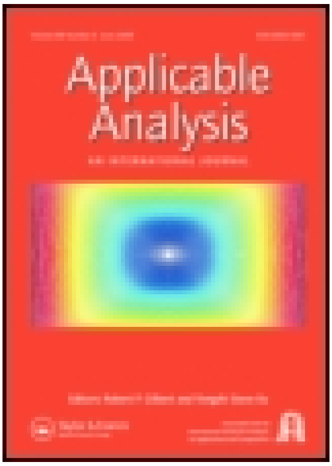

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Spaces of harmonic functions with boundary values in $Q^{\alpha}_{p,q}$

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In this paper, we apply wavelets to study two classes of function spaces of harmonic functions: the weighted Besov spaces $H^{\alpha,\lambda}_{p,q}(\mathbb{R}^{n+1}_{+})$ and Carleson spaces $C^{\alpha}_{p,q}(\mathbb{R}^{n+1}_{+})$. By a reproducing formula, we prove that the elements in these harmonic function spaces can be characterized by the Poisson integral of the functions in the Besov-Q spaces $Q^{\alpha}_{p,q}(\mathbb{R}^{n})$.

Keywords: wavelet; weighted Besov spaces; Carleson measures; Besov-Morrey space

AMS Subject Classifications: 42B35; 42C40

1. Introduction

In this paper, we use wavelets to study the spaces of harmonic functions with boundary values in Besov-Q spaces $Q^{\alpha}_{p,q}(\mathbb{R}^{n})$. It is well known that for a measurable function on $\mathbb{R}^{n}$, the Poisson integral $P_t f$ gives a harmonic extension of f. In the literature, Poisson integral is used to describe the relation between the harmonic function spaces on $\mathbb{R}^{n+1}_{+}$ and their boundary values. Fabes et al. [1] characterized the spaces $HMO(\mathbb{R}^{n+1}_{+})$ with trace in $BMO(\mathbb{R}^{n})$. Precisely, they proved the following result:

$$u \in HMO(\mathbb{R}^{n+1}_{+}) \Longleftrightarrow u = P_t * f \text{ for some } f \in BMO(\mathbb{R}^{n}). \tag{1.1}$$

See [1, Theorem 1.0].

By wavelet methods, we will establish the following relations among the Besov-Q spaces $Q^{\alpha}_{p,q}(\mathbb{R}^{n})$, the wavelet spaces $W^{\alpha}_{p,q}(\mathbb{R}^{n})$, the weighted Besov spaces $H^{\alpha,\lambda}_{p,q}(\mathbb{R}^{n+1}_{+})$ and the Carleson spaces $C^{\alpha}_{p,q}(\mathbb{R}^{n+1}_{+})$:

$$Q^{\alpha}_{p,q} \overset{\text{Theorem 2.8}}{=\!=\!=} W^{\alpha}_{p,q} \ni f(x) \underset{\text{Theorem4.2}}{\overset{\text{Theorem4.4}}{\Longleftrightarrow}} f(x,t) \in C^{\alpha}_{p,q} \overset{\text{Theorem 3.4}}{=\!=\!=} H^{\alpha,\lambda}_{p,q}. \tag{1.2}$$

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We give the definitions of $Q^{\alpha}_{p,q}(\mathbb{R}^n)$ and $W^{\alpha}_{p,q}(\mathbb{R}^n)$ in Section 2. The definitions of the harmonic function spaces $H^{\alpha,\lambda}_{p,q}(\mathbb{R}^{n+1}_+)$ and $C^{\alpha}_{p,q}(\mathbb{R}^{n+1}_+)$ can be found in Section 3. Precisely, in this paper, we show the following results.

THEOREM 1.1 *Let* $1 \le q \le p < \infty$ *and* $0 \le \alpha < \min(1, \frac{n}{q})$. *The following five statements are equivalent:*

(i) $f(x) \in Q^{\alpha}_{p,q}(\mathbb{R}^n)$.
(ii) $f(x) \in W^{\alpha}_{p,q}(\mathbb{R}^n)$.
(iii) $P_t f(x) \in H^{\alpha,\lambda}_{p,q}(\mathbb{R}^{n+1}_+)$, $\exists \lambda > n(1 - \frac{q}{p})$.
(iv) $P_t f(x) \in H^{\alpha,\lambda}_{p,q}(\mathbb{R}^{n+1}_+)$, $\forall \lambda > n(1 - \frac{q}{p})$.
(v) $P_t f(x) \in C^{\alpha}_{p,q}(\mathbb{R}^{n+1}_+)$.

The significance of these spaces is that for particular choices of the parameters p, q and α, one obtains various classical function spaces, such as the Bergman spaces, the Bloch spaces, the Besov spaces, the BMO spaces and the Q spaces. We give the following space structure table to clarify the relation between these spaces and $Q^{\alpha}_{p,q}(\mathbb{R}^n)$:

$\alpha \in [0, 1)$, $1 \le p = q \le \infty$,	Besov spaces [2]
$\alpha = 0$, $p = \infty$, $q = 2$,	BMO space [3]
$\alpha \in (0, \min(1, \frac{n}{2}))$, $p = 2n/\alpha$, $q = 2$,	Q-spaces Q_α [4]
$\alpha = 0$, $p > 2$, $q = 2$,	Morrey spaces $L^{2,\lambda}$ [5]
$\alpha = 0$, $p = q = 1$,	real Bergman space [6,7]
$\alpha = 0$, $p = q = \infty$,	real Bloch space [6]

In the proof of Theorem 1.1, we need to overcome two difficulties:

On one hand, for a harmonic function $F(x, t)$, its boundary value may not be obtained via pointwise limits. In the paper, we use an alternative way to define the boundary value of functions in $H^{\alpha,\lambda}_{p,q}(\mathbb{R}^{n+1}_+)$. By a reproducing formula (2.3), we define the boundary value of $f(x, t)$ via (2.4).

On the other hand, to characterize $Q^{\alpha}_{p,2}(\mathbb{R}^n)$ by the Poisson kernel and the heat semi-groups, one of the main methodologies is the Fourier transform. See [8]. However for the spaces $Q^{\alpha}_{p,q}$ with $q \ne 2$, Fourier transform does not work. For functions $f \in Q^{\alpha}_{p,q}$, to surmount this obstacle, we use regular wavelets to estimate the Poisson kernel $P_t(x)$.

Now we give an outline of the proof of Theorem 1.1.

(1) The equivalence (i) and (ii) is well known. We list it as Theorem 2.8. See Section 2.4 and the references.[9–11]
(2) In Section 3, we prove that $H^{\alpha,\lambda}_{p,q}(\mathbb{R}^{n+1}_+) = C^{\alpha}_{p,q}(\mathbb{R}^{n+1}_+)$. In fact, our result implies that $f(x, t) \in H^{\alpha,\lambda}_{p,q}(\mathbb{R}^{n+1}_+)$ if and only if

$$d\mu =: |\nabla f(x, t)|^q t^{q-1-q\alpha} dx dt$$

is a $(1 - q/p)$-Carleson measure. See Theorem 3.4 for the equivalence of (iii), (iv) and (v).

(3) Let P_t be the Poisson kernel. In Lemma 2.5, we estimate the wavelet coefficients of the function $\frac{\partial}{\partial x_i} P_t(x-y)$. With the help of Lemma 2.5 and Theorem 2.8, we can get the following inclusion relation in Section 4.1:

$$P_t * \left(Q^{\alpha}_{p,q}(\mathbb{R}^n)\right) \subseteq C^{\alpha}_{p,q}(\mathbb{R}^{n+1}_+).$$

This gives (ii)⇒(v). In Section 4.2, we prove that $f(x,t) \in C^{\alpha}_{p,q}(\mathbb{R}^{n+1}_+)$ can be represented as the Poisson integral $P_t * f(x)$, where f is an element in $W^{\alpha}_{p,q}(\mathbb{R}^n)$. See Theorem 4.2 for a proof of (v)⇒(ii).

Remark: In a recent paper, by a different method, Wang-Xiao [24] obtain a extension of Campanato-Sobolev spaces $Q^s_{\lambda,2}$ via the fractional heat semigroups. We also refer the reader to Jiang-Xiao-Yang [25] for further information on this topic.

Some notations:

- $\mathsf{U} \approx \mathsf{V}$ represents that there is a constant $c > 0$ such that $c^{-1}\mathsf{V} \le \mathsf{U} \le c\mathsf{V}$ whose right inequality is also written as $\mathsf{U} \lesssim \mathsf{V}$. Similarly, one writes $\mathsf{V} \gtrsim \mathsf{U}$ for $\mathsf{V} \ge c\mathsf{U}$.
- For convenience, the positive constants C may change from one line to another and usually depend on the dimension n, α, β and other fixed parameters. The Schwartz class of rapidly decreasing functions and its dual will be denoted by $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$, respectively. For $f \in \mathscr{S}(\mathbb{R}^n)$, $\widehat{f}$ means the Fourier transform of f.

