Chin. Ann. Math.
36B(1), 2015, 141–146
DOI: 10.1007/s11401-014-0870-8

Chinese Annals of Mathematics, Series B

A Constructive Proof of Beurling-Lax Theorem*

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Abstract This paper deals with an alternative proof of Beurling-Lax theorem by adopting a constructive approach instead of the isomorphism technique which was used in the original proof.

Keywords Beurling-Lax theorem, Shift operator, Inner function
2000 MR Subject Classification 30K99, 42A36

1 Introduction

In 1948, Beurling [2] first investigated the shift-operator-invariant subspace $\mathbb{M}$ in Hardy space $H^2(\mathbb{D})$ in the unit disc context for the shift operator defined by

$$S^*f(z) = zf(z), \quad z \in \mathbb{D}.$$

Here a subspace $\mathbb{M}$ in $H^2(\mathbb{D})$ is said to be a shift-operator-invariant-subspace if

$$S^*\mathbb{M} \subset \mathbb{M}.$$

The following celebrated result is called Beurling theorem.

Theorem 1.1 *Suppose that $\mathbb{M}$ is a non-zero closed subspace of $H^2(\mathbb{D})$. Then $\mathbb{M}$ is a shift-operator-invariant space if and only if*

$$\mathbb{M} = IH^2(\mathbb{D}),$$

where I is an inner function in $H^\infty(\mathbb{D})$.

Here we say a function $f \in H^\infty(\mathbb{D})$ is an inner function if $|f| = 1$ almost everywhere on the boundary $\partial\mathbb{D}$. Similarly, a function $f \in H^\infty(\mathbb{C}^+)$ is an inner function if $|f| = 1$ almost everywhere on the real line $\mathbb{R}$.

As a consequence of this theorem, a subspace is a backward-shift-invariant space ($S\mathbb{M} \subset \mathbb{M}$) if and only if (see [3])

$$\mathbb{M} = H^2(\mathbb{D}) \cap I\overline{H^2}(\mathbb{D}),$$

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*This work was supported by the Multi-Year Research Grant (No. MYRG115(Y1-L4)-FST13-QT), the Multi-Year Research Grant (No. MYRG116(Y1-L3)-FST13-QT), Macau Government (No. FDCT 098/2012/A3), and the Natural Science Foundation of Guangdong Province (No. S2011010004986).

where the backward-shift operator is defined by

$$Sf(z) = \frac{f(z) - f(0)}{z}, \quad z \in \mathbb{D}.$$

The Beurling theorem was extended to the upper-half complex plane by Lax in 1959, which is now named the Beurling-Lax theorem (see [7]). In the upper-half plane case, by using the same notation, the shift operator and the backward-shift operator are defined, respectively, by

$$S^* f(w) = \mathrm{e}^{\mathrm{i}\lambda w} f(w), \quad Sf(w) = \mathrm{e}^{-\mathrm{i}\lambda w} f(w), \quad \lambda > 0, \ w \in \mathbb{C}^+.$$

Theorem 1.2 *Let $\mathbb{M}$ be a non-zero closed subspace of $H^2(\mathbb{C}^+)$. Then $\mathbb{M}$ is a shift-operator-invariant subspace if and only if there exists an inner function $F \in H^\infty(\mathbb{C}^+)$ such that $\mathbb{M} = FH^2(\mathbb{C}^+)$.*

There is a large number of documents for the studies of forward shift and backward-shift invariant subspaces (see [1, 3–5]) mainly in theoretical aspects. As an important tool, the shift invariant subspace is useful in bandwidth keeping and phase retrieval, which are two topics in optical and signal processing (see [6]). Recently, Tan and Qian used techniques of forward and backward operators to solve some questions in the theory of analytic signals, especially the characterization of Bedrosian identity. We will offer a survey in Section 2 about Tan and Qian's results. We are inspired to revisit the proof of the Beurling-Lax theorem.

