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Complex Analysis and Operator Theory

Corner Boundary Value Problems

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Abstract Boundary value problems on a manifold with smooth boundary are closely related to the edge calculus where the boundary plays the role of an edge. The problem of expressing parametrices of Shapiro–Lopatinskij elliptic boundary value problems for differential operators gives rise to pseudo-differential operators with the transmission property at the boundary. However, there are interesting pseudo-differential operators without the transmission property, for instance, the Dirichlet-to-Neumann operator. In this case the symbols become edge-degenerate under a suitable quantisation, cf. Chang et al. (J Pseudo-Differ Oper Appl 5(2014):69–155, 2014). If the boundary itself has singularities, e.g., conical points or edges, then the symbols are corner-degenerate. In the present paper we study elements of the corresponding corner pseudo-differential calculus.

Keywords Corner pseudo-differential operators · Ellipticity of corner-degenerate operators · Meromorphic operator-valued symbols

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Contents

1 Introduction

Elliptic operators on a smooth manifold with boundary are determined by a principal symbolic hierarchy $\sigma = (\sigma_0, \sigma_1)$ where $\sigma_0 = \sigma_0(A)$ is the homogeneous principal symbol of the given elliptic operator A and $\sigma_1 = \sigma_1(A)$ the twisted homogeneous boundary symbol which is responsible for the boundary conditions. For instance, if $A = \Delta = \sum_{j=1}^{n} \partial^2/\partial^2_{x_j}$ is the Laplacian in the half-space $\mathbb{R}^n_+ = \{x = (x', x_n) : x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n > 0\}$, then we have $\sigma_0(A)(\xi) = -|\xi|^2$, considered for $\xi \neq 0$, and

$$\sigma_1(A)(\xi') = -|\xi'|^2 + \partial^2/\partial^2_{x_n} : H^s(\mathbb{R}_+) \to H^{s-2}(\mathbb{R}_+) \tag{1.1}$$

for $\xi' \neq 0$. Here ξ and ξ' are the covariables belonging to x and x', respectively; clearly, if A has variable coefficients, then we have $\sigma_0(A) = \sigma_0(A)(x, \xi)$ and $\sigma_1(A) = \sigma_1(A)(x', \xi')$. In (1.1) we assume an arbitrary $s > 3/2$. Then (1.1) is a family of Fredholm operators, even surjective in this case, and there are many choices of operator families

$$\sigma_1(T)(\xi') : H^s(\mathbb{R}_+) \to \mathbb{C}$$

which fill up (1.1) to a column matrix of isomorphisms

$$\sigma(\mathcal{A})(\xi') := \begin{pmatrix} \sigma_1(A) \\ \sigma_1(T) \end{pmatrix} (\xi') : H^s(\mathbb{R}_+) \to \begin{matrix} H^{s-2}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C} \end{matrix} .$$

For instance, for T we can take T_k, defined by $T_k u := (\partial/\partial_{x_n})^k u|_{x_n=0}$, corresponding to Dirichlet (for $k = 0$) or Neumann (for $k = 1$) conditions. There is also the famous category of mixed elliptic problems where the boundary is subdivided into submanifolds with smooth boundary, e.g.,

$$\mathbb{R}^{n-1} = \overline{\mathbb{R}}_-^{n-1} \cup \overline{\mathbb{R}}_+^{n-1} \tag{1.2}$$

for $\overline{\mathbb{R}}_-^{n-1} := \{x' = (x'', x_{n-1}) \in \mathbb{R}^{n-1} : x_{n-1} \leq 0\}$, $x'' = (x_1, \dots, x_{n-2})$, and $\overline{\mathbb{R}}_+^{n-1}$ determined by $x_{n-1} \geq 0$. Then $\mathbb{R}^{n-2} = \overline{\mathbb{R}}_-^{n-1} \cap \overline{\mathbb{R}}_+^{n-1}$ is the common boundary. In mixed boundary value problems we assume boundary conditions with a jump across $\mathbb{R}^{n-2}$, for instance, Dirichlet conditions on the minus and Neumann conditions on the plus side.

Reducing the Neumann problem to the boundary by means of the Dirichlet problem gives rise to a classical elliptic first order pseudo-differential operator on the Neumann side of the boundary which has not the transmission property at $\mathbb{R}^{n-2}$, see, for instance, [4]. A rigorous pseudo-differential calculus of boundary value problems in this case requires the edge calculus which treats the interface on the boundary as an edge. However, if the edge itself has singularities, then we have a case of corner singularities, and this is just the situation of the present paper. For instance, instead of (1.2) we can consider a decomposition

$$\mathbb{R}^{n-1} = M_- \cup M_+ \quad \text{for} \quad M_+ := \mathbb{R}^{n-3}_{x_1,\dots,x_{n-3}} \times I^\Delta,\ M_- := \mathbb{R}^{n-1} \backslash \operatorname{int} M_+, \tag{1.3}$$

where I^Δ is a cone in the (x_{n-2}, x_{n-1})-plane, for $(t, r) := (x_{n-2}, x_{n-1})$ defined by

$$I := \{(t, r) \in \mathbb{R}^2 : t = 1, 0 \leq r \leq 1\} \quad \text{and} \quad I^\Delta := \{(t, tr) : t \in \overline{\mathbb{R}}_+, 0 \leq r \leq 1\}. \tag{1.4}$$

