(f)(x).$$

Here we have used the projection operators P_k and Q_k given by

$$P_k(f)(x) = \frac{1}{\Omega_n} \int_{\Sigma} |y^{-1}x|^k C^+_{n+1,k}(\xi,\eta) E(y) n(y) f(y) d\sigma(y),$$
$$Q_k(f)(x) = \frac{1}{\Omega_n} \int_{\Sigma} |y^{-1}x|^{-n-k} C^-_{n+1,k}(\xi,\eta) E(y) n(y) f(y) d\sigma(y),$$

where $x = |x|\xi$ and $y = |y|\eta$. Here $C^+_{n+1,k}(\xi,\eta)$ and $C^-_{n+1,k}(\xi,\eta)$ are the functions defined by

$$C^+_{n+1,k}(\xi,\eta) \frac{1}{1-n} \left[-(n+k-1) C_k^{(n-1)/2}(\langle\xi,\eta\rangle) + (1-n) C_{k-1}^{(n+1)/2}(\langle\xi,\eta\rangle)(\langle\xi,\eta\rangle - \overline{\xi}\eta)\right]$$

and

$$C^-_{n+1,k}(\xi,\eta) \frac{1}{n-1} \left[(k+1) C_{k+1}^{(n-1)/2}(\langle\xi,\eta\rangle) + (1-n) C_k^{(n+1)/2}(\langle\eta,\xi\rangle)(\langle\eta,\xi\rangle - \overline{\eta}\xi)\right],$$

where C^{ν}_k is the Gegenbauer polynomial of degree k associated with ν.

Let $b \in H^s(S^c_{\omega})$. Now we introduce the unbounded Fourier multiplier operator M_b on a starlike Lipschitz surface Σ. Note that the functions ϕ obtained in Theorem 3.9 satisfy $|\phi(x)| \le C_{\mu}/|1-x|^{n+s}$ for $s > 0$. In order to compensate the role of s, we need to restrict our multipliers into some subspaces of $L^2(\Sigma)$. Hence we introduce the following Sobolev space on the starlike Lipschitz surface Σ.

Definition 3.14. Let $s \in \mathbb{Z}^+ \cup \{0\}$ and Σ be a starlike Lipschitz surface. Define the Sobolev norm $\|\cdot\|_{W^{p,s}_{\Gamma_\xi}(\Sigma)}$, $1 \le p < \infty$, as

$$\|\cdot\|_{W^{p,s}_{\Gamma_\xi}(\Sigma)} = \|f\|_{L^p(\Sigma)} + \sum_{j=0}^{s} \|\Gamma_\xi^j f\|_{L^p(\Sigma)},$$

where Γ_ξ denotes the spherical Dirac operator. The Sobolev space associated with Γ_ξ is defined as the closure of the class $\mathcal{A}$ under the norm $\|\cdot\|_{W^{p,s}_{\Gamma_\xi}(\Sigma)}$, that is, $\overline{\mathcal{A}}^{\|\cdot\|_{W^{p,s}_{\Gamma_\xi}(\Sigma)}}$.

Now we give the definition of the Fourier multiplier operators. By Definition 3.14, $\mathcal{A}$ is dense in $W^{s,p}_{\Gamma_\xi}$. In the next definition, we assume $f \in \mathcal{A}$.

Definition 3.15. Let $\{b_k\}_{k\in\mathbb{Z}}$ be a sequence satisfying $|b_k| \le k^s$. We define the Fourier multiplier operator $M_{(b_k)}$ as follows.

$$M_{(b_k)} f(x) = \sum_{k=0}^{\infty} b_k P_k(f)(x) + \sum_{k=0}^{\infty} b_{-k-1} Q_k(f)(x).$$

Now for $k \ge 0$, we define

$$\tilde{P}^{(k)}(y^{-1}x) = |y^{-1}x|^k C^+_{n+1,k}(\xi,\eta) \quad \text{and} \quad \tilde{P}^{(-k-1)}(y^{-1}x) = |y^{-1}x|^{-k-n} C^-_{n+1,k}(\xi,\eta).$$