We remark that, in the unit disc case, Beurling's proof is constructive for the choice of an inner function for any invariant subspace, which is important for the calculation and characterization of analytic signals from the Bedrosian identity. But in the upper half plane case, the proof of the Beurling-Lax theorem (see [5, 7]) is complicated by involving the technique of isomorphism between $H^2(\mathbb{D})$ and $H^2(\mathbb{C}^+)$. Many applications are dependent on the concrete form of the factor inner function inducing the shift invariant subspace. In this note, we choose the special function $\frac{1}{\mathrm{i}+\cdot} \in H^2(\mathbb{C}^+)$ and use its orthogonal projection into $\mathbb{M}$ to construct the inner function F appearing in Theorem 1.2.

2 Preliminaries and Surveys

Related to the Hardy space in the unit disc, we say $f \in H^p(\mathbb{D})$, if

$$\sup_r \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta = \|f\|^p_{H^p(\mathbb{D})} < \infty$$

for $p \in (0, \infty)$. When $p = \infty$, we say $f \in H^\infty(\mathbb{D})$ if f is a bounded analytic function on $\mathbb{D}$ and we write

$$\|f\|_{H^\infty(\mathbb{D})} = \sup_{z \in \mathbb{D}} |f(z)|.$$

In the upper half plane case, we say $f \in H^p(\mathbb{C}^+)$ if f is analytic on $\mathbb{C}^+$ and

$$\sup_{y>0} \int_{-\infty}^{\infty} |f(x + \mathrm{i}y)|^p \mathrm{d}x = \|f\|^p_{H^p(\mathbb{C}^+)} < \infty.$$

When $p=\infty$, we write $f\in H^\infty(\mathbb{C}^+)$ for the bounded analytic functions on $\mathbb{C}^+$, and we give $H^\infty(\mathbb{C}^+)$ the norm $\|f\|_{H^\infty(\mathbb{C}^+)}=\sup\limits_{w\in\mathbb{C}^+}|f(w)|$.

In the case $p=2$, as Hilbert spaces, $H^2(\mathbb{D})$ and $H^2(\mathbb{C}^+)$ are equipped with the inner product

$$\langle f,g\rangle_{\mathbb{D}}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\mathrm{e}^{\mathrm{i}\theta})\overline{g(\mathrm{e}^{\mathrm{i}\theta})}\mathrm{d}\theta,\quad f,g\in H^2(\mathbb{D})$$

and

$$\langle f,g\rangle_{\mathbb{C}^+}=\int_{-\infty}^{\infty}f(t)\overline{g(t)}\mathrm{d}t,\quad f,g\in H^2(\mathbb{C}^+),$$

respectively.

The linear mapping which takes a function $f\in H^p(\mathbb{C}^+)$ to its non-tangential boundary limit function $f\in L^p(\mathbb{R})$ is an isometric isomorphism from $H^p(\mathbb{C}^+)$ onto a closed subspace of $L^p(\mathbb{R})$ which we shall denote by $H^p(\mathbb{R})$.

For any function $f\in H^p(\mathbb{C}^+)$, by the well-known Nevanlinna factorization theorem, f can be factorized as

$$f(z)=O_f(z)I_f(z),\quad z\in\mathbb{C}^+,$$

where

(1) O_f, the outer factor of f, is given by

$$O_f(z)=\exp\Big(\frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}\frac{1+tz}{z-t}\frac{\ln|f(t)|}{1+t^2}\mathrm{d}t\Big);$$

(2) I_f, the inner factor of f, can be further factorized into

$$I_f(z)=\mathrm{e}^{\mathrm{i}(az+b)}B_f(z)S_f(z),$$

where

(i) b is a real number and a is a nonnegative number;