2. Preliminaries

2.1. *Regular Daubechies wavelets*

We present some preliminaries on Daubechies' wavelets Φ^ϵ, $\epsilon = 0$ or 1, and refer the reader to [6,12] and [13] for further information. Let

$$\begin{cases} E_n &= \{0,1\}^n \backslash \{0\}; \\ F_n &= \{(\epsilon,k) : \epsilon \in E_n, k \in \mathbb{Z}^n\}; \\ \Lambda_n &= \{(\epsilon,j,k), \epsilon \in E_n, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}, \end{cases}$$

We will use the real-valued regular Daubechies' wavelets. Let C^m denote the smooth function spaces with all the derivatives up to the order m, and being bounded. In this paper, we assume there exist two sufficiently large integers m and M such that

(i) For any $\epsilon \in E_n$, $\operatorname{supp}\Phi^\epsilon \subset [-2^M, 2^M]^n$;
(ii) $\Phi^\epsilon \in C^m([-2^M, 2^M]^n)$;
(iii) For $|\alpha| \le m$, $\int x^\alpha \Phi^\epsilon(x)dx = 0$.

For $(\epsilon, j, k) \in \Lambda_n$, let

$$\Phi^\epsilon_{j,k}(x) = 2^{jn/2}\Phi^\epsilon(2^j x - k).$$

The set $\{\Phi^\epsilon_{j,k}, (\epsilon,j,k) \in \Lambda_n\}$ forms a wavelet basis. For any $\epsilon \in \{0,1\}^n$, $k \in \mathbb{Z}^n$ and a function f on $\mathbb{R}^n$, we write $f^\epsilon_{j,k} = \langle f, \Phi^\epsilon_{j,k}\rangle$. The following result is well known.

LEMMA 2.1 *The Daubechies wavelets* $\{\Phi^{\epsilon}_{j,k}\}_{(\epsilon,j,k)\in\Lambda_n}$ *form an orthogonal basis of* $L^2(\mathbb{R}^n)$. *Consequently, for any* $f \in L^2(\mathbb{R}^n)$, *the following wavelet decomposition holds in the* L^2 *convergence sense:*

$$f = \sum_{(\epsilon,j,k)\in\Lambda_n} f^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}.$$

2.2. *Poisson extension and boundary value*

We first construct some functions with special compact supports in order to define boundary limits of harmonic functions.

LEMMA 2.2 *Fix* $m \in \mathbb{N}$. *There exist a constant* $C_0 > 0$ *and two radial real-valued functions* $\phi \in C^{2m+8}(B(0,1))$ *and* $\Phi \in C^{4m+8}(B(0,1))$ *such that*

(i) $\phi(x) = (-\Delta)^m \Phi(x)$;
(ii) $\int_0^{\infty}(\widehat{\phi}(t\xi))^2 \frac{dt}{t} = 1, \forall \xi \neq 0$;
(iii) $C_0 \int_0^{\infty} \widehat{\phi}(t) e^{-t} \frac{dt}{t} = 1$.

Proof It is easy to choose a radial real-valued function $\Psi \in C^{4m+8}(B(0,1))$ such that

$$\int_0^{\infty} t^{2m}\widehat{\Psi}(t)e^{-t}\frac{dt}{t} \neq 0.$$

Let

$$C_{\Psi} = \int_0^{\infty} t^{4m}|\xi|^{4m}|\widehat{\Psi}(t\xi)|^2\frac{dt}{t}$$

and

$$\Phi(x) = (C_{\Psi})^{-\frac{1}{2}}\Psi(x).$$

□

Let $C_n = \Gamma((n+1)/2)/\pi^{(n+1)/2}$ and let

$$\begin{cases} P(x) = \frac{C_n}{(1+|x|^2)^{(n+1)/2}}; \\ P_t(x) = t^{-n}P\left(\frac{x}{t}\right) = \frac{C_n t}{(t^2+|x|^2)^{(n+1)/2}}. \end{cases}$$

Let f be any measurable function on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{1+|x|^{n+1}}dx < \infty. \tag{2.1}$$

The Poisson integral of f is defined by

$$f(x,t) = \int_{\mathbb{R}^n} P_t(x-y)f(y)dy.$$

Let $A(\mathbb{R}^n) = \{f(x) : (1+|x|)^{n+1}\partial^{\alpha}_x f \in L^{\infty}, \forall \alpha \in \mathbb{N}^n\}$ and denote $A'(\mathbb{R}^n)$ be the dual space of $A(\mathbb{R}^n)$. For any function $g \in \mathscr{S}(\mathbb{R}^n)$, we know that $P_t g(x) \in A(\mathbb{R}^n)$ $\forall t \geq 0$.

Hence, for any distribution $f \in A'(\mathbb{R}^n)$, f can be extended formally to a harmonic function $P_t f(x)$ as

$$f(x,t) = e^{-t(-\Delta)^{\frac{1}{2}}} f(x) = P_t * f(x), \tag{2.2}$$

For any $t \geq 0$, $f(x,t)$ is a distribution. That is to say, if f does not satisfy (2.1), we can still define the Poisson extension $P_t f$.

Example 2.3 Let δ be the Dirac function. It is well known that δ is not measurable. However, it is obvious that

$$P_t\delta(x) = P_t(x).$$

For the harmonic function $f(x,t) =: P_t(x)$, we have

$$\underline{\lim}_{t\to 0} f(x,t) = 0, \quad \forall x \neq 0.$$

But we know, as $t \to 0$, $P_t(x)$ converges to $\delta(x)$ in the sense of distribution.

Example 2.3 implies that for a general harmonic function $f(x,t)$, its boundary value may not be defined in the sense of the pointwise limit as $t \to 0$. In Lemma 2.2, we use some compactly supported function to pull back the harmonic functions to some boundary functions.

Let ϕ be the function obtained in Lemma 2.2. Write $\phi_t(x) = t^{-n}\phi(\frac{x}{t})$ with $\widehat{\phi_t}(\xi) = \widehat{\phi}(t\xi)$. From (iii) of Lemma 2.2, we can deduce that

$$\widehat{f}(\xi) = C_0 \int_0^\infty \widehat{\phi}(t)e^{-t}\frac{dt}{t}\widehat{f}(\xi) = C_0 \int_0^\infty \widehat{\phi}(t\xi)e^{-t|\xi|}\widehat{f}(\xi)\frac{dt}{t}. \tag{2.3}$$

By the inverse Fourier transform, we can get the following result.

Proposition 2.4 *If $f \in \mathscr{S}(\mathbb{R}^n)$, then the following two identities hold point by point.*

$$f(x) = \underline{\lim}_{t\to 0} P_t f(x) = C_0 \int_0^\infty \int_{\mathbb{R}^n} P_t f(y)\phi_t(x-y)\frac{dt}{t}dy.$$

By Proposition 2.4, harmonic function $f(x,t)$ can be pulled back to the trace function $f(x)$ in the sense of distribution

$$f(x) = C_0 \int_0^\infty \int_{\mathbb{R}^n} f(x-y,t)\phi_t(y)\frac{dt}{t}dy.$$

2.3. *Wavelet estimates on the Poisson kernel*

Let

$$P_i(x) = \frac{-(n+1)C_n x_i}{(1+|x|^2)^{(n+3)/2}}.$$

For $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n \setminus \{0\}$, let τ_ϵ be the smallest index s such that $\epsilon_s \neq 0$. Let $P_{i,\epsilon}(x) = \partial_{x_{\tau_\epsilon}} P_i(x)$. We can see that

$$P_{i,\epsilon}(x) = \begin{cases} \frac{-(n+1)C_n(1+|x|^2-(n+3)x_i^2)}{(1+|x|^2)^{(n+5)/2}}, & i = \tau_\epsilon; \\ \frac{(n+1)(n+3)C_n x_i x_{\tau_\epsilon}}{(1+|x|^2)^{(n+5)/2}}, & i \neq \tau_\epsilon. \end{cases}$$

Let $\Phi^{\epsilon,i}(x) = \frac{\partial \Phi^\epsilon(x)}{\partial x_i}$ and

$$I_\epsilon \Phi^\epsilon(x) = \int_{-\infty}^{x_{\tau_\epsilon}} \Phi^\epsilon(x_1, \ldots, x_{-1+\tau_\epsilon}, y, x_{1+\tau_\epsilon}, \ldots, x_n)dy.$$

For $i = 1, 2, \ldots, n$, let

$$P_{i,t}(x) = -(n+1)C_n \frac{tx_i}{(t^2+|x|^2)^{(n+3)/2}}.$$

For $i = 1, \ldots, n$ and $(\epsilon, j, k) \in \Lambda_n$, define

$$\begin{aligned} I(i,t,x,\epsilon,j,k) &= \frac{\partial}{\partial x_i} \int P_t(x-y)\Phi^\epsilon_{j,k}(y)dy \\ &=: \int_{\mathbb{R}^n} P_{i,t}(x-y)\Phi^\epsilon_{j,k}(y)dy. \end{aligned} \tag{2.4}$$

We estimate $I(i,t,x,\epsilon,j,k)$ by wavelets.