In this case M_+ is a domain with boundary $\mathbb{R}^{n-3} \times \partial I^\Delta$ and edge $\mathbb{R}^{n-3}$. The cone I^Δ is regarded as a corner with two axial variables $t \in \overline{\mathbb{R}}_+$ and $0 \leq r \leq 1$, see also notation below in Sect. 2. The interval I is treated as a manifold with conical singularities $r = 0$ and $r = 1$. The task to establish an algebra of pseudo-differential operators with ellipticity and parametrices is voluminous. Therefore in this article we develop some typical elements of the general calculus. Examples and special cases will be investigated in a forthcoming paper. More ideas and motivation may also be found in [11].

This article is organised as follows. The material in Sects. 1 and 2 consists of necessary preparations of the iterative process of establishing pseudo-differential structures on higher singular configurations. In Sect. 2.1 we define a category of manifolds with second order singularities which contains, in particular, domains with non-smooth boundary, e.g., wedges as sketched before. In Sect. 2.2 we establish necessary tools on weighted Sobolev spaces with double weights, based on the Mellin transform and with a control at conical exits to infinity of the underlying configuration. Section 2.3 treats subspaces with iterated asymptotics, and we introduce Green symbols which play a role as specific operator-valued symbols in the corner pseudo-differential calculus. Section 3 is devoted to one of the crucial ingedients of the corner calculus, namely, operator-valued Mellin symbols with a control of asymptotics in corner axis direction, combined with asymptotics close to the conical singularities on the base I of the model cone of the wedge. In Sect. 4.1 we pass to the non-smoothing elements of the corner calculus, first to corner-degenerate differential operators and their principal symbolic hierarchies associated with the stratification of the underlying corner configuration. After that we consider corner-degenerate pseudo-differential symbols and construct various quantisations in form of operator-valued symbols with twisted homogeneity, referring to the spaces on the infinite stretched cone $I^\wedge$ from Sects. 2.2 and 2.3. The main new results of Sect. 4.1 are Proposition 4.7 and Theorem 4.8. Owing to the ideas of the iterative program they appear as natural generalisations of the first order edge calculus. In Sect. 4.2 we establish other essential structures of the corner pseudo-differential calculus, in particular, Theorems 4.9 and 4.12.

After the experience with pseudo-differential operators on manifolds with conical or edge singularities, see [16,20], or the monographs [21,22], the program of expressing parametrices to elliptic differential operators with some typical degenerate behaviour in stretched coordinates, creates a number of additional types of operators referring to the singularities or strata of the underlying configuration. Those are, for instance, Green, trace, and potential operators as they already appear in the solution process of classical elliptic boundary value problems, see, Boutet der Monvel [1], or Rempel and Schulze [14]. Another important class are Mellin operators. Specific operators of that kind have been discovered by Eskin [7] in connection with a pseudo-differential algebra generated by truncated operators on the half-axis. Mellin operators in more general form have been established in cone theories, cf. [16,23], and boundary value problems without the transmission property at the boundary, cf. [15,25], and later on in edge theories, see [20,22].

Another specific point are weighted cone and edge spaces and subspaces with asymptotics where the above-mentioned operators act in a natural way. In the edge situation the exponents in $r^p,\ p \in \mathbb{C}$, for the distance variable r to the singularity may be variable, and this requires adequate singular functions of such edge asymptotics and new elements of the Green and Mellin calculus. Variable asymptotics in that sense have been studied in general form in [21]. Since then this concept is integrated in the subsequent development under the key-words variable discrete and continuous asymptotics, see, in particular, [26,28], and the references there.

All these aspects formulate in advance the structure of parametrices and regularity properties of solutions to elliptic equations on a singular manifold, also on manifolds with higher edges and corners. Because of the extent of such a program here we confine ourselves to a part of the new structures that participate in parametrices and regularitiy for boundary value problems on corner manifolds.