Then the projection operators P_k and Q_k can be represented as

$$P_k(f)(x) = \frac{1}{\Omega_n}\int_\Sigma \tilde{P}^{(k)}(y^{-1}x)E(y)n(y)f(y)d\sigma(y),$$
$$Q_k(f)(x) = \frac{1}{\Omega_n}\int_\Sigma \tilde{P}^{(-k-1)}(y^{-1}x)E(y)n(y)f(y)d\sigma(y).$$

For $f \in \mathcal{A}$, the above introduced multiplier $M_{(b_k)}$ is well defined. For $b \in H^s(S^c_\omega)$, we consider the following multiplier:

$$M^r_{(b_k)}(f)(x) = \sum_{k=0}^{\infty} b_k P_k(f)(rx) + \sum_{k=0}^{\infty} b_{-k-1} Q_k(f)(r^{-1}x), \quad \rho - s < |x| < l + s,$$

where $x \in \Sigma$, $r \approx 1$ and $r < 1$.

We denote by M_1 and M_2 the two sums in the above expression of $M^r_{(b_k)}$. For M_1, $|b_k| = |b(k)| \le k^{s_1}$, we take $b_1(z) = z^{-s_1} b(z)$. It is easy to see that b_1 is also holomorphic in S^c_ω. Then we have

$$M_1 = \sum_{k=0}^{\infty} b_k P_k(f)(rx) = \sum_{k=0}^{\infty} b_{1,k} k^{s_1} P_k(f)(rx),$$

where $b_{1,k} = b_1(k) = \frac{b_k}{k^{s_1}}$. Because the spaces M_k is the eigenspace of the left spherical Dirac operator Γ_ξ, we have $\Gamma_\xi P_k(f)(rx) = kP_k(f)(rx)$ and

$$M_1 = \sum_{k=0}^{\infty} b_{1,k} \Gamma_\xi^{s_1} P_k(f)(rx) = \Gamma_\xi^{s_1}\left(\sum_{k=0}^{\infty} b_{1,k} P_k(f)(rx)\right).$$

According to a result of [17], we give another expression of $P_k(f)$.

$$P_k(f)(x) = \frac{1}{\Omega_n}\int_\Sigma \tilde{P}^k(y^{-1}rx)E(y)n(y)f(y)d\sigma(y)$$
$$= \frac{1}{\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(rx)W_{\underline{\alpha}}(y)n(y)f(y)d\sigma(y),$$

where we used the Cauchy–Kovalevskaya extension

$$\tilde{P}^{(k)}(y^{-1}x)E(y) = \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x)W_{\underline{\alpha}}(y),$$

where $V_{\underline{\alpha}} \in M_k$ and $W_{\underline{\alpha}} \in M_{-n-k}$ (see [17, Chapter 2, (1.15)]). The above relation implies

$$\begin{aligned}
\Gamma_\xi P_k(f)(x) &= \frac{1}{\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} kV_{\underline{\alpha}}(x)W_{\underline{\alpha}}(y)n(y)f(y)d\sigma(y)\\
&= \frac{1}{\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} \frac{k}{n+k-2}V_{\underline{\alpha}}(x)(n+k-2)W_{\underline{\alpha}}(y)n(y)f(y)d\sigma(y)\\
&= \frac{k}{n+k-2}\frac{1}{\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x)(\Gamma_\eta W_{\underline{\alpha}})(y)n(y)f(y)d\sigma(y).
\end{aligned}$$