Lemma 2.5

(i) *If* $2^j t > 1$, *then*

$$|I(i,t,x,\epsilon,j,k)| \lesssim \frac{2^{(\frac{n}{2}+2)j}t}{(4^j t^2 + |2^j x - k|^2)^{\frac{n+3}{2}}}. \tag{2.5}$$

(ii) *If* $2^j t \le 1$, *then*

$$|I(i,t,x,\epsilon,j,k)| \lesssim \begin{cases} \frac{2^{(\frac{n}{2}+2)j}t}{(1+|2^j x-k|^2)^{\frac{n+1}{2}}}, & |2^j x - k| \ge C_\Phi; \\ 2^{(\frac{n}{2}+1)j}, & |2^j x - k| < C_\Phi. \end{cases} \tag{2.6}$$

Proof

(i) For $2^j t > 1$, by the change of variable, we have

$$\begin{aligned} |I(i,t,x,\epsilon,j,k)| &\lesssim t^{-2}2^{-j} \int_{\mathbb{R}^n} |P_{i,\epsilon,t}(x-y)||(I_\epsilon \Phi^\epsilon)_{j,k}(y)|dy \\ &\lesssim t2^{2j+nj/2} \int_{\mathbb{R}^n} \frac{1}{(4^j t^2 + |2^j x - k - u|^2)^{\frac{n+3}{2}}} |\Phi^\epsilon(u)|du. \end{aligned}$$

Because supp $\Phi^\epsilon \subset B(0, 2^M)$, we have $|2^j x - k - u| \lesssim |2^j x - k| + 1$. On the other hand, by $2^j t > 1$, we can see that there exists a constant C large enough such that

$$C[4^j t^2 + |2^j x - k - u|^2] \ge C4^j t^2 + |2^j x - k|^2 - 2|u|^2 \ge 4^j t + |2^j x - k|^2.$$

Then, we obtain

$$|I(i,t,x,\epsilon,j,k)| \lesssim t2^{2j+nj/2} \frac{1}{(4^j t^2 + |2^j x - k|^2)^{\frac{n+3}{2}}}.$$

Now we prove (ii). If $2^j t \le 1$, applying integration by parts, we can get

$$I(i,t,x,\epsilon,j,k) = \int_{\mathbb{R}^n} P_t(x-y)2^j \Phi^{\epsilon,i}_{j,k}(y)dy$$

Hence, we have

$$\begin{aligned}|I(i,t,x,\epsilon,j,k)| &\lesssim \int_{\mathbb{R}^n} P_t(x-y)2^j |\Phi^{\epsilon,i}_{j,k}(y)|dy\\ &\lesssim \int_{\text{supp }\Phi^\epsilon} \frac{2^{(2+\frac{n}{2})j}t}{(4^j t^2 + |2^j x - k - u|^2)^{\frac{n+1}{2}}} |\Phi^{\epsilon,i}(u)|du.\end{aligned}$$

We distinguish two cases. If $|2^j x - k| \ge C_\Phi$, we can get

$$|2^j x - k - u| \ge |2^j x - k| - |u| \ge \frac{1}{2}|2^j x - k| \ge C_\Phi/2.$$

On the other hand, by $2^j t \le 1$, $4^j t^2 \lesssim |2^j x - k|^2$. The above estimates imply that

$$|I(i,t,x,\epsilon,j,k)| \lesssim \frac{2^{(2+\frac{n}{2})j}t}{(|2^j x-k|^2 + C_\Phi^2)^{\frac{n+1}{2}}} \lesssim \frac{2^{(\frac{n}{2}+2)j}t}{(1+|2^j x - k|^2)^{\frac{n+1}{2}}}$$

If $|2^j x - k| \le C_\Phi$, because $|\Phi^{\epsilon,i}(2^j y - k)| \le C$, a direct computation gives

$$\begin{aligned}|I(i,t,x,\epsilon,j,k)| &\lesssim 2^{j(1+\frac{n}{2})} \int_{\mathbb{R}^n} \frac{t}{(t^2+|x-y|^2)^{\frac{n+1}{2}}} |\Phi^{\epsilon,i}(2^j y - k)|dy\\ &\lesssim 2^{j(1+\frac{n}{2})}.\end{aligned}$$

This completes the proof of Lemma 2.5. □

2.4. *Besov-Q spaces and wavelet characterization*

Besov-Q spaces $Q^\alpha_{p,q}(\mathbb{R}^n)$ are studied in [11].

Definition 2.6 Let $1 \le q \le p < \infty$ and $0 \le \alpha < \min(1, \frac{n}{q})$. The Besov-Q space $Q^\alpha_{p,q}(\mathbb{R}^n)$ is defined to be the set of all functions with

$$\sup_I(f, Q^\alpha_{p,q})(I) =: \sup_I |I|^{\frac{q}{p}-1} \int_I \int_I \frac{|f(x)-f(y)|^q}{|x-y|^{n+q\alpha}} dxdy < +\infty,$$

where the supremum is taken over all cubes I with the edge length $\ell(I)$ and the edges parallel to the coordinate axes in $\mathbb{R}^n$.

For $\alpha \in (0,1)$, $p = n/\alpha, q = 2$, $Q^\alpha_{n/\alpha,\,2}(\mathbb{R}^n) = Q_\alpha(\mathbb{R}^n)$. Q spaces $Q_\alpha(\mathbb{R}^n)$ were studied extensively. For further information on $Q_\alpha(\mathbb{R}^n)$, we refer the reader to Dafni–Xiao [14,15], Essen et al. [4], Wu–Xie [5] and the reference therein. The space $Q^\alpha_{p,q}(\mathbb{R}^n)$ with $\alpha \in (0,1)$ and $2 \le q < p < \infty$ was introduced by Cui–Yang [9]. Yang–Yuan [11] established the Littlewood-Paley characterization of $Q^\alpha_{p,q}(\mathbb{R}^n)$ with the full indices as in Definition 2.6.

Let $\{\Phi^{\varepsilon}_{j,k}\}$ be a wavelet basis defined in Section 2.1. For any function f, let $\{f^{\epsilon}_{j,k}\}$ be the wavelet coefficients of f. By Lemma 2.1, formally

$$f = \sum_{(\epsilon,j,k)\in\Lambda_n} f^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}.$$

We introduce a space which consists of $\{f^{\epsilon}_{j,k}\}$ as follows.

Definition 2.7 Let $1 \le q \le p < \infty$ and $0 \le \alpha < \min(1, \frac{n}{q})$. The space $W^{\alpha}_{p,q}(\mathbb{R}^n)$ is defined to be the set of all functions with the wavelet coefficients satisfying

$$\sup_I \left\{(f, W^{\alpha}_{p,q})(I)\right\}^{\frac{1}{q}} =: \sup_I \left\{|I|^{\frac{q}{p}-1} \sum_{(\varepsilon,j,k)\in\Lambda_n : I_{j,k}\subset I} 2^{qj(\alpha+\frac{n}{2})-nj}|f^{\epsilon}_{j,k}|^q\right\}^{1/q} < \infty,$$

where the supremum is taken over all dyadic cubes I.

For $\alpha \in (0, 1)$ and $2 \le q \le p < \infty$, the wavelet characterization of $Q^{\alpha}_{p,q}(\mathbb{R}^n)$ is obtained by Cui–Yang [9]. By different methods, Lin–Yang [10] and Yang–Yuan [11] improved the scope to $\alpha \in [0, \infty)$ and $1 \le q \le p \le \infty$. See also [16] and [17].

THEOREM 2.8 *Let* $1 \le q \le p < \infty$ *and* $0 \le \alpha < \min(1, \frac{n}{q})$. *Then*

$$Q^{\alpha}_{p,q}(\mathbb{R}^n) = W^{\alpha}_{p,q}(\mathbb{R}^n).$$

3. Weighted Besov spaces and Carleson measures

On the unit disc, Zhao [18] introduced a family of analytic functions on the open unit disk, denoted by $F(p, q, s)$. The spaces $F(p, q, s)$ cover many known function spaces of analytic functions: the Bloch space, Bergman spaces and weighted Dirichlet spaces. Such spaces have been studied heavily by different authors. In the latest decades, $F(p, q, s)$ have been studied extensively. We refer the reader to [5,19–23] and the reference therein.

For $1 \le q \le p < \infty$, $0 \le \alpha < \min(1, \frac{n}{q})$ and $\lambda > n(1 - \frac{q}{p})$, replacing the analytic functions by harmonic functions, we introduce a class of spaces of harmonic functions on $\mathbb{R}^{n+1}_+$. For $1 \le q < \infty$, we define the gradient of $f(x, t)$ by

$$|\nabla f(x, t)|^q = \sum_{i=1}^{n} \left|\frac{\partial f(x, t)}{\partial x_i}\right|^q.$$

Definition 3.1 Let $1 \le q \le p < \infty$, $0 \le \alpha < \min(1, \frac{n}{q})$ and $\lambda > n(1 - \frac{q}{p})$. The weighted Besov spaces $H^{\alpha,\lambda}_{p,q}(\mathbb{R}^{n+1}_+)$ is defined as the space of all harmonic functions such that

$$\|f\|_{H^{\alpha,\lambda}_{p,q}} = \sup_{(y,u)\in\mathbb{R}^{n+1}_+} \left\{(f, H^{\alpha,\lambda}_{p,q})(y, u)\right\}^{\frac{1}{q}} < +\infty,$$

where

$$
\begin{aligned}
(f, H^{\alpha,\lambda}_{p,q})(y,u) &= \int_{(x,t)\in\mathbb{R}^{n+1}_+} \frac{|\nabla f(x,t)|^q t^{q-1-q\alpha} u^{\frac{nq}{p}-n+\lambda}}{|(x-y)^2+(u+t)^2|^{\frac{\lambda}{2}}} dxdt \\
&= \sum_{i=1}^{n} \int_{(x,t)\in\mathbb{R}^{n+1}_+} \frac{\left|\frac{\partial f(x,t)}{\partial x_i}\right|^q t^{q-1-q\alpha} u^{\frac{nq}{p}-n+\lambda}}{|(x-y)^2+(u+t)^2|^{\frac{\lambda}{2}}} dxdt \\
&\equiv \sum_{i=1}^{n} (f, H^{\alpha,\lambda}_{p,q})_i(y,u).
\end{aligned}
$$

The Carleson box based on a cube I is defined by

$$S(I) = I \times (0, \ell(I)] = \left\{(x,t) \in \mathbb{R}^{n+1}_+ : x \in I, t \in (0, \ell(I)]\right\}.$$

A positive measure μ is called a p-Carleson measure on $\mathbb{R}^{n+1}_+$ if

$$\sup_I \frac{\mu(S(I))}{|I|^p} < \infty.$$

Here, $\sup_I$ indicates the supremum take over all cubes in $\mathbb{R}^n$. Note that $p = 1$ gives the classical Carleson measure.