2 Weighted Spaces on Manifolds with Boundary and Edge

2.1 Singular Manifolds

Let M be a stratified space, in our case a disjoint union

$$M = s_0(M) \cup s_1(M) \cup s_2(M)$$

of strata $s_j(M) \subset M,\ j = 0, 1, 2,$ which are embedded smooth manifolds,

$$\dim s_0(M) = 2 + d, \dim s_1(M) = 1 + d, \dim s_2(M) = d$$

for some $d \in \mathbb{N}\backslash\{0\}$. Here $M\backslash s_2(M)$ is a smooth manifold with boundary $\partial(M\backslash s_2(M)) = s_1(M)$, $s_0(M) = \text{int}\,(M\backslash s_2(M))$, and $s_2(M) =: Z$ is an edge of M. We assume that Z has a neighbourhood V in M with the structure of a locally trivial I^Δ-bundle over Z. Here $I := \{r \in \overline{\mathbb{R}}_+ : 0 \leq r \leq 1\}$ is the unit interval and

$$I^\Delta := (\overline{\mathbb{R}}_+ \times I)/(\{0\} \times I\})$$

the infinite straight cone with base I. The assumed length of the interval is unessential; we could take an interval $\{c_0 \leq r \leq c_1\}$ for any $c_0 < c_1$ as well. We often consider the stretched cones

$$I^\wedge := \mathbb{R}_+ \times I, \quad \overline{I^\wedge} := \overline{\mathbb{R}}_+ \times I$$

with the splitting of variables (t, r) and the stretched wedges $I^\wedge \times \mathbb{R}^d$, $\overline{I^\wedge} \times \mathbb{R}^d$ in the variables (t, r, z).

Incidentally the stratification of M will be indicated by the sequence of strata

$$s(M) := (s_0(M), s_1(M), s_2(M)). \tag{2.1}$$

An example is the wedge $M = I^\Delta \times \mathbb{R}^d$. In this case we have $s_0(M) = \text{int}\, I^\wedge \times \mathbb{R}^d$, $s_1(M) = \partial I^\wedge \times \mathbb{R}^d$, and $s_2(M) = \mathbb{R}^d$. The boundary $\partial I^\wedge$ has two components

$$\partial^0 I^\wedge, \quad \partial^1 I^\wedge \tag{2.2}$$

that are copies of $\mathbb{R}_+$, associated with $\partial I = \{0, 1\}$.

With the above-mentioned V we can also associate an $\overline{I^\wedge}$-bundle over Z, i.e., a locally trivial bundle with fibre $\overline{I^\wedge}$. This contains corresponding $I^\wedge$- and I-subbundles. The transitions of fibres of the $\overline{I^\wedge}$-bundle are defined as homeomorphisms $\overline{\mathbb{R}}_+ \times I \to \overline{\mathbb{R}}_+ \times I$ that are restrictions of diffeomorphisms $\mathbb{R} \times I \to \mathbb{R} \times I$ (as smooth manifolds with boundary) to $\overline{\mathbb{R}}_+ \times I$.

In the case $M = I^\Delta \times \mathbb{R}^d$ the $\overline{I^\wedge}$-bundle is trivial, namely, $\overline{I^\wedge} \times \mathbb{R}^d$, and it contains the trivial subbundles $I^\wedge \times \mathbb{R}^d$ and $I \times \mathbb{R}^d$. The space $\mathbb{M} := \overline{I^\wedge} \times \mathbb{R}^d$ plays the role of the stretched manifold associated with M. It is obtained from M by attaching the I-bundle $I \times \mathbb{R}^d$ to $M\backslash Z$.

For general M we obtain the stretched manifold $\mathbb{M}$ by invariantly attaching the above-mentioned I-bundle V over Z to $M\backslash Z$.

For purposes below we call a trivialisation of V over a coordinate neighbourhood $D \subset Z$ a singular chart

$$\chi : V|_D \to I^\Delta \times \mathbb{R}^d.$$

This is considered together with a chart $\chi_0 : D \to \mathbb{R}^d$ on Z such that $\chi_0 \circ \pi = \pi \circ \chi$ with π being the respective bundle projection. The restriction of χ to $V|_D\backslash Z$ gives rise to a map

$$\chi_{\text{st}} : V|_D\backslash Z \to \mathbb{R}_+ \times I \times \mathbb{R}^q \tag{2.3}$$

and to a local splitting of variables $(t, r, z) \in \mathbb{R}_+ \times I \times \mathbb{R}^q$.

Remark 2.1 Our space M will also be interpreted as a manifold with boundary $\partial M := \partial(M \backslash Z) \cup Z$ where ∂M is a manifold with edge Z. The program of the analysis here is to perform a calculus of boundary value problems for pseudo-differential operators that do not necessarily have the transmission property at $\partial(M \backslash Z)$. This requires a suitable corner pseudo-differential approach. According to (2.1) the operators A in this calculus have a principal symbolic hierarchy

$$\sigma(A) := (\sigma_0(A), \sigma_1(A), \sigma_2(A)). \tag{2.4}$$

This will be developed below.