We denote $b_{1,k}(\frac{k}{n+k-2})^{s_1}$ by $b_{1,k}$ again and see that $|b_{1,k}(\frac{k}{n+k-2})^{s_1}| \le C$. Because of the fast decay of the Fourier expansions of functions in $\mathcal{A}$, we have, by integration by parts,

$$\begin{aligned}
M_1 &= \sum_{k=1}^{\infty} b_{1,k}\left(\frac{k}{n+k-2}\right)^{s_1}\frac{r^k}{\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x)W_{\underline{\alpha}}(y)n(y)(\Gamma_\eta^{s_1}f)(y)d\sigma(y)\\
&= \sum_{k=1}^{\infty} b_{1,k}\frac{1}{\Omega_n}\int_\Sigma \tilde{P}^k(y^{-1}rx)E(y)n(y)(\Gamma_\eta^{s_1}f(y))d\sigma(y)\\
&= \frac{1}{\Omega_n}\int_\Sigma \left(\sum_{k=1}^{\infty} b_{1,k}\tilde{P}^k(y^{-1}rx)\right)E(y)n(y)(\Gamma_\eta^{s_1}f(y))d\sigma(y)\\
&=: \frac{1}{\Omega_n}\int_\Sigma \tilde{\phi}_1(y^{-1}rx)E(y)n(y)(\Gamma_\eta^{s_1}f(y))d\sigma(y).
\end{aligned}$$

Similarly, for the term M_2, by the Cauchy–Kovalevskaya extension again ([17, Chapter II, (1.16)]), we have

$$\begin{aligned}
M_2 &= \sum_{k=0}^{\infty}\frac{b_{-k-1}}{(k+1)^{s_1}}\left(\frac{k+1}{k}\right)^{s_1}\frac{1}{\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} W_{\underline{\alpha}}(r^{-1}x)k^{s_1}\overline{V}_{\underline{\alpha}}(y)n(y)f(y)d\sigma(y)\\
&= \sum_{k=0}^{\infty}\frac{b_{-k-1}}{(k+1)^{s_1}}\left(\frac{k+1}{k}\right)^{s_1}\frac{1}{\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} W_{\underline{\alpha}}(r^{-1}x)(\Gamma_\eta^{s_1}\overline{V}_{\underline{\alpha}})(y)n(y)f(y)d\sigma(y)\\
&= \sum_{k=0}^{\infty}\frac{b_{-k-1}}{(k+1)^{s_1}}\left(\frac{k+1}{k}\right)^{s_1}\frac{1}{\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} W_{\underline{\alpha}}(r^{-1}x)\overline{V}_{\underline{\alpha}}(y)n(y)(\Gamma_\eta^{s_1}f)(y)d\sigma(y).
\end{aligned}$$

As above, we still write the term $\frac{b_{-k-1}}{(k+1)^{s_1}}\left(\frac{k+1}{k}\right)^{s_1}$ as b_{-1-k} and get the singular integral expression of M_2 as follows.

$$\begin{aligned}
M_2 &= \frac{1}{\Omega_n}\int_\Sigma \left(\sum_{k=0}^{\infty} b_{-k-1}\tilde{P}^{-k-1}(y^{-1}r^{-1}x)\right)E(y)n(y)(\Gamma_\eta^{s_1}f)(y)d\sigma(y)\\
&=: \frac{1}{\Omega_n}\int_\Sigma \tilde{\phi}_2(y^{-1}r^{-1}x)E(y)n(y)(\Gamma_\eta^{s_1}f(y))d\sigma(y).
\end{aligned}$$

Finally we rewrite the multiplier $M^r_{(b_k)}(f)$ as

$$M^r_{(b_k)}(f)(x) = \lim_{r\to 1-}\frac{1}{\Omega_n}\int_\Sigma (\widetilde{\phi_1}(y^{-1}rx) + \widetilde{\phi_2}(y^{-1}r^{-1}x))E(y)n(y)(\Gamma_\xi^{s_1}f)(y)d\sigma(y),$$

where we have used the fact that the series defining $M^r_{b_k}(f)$ uniformly converges as $r \to 1-$ for $f \in \mathcal{A}$.

Similar to the case of a singular Cauchy integral, we could deduce a Plemelj type formula for such Fourier multiplier operators with $s > 0$. We refer [16] for the proof.