Definition 3.2 Let $1 \le q \le p < \infty$ and $0 \le \alpha < \min(1, \frac{n}{q})$. We define Carleson spaces $C^{\alpha}_{p,q}(\mathbb{R}^{n+1}_+)$ as the space of all harmonic functions such that

$$\|f\|_{C^{\alpha}_{p,q}} = \sup_I \left\{(f, C^{\alpha}_{p,q})(I)\right\}^{\frac{1}{q}} < +\infty,$$

where

$$(f, C^{\alpha}_{p,q})(I) = |I|^{\frac{q}{p}-1} \int_{S(I)} |\nabla f(x,t)|^q t^{q-1-q\alpha} dxdt \equiv \sum_{i=1}^{n} (f, C^{\alpha}_{p,q})_i(I).$$

Remark 3.3

(i) In the above definition 3.1, for $1 \le p = q \le \infty$, we can take $\lambda = 0$. Then the definition of $H^{\alpha,0}_{q,q}(\mathbb{R}^{n+1}_+)$ coincides with that of $C^{\alpha}_{q,q}(\mathbb{R}^{n+1}_+)$ in the definition 3.2. In particularly, $H^{0,0}_{1,1}(\mathbb{R}^{n+1}_+)$ is the classic definition of Bergman spaces and $H^{0,0}_{\infty,\infty}(\mathbb{R}^{n+1}_+)$ is the classic definition of Bloch spaces. See Section 8 of chapter 6 in [6].

(ii) For $\alpha = 0$, $p = \infty$ and $q = 2$, $C^{0}_{\infty,2}(\mathbb{R}^{n+1}_+)$ becomes the space $HMO(\mathbb{R}^{n+1}_+)$ introduced by Fabes et al. [1].

We characterize the space $H^{\alpha,\lambda}_{p,q}(\mathbb{R}^{n+1}_+)$ by Carleson measure.

THEOREM 3.4 *Let* $1 \le q \le p < \infty$, $0 \le \alpha < \min(1, \frac{n}{q})$ *and* $\lambda > n - \frac{nq}{p}$.

$$C^{\alpha}_{p,q}(\mathbb{R}^{n+1}_+) = H^{\alpha,\lambda}_{p,q}(\mathbb{R}^{n+1}_+).$$

Proof Let I be the cube parallel to the coordinate axes with centre y and the edge length $\ell(I)$, and let $u = \ell(I)/2$. When $(x,t) \in S(I)$, we have $(x-y)^2 + (u+t)^2 \sim \ell(I)^2$. Therefore,

$$|I|^{\frac{q}{p}-1}\int_{S(I)} |\nabla f(x,t)|^q t^{q-1-q\alpha} dxdt \lesssim \int_{(x,t)\in\mathbb{R}^{n+1}_+} \frac{|\nabla f(x,t)|^q t^{q-1-q\alpha} u^{\frac{nq}{p}-n+\lambda}}{|(x-y)^2+(u+t)^2|^{\frac{\lambda}{2}}} dxdt.$$

Conversely, for any fixed (y,u), let I be the cube parallel to the coordinate axes with centre y and the edge length $2u$. For nonnegative integer m, we use I_m to denote the cubes with the same centre as I and the length $2^m\ell(I)$.

$$\begin{aligned} I_{y,u} &\equiv \int_{(x,t)\in\mathbb{R}^{n+1}_+} \frac{|\nabla f(x,t)|^q t^{q-1-q\alpha} u^{\frac{nq}{p}-n+\lambda}}{|(x-y)^2+(u+t)^2|^{\frac{\lambda}{2}}} dxdt \\ &\lesssim \int_{(x,t)\in S(I)} \frac{|\nabla f(x,t)|^q t^{q-1-q\alpha} u^{\frac{nq}{p}-n+\lambda}}{|(x-y)^2+(u+t)^2|^{\frac{\lambda}{2}}} dxdt \\ &\quad + \sum_{m=0}^{\infty} \int_{S(I_{m+1})\backslash S(I_m)} \frac{|\nabla f(x,t)|^q t^{q-1-q\alpha} u^{\frac{nq}{p}-n+\lambda}}{|(x-y)^2+(u+t)^2|^{\frac{\lambda}{2}}} dxdt. \end{aligned}$$

When $(x,t)\in S(I)$, we have $(x-y)^2+(u+t)^2 \sim \ell(I)^2$. If $(x,t)\in S(I_{m+1})\backslash S(I_m)$, then $(x-y)^2+(u+t)^2 \sim 2^{2m}\ell(I)^2$. Therefore,

$$\begin{aligned} I_{y,u} &\lesssim |I|^{\frac{q}{p}-1}\int_{S(I)} |\nabla f(x,t)|^q t^{q-1-q\alpha} dxdt \\ &\quad + \sum_{m=0}^{\infty} 2^{m(n-\frac{nq}{p}-\lambda)} |I_{m+1}|^{\frac{q}{p}-1} \int_{S(I_{m+1})} |\nabla f(x,t)|^q t^{q-1-q\alpha} dxdt \\ &\lesssim 1 + \sum_{m=0}^{\infty} 2^{m(n-\frac{nq}{p}-\lambda)} \lesssim 1. \end{aligned}$$

□

The following result is easily deduced from Theorem 3.4.

COROLLARY 3.5 *Let $0 \le \alpha < \min(1, \frac{n}{q})$, $1 \le q \le p < \infty$ and $\lambda > n - \frac{nq}{p}$. The definitions of $H^{\alpha,\lambda}_{p,q}(\mathbb{R}^{n+1}_+)$ are independent of the index λ.*

4. Harmonic function and Besov-Q spaces

In this section, by Theorems 2.8, 3.4 and 4.1, we extend the functions in $Q^{\alpha}_{p,q}(\mathbb{R}^n)$ to harmonic functions on $\mathbb{R}^{n+1}_+$. By Proposition 2.4, Theorems 3.4 and 4.2, we pull back the harmonic functions in $C^{\alpha}_{p,q}(\mathbb{R}^{n+1}_+)$ to their relative trace function in $Q^{\alpha}_{p,q}(\mathbb{R}^n)$.

4.1. *Poisson extension*

In this subsection, we extend the functions in $Q^{\alpha}_{p,q}(\mathbb{R}^n)$ to harmonic functions in Carleson spaces. In fact, THEOREM 4.1 *Let* $1 \leq q \leq p < \infty$ *and* $0 \leq \alpha < \min(1, \frac{n}{q})$. *For any* $f \in W^{\alpha}_{p,q}(\mathbb{R}^n)$, *we have*

$$f(x,t) =: P_t * f(x) \in C^{\alpha}_{p,q}(\mathbb{R}^{n+1}_+).$$

Proof By Theorem 3.4, it is enough to verify for $f \in W^{\alpha}_{p,q}(\mathbb{R}^n)$,

$$\sup_I |I|^{\frac{q}{p}-1} \int_{S(I)} |\nabla(P_t * f)(x)|^q \, t^{q-1-q\alpha} dxdt \lesssim \|f\|_{W^{\alpha}_{p,q}}.$$

For $i = 1, \ldots, n$, define

$$C_{I,i} = |I|^{\frac{q}{p}-1} \int_{S(I)} \left| \frac{\partial f(x,t)}{\partial x_i} \right|^q t^{q-1-q\alpha} dxdt.$$

We only need to prove that

$$\sup_I C_{I,i} \lesssim \|f\|^q_{W^{\alpha}_{p,q}}, i = 1, 2, \ldots, n.$$

The kernel of $\frac{\partial f(x,t)}{\partial x_i}$ is $P_{i,t}(x)$. Let

$$\begin{cases} f_{\epsilon,j}(x) = \sum\limits_{k \in \mathbb{Z}} a^{\epsilon}_{j,k} \Phi^{\epsilon}_{j,k}(x); \\ \frac{\partial f_{\epsilon,j}}{\partial x_i}(x,t) = P_{i,t} * f_{\epsilon,j}(x). \end{cases}$$

We obtain

$$\frac{\partial f(x,t)}{\partial x_i} = \sum_{\epsilon,j} \frac{\partial f_{\epsilon,j}(x,t)}{\partial x_i}.$$