The space M with the stratification (2.1) belongs to the category $\mathfrak{M}_2$ of manifolds with second order singularities, in the terminology of [24]. While $\mathfrak{M}_0$ indicates smoothness, $\mathfrak{M}_1$ is the category of manifolds with conical singularities or edge. The elements $B \in \mathfrak{M}_1$ have a stratification

$$s(B) = (s_0(B), s_1(B))$$

with $Y := s_1(B) \in \mathfrak{M}_0$ being the conical singularity or edge of B and $s_0(B) := B \backslash s_1(B) \in \mathfrak{M}_0$ the main stratum. It is assumed that Y has a neighbourhood $W \subset B$ with the structure of a locally trivial X^Δ-bundle over Y for some $X \in \mathfrak{M}_0$. Let $\pi : W \to Y$ be the bundle projection. Trivialisations

$$\chi : W|_G \to X^\Delta \times \mathbb{R}^q,$$

$q := \dim Y$, belonging to charts $\chi_0 : G \to \mathbb{R}^q$ on Y (where $\chi_0 \circ \pi = \pi \circ \chi$) will be referred to as singular charts on B. The restriction of χ to $W|_G \backslash Y$ gives rise to a map

$$\chi_{\mathrm{st}} : W|_G \backslash Y \to \mathbb{R}_+ \times X \times \mathbb{R}^d \tag{2.5}$$

and to a local splitting of variables $(r, x, y) \in \mathbb{R}_+ \times X \times \mathbb{R}^d$.

Similarly as before the X^Δ-bundle over Y can be considered together with an $\overline{\mathbb{R}}_+ \times X$ bundle over Y. This contains an X-bundle W' over Y as a subbundle. It can be invariantly attached to $B \backslash Y$, and we then obtain the stretched manifold $\mathbb{B}$ associated with B. Then $\mathbb{B}$ is a manifold with smooth boundary $\partial\mathbb{B} = W'$. An example is the case $B := X^\Delta \times \mathbb{R}^q$ which can be identified with W. Moreover, $\mathbb{B} = \overline{\mathbb{R}}_+ \times X \times \mathbb{R}^q$, and $W' = X \times \mathbb{R}^q$.

2.2 Weighted Corner Spaces

Let us now establish some tools on weighted corner Sobolev spaces. Consider the Mellin transform

$$Mu(w) := \int_0^\infty r^{w-1} u(r) dr,$$

first for $u \in C_0^\infty(\mathbb{R}_+)$, with the inverse $(M^{-1}g)(r) = \int_{\Gamma_\beta} r^{-w} g(w) đw$, $đw := (2\pi i)^{-1} dw$. Here

$$\Gamma_\beta := \{w \in \mathbb{C} : \operatorname{Re} w = \beta\}$$

for some real β. Incidentally, in order to indicate the variable r and its covariable $w \in \mathbb{C}$ in the Mellin transform we also write $M_{r\to w}$ rather than M. Extending the Mellin transform to, say, $r^\gamma L^2(\mathbb{R}_+)$, $\gamma \in \mathbb{R}$, then we take $\beta = 1/2 - \gamma$. In this case M induces an isomorphism

$$M_\gamma : r^\gamma L^2(\mathbb{R}_+) \to L^2(\Gamma_{1/2-\gamma}),$$

and M_γ is called the weighted Mellin transform with weight γ. The weighted Mellin Sobolev space $\mathcal{H}^{s,\gamma_1}(\mathbb{R}_+)$ of smoothness s and weight γ_1 is defined as the completion of $C_0^\infty(\mathbb{R}_+)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma_1}(\mathbb{R}_+)} = \left\{ \int_{\Gamma_{1/2-\gamma_1}} \langle w\rangle^{2s} |(M_{r\to w}u)(w)|^2 đw \right\}^{1/2},$$

$s, \gamma_1 \in \mathbb{R}$.

Similar spaces will play a role with respect to a second half-axis variable t and its Mellin covariable $v \in \mathbb{C}$, and a weight γ_2. We define the space

$$\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+ \times \mathbb{R}^n)$$

for some $n \in \mathbb{N}$ as the completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+\times\mathbb{R}^n)} = \left\{ \int_{\mathbb{R}^n} \int_{\Gamma_{(1+n)/2-\gamma_2}} \langle v, \xi\rangle^{2s} |(M_{t\to v} F_{x\to\xi} u)(v, \xi)|^2 đv đ\xi \right\}^{1/2} \tag{2.6}$$

with $F_{x\to\xi}$ being the Fourier transform in $\mathbb{R}^n \ni x$. Then for any closed C^∞ manifold X we have the space $\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+ \times X)$ with the norm

$$\|u\|_{\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+\times X)} = \left\{ \sum_{j=1}^N \|\varphi_j u \circ (\mathrm{id}_{\mathbb{R}_+} \times \chi_j)^{-1}\|^2_{\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+\times\mathbb{R}^n)} \right\}^{1/2}.$$

Here $\chi_j : U_j \to \mathbb{R}^n$, $j = 1, \dots, N$, are charts for an open covering of X by coordinate neighbourhoods $\{U_1, \dots, U_N\}$, and $\{\varphi_1, \dots, \varphi_N\}$ is a subordinate partition of unity.