Theorem 3.16 *([16, Theorem 4.6]). Let $b \in H^s(S^c_\omega)$ with $s > 0$ and $s_1 = [s] + 1$. For $f \in \mathcal{A}$ and $x \in \Sigma$, we have*

$$\begin{aligned}
M_bf(x) &= \lim_{r\to 1-}\frac{1}{2\pi^2}\int_\Sigma \left(\phi_1(y^{-1}rx) + \phi_2(y^{-1}r^{-1}x)\right)E(y)n(y)f(y)d\sigma(y)\\
&= \lim_{\varepsilon\to 0}\frac{1}{2\pi^2}\left\{\int_{|1-p^{-1}q|>\varepsilon,\ p\in\Sigma}\left(\phi_1(y^{-1}x) + \phi_2(y^{-1}x)\right)E(y)n(y)(\Gamma_\xi^{s_1}f)(y)d\sigma(y)\right.\\
&\quad \left. + \left(\tilde{\phi}_1(\varepsilon, x) + \tilde{\phi}_2(\varepsilon, x)\right)(\Gamma_\xi^{s_1}f)(x)\right\},
\end{aligned}$$

where

$$\begin{cases}\tilde{\phi}_1(\varepsilon,x)=\displaystyle\int_{S(\varepsilon,y,+)}\phi_1(y)E(y)n(y)d\sigma(y),\\ \tilde{\phi}_2(\varepsilon,x)=\displaystyle\int_{S(\varepsilon,y,-)}\phi_2(y)E(y)n(y)d\sigma(y),\end{cases}$$

where $S(\varepsilon,x,\pm)$ is the part of the surface $|1-y^{-1}x|=\varepsilon$ inside or outside Σ, depending on $\pm$ taking $+$ or $-$.

4. Applications to Sobolev spaces on Lipschitz surfaces

4.1. Boundedness on Sobolev spaces

Now we consider the boundedness of the operators M_b on Sobolev spaces. Our proof is based on the Hardy space theory on starlike Lipschitz surfaces established by Jerison–Kenig [5], Kenig [19] and Mitrea [20].

Let Δ and Δ^c be the bounded and unbounded connected components of $\mathbb{R}_1^n\setminus\Sigma$. For $\alpha>0$, define the non-tangential approach regions Λ_α and $\Lambda_\alpha^c(x)$ to a point $q\in\Sigma$ to be

$$\Lambda_\alpha(x)=\Lambda_\alpha(x,\Delta)=\Big\{y\in\Delta:\ |y-x|<(1+\alpha)\,\mathrm{dist}\,(y,\Sigma)\Big\}$$

and

$$\Lambda_\alpha^c(x)=\Lambda_\alpha(x,\Delta^c)=\Big\{y\in\Delta^c:\ |y-x|<(1+\alpha)\,\mathrm{dist}\,(y,\Sigma)\Big\}.$$

Let f be defined on Δ. The interior non-tangential maximal function $N_\alpha(f)$ is defined by

$$N_\alpha(f)(x)=\sup\Big\{|f(y)|:y\in\Lambda_\alpha(x)\Big\},\quad x\in\Sigma.$$

The exterior non-tangential maximal function can be defined similarly.

For $0<p<\infty$, the left-Hardy space $\mathcal{H}^p(\Delta)$ is defined by

$$\mathcal{H}^p(\Delta)=\Big\{f:f\text{ is left regular in }\Delta,\ \text{and }N_\alpha(f)\in L^p(\Sigma)\Big\}.$$

We can define the space $\mathcal{H}^p(\Delta^c)$ similarly, except that the functions in $\mathcal{H}^p(\Delta^c)$ are assumed to vanish at the infinity. See Mitrea [20] for further properties of $\mathcal{H}^p(\Delta)$ and $\mathcal{H}^p(\Delta^c)$, $p>1$. For the special case $p=2$, $\mathcal{H}^2(\Delta)$ and $\mathcal{H}^2(\Delta^c)$ have equivalent characterizations of the higher order Littlewood–Paley g-functions. Taking $\mathcal{H}^2(\Delta)$ for example, we have the following.