(ii) B_f is the Blaschke product formed with the zero sequence $E_f=\{\alpha_k\}$ of f in $\mathbb{C}^+$ (the zeros repeat according to their respective multiples) which can be represented by

$$B_f(z)=\prod_{\alpha_k\in E_f}\frac{|\alpha_k^2+1|}{\alpha_k^2+1}\frac{z-\alpha_k}{z-\overline{\alpha_k}},$$

where $\frac{|\alpha_k^2+1|}{\alpha_k^2+1}$ is understood whenever $\alpha_k=\mathrm{i}$;

(iii) S_f, the singular inner function, is given by

$$S_f(z)=\exp\Big(\frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}\frac{1+tz}{t-z}\mathrm{d}\mu(t)\Big),$$

where μ is a real, bounded, increasing function with derivative $\mu'(t)=0$ almost everywhere on $\mathbb{R}$.

Two classical problems of long interest in a number of practical areas, including optics, antenna theory and physics, are formulated as follows: The first is to find all functions g such that $\mathrm{Band}(fg)\subset\mathrm{Band}(f)$; the second is to find all-pass filters $\mathrm{e}^{\mathrm{i}\theta(t)}$ such that $\mathrm{Band}(f\mathrm{e}^{\mathrm{i}\theta})\subset$

Band(f). Here, Band(f) is the support of the bandlimited signal $f \in L^2(\mathbb{R})$ whose Fourier transform has compact support. The Lebesgue measure of Band(f) is called the bandwidth of f. They are referred to as the band keeping problem and the phase retrieval problem, respectively. Obviously, the phase retrieval problem is closely related to the band keeping problem. In [10], the authors give a characterization of function g such that Band(fg) $\subset$ Band(f).

Proposition 2.1 *Suppose that f, g are nonzero functions, $f \in L^2(\mathbb{R})$ is bandlimited with* Band(f) $= [0, A]$ *and $\overline{g} \in H^p(\mathbb{R})$, $p \in [1, \infty]$. Then $fg \in L^2(\mathbb{R})$ is bandlimited with* Band(fg) $= [0, A]$ *if and only if $\overline{g} \in H^p(\mathbb{R}) \cap I_f \overline{H^p(\mathbb{R})}$, where I_f is the inner factor of f which does not contain the singular inner function.*

Noting that $H^p(\mathbb{R}) \cap I_f \overline{H^p(\mathbb{R})}$ is a backward-shift invariant subspace, the authors in [10] offer a characterization of this subspace.

Proposition 2.2 *Suppose that $\{\alpha_k\}_{k=1}^{\infty}$ is a sequence in $\mathbb{C}^+$ which can define a Blaschke product*

$$B(z) = \prod_{k=1}^{\infty} \frac{|\alpha_k^2 + 1|}{\alpha_k^2 + 1} \frac{z - \alpha_k}{z - \overline{\alpha_k}}.$$

Then for $p \in (1, \infty)$,

$$H^p(\mathbb{R}) \cap B\overline{H^p(\mathbb{R})} = (BH^{p'})^{\perp} = \overline{\text{span}}^p \{e_n\}_{n=1}^{\infty},$$

where the closure $\overline{\text{span}}^p$ is in the L^p topology, $p' = \frac{p}{p-1}$, $(BH^{p'})^{\perp} = \{f \in H^p : \langle f, Bg \rangle = 0, \forall g \in H^{p'}\}$ and the Takenaka-Malmquist (TM, for short) system $\{e_n\}_{n=1}^{\infty}$ is defined by

$$e_n(z) = \frac{\sqrt{\frac{2}{\pi}\mathrm{Im}\{\alpha_n\}}}{z - \overline{\alpha_n}} \prod_{k=1}^{n-1} \frac{z - \alpha_k}{z - \overline{\alpha_k}}.$$

Moreover, in [8], the authors prove the following result.

Proposition 2.3 *The TM system $\{e_n\}$ is a Schauder basis in $\overline{\text{span}}^p\{e_n\}, 1 < p < \infty$. In other words, for any $f \in \overline{\text{span}}^p\{e_n\}$, we have*

$$f = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n \quad \text{in the } H^p(\partial\mathbb{D})\text{-norm sense.}$$

It consequently concludes that the TM system $\{e_n\}$ is a Schauder basis of $H^p(\mathbb{R})$ if $\{\alpha_k\}$ can not form a Blaschke product.