If a Fréchet space E is a left module over an algebra A then we set $[a]E :=$ closure of $\{ae : e \in E\}$ in E. Moreover, if E_0, E_1 are Fréchet spaces, embedded in a Hausdorff topologial vector space, we define the non-direct sum $E_0 + E_1$ in the Fréchet topology from the identification $E_0 + E_1 \cong E_0 \oplus E_1/\Delta$ for $\Delta := \{(e, -e) :$

$e \in E_0 \cap E_1\}$. In particular, the non-direct sum of Hilbert spaces is again a Hilbert space as the orthogonal complement of Δ in the direct sum.

We define

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_+) := \big\{\omega u + (1-\omega)v : u \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+),\, v \in H^s(\mathbb{R}_+)\big\}, \tag{2.7}$$

$s, \gamma \in \mathbb{R}$, where ω is a cut-off function on the r half-axis, i.e., $\omega \in C^\infty(\overline{\mathbb{R}}_+)$ real-valued, $\omega = 1$ close to $r = 0$, $\omega = 0$ for r off some neighbourhood of $r = 0$. The choice of ω is not essential for (2.7). However, we fix ω and endow the space with the Hilbert space structure of the non-direct sum

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_+) = [\omega]\mathcal{H}^{s,\gamma}(\mathbb{R}_+) + [1-\omega]H^s(\mathbb{R}_+).$$

Moreover, let

$$\mathcal{K}^{s,\gamma;e}(\mathbb{R}_+) := \langle r\rangle^{-e}\mathcal{K}^{s,\gamma}(\mathbb{R}_+) \tag{2.8}$$

for any $s, \gamma, e \in \mathbb{R}$. For $s = \gamma = e = 0$ we have natural identifications

$$\mathcal{K}^{0,0;0}(\mathbb{R}_+) = \mathcal{K}^{0,0}(\mathbb{R}_+) = \mathcal{H}^{0,0}(\mathbb{R}_+) = L^2(\mathbb{R}_+). \tag{2.9}$$

The $\mathcal{K}^{0,0;0}(\mathbb{R}_+)$-scalar product induces non-degenerate sesquilinear pairings

$$\mathcal{K}^{s,\gamma;e}(\mathbb{R}_+) \times \mathcal{K}^{-s,-\gamma;-e}(\mathbb{R}_+) \to \mathbb{C} \quad \text{and} \quad \mathcal{H}^{s,\gamma}(\mathbb{R}_+) \times \mathcal{H}^{-s,-\gamma}(\mathbb{R}_+) \to \mathbb{C} \tag{2.10}$$

for every $s, \gamma, e \in \mathbb{R}$.

Note that the dilation operator $\iota_\delta : u(r) \mapsto u(\delta r)$, $\delta \in \mathbb{R}_+$, acts both on $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$, $H^s(\mathbb{R}_+)$, and $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ or $\mathcal{K}^{s,\gamma;e}(\mathbb{R}_+)$. Moreover, $\partial_r^j = (\partial/\partial r)^j$ induces continuous operators

$$\begin{aligned}\partial_r^j : \mathcal{H}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{H}^{s-j,\gamma-j}(\mathbb{R}_+),\ H^s(\mathbb{R}_+) \to H^{s-j}(\mathbb{R}_+),\\ \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}^{s-j,\gamma-j}(\mathbb{R}_+)\end{aligned}$$

where

$$\partial_r^j = \delta^j \iota_\delta\, \partial_r^j\, \iota_\delta^{-1},\ \delta \in \mathbb{R}_+.$$

For $s \in \mathbb{N}$ we have an equivalence of norms

$$\|u\|_{\mathcal{K}^{s,\gamma}(\mathbb{R}_+)} \sim \big\{\|u\|^2_{\mathcal{K}^{0,\gamma}(\mathbb{R}_+)} + \|\partial_r^s u\|^2_{\mathcal{K}^{0,\gamma-s}(\mathbb{R}_+)}\big\}^{1/2}. \tag{2.11}$$

More generally, if X is a closed C^∞ manifold we define

$$\mathcal{K}^{s,\gamma}(X^\wedge) := [\omega]\mathcal{H}^{s,\gamma}(X^\wedge) + [1-\omega]H^s_{\text{cone}}(X^\wedge). \tag{2.12}$$

Here $H^s_{\text{cone}}(X^\wedge)$ is the set of all $u \in H^s_{\text{loc}}(\mathbb{R} \times X)|_{\mathbb{R}_+\times X}$ such that for any chart $\chi : U \to \mathbb{R}^n$ on X and $\beta : \mathbb{R}_+ \times U \to \mathbb{R}^{1+n}$ defined by $\beta(r, x) = (r, r\chi(x))$ we have $(1-\omega)\varphi u \circ \beta^{-1} \in H^s(\mathbb{R}^{1+n})$, for any $\varphi \in C_0^\infty(U)$ and a cut-off function ω on the r half-axis. There is an anologue of the relation (2.11) for the spaces (2.12), cf. [27, Proposition 1.2].