Proposition 4.1 (*Mitrea [20]*)**.** *Suppose that $f\in\mathcal{H}^2(\Delta)$. Then the norm $\|f\|_{\mathcal{H}^2(\Delta)}$ is equivalent to the norm*

$$\left(\int_0^1\int_\Sigma\left|\left(\Gamma_\xi^j f\right)(ry)\right|^2(1-r)^{2j-1}d\sigma(y)\frac{dr}{r}\right)^{1/2},\quad j=1,2,3,\ldots.$$

As two subspaces of $L^2(\Sigma)$, $\mathcal{H}^2(\Delta)$ and $\mathcal{H}^2(\Delta^c)$ are orthogonal with each other, we state this property in the following proposition.

Proposition 4.2. *Suppose that $f\in L^2(\Sigma)$. Then there exist $f^+\in\mathcal{H}^2(\Delta)$ and $f^-\in\mathcal{H}^2(\Delta^c)$ such that their non-tangential boundary limits, still denoted by f^+,f^-, respectively, lie in $L^2(\Sigma)$ and $f=f^++f^-$. The mapping $f\to f^\pm$ is bounded on $L^2(\Sigma)$.*

By an example given in Eelbode [21] and Eelbode–Sommen [22], we can see that for $b\in H^s(S_\omega^c)$, $s\neq 0$, the Fourier multiplier operators are unbounded on $L^2(\Sigma)$. See also [16, Example 1.1]. The occurrence of the terms $k^sP_k(f)$ implies that we should assume f owns some regularity to justify the boundedness. This observation hints us to consider the boundedness of the Fourier multipliers on some Sobolev spaces.

Theorem 4.3. *Let $\omega\in(\arctan N,\frac{\pi}{2})$, $s\geq 0$ and $r\in\mathbb{Z}_+\cup\{0\}$. If $b\in H^s(S_\omega^c)$, then with the convention $b(0)=0$, the Fourier multiplier operator introduced in Section 3.3 can be extended to a bounded operator from $W_{\Gamma_\xi}^{2,\,r+s_1}(\Sigma)$ to $W_{\Gamma_\xi}^{2,\,r}(\Sigma)$, where s_1 is the constant defined by*

$$s_1=\begin{cases}s, & s\text{ is an integer },\\ [s]+1, & \text{otherwise }.\end{cases}$$

Moreover, for the operator norm $\|\cdot\|_{op}$, we have

$$\|M_b\|_{op}\leq C_\nu\left\|\frac{b}{|z+1|^{s_1}}\right\|_{L^\infty(S_\nu^c)},\quad\arctan N<\nu<\omega.$$

Proof. It is obvious that $W^{2,s_1}_{\Gamma_\xi}(\Sigma) \subset L^2(\Sigma)$. For any $f \in W^{2,s}_{\Gamma_\xi}(\Sigma)$, Proposition 4.2 implies that $f = f^+ + f^-$, where $f^+ \in \mathcal{H}^2(\Delta)$ and $f^- \in \mathcal{H}^2(\Delta^c)$ such that $\|f^\pm\|_{L^2(\Sigma)} \le C\|f\|_{W^{2,s_1}_{\Gamma_\xi}(\Sigma)}$. Applying Theorem 3.16, we have $M_b(f) = M_{b^+}(f^+) + M_{b^-}(f^-)$, where

$$M_{b^\pm}(f^\pm)(x) = \lim_{r\to 1-}\int_\Sigma \phi_\pm(r^{\pm 1}y^{-1}x)E(y)n(y)f(y)d\sigma(y), \quad x \in \Sigma.$$

Hence it suffices to prove

$$\|\Gamma_\xi^k M_{b^\pm}(f^\pm)\|_{\mathcal{H}^2} \le C_N\|\Gamma_\xi^{k+s_1}f^\pm\|_{\mathcal{H}^2}, \quad \forall k = 1, 2, \dots, r. \tag{4.1}$$