By (2.5) and (2.6), we estimate $\partial f_{\epsilon,j}(x,t)/\partial x_i$ as follows. If $2^j t > 1$, then by integration by parts, we have

$$\frac{\partial f_{\epsilon,j}(x,t)}{\partial x_i} = \int_{\mathbb{R}^n} P_{i,\epsilon,t}(x-y) \sum_{k \in \mathbb{Z}} 2^{-j} a^{\epsilon}_{j,k} (I_\epsilon \Phi^{\epsilon})_{j,k}(y) dy.$$

Hence, by (2.5), we can get

$$\begin{aligned} \left| \frac{\partial f_{\epsilon,j}(x,t)}{\partial x_i} \right| &\lesssim \int_{\mathbb{R}^n} |P_{i,t}(x-y)| \sum_{k \in \mathbb{Z}^n} |a^{\epsilon}_{j,k}| |\Phi^{\epsilon}_{j,k}(y)| dy \\ &\lesssim \sum_{k \in \mathbb{Z}^n} |a^{\epsilon}_{j,k}| \frac{2^{(\frac{n}{2}+2)j} t (1 + |2^j x - k|)}{(4^j t^2 + |2^j x - k|^2)^{\frac{n+3}{2}}}. \end{aligned}$$

If $2^j t \leq 1$, then

$$\frac{\partial f_{\epsilon,j}(x,t)}{\partial x_i} = \int_{\mathbb{R}^n} P_t(x-y) \sum_{k \in \mathbb{Z}^n} 2^j a^{\epsilon}_{j,k} \Phi^{\epsilon,i}_{j,k}(y) dy.$$

Hence, we have

$$\left|\frac{\partial f_{\epsilon,j}(x,t)}{\partial x_i}\right| \lesssim \int_{\mathbb{R}^n} P_t(x-y)\sum_{k\in\mathbb{Z}^n} 2^j |a^{\epsilon}_{j,k}||\Phi^{\epsilon,i}_{j,k}(y)|dy$$
$$\lesssim \sum_{k\in\mathbb{Z}^n} |a^{\epsilon}_{j,k}|\frac{2^{(\frac{n}{2}+1)j}}{(1+|2^jx-k|^2)^{\frac{n+1}{2}}}.$$

For $i=1,\ldots,n$ and $(\epsilon,j,k)\in\Lambda_n$, let

$$I(i,t,x,\epsilon,j,k)=\frac{\partial}{\partial x_i}\int P_t(x-y)\Phi^{\epsilon}_{j,k}(y)dy \tag{4.1}$$

be the function defined by (2.4). Then, we regroup the indices (ϵ, j, k) by I. Let $I_1 = 8I = \tilde{I}$ and $|I_1| = 2^{-nj_I}$. For $\tau \geq 1$, let I_τ be the cube which contains I_1 with $|I_\tau| = 2^{n\tau}|I_1|$. We divide the indices (ϵ, j, k) into three cases.

Case 1 $2^jt\geq 1$. For $l\in\mathbb{Z}^n$, define

$$\begin{cases} S_{-1,l}=\left\{(\epsilon,j,k):\ 2^{-j}k\in 2^{-j_I}l+I_1, 2^jt\geq 1\right\};\\ I_{-1,l}(i,t,x,I)=\sum\limits_{(\epsilon,j,k)\in S_{-1,l}}|I(i,t,x,\epsilon,j,k)||a^{\epsilon}_{j,k}|.\end{cases} \tag{4.2}$$

Case 2 $2^jt<1\leq 2^j\ell(I)$. For $l\in\mathbb{Z}^n$, define

$$\begin{cases} S_{0,l}=\left\{(\epsilon,j,k):\ 2^{-j}k\in 2^{-j_I}l+I_1, 2^jt<1\leq 2^j\ell(I)\right\};\\ I_{0,l}(i,t,x,I)=\sum\limits_{(\epsilon,j,k)\in S_{0,l}}|I(i,t,x,\epsilon,j,k)||a^{\epsilon}_{j,k}|.\end{cases} \tag{4.3}$$

Case 3 $2^j\ell(I)<1$. For $l\in\mathbb{Z}^n$, define

$$\begin{cases} S_{\tau,l}=\left\{(\epsilon,j,k):\ 2^{-j}k\in 2^{\tau-j_I}l+I_\tau, 2^j\ell(I)<1\right\};\\ I_{\tau,l}(i,t,x,I)=\sum\limits_{(\epsilon,j,k)\in S_{\tau,l}}|I(i,t,x,\epsilon,j,k)||a^{\epsilon}_{j,k}|.\end{cases} \tag{4.4}$$

Hence, we obtain that

$$\left|\frac{\partial f(x,t)}{\partial x_i}\right|\lesssim\sum_{\tau\geq-1,l\in\mathbb{Z}^n} I_{\tau,l}(i,t,x,I).$$

Now, we estimate the terms:

$$I_{i,\tau,l,I}=|I|^{\frac{q}{p}-1}\int_{S(I)}|I_{\tau,l}(i,t,x,I)|^q t^{q-1-q\alpha}dt.$$

We first estimate the case $\tau=-1$. At first, we assume $|l|\leq C$. We can see that colourredwrong

$$|I|^{q/p-1}\int_{S(I)}\left\{\sum_{(\varepsilon,j,k)\in S_{-1,l}}|a^{\varepsilon}_{j,k}|\frac{2^{(\frac{n}{2}+2)j}t(1+|2^jx-k|)}{(4^jt^2+|2^jx-k|^2)^{\frac{n+3}{2}}}\right\}^q t^{q-1-q\alpha}dxdt$$
$$\lesssim |I|^{q/p-1}\int_{S(I)}\left\{\sum_{(\varepsilon,j,k)\in S_{-1,l}}|a^{\varepsilon}_{j,k}|\frac{2^{(\frac{n}{2}+1)j}(2^jt)(1+|2^jx-k|)}{(4^jt^2+|2^jx-k|^2)^{\frac{n+3}{2}}}\right\}^q t^{q-1-q\alpha}dxdt.$$

Because $2^j t \geq 1$,

$$\frac{2^j t}{(4^j t^2 + |2^j x - k|^2)^{\frac{n+3}{2}}} \lesssim \frac{1}{(4^j t^2 + |2^j x - k|^2)^{\frac{n+1}{2}}}.$$

This gives

$$\begin{aligned}
&\sum_{(\varepsilon,j,k)\in S_{-1,l}} |a^{\varepsilon}_{j,k}| \frac{2^{(\frac{n}{2}+1)j}}{(4^j t^2 + |2^j x - k|^2)^{\frac{n+1}{2}}} \\
&\lesssim \sum_{j\geq -\log_2 t} 2^{(\frac{n}{2}+1)j} \left(\sum_{2^{-j}k \in 2^{-j_I} l + I_1} \frac{|a^{\varepsilon}_{j,k}|^q}{(4^j t^2 + |2^j x - k|^2)^{\frac{n+1}{2}}} \right)^{1/q} \\
&\quad \times \left(\sum_{2^{-j}k \in 2^{-j_I} l + I_1} \frac{1}{(4^j t^2 + |2^j x - k|^2)^{\frac{n+1}{2}}} \right)^{\frac{q-1}{q}} \\
&\lesssim \sum_{j\geq -\log_2 t} 2^{(\frac{n}{2}+1)j} (2^j t)^{-\frac{q-1}{q}} \left(\sum_{2^{-j}k \in 2^{-j_I} l + I_1} \frac{|a^{\varepsilon}_{j,k}|^q}{(4^j t^2 + |2^j x - k|^2)^{\frac{n+1}{2}}} \right)^{1/q}.
\end{aligned} \tag{4.5}$$

We obtain that

$$\begin{aligned}
&|I|^{q/p-1} \int_{S(I)} \left\{ \sum_{(\varepsilon,j,k)\in S_{-1,l}} |a^{\varepsilon}_{j,k}| \frac{2^{(\frac{n}{2}+2)j} t (1 + |2^j x - k|)}{(4^j t^2 + |2^j x - k|^2)^{\frac{n+3}{2}}} \right\}^q t^{q-1-q\alpha} dx dt \\
&\lesssim |I|^{q/p-1} \int_I \int_0^{\ell(I)} \sum_{\varepsilon, j\geq -\log_2 t} 2^{q(\frac{n}{2}+1)j} 2^{-j(q-1)} \\
&\quad \times \Big(\sum_{k\in 2^{j-j_I} l + 2^j I_1} \frac{|a^{\varepsilon}_{j,k}|^q (2^j t)^{\delta}}{(4^j t^2 + |2^j x - k|^2)^{\frac{n+1}{2}}} \Big) t^{-q\alpha} dx dt.
\end{aligned}$$