For a function in $(r, x) \in X^\wedge$ we set

$$({}^1\kappa_\delta)u(r,x) := \delta^{(n+1)/2} u(\delta r, x), \quad \delta \in \mathbb{R}_+. \tag{2.13}$$

This is a group action on the space $\mathcal{K}^{s,\gamma}(X^\wedge)$ in the following sense. A Hilbert space H is said to be endowed with a group action $\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$, if $\kappa_\delta : H \to H$ is an isomorphism for every δ, moreover, $\kappa_\delta \kappa_\nu = \kappa_{\delta\nu}$, $\delta, \nu \in \mathbb{R}_+$, and if $\delta \to \kappa_\delta h$ defines an element of $C(\mathbb{R}_+, H)$ for every $h \in H$.

Now if H is a Hilbert space with group action, then

$$\mathcal{W}^s(\mathbb{R}^q, H), \tag{2.14}$$

$s \in \mathbb{R}$, is defined as the completion of $C_0^\infty(\mathbb{R}^q, H)$ with respect to the norm

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q,H)} = \left\{ \int \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle} \hat{u}(\eta)\|^2_H đ\eta \right\}^{1/2},$$

for $đ\eta := (2\pi)^{-q} d\eta$ and the Fourier transform $\hat{u}(\eta) = F_{y\to\eta} u(\eta)$ in $\mathbb{R}^q$.

Clearly the spaces (2.14) depend on the choice of κ. If necessary we write

$$\mathcal{W}^s(\mathbb{R}^q, H)_\kappa$$

rather than (2.14).

It can be easily verified that $\mathcal{W}^s(\mathbb{R}^q, H)_\kappa \subset \mathcal{S}'(\mathbb{R}^q, H)$. Analogously as in notation for standard Sobolev spaces for any open set $\Omega \subseteq \mathbb{R}^q$ we have the spaces

$$\mathcal{W}^s_{\mathrm{comp}}(\Omega, H)_\kappa \quad \text{and} \quad \mathcal{W}^s_{\mathrm{loc}}(\Omega, H)_\kappa$$

where $\mathcal{W}^s_{\mathrm{comp}}(\Omega, H)_\kappa$ consists of all elements of $\mathcal{W}^s(\mathbb{R}^q, H)_\kappa$ which have compact support in Ω, while $\mathcal{W}^s_{\mathrm{loc}}(\Omega, H)_\kappa$ is the space of those $u \in \mathcal{D}'(\Omega, H)$ such that $\varphi u \in \mathcal{W}^s(\mathbb{R}^q, H)_\kappa$ for every $\varphi \in C_0^\infty(\Omega)$.

Recall from [21] that a motivation of the definition of (2.14) is the anisotropic reformulation of standard Sobolev spaces $H^s(\mathbb{R}^m \times \mathbb{R}^q)$ over a Cartesian product $\mathbb{R}^m \times \mathbb{R}^q \ni (x, y)$ as

$$H^s(\mathbb{R}^m \times \mathbb{R}^q) = \mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^m))_\kappa \quad \text{for} \quad (\kappa_\delta u)(x) = \delta^{m/2} u(\delta x), \ \delta \in \mathbb{R}_+. \tag{2.15}$$

More generally we have the following iterative property.

Proposition 2.2 [21] *Let H be a Hilbert space with group action $\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$. Then also $\mathcal{W}^s(\mathbb{R}^q, H)_\kappa$ is a Hilbert space with group action $\chi = \{\chi_\delta\}_{\delta\in\mathbb{R}_+}$ for $(\chi_\delta u)(y) := \delta^{q/2}\kappa_\delta u(\delta y)$ where κ_δ acts on the values of u in H, and for every $p \in \mathbb{N}$ we have*

$$\mathcal{W}^s(\mathbb{R}^p, \mathcal{W}^s(\mathbb{R}^q, H)_\kappa)_\chi = \mathcal{W}^s(\mathbb{R}^{p+q}, H)_\kappa.$$