We only prove (4.1) for the part f^+ and omit "+" in the sequel for simplicity. The treatment for f^- is similar. By Theorem 3.9, for $b \in H^s(S^c_\omega)$, we have

$$|\phi(y^{-1}x)| \le \frac{C}{|1-y^{-1}x|^{3+s}}.$$

Hölder's inequality implies that

$$\begin{aligned}
|\Gamma_\xi^{1+s_1+k}M_bf(x)| &\le \left(\int_{\Sigma_{\sqrt{t}}}|\phi(x^{-1}y)|\frac{d\sigma(y)}{|y|^n}\right)^{1/2}\left(\int_{\Sigma_{\sqrt{t}}}|\phi(x^{-1}y)||\Gamma_\xi^{s_1+1+k}f(y)|^2\frac{d\sigma(y)}{|y|^n}\right)^{1/2}\\
&\le C\left(\int_{\Sigma_{\sqrt{t}}}\frac{1}{|1-y^{-1}x|^{n+s}}\frac{d\sigma(y)}{|y|^n}\right)^{1/2}\left(\int_{\Sigma_{\sqrt{t}}}\frac{|\Gamma_\xi^{s_1+1+k}f(y)|^2}{|1-y^{-1}x|^{n+s}}\frac{d\sigma(y)}{|y|^n}\right)^{1/2}\\
&\le C\left(\int_\Sigma\frac{1}{\left[(1-\sqrt{t})^2+\theta_0^2\right]^{\frac{s+n}{2}}}d\sigma(y)\right)^{1/2}\left(\int_\Sigma\frac{|\Gamma_\xi^{s_1+1+k}f(y)|^2}{\left[(1-\sqrt{t})^2+\theta_0^2\right]^{\frac{s+n}{2}}}d\sigma(y)\right)^{1/2}\\
&\le C\left(\frac{1}{(1-\sqrt{t})^s}\right)^{1/2}\left(\int_\Sigma\frac{|\Gamma_\xi^{s_1+1+k}f(y)|^2}{\left[(1-\sqrt{t})^2+\theta_0^2\right]^{\frac{s+n}{2}}}d\sigma(y)\right)^{1/2}.
\end{aligned}$$

Finally by Proposition 4.1, we have

$$\begin{aligned}
\|\Gamma_\xi^kM_bf\|^2_{\mathcal{H}^2(\Delta)} &\le \int_0^1\left|\Gamma_\xi^{s_1+1+k}M_bf(ty)\right|^2(1-t)^{2s_1+1}d\sigma(y)\frac{dt}{t}\\
&\le \int_0^1\int_\Sigma\left|\Gamma_\xi^{s_1+1+k}f(\sqrt{t}x)\right|^2\left(\int_\Sigma\frac{(1-\sqrt{t})^{s_1}}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{s+3}{2}}}d\sigma(y)\right)(1-\sqrt{t})d\sigma(x)\frac{dt}{t}\\
&\le \int_0^1\int_\Sigma\left|\Gamma_\xi(\Gamma_\xi^{s_1+k}f)(\sqrt{t}x)\right|^2(1-\sqrt{t})d\sigma(x)\frac{dt}{t}\\
&\le \|\Gamma_\xi^{s_1+k}f\|^2_{\mathcal{H}^2(\Delta)},
\end{aligned}$$

where in the fourth inequality, we have used the facts:

$$(1-\sqrt{t})^{2s_1+1-s} = (1-\sqrt{t})^{1+s+(2s_1-2s)} \le (1-\sqrt{t})^{1+s} \quad \text{for } t \in (0,1)$$

and the integral

$$\int_\Sigma\frac{(1-\sqrt{t})^s}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{s+3}{2}}}d\sigma(y) \le C.$$

This completes the proof of Theorem 4.3. □

4.2. Equivalence between Hardy–Sobolev spaces

In the proof of Theorem 4.3, we use the Hardy decomposition of $L^2(\Sigma)$. For $f \in L^2(\Sigma)$, $f = f^+ + f^-$, where $f^+ \in \mathcal{H}^2(\Delta)$ and $f^- \in \mathcal{H}^2(\Delta^c)$. If $f \in W^{2,s}_{\Gamma_\xi}(\Sigma)$, f^+ and f^- belong to the so called Hardy–Sobolev spaces. For these spaces, there exist two methods to give the definitions.