We change the order of summation and integration to get

$$\begin{aligned}
&|I|^{q/p-1} \int_{S(I)} \left\{ \sum_{(\varepsilon,j,k)\in \Lambda_n} |a^{\varepsilon}_{j,k}| \frac{2^{(\frac{n}{2}+2)j} t (1 + |2^j x - k|)}{(4^j t^2 + |2^j x - k|^2)^{\frac{n+3}{2}}} \right\}^q t^{q-1-q\alpha} dx dt \\
&\lesssim |I|^{q/p-1} \sum_{j\geq -\log_2 \ell(I)} 2^{q(\frac{n}{2}+1)j} 2^{-j(q-1)} 2^{j\delta} \sum_{\varepsilon,k} |a^{\varepsilon}_{j,k}|^q \\
&\quad \times \int_I \int_{2^{-j}}^{\ell(I)} \frac{1}{(4^j t^2 + |2^j x - k|^2)^{\frac{n+1}{2}}} t^{\delta - q\alpha} dx dt \\
&\lesssim |I|^{q/p-1} \sum_{j\geq -\log_2 \ell(I)} 2^{q(\frac{n}{2}+1)j} 2^{-j(q-1)} 2^{j\delta} 2^{-jn-j} 2^{-j(\delta-q\alpha)} \sum_{\varepsilon,k} |a^{\varepsilon}_{j,k}|^q \\
&\lesssim \|f\|_{W^{\alpha}_{p,q}}.
\end{aligned}$$

If $|l| > C$, for $x \in I$, $|x - x_I| < \ell(I)$. On the other hand, $k \in 2^{j-j_I}l + 2^j I_1$ implies that $|k - 2^{j-j_I}l| \le 2^j \ell(I_1) = 2^{j-j_I}$. We can see that $|2^j x - k| \sim 2^{j-j_I}(1 + |l|)$. Also for any fixed l, $\sharp\{k : k \in 2^{j-j_I}l + 2^j I_1\} = 2^{(j-j_I)n}$. From these estimates, we can deduce that

$$\sum_{k \in 2^{j-j_I}l + 2^j I_1} \frac{1}{(4^j t^2 + |2^j x - k|^2)^{\frac{n+1}{2}}} \lesssim \frac{1}{2^j t(1 + |l|)^{n+1}},$$

where in the last inequality, we have used the facts that $8\ell(I) = 2^{-j_I}$ and $1 < 2^j t < 2^j \ell(I)$. Similar to the case of $|l| \le C$, we still have

$$|I|^{q/p-1} \int_{S(I)} \left\{ \sum_{(\varepsilon,j,k) \in \Lambda_n} |a^{\varepsilon}_{j,k}| \frac{2^{(\frac{n}{2}+2)j} t(1 + |2^j x - k|)}{(4^j t^2 + |2^j x - k|^2)^{\frac{n+3}{2}}} \right\}^q t^{q-1-q\alpha} dx dt$$
$$\lesssim (1 + |l|)^{-q(n+1)} \|f\|_{W^{\alpha}_{p,q}}.$$

We now estimate the case $\tau = 0$. We consider the case $|l| \le C$.

$$I_{i,0,l,I} \lesssim |I|^{\frac{q}{p}-1} \int_{S(I)} \left\{ \sum_{(\epsilon,j,k) \in S_{0,l}} |a^{\epsilon}_{j,k}| \frac{2^{(\frac{n}{2}+1)j}}{(1 + |2^j x - k|^2)^{\frac{n+1}{2}}} \right\}^q t^{q-1-q\alpha} dx dt.$$

Take $\delta > q - q\alpha > 0$. Applying Cauchy-Schwartz's inequality to k and j, respectively, we can obtain

$$\left\{ \sum_{(\epsilon,j,k) \in S_{0,l}} |a^{\epsilon}_{j,k}| \frac{2^{(\frac{n}{2}+1)j}}{(1 + |2^j x - k|^2)^{\frac{n+1}{2}}} \right\}^q$$
$$\lesssim \left\{ \sum_{\epsilon, -\log_2 \ell(I) \le j < -\log_2 t} 2^{(\frac{n}{2}+1)j} \left(\sum_{k:(\epsilon,j,k) \in S_{0,l}} \frac{|a^{\epsilon}_{j,k}|^q}{(1 + |2^j x - k|^2)^{\frac{n+1}{2}}} \right)^{\frac{1}{q}} \right\}^q$$
$$\lesssim \sum_{(\epsilon,j,k) \in S_{0,l}} 2^{q(\frac{n}{2}+1)j} \frac{|a^{\epsilon}_{j,k}|^q (2^j t)^{-\delta}}{(1 + |2^j x - k|^2)^{\frac{n+1}{2}}}.$$

The above estimate gives

$$I_{i,0,l,I} \lesssim |I|^{\frac{q}{p}-1} \int_{S(I)} \sum_{(\epsilon,j,k) \in S_{0,l}} |a^{\epsilon}_{j,k}|^q \frac{2^{(\frac{n}{2}+1)qj} (2^j t)^{-\delta}}{(1 + |2^j x - k|^2)^{\frac{n+1}{2}}} t^{q-1-q\alpha} dx dt$$
$$\lesssim |I|^{\frac{q}{p}-1} \sum_{(\epsilon,j,k) \in S_{0,l}} 2^{qj(\alpha+\frac{n}{2})-nj} (2^j l(I))^{q-q\alpha-\delta} |a^{\epsilon}_{j,k}|^q$$
$$\lesssim \|f\|_{W^{\alpha}_{p,q}},$$

where in the last inequality, we have used the fact that $(\epsilon, j, k) \in S_{0,l}$ implies that $2^j \ell(I) \le 1$.

We consider then the case $|l| > C$. At first, Cauchy-Schwartz's inequality gives

$$\left\{ \sum_{(\epsilon,j,k)\in S_{0,l}} |a^{\epsilon}_{j,k}| \frac{2^{(\frac{n}{2}+1)j}}{(1+|2^j x-k|^2)^{\frac{n+1}{2}}} \right\}^q \lesssim \left[\sum_{\varepsilon, j<-\log_2 t} 2^{j(\frac{n}{2}+1)} \left(\sum_{k\in\mathbb{Z}^n} \frac{|a^{\varepsilon}_{j,k}|^q}{(1+|2^j x-k|)^{\frac{n+1}{2}}} \right)^{\frac{1}{q}} \left(\sum_{k\in\mathbb{Z}^n} \frac{1}{(1+|2^j x-k|)^{\frac{n+1}{2}}} \right)^{\frac{q-1}{q}} \right]^q.$$

Because $(\varepsilon, j, k) \in S_{0,l}$, we can see that $|2^j x - k| \sim 2^{j-j_I}(1+|l|)$ which implies that

$$\sum_{k\in\mathbb{Z}^n} \frac{1}{(1+|2^j x-k|)^{\frac{n+1}{2}}} \sim (2^{j-j_I}(1+|l|)^{n+1})^{-1}.$$

By the above estimate, we apply Cauchy-Schwartz's inequality to j and get

$$\left\{ \sum_{(\epsilon,j,k)\in S_{0,l}} |a^{\epsilon}_{j,k}| \frac{2^{(\frac{n}{2}+1)j}}{(1+|2^j x-k|^2)^{\frac{n+1}{2}}} \right\}^q \lesssim \sum_{(\varepsilon,j,k)\in S_{0,l}} 2^{qj(\frac{n}{2}+1)} \frac{|a^{\varepsilon}_{j,k}|^q}{(1+|2^j x-k|)^{\frac{n+1}{2}}} \frac{(2^j t)^{-\delta}}{[2^{j-j_I}(1+|l|)^{n+1}]^{q-1}}. \tag{4.6}$$

By (4.6), we can obtain

$$\begin{aligned} I_{i,0,l,I} &\lesssim |I|^{\frac{q}{p}-1} \int_{S(I)} \left\{ \sum_{(\epsilon,j,k)\in S_{0,l}} |a^{\epsilon}_{j,k}| \frac{2^{(\frac{n}{2}+1)j}}{(1+|2^j x-k|^2)^{\frac{n+1}{2}}} \right\}^q t^{q-1-q\alpha} dx dt \\ &\lesssim |I|^{\frac{q}{p}-1} \int_{S(I)} \sum_{(\epsilon,j,k)\in S_{0,l}} \frac{|a^{\epsilon}_{j,k}|^q}{[2^{j-j_I}(1+|l|)^{n+1}]^{q-1}} \frac{2^{(\frac{n}{2}+1)qj}(2^j t)^{-\delta}}{(1+|2^j x-k|^2)^{\frac{n+1}{2}}} t^{q-1-q\alpha} dx dt \\ &\lesssim (1+|l|)^{-q(n+1)} |I|^{\frac{q}{p}-1} \sum_{(\epsilon,j,k)\in S_{0,l}} 2^{qj(\alpha+\frac{n}{2})-nj} |a^{\epsilon}_{j,k}|^q \\ &\lesssim (1+|l|)^{-q(n+1)} \|f\|_{W^{\alpha}_{p,q}}. \end{aligned}$$

Thirdly, we estimate the case $\tau \geq 1$ where the number of (ϵ, j, k) is finite. We consider the case $|l| \leq C$.

$$\begin{aligned} I_{i,\tau,l,I} &\lesssim |I|^{\frac{q}{p}-1} \int_{S(I)} \left\{ \sum_{(\epsilon,j,k)\in S_{\tau,l}} |a^{\epsilon}_{j,k}| \frac{2^{(\frac{n}{2}+1)j}}{(1+|2^j x-k|^2)^{\frac{n+1}{2}}} \right\}^q t^{q-1-q\alpha} dx dt \\ &\lesssim 2^{-q\delta\tau} |I_\tau|^{\frac{q}{p}-1} \sum_{(\epsilon,j,k)\in S_{\tau,l}} 2^{qj(\alpha+\frac{n}{2})-nj} |a^{\epsilon}_{j,k}|^q \\ &\lesssim 2^{-q\delta\tau} \|f\|_{W^{\alpha}_{p,q}}. \end{aligned}$$