Remark 2.3 Let $\mathbb{R}^q_+$ be the half-space in $\mathbb{R}^q \ni y = (y_1, \dots, y_q)$, defined by $y_q > 0$. Analogously we define $\overline{\mathbb{R}}^q_+$, $\mathbb{R}^q_-$ and $\overline{\mathbb{R}}^q_-$ by $y_q \geq 0$, $y_q < 0$, and $y_q \leq 0$, respectively. Setting

$$\mathcal{W}^s(\mathbb{R}^q_+, H) := \mathcal{W}^s(\mathbb{R}^q, H)|_{\mathbb{R}^q_+},\ \mathcal{W}^s_0(\overline{\mathbb{R}}^q_-, H)) := \{u \in \mathcal{W}^s(\mathbb{R}^q, H)) : \operatorname{supp} u \subseteq \overline{\mathbb{R}}^q_-\}$$

we have a natural identification

$$\mathcal{W}^s(\mathbb{R}^q_+, H) = \mathcal{W}^s(\mathbb{R}^q, H)/\mathcal{W}^s_0(\overline{\mathbb{R}}^q_-, H),$$

and both $\mathcal{W}^s(\mathbb{R}^q_+, H)$ and $\mathcal{W}^s_0(\overline{\mathbb{R}}^q_-, H)$ are Hilbert spaces with group action, induced by $\chi = \{\chi_\delta\}_{\delta \in \mathbb{R}_+}$ of Proposition 2.2. The group action $\mathcal{W}^s_0(\overline{\mathbb{R}}^q_-, H)$ is simply the restriction of χ to the subspace of elements supported by $\overline{\mathbb{R}}^q_-$, while that on $\mathcal{W}^s(\mathbb{R}^q_+, H)$ is the corresponding quotient map.

Remark 2.4 It is necessary to formulate more results on abstract wedge spaces $\mathcal{W}^s(\mathbb{R}^q, H)$ for Hilbert spaces H with group action κ in general. In our applications we have in mind more specific spaces, such as weighted cone Sobolev spaces $H := \mathcal{K}^{s,\gamma}(X^\wedge)$, etc. Also Fréchet subspaces with group action will be of interest. The following invariance property under diffeomorphisms is valid for the concrete spaces of our applications, cf. [22, Theorem 3.1.29]. Let $\Omega, \tilde{\Omega} \subseteq \mathbb{R}^q$ be open sets and $\chi : \Omega \to \tilde{\Omega}$ a diffeomorphism. Then the pull back χ^* induces isomorphisms

$$\chi^* : \mathcal{W}^s_{\mathrm{comp}}(\tilde{\Omega}, H) \to \mathcal{W}^s_{\mathrm{comp}}(\Omega, H), \quad \mathcal{W}^s_{\mathrm{loc}}(\tilde{\Omega}, H) \to \mathcal{W}^s_{\mathrm{loc}}(\Omega, H)$$

for every $s \in \mathbb{R}$.

Let us consider the space (2.14) for $q = 1$. Since $\mathcal{W}^s(\mathbb{R}, H) \subset \mathcal{S}'(\mathbb{R}, H)$ it makes sense to form $\mathcal{W}^s(\mathbb{R}_+, H) := \mathcal{W}^s(\mathbb{R}, H)|_{\mathbb{R}_+}$. Moreover, let $\mathcal{W}^s_0(\overline{\mathbb{R}}_-, H) := \{u \in \mathcal{W}^s(\mathbb{R}, H) : \operatorname{supp} u \subseteq \overline{\mathbb{R}}_-\}$. The latter space is closed in $\mathcal{W}^s(\mathbb{R}, H)$, and we have a canonical identification

$$\mathcal{W}^s(\mathbb{R}_+, H) = \mathcal{W}^s(\mathbb{R}, H)/\mathcal{W}^s_0(\overline{\mathbb{R}}_-, H). \tag{2.16}$$

Notation with calligraphic letters such as $\mathcal{H}^{s,\gamma_1}(\mathbb{R}_+)$, $\mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)$, $\mathcal{W}^s(\mathbb{R}^q, H)$, etc., indicate a situation where the underlying manifold such as $\mathbb{R}_+$ or $\mathbb{R}^q$ affects properties 'up to the non-compacts ends' of the configuration, e.g., up to $r \to 0$, $r \to \infty$, or $|y| \to \infty$. However, if such aspects are not in the focus of considerations we prefer notation similar to standard Sobolev spaces.

An example are the following spaces on the interval I, regarded as a compact manifold with conical singularities $r = 0$ and $r = 1$, namely,

$$H^{s,\gamma_{1,0},\gamma_{1,1}}(I) := [\omega_0]\mathcal{H}^{s,\gamma_{1,0}}(\mathbb{R}_+) + \vartheta^*[\omega_1]\mathcal{H}^{s,\gamma_{1,1}}(\mathbb{R}_+) \ \text{ for } s, \gamma_{1,0}, \gamma_{1,1} \in \mathbb{R}, \tag{2.17}$$

defined by

$$\vartheta : {}^1\overline{\mathbb{R}}_- := \{r \in \mathbb{R} : -\infty < r \leq 1\} \to \overline{\mathbb{R}}_+,\ \vartheta(r) = -r + 1, \tag{2.18}$$

and cut-off functions ω_0, ω_1 on the half-axis such that $\omega_0(r) + \omega_1(-r+1) = 1$ for all $r \in I$. For convenience from now on we assume that the weights at the end points of I are equal, i.e., $\gamma_{1,0} = \gamma_{1,1}$. The generalisation to different $\gamma_{1,0}, \gamma_{1,1}$ is straightforward.