Method I. For $f \in L^2(\Sigma), f = f^+ + f^-$, where $f^+ \in \mathcal{H}^{2,+}$ and $f^- \in \mathcal{H}^{2,-}$. That is, f^+ belongs to the Hardy space, while f^- belongs to the conjugate Hardy space. We define the Hardy–Sobolev spaces on Σ as

$$\mathcal{H}^{2,s}_{+,1}(\Sigma) = \left\{f : \text{ there exists a function } g \in L^2(\Sigma) \text{ such that } f = g^+ \in L^2(\Sigma) \text{ and } \Gamma_\xi^j(g^+) \in L^2(\Sigma),\ j = 1, 2, \ldots, s\right\}$$

and

$$\mathcal{H}^{2,s}_{-,1}(\Sigma) = \Big\{f :\in L^2(\Sigma) \text{ there exists a function } g \in L^2(\Sigma) \text{ such that}$$
$$f = g^- \in L^2(\Sigma) \text{ and } \Gamma_\xi^j(g^-) \in L^2(\Sigma),\ j = 1, 2, \ldots, s\Big\}.$$

Method II. For every $f \in W^{2,s}_{\Gamma_\xi}$, $\Gamma_\xi^j f \in L^2(\Sigma), j = 1, 2, \ldots, s$. We get the decomposition $\Gamma_\xi^j f = (\Gamma_\xi^j f)^+ + (\Gamma_\xi^j f)^-$, where $(\Gamma_\xi^j f)^+ \in H^{2,+}$ and $(\Gamma_\xi^j f)^- \in H^{2,-}$. The Hardy–Sobolev space is defined as follows.

$$\mathcal{H}^{2,s}_{+,2}(\Sigma) = \left\{f : \text{ there exists a functiong } \in L^2(\Sigma) \text{ such that } f = g^+ \in L^2(\Sigma) \text{ and } (\Gamma_\xi^j g)^+ \in L^2(\Sigma),\ j = 1, 2, \ldots, s\right\}$$

and

$$\mathcal{H}^{2,s}_{-,2}(\Sigma) = \Big\{f : \text{ there exists a functiong } \in L^2(\Sigma) \text{ such that}$$
$$f = g^- \in L^2(\Sigma) \text{ and } (\Gamma_\xi^j g)^- \in L^2(\Sigma),\ j = 1, 2, \ldots, s\Big\}.$$

On the unit sphere, the order of Riesz transforms and the Dirac operator can be changed. The above two Hardy–Sobolev spaces are the same one obviously. On a general starlike Lipschitz surface, we prove the two spaces are equivalent by the Fourier multiplier theory.

Theorem 4.4. *Let Σ be a starlike Lipschitz surface and s be a positive integer. The Hardy–Sobolev spaces $\mathcal{H}^{2,s}_{\pm,\,1}(\Sigma)$ and $\mathcal{H}^{2,s}_{\pm,\,2}(\Sigma)$ are equivalent.*

Proof. Because $\mathcal{A}$ is dense in $L^2(\Sigma)$, without loss of generalization, we assume $f \in \mathcal{A}$. By spherical monogenic extension, we have

$$f = \sum_{k=1}^{\infty} P_k(f)(x) + \sum_{k=1}^{\infty} Q_k(f)(x).$$

Let $f^+ = \sum_{k=1}^{\infty} P_k(f)(x)$ and $f^- = \sum_{k=1}^{\infty} Q_k(f)(x)$. We can get

$$\Gamma_\xi(f^+) = \Gamma_\xi\left(\sum_{k=1}^{\infty} P_k(f)(x)\right).$$

Because $P_k(f) \in \mathcal{M}_k$, the homogeneous eigenspace of order k, we have

$$\Gamma_\xi(f^+)(x) = \sum_{k=1}^{\infty} kP_k(f)(x) \quad \text{for } f \in \mathcal{A}.$$