3 A Constructive Proof of Lax's Theorem

The sufficiency is trivial since $FH^2(\mathbb{C}^+)$ is an invariant subspace under the multiplication by $e^{i\lambda w}$. Now we show the necessity. Suppose that $\mathbb{M}$ is a non-zero closed subspace of $H^2(\mathbb{C}^+)$. Let $G \in H^2(\mathbb{C}^+)$ be the orthogonal projection of the function $\frac{1}{w+i} \in H^2(\mathbb{C}^+)$ onto $\mathbb{M}$. Therefore,

$$\frac{1}{w + i} = G(w) + \Big(\frac{1}{w + i} - G(w)\Big),$$

where $G \in \mathbb{M}$ and $\frac{1}{w+\mathrm{i}} - G(w)$ is orthogonal to $\mathbb{M}$.

Now we prove that $|G(t)|^2 = \frac{C}{1+t^2}$ for $t \in \mathbb{R}$, where C is some positive constant. Since G is in $\mathbb{M}$, we know that $\mathrm{e}^{\mathrm{i}\lambda w}G$ is in $\mathbb{M}$ for any $\lambda \geq 0$ due to the invariance of $\mathbb{M}$ under the multiplication by $\mathrm{e}^{\mathrm{i}\lambda w}$. By the orthogonality between $\mathbb{M}$ and $\frac{1}{w+\mathrm{i}} - G$, we have

$$\int_{-\infty}^{\infty} \Big(\frac{1}{t-\mathrm{i}} - \overline{G(t)}\Big)\mathrm{e}^{\mathrm{i}\lambda t}G(t)\mathrm{d}t = 0, \quad \lambda \geq 0.$$

It follows that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |G(t)|^2\mathrm{e}^{\mathrm{i}\lambda t}\mathrm{d}t = \frac{2\pi\mathrm{i}}{\sqrt{2\pi}}\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\lambda t}G(t)}{t-\mathrm{i}}\mathrm{d}t.$$

Since $\mathrm{e}^{\mathrm{i}\lambda z}G(z)$ is analytic on the upper half plane when $\lambda > 0$, we apply Cauchy's formula to get that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |G(t)|^2\mathrm{e}^{\mathrm{i}\lambda t}\mathrm{d}t = \sqrt{2\pi}\mathrm{i}G(\mathrm{i})\mathrm{e}^{-\lambda}.$$

By noting the Hermitian property of the Fourier transform of a real function, using the Laplace integral $\displaystyle\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\pm\mathrm{i}\lambda t}}{t^2+y^2}\mathrm{d}t = \frac{\pi}{y}\mathrm{e}^{-y|\lambda|}$, $y > 0$ (see [10]) and utilizing the uniqueness of Fourier transform of function in L^2, we obtain that

$$|G(t)|^2 = \frac{2\mathrm{i}G(\mathrm{i})}{1+t^2}, \quad t \in \mathbb{R}.$$

Since $G \in H^2(\mathbb{C}^+)$, by Nevanlinna factorization theorem, we have $G = I_G O_G$ with $I_G \in H^\infty(\mathbb{C}^+)$ and $O_G \in H^2(\mathbb{C}^+)$ being inner function and outer function, respectively. Now we define the subspace $\widetilde{\mathbb{M}}$ of $H^2(\mathbb{C}^+)$ by

$$\widetilde{\mathbb{M}} = I_G H^2(\mathbb{C}^+).$$

Obviously, $\widetilde{\mathbb{M}}$ is an invariant subspace under the multiplication by $\mathrm{e}^{\mathrm{i}\lambda w}$ for any $\lambda \geq 0$. We observe that G is also in $\widetilde{M}$ since $O_G \in H^2(\mathbb{C}^+)$. By using the density of $\{\mathrm{e}^{\mathrm{i}\lambda\cdot} : \lambda > 0\}$ in $H^2(\partial\mathbb{C}^+)$ and the invariance of $\mathbb{M}$ under multiplication by $\mathrm{e}^{\mathrm{i}\lambda w}$, we conclude that $\widetilde{\mathbb{M}} \subset \mathbb{M}$.