We consider then the case $|l| > C$. Similar to the estimate (4.6), taking $0 < \delta < q - q\alpha$, we can get

$$\begin{aligned}
I_{i,\tau,l,I} &\lesssim |I|^{\frac{q}{p}-1} \int_{S(I)} \left\{ \sum_{(\epsilon,j,k)\in S_{\tau,l}} |a^{\epsilon}_{j,k}| \frac{2^{(\frac{n}{2}+1)j}}{(1+|2^j x - k|^2)^{\frac{n+1}{2}}} \right\}^q t^{q-1-q\alpha} dxdt \\
&\lesssim |I|^{\frac{q}{p}-1} \int_{S(I)} \sum_{(\epsilon,j,k)\in S_{\tau,l}} \frac{|a^{\epsilon}_{j,k}|^q}{[2^{j-j_I}(1+|l|)^{n+1}]^{q-1}} \frac{2^{(\frac{n}{2}+1)qj}(2^j t)^{-\delta}}{(1+|2^j x - k|^2)^{\frac{n+1}{2}}} t^{q-1-q\alpha} dxdt \\
&\lesssim 2^{-q\delta\tau}(1+|l|)^{-q(n+1)} |I_\tau|^{\frac{q}{p}-1} \sum_{(\epsilon,j,k)\in S_{\tau,l}} 2^{qj(\alpha+\frac{n}{2})-nj} |a^{\epsilon}_{j,k}|^q \\
&\lesssim 2^{-q\delta\tau}(1+|l|)^{-q(n+1)} \|f\|_{W^{\alpha}_{p,q}},
\end{aligned}$$

where we have used the fact that $(\epsilon, j, k) \in S_{\tau,l} \Leftrightarrow 2^{-j}k \in 2^{\tau-j_I} l + I_\tau$ and $2^j \ell(I) < 1$. Take a positive number δ small enough. We repeat applying Cauchy-Schwartz's inequality

$$\begin{aligned}
C_{I,i} &\lesssim \left\{ \sum_{\tau\geq -1, l\in\mathbb{Z}^n} 2^{-\tau\delta}(1+|l|)^{-(n+1)} \right\}^{q-1} \\
&\quad \times \sum_{\tau\geq -1, l\in\mathbb{Z}^n} 2^{(q-1)\tau\delta}(1+|l|)^{(q-1)(n+1)} |I|^{\frac{q}{p}-1} \int_{S(I)} |I_{\tau,l}(i,t,x,I)|^q t^{q-1-q\alpha} dt \\
&\lesssim \sum_{\tau\geq -1, l\in\mathbb{Z}^n} 2^{-\tau\delta}(1+|l|)^{-(n+1)} \|f\|_{W^{\alpha}_{p,q}}.
\end{aligned}$$

□

4.2. *Boundary value*

The boundary value of a harmonic function in $C^{\alpha}_{p,q}(\mathbb{R}^{n+1}_+)$ may not be locally integrable. But we have

THEOREM 4.2 *Let* $1 \leq q \leq p < \infty$ & $0 \leq \alpha < \min(1, \frac{n}{q})$. *For any* $f(x,t) \in C^{\alpha}_{p,q}(\mathbb{R}^{n+1}_+)$, *there exists a function* $f \in W^{\alpha}_{p,q}(\mathbb{R}^n)$ *such that*

$$f(x,t) = P_t * f(x).$$

Proof For simplicity, for any ϵ, let

$$(f, W^{\alpha}_{p,q})_\epsilon(I) =: |I|^{\frac{q}{p}-1} \sum_{(j,k):I_{j,k}\subset I} 2^{qj(\alpha+\frac{n}{2})-nj} |f^{\epsilon}_{j,k}|^q.$$

We write

$$(f, W^{\alpha}_{p,q})(I) = |I|^{\frac{q}{p}-1} \sum_{(\varepsilon,j,k)\in\Lambda_n:I_{j,k}\subset I} 2^{qj(\alpha+\frac{n}{2})-nj} |f^{\epsilon}_{j,k}|^q \equiv \sum_{\epsilon} (f, W^{\alpha}_{p,q})_\epsilon(I).$$

For $i = 1, \ldots, n$ and any function f define

$$I_i f(x) = \int_{-\infty}^{x_i} f(x_1, \ldots, x_{-1+i_\epsilon}, y, x_{1+i_\epsilon}, \ldots, x_n) dy.$$

For $m \geq n+8$, define $I_i^m f(x) = I_i I_i^{m-1} f(x)$. Let ϕ be a function in Lemma 2.2, we know $I_i^m \phi(x)$ is a C^{2n+8} function with compact support. For $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in E_n$, denote by i_ϵ the smallest index such that $\epsilon_{i_\epsilon} = 1$. Let $\partial_\epsilon = \partial_{x_{i_\epsilon}}$ and $I_\epsilon \Phi^\epsilon(x) = I_{i_\epsilon} \Phi^\epsilon(x)$. Hence, we have

$$\begin{aligned}(f, W^\alpha_{p,q})_\epsilon(I) &= |I|^{\frac{q}{p}-1} \sum_{I_{j,k} \subset I} 2^{qj(\alpha+\frac{n}{2})-nj} |f^\epsilon_{j,k}|^q \\ &= |I|^{\frac{q}{p}-1} \sum_{I_{j,k} \subset I} 2^{qj(\alpha+\frac{n}{2})-nj} \left| \left\langle \int_0^\infty \int_{\mathbb{R}^n} f(y,t)\phi_t(x-y)\frac{dt}{t} dy, \Phi^\epsilon_{j,k} \right\rangle \right|^q.\end{aligned}$$

We divide the integration on $(0, \infty)$ into two parts.

$$\begin{aligned}(f, W^\alpha_{p,q})_\epsilon(I) \lesssim\ & |I|^{\frac{q}{p}-1} \sum_{I_{j,k} \subset I} 2^{qj(\alpha+\frac{n}{2})-nj} 2^{mqj} \\ & \times \left| \left\langle \int_0^{2^{-j}} \int_{\mathbb{R}^n} \partial_1 f(y,t)(I_1^{m+1}\phi)_t(x-y)t^m dt dy,\ (\partial_1^m \Phi^\epsilon)_{j,k} \right\rangle \right|^q \\ & + |I|^{\frac{q}{p}-1} \sum_{I_{j,k} \subset I} 2^{qj(\alpha+\frac{n}{2})-nj} 2^{-(m+1)qj} \\ & \times \left| \left\langle \int_{2^{-j}}^\infty \int_{\mathbb{R}^n} \partial_\epsilon f(y,t)(\partial_\epsilon^m \phi)_t(x-y)\frac{dt}{t^{m+1}} dy,\ (I_\epsilon^{m+1}\Phi^\epsilon)_{j,k} \right\rangle \right|^q \\ =:\ & J_0 + J_1,\end{aligned}$$

where

$$\begin{aligned}J_0 = |I|^{\frac{q}{p}-1} \sum_{I_{j,k} \subset I} 2^{qj(\alpha+n)-nj} 2^{mqj} \Bigg\{ \int_0^{2^{-j}} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\partial_1 f(y,t)| \left| (I_1^{m+1}\phi)(\frac{x-y}{t}) \right| \\ \times |(\partial_1^m \Phi^\epsilon)(2^j x - k)| t^{m-n} dt dx dy \Bigg\}^q\end{aligned}$$

and

$$\begin{aligned}J_1 = |I|^{\frac{q}{p}-1} \sum_{I_{j,k} \subset I} 2^{qj(\alpha+n)-nj} 2^{-(m+1)qj} \left\{ \int_{2^{-j}}^\infty \int_{\mathbb{R}^n \times \mathbb{R}^n} |\partial_\epsilon f(y,t)| \left| (\partial_\epsilon^m \phi) \left(\frac{x-y}{t} \right) \right| \right. \\ \left. \times \left| \left(I_\epsilon^{m+1} \Phi^\epsilon \right) \left(2^j x - k \right) \right| \frac{dt}{t^{m+n+1}} dx dy \right\}^q.\end{aligned}$$

We can see that

$$\begin{aligned}&\left\{\int_{\mathbb{R}^n\times\mathbb{R}^n}|\partial_1 f(y,t)|\left|\left(I_1^{m+1}\phi\right)\left(\frac{x-y}{t}\right)\right|\left|\left(\partial_1^m\Phi^\epsilon\right)(2^jx-k)\right|dxdy\right\}^q\\&\lesssim 2^{-\frac{qnj}{q'}}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}|\partial_1 f(y,t)|\left|\left(I_1^{m+1}\phi\right)\left(\frac{x-y}{t}\right)\right|dy\right\}^q\left|\left(\partial_1^m\Phi^\epsilon\right)\left(2^jx-k\right)\right|dx\\&\lesssim t^{n(q-1)}2^{-n(q-1)j}\int_{\mathbb{R}^{2n}}|\partial_1 f(y,t)|^q\left|\left(I_1^{m+1}\phi\right)\left(\frac{x-y}{t}\right)\right|\left|\left(\partial_1^m\Phi^\epsilon\right)\left(2^jx-k\right)\right|dxdy.\end{aligned}$$