On the other hand,

$$\begin{aligned} P_k(f)(x) &= \frac{1}{\Omega_n}\int_\Sigma \widetilde{P}^k(x^{-1}y)E(y)n(y)f(y)d\sigma(y) \\ &= \frac{1}{\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(y)W_{\underline{\alpha}}(y)n(y)f(y)d\sigma(y), \end{aligned}$$

where we have used the Cauchy–Kovalevskaya extension again

$$\widetilde{P}^k(x^{-1}y)E(y) = \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(y)W_{\underline{\alpha}}(y),$$

where $V_{\underline{\alpha}} \in \mathcal{M}_k$ and $W_{\underline{\alpha}} \in \mathcal{M}_{-n-k}$. Therefore we can get

$$\begin{aligned} \Gamma_\xi(f^+)(x) &= \frac{1}{\Omega_n}\sum_{k=1}^{\infty}\int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x)\frac{k}{k+n-1}(k+n-1)W_{\underline{\alpha}}(y)n(y)f(y)d\sigma(y) \\ &= \frac{1}{\Omega_n}\sum_{k=1}^{\infty}\frac{k}{k+n-1}\int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x)\Gamma_\eta W_{\underline{\alpha}}(y)n(y)f(y)d\sigma(y). \end{aligned}$$

Because $f \in \mathcal{A}$, f decays fast enough. By integration by parts, we have

$$\begin{aligned}\Gamma_\xi(f^+)(x) &= \frac{1}{\Omega_n}\sum_{k=1}^{\infty}\frac{k}{k+n-1}\int_\Sigma \widetilde{P}^{(k)}(y^{-1}x)E(y)n(y)(\Gamma_\eta f)(y)d\sigma(y)\\ &= \frac{1}{\Omega_n}\sum_{k=1}^{\infty}\frac{k}{k+n-1}P_k(\Gamma_\xi f)(x).\end{aligned}$$

Let $b_k = \frac{k}{k+n-1}$. We have $\Gamma_\xi(f^+) = M_{(b_k)}((\Gamma_\xi f)^+)$. It is easy to see that $|b_k| \leq C$. Take $s = 0$ in Theorem 4.3. We obtain that $M_{(b_k)}$ is bounded on $L^2(\Sigma)$ (we can also deduce this result by the H^∞-Fourier multiplier theory on Σ). There exists a constant C_1 such that

$$\|(\Gamma_\xi f^+)\|_{L^2(\Sigma)} \leq C_1\|(\Gamma_\xi f)^+\|_{L^2(\Sigma)}.$$

For the converse, let $b'_k = \frac{k+1}{k}$. Similarly, we have

$$(\Gamma_\xi f)^+(x) = \frac{1}{\Omega_n}\sum_{k=1}^{\infty}\frac{k+n-1}{k}(\Gamma_\xi P_k(f))(x) = M_{(b'_k)}(\Gamma_\xi(f^+))(x).$$

So there exists another constant C_2 such that

$$\|(\Gamma_\xi f)^+\|_{L^2(\Sigma)} \leq C_1\|\Gamma_\xi(f^+)\|_{L^2(\Sigma)}.$$

This completes the proof of Theorem 4.4. □

Acknowledgments

The authors would like to thank sincerely the editor and the referees for providing advice to improve this paper.

This project was supported by NSFC No. 11171203, 11201280; New Teacher's Fund for Doctor Stations, Ministry of Education No. 20114402120003; University of Macao research grant MYRG 115 (Y1-L4)-FST13-QT; Guangdong Natural Science Foundation S2011040004131; Foundation for Distinguished Young Talents in Higher Education of Guangdong, China, LYM11063.

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