Finally, we want to show that $\widetilde{\mathbb{M}} = \mathbb{M}$. To this end, it suffices to show that the conditions $f \in \mathbb{M}$ and $f \perp \widetilde{\mathbb{M}}$ imply that $f = 0$. On the one hand, from the invariance of $\widetilde{\mathbb{M}}$ under multiplication by $\mathrm{e}^{\mathrm{i}\lambda w}$ and $f \perp \widetilde{\mathbb{M}}$, we have

$$\int_{-\infty}^{\infty} f(t)\overline{G(t)}\mathrm{e}^{-\mathrm{i}\lambda t}\mathrm{d}t = 0, \quad \lambda \geq 0. \tag{3.1}$$

On the other hand, from the facts that $\frac{1}{w+\mathrm{i}} - G \perp \mathbb{M}$ and the invariance of $\mathbb{M}$ under the multiplication by $\mathrm{e}^{\mathrm{i}\lambda w}$, we obtain that

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda t}f(t)\Big(\frac{1}{t-\mathrm{i}} - \overline{G(t)}\Big)\mathrm{d}t = 0,$$

which is equivalent to

$$\int_{-\infty}^{\infty} f(t)\overline{G(t)}\mathrm{e}^{\mathrm{i}\lambda t}\mathrm{d}t = 2\pi\mathrm{i}f(\mathrm{i})\mathrm{e}^{-\lambda}, \quad \lambda \geq 0. \tag{3.2}$$

Setting $\lambda = 0$ in Equations (3.1)–(3.2), we conclude that $f(\mathrm{i}) = 0$. Therefore, (3.1)–(3.2) indicate that the Fourier transform of $f\overline{G}$ is zero. Thus $f\overline{G} = 0$. Noting that we have already shown $G \neq 0$, we therefore conclude $f = 0$. The proof of this theorem is completed.

Acknowledgement The authors are grateful to the anonymous referees for their helpful comments and suggestions.

References

[1] Aleksandrov, A. B., Embedding theorem for coinvariant subspaces of the shift operator, *Zap. Nauchn. Sem. S. -Peterburg, Otel. Math. Inst. Steklov,* **262**, 1999, 5–48.

[2] Beurling, A., On two problems concerning linear transformations in Hilbert space, *Acta Math.,* **81**, 1949, 239–255.

[3] Cima, J. and Ross, W., The Backward Shift on the Hardy Space, American Mathematical Society, Providence, RI, 2000.

[4] Douglas, R. G., Shapiro, H. S. and Shields, A. L., Cyclic vectors and invariant subspace for the backward shift invariant subspace, Annales de l'Institut Fourier, **20**, 1970, 37–76.

[5] Hoffman, K., Banach Spaces of Analytic Functions, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

[6] Jaming, P., Phase retrieval techniques for radar ambiguity problems, *Journal of Fourier Analysis and Applications,* **5**, 1999, 309–329.

[7] Lax, P. D., Translation invariant spaces, *Acta Mathematic,* **101**, 1959, 163–178.

[8] Qian, T., Chen, Q. and Tan, L., The rational orthogonal systems are Schauder bases, *Complex Variable and Elliptic Equations,* DOI: 10.1080/17476933.2013.787532, 2013.

[9] Stein, E. M. and Weiss, G., Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, Princeton, New Jersey, 1975.

[10] Tan, L. and Qian, T., Backward shift invariant subspace of band-limited signals and generalizations of Titchmarsh's theorem, manuscript, 2012.