We first estimate the term J_0. By Hölder's inequality, we can deduce that

$$\begin{aligned}J_0\lesssim|I|^{\frac{q}{p}-1}\sum_{I_{j,k}\subset I}2^{qj(m+\alpha)-j(q-1)}\int_0^{2^{-j}}\int_{\mathbb{R}^{2n}}|\partial_1 f(y,t)|^q\left|\left(I_1^{m+1}\phi\right)\left(\frac{x-y}{t}\right)\right|\\\times\left|\left(\partial_1^m\Phi^\epsilon\right)\left(2^jx-k\right)\right|t^{mq-n}dxdydt.\end{aligned}$$

Notice that

$$u(y,t)=t^{-n}\sum_{I_{j,k}\subset I}\int_{\mathbb{R}^n}\left|\left(I_1^{m+1}\phi\right)\left(\frac{x-y}{t}\right)\right|\left|\left(\partial_1^m\Phi^\epsilon\right)\left(2^jx-k\right)\right|dx\in L^\infty(\tilde{I}).$$

Finally, we obtain

$$\begin{aligned}J_0&\lesssim|I|^{\frac{q}{p}-1}\sum_{2^{nj}|I|\geq 1}2^{qj(m+\alpha)-j(q-1)}\int_0^{2^{-j}}\int_{\tilde{I}}|\partial_1 f(y,t)|^q t^{mq}dydt\\&\lesssim|I|^{\frac{q}{p}-1}\int_{S(\tilde{I})}|\partial_1 f(y,t)|^q t^{q-1-q\alpha}dydt.\end{aligned}$$

For sufficient small positive real number δ, we have

$$J_1\lesssim\sum_{\tau=0}^{\infty}2^{\delta\tau}J_{1,\tau},$$

where

$$\begin{aligned}J_{1,0}=|I|^{\frac{q}{p}-1}\sum_{I_{j,k}\subset I}2^{qj(\alpha+n)-nj}2^{-(m+1)qj}\left\{\int_{2^{-j}}^{\ell(I)}\int_{\mathbb{R}^{2n}}\left|\partial_\epsilon f(y,t)\right|\right.\\\left.\times\left|\left(\partial_\epsilon^m\phi\right)\left(\frac{x-y}{t}\right)\right|\left|\left(I_\epsilon^{m+1}\Phi^\epsilon\right)\left(2^jx-k\right)\right|\frac{dt}{t^{m+n+1}}dxdy\right\}^q\end{aligned}$$

and

$$\begin{aligned}J_{1,\tau}=|I|^{\frac{q}{p}-1}\sum_{I_{j,k}\subset I}2^{qj(\alpha+n)-nj}2^{-(m+1)qj}\left\{\int_{2^{\tau-1}\ell(I)}^{2^\tau\ell(I)}\int_{\mathbb{R}^{2n}}\left|\partial_\epsilon f(y,t)\right|\right.\\\left.\times\left|\left(\partial_\epsilon^m\phi\right)\left(\frac{x-y}{t}\right)\right|\left|\left(I_\epsilon^{m+1}\Phi^\epsilon\right)\left(2^jx-k\right)\right|\frac{dt}{t^{m+n+1}}dxdy\right\}^q.\end{aligned}$$

Via Hölder's inequality, a simple computation implies that

$$\left\{\int_{\mathbb{R}^n\times\mathbb{R}^n} |\partial_\epsilon f(y,t)| \left|\left(\partial_\epsilon^m \phi\right)\left(\frac{x-y}{t}\right)\right| \left|\left(I_\epsilon^{m+1}\Phi^\epsilon\right)\left(2^j x-k\right)\right| dxdy\right\}^q$$
$$\lesssim t^{n(q-1)}2^{-n(q-1)j}\int_{\mathbb{R}^{2n}} |\partial_\epsilon f(y,t)|^q \left|\left(\partial_\epsilon^m \phi\right)\left(\frac{x-y}{t}\right)\right| \left|\left(I_\epsilon^{m+1}\Phi^\epsilon\right)\left(2^j x-k\right)\right| dxdy.$$

Then, we can get

$$J_{1,0} \lesssim |I|^{\frac{q}{p}-1}\sum_{I_{j,k}\subset I} 2^{qj(\alpha+n)-nj}2^{-(m+1)qj}2^j\int_{2^{-j}}^{\ell(I)}\left\{\int_{\mathbb{R}^{2n}} |\partial_\epsilon f(y,t)| \left|\left(\partial_\epsilon^m \phi\right)\left(\frac{x-y}{t}\right)\right|\right.$$
$$\left.\times\left|\left(I_\epsilon^{m+1}\Phi^\epsilon\right)\left(2^j x-k\right)\right| dxdy\right\}^q \frac{dt}{t^{q(m+n)}}$$
$$\lesssim |I|^{\frac{q}{p}-1}\sum_{I_{j,k}\subset I} 2^{qj\alpha}2^{j-(m+1)qj}\int_{2^{-j}}^{\ell(I)}\int_{\mathbb{R}^{2n}} |\partial_\epsilon f(y,t)|^q \left|\left(\partial_\epsilon^m \phi\right)\left(\frac{x-y}{t}\right)\right|$$
$$\times\left|\left(I_\epsilon^{m+1}\Phi^\epsilon\right)\left(2^j x-k\right)\right| t^{-mq-n}dxdydt.$$

Notice that

$$u(y,t)=t^{-n}\sum_{I_{j,k}\subset I}\int_{\mathbb{R}^n}\left|\left(\partial_\epsilon^m \phi\right)\left(\frac{x-y}{t}\right)\right| \left|\left(I_\epsilon^{m+1}\Phi^\epsilon\right)\left(2^j x-k\right)\right| dx \in L^\infty(\tilde{I}).$$

We can obtain that

$$J_{1,0} \lesssim |I|^{\frac{q}{p}-1}\sum_{2^{nj}|I|\geq 1} 2^{qj\alpha}2^{j-(m+1)qj}\int_{2^{-j}}^{\ell(I)}\int_{\tilde{I}} |\partial_\epsilon f(y,t)|^q\, t^{-mq}dydt$$
$$\lesssim |I|^{\frac{q}{p}-1}\int_{S(\tilde{I})} |\partial_\epsilon f(y,t)|^q\, t^{q-1-q\alpha}dydt \lesssim C.$$

By a similar method, for $J_{1,\tau}$, we have

$$J_{1,\tau} \lesssim |I|^{\frac{q}{p}-1}\sum_{I_{j,k}\subset I} 2^{qj(\alpha+n)-nj}2^{-(m+1)qj}\left[2^\tau \ell(I)\right]^{-1}\int_{2^{\tau-1}\ell(I)}^{2^\tau \ell(I)}\left\{\int_{\mathbb{R}^{2n}}\left|\partial_\epsilon f(y,t)\right|\right.$$
$$\left.\times\left|\left(\partial_\epsilon^m \phi\right)\left(\frac{x-y}{t}\right)\right| \left|\left(I_\epsilon^{m+1}\Phi^\epsilon\right)\left(2^j x-k\right)\right| dxdy\right\}^q \frac{dt}{t^{q(m+n)}}$$
$$\lesssim |I|^{\frac{q}{p}-1}\sum_{I_{j,k}\subset I} 2^{qj\alpha}2^{-(m+1)qj}\{2^\tau l(I)\}^{-1}\int_{2^{\tau-1}\ell(I)}^{2^\tau \ell(I)}\int_{\mathbb{R}^{2n}} |\partial_\epsilon f(y,t)|^q$$
$$\times\left|\left(\partial_\epsilon^m \phi\right)\left(\frac{x-y}{t}\right)\right| \left|\left(I_\epsilon^{m+1}\Phi^\epsilon\right)\left(2^j x-k\right)\right| t^{-mq-n}dxdydt.$$

We can see that

$$u(y,t)=t^{-n}\sum_{I_{j,k}\subset I}\int_{\mathbb{R}^n}\left|\left(\partial_\epsilon^m \phi\right)\left(\frac{x-y}{t}\right)\right| \left|\left(I_\epsilon^{m+1}\Phi^\epsilon\right)\left(2^j x-k\right)\right| dx \in L^\infty\left(\tilde{I}_\tau\right).$$

The above estimate gives

$$\begin{aligned} J_{1,\tau} &\lesssim |I|^{\frac{q}{p}-1} \sum_{2^{nj}|I|\geq 1} 2^{qj\alpha} 2^{-(m+1)qj} \left[2^{\tau}\ell(I)\right]^{-1} \int_{2^{\tau-1}\ell(I)}^{2^{\tau}\ell(I)} \int_{\tilde{I_\tau}} |\partial_\epsilon f(y,t)|^q t^{-mq} dydt \\ &\lesssim 2^{n\tau(1-\frac{q}{p})+\tau(q\alpha-mq-q)} |I_\tau|^{\frac{q}{p}-1} \int_{S(\tilde{I_\tau})} |\partial_\epsilon f(y,t)|^q t^{q-1-q\alpha} dydt \\ &\lesssim 2^{n\tau(1-\frac{q}{p})+\tau(q\alpha-mq-q)}. \end{aligned}$$

□

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