Definition 2.5 Let B be a manifold with edge Y (not necessarily compact). Then $H^{s,\gamma_1}_{[\mathrm{loc})}(B)$ for $s, \gamma_1 \in \mathbb{R}$ is defined as the set of all $u \in H^s_{\mathrm{loc}}(B\setminus Y)$ such that for any singular chart

$$\chi : W|_G \to X^\Delta \times \mathbb{R}^q$$

belonging to a chart $\chi_0 : G \to \mathbb{R}^q$ on Y and

$$\chi_{\mathrm{st}} := \chi|_{W|_G\setminus Y} : W|_G\setminus Y \to X^\wedge \times \mathbb{R}^q$$

we have

$$(\chi_{\mathrm{st}}^{-1})^* \sigma u \in \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma_1}(X^\wedge))_{1\,\kappa}$$

for any $\sigma \in C^\infty(\mathbb{B})$ of the form $\sigma = \chi^*_{\mathrm{st}}\sigma_0$ for some cut-off function σ_0 on the half-axis.

Let us now recall a few notions on operator-valued symbols with twisted symbolic estimates that we also need later on in connection with edge amplitude functions of second singularity order.

Given Hilbert spaces H and $\widetilde{H}$ with group action κ and $\tilde{\kappa}$, respectively, by $S^\mu(U \times \mathbb{R}^q; H, \widetilde{H})$ for $\mu \in \mathbb{R}$ and open $U \subseteq \mathbb{R}^p$ we denote the set of all $a(y,\eta) \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(H, \widetilde{H}))$ such that

$$\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})} \leq c\langle\eta\rangle^{\mu-|\beta|} \tag{2.19}$$

for all $(y,\eta) \in K \times \mathbb{R}^q,\ K \Subset U,\ \alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^q$, for constants $c = c(K,\alpha,\beta) > 0$. Moreover, let

$$S^{(\nu)}(U \times (\mathbb{R}^q\setminus\{0\}); H, \widetilde{H}), \tag{2.20}$$

$\nu \in \mathbb{R}$, be the space of all $a_{(\nu)}(y,\eta) \in C^\infty(U \times (\mathbb{R}^q\setminus\{0\}), \mathcal{L}(H,\widetilde{H}))$ such that

$$a_{(\nu)}(y,\delta\eta) = \delta^\nu \tilde{\kappa}_\delta a_{(\nu)}(y,\eta)\kappa^{-1}_\delta$$

for all $\delta \in \mathbb{R}_+$. Then $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; H, \widetilde{H}) \subset S^\mu(U \times \mathbb{R}^q; H, \widetilde{H})$, the set of classical elements $a(y,\eta)$, is defined by the condition

$$a(y,\eta) - \sum_{j=0}^{N} \chi(\eta) a_{(\mu-j)}(y,\eta) \in S^{\mu-(N+1)}(U \times \mathbb{R}^q; H, \widetilde{H})$$

for suitable $a_{(\mu-j)}(y,\eta) \in S^{(\mu-j)}(U \times (\mathbb{R}^q \setminus \{0\}); H, \widetilde{H}),\ j \in \mathbb{N}$, for every $N \in \mathbb{N}$. Here χ is an excision function (i.e., an element of $C^\infty(\mathbb{R}^q_\eta)$ which is equal to 0 for $|\eta| < \varepsilon_0$ and equal to 1 for $|\eta| > \varepsilon_1$, for some $0 < \varepsilon_0 < \varepsilon_1$). Clearly the spaces $S^\mu(U \times \mathbb{R}^q; H, \widetilde{H})$ depend on the choice of $\kappa, \tilde{\kappa}$. Also the notion of homogeneous components in classical symbols can depend on the group actions.

Remark 2.6 Let $a(y,\eta) \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(H, \widetilde{H}))$ and $a(y, \delta\eta) = \delta^\mu \tilde{\kappa}_\delta a(y,\eta) \kappa_\delta^{-1}$ for all $\delta \geq 1$ and $|\eta| \geq C$ for some $C > 0$. Then we have $a(y,\eta) \in S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; H, \widetilde{H})$.

For any $a(y, y', \eta) \in S^\mu(\Omega \times \Omega \times \mathbb{R}^q; H, \widetilde{H}),\ \Omega \subseteq \mathbb{R}^q$ open, we set

$$\mathrm{Op}(a)u(y) := \iint e^{i(y-y')\eta} a(y, y', \eta) u(y') dy' đ\eta,$$

for $u \in C_0^\infty(\Omega, H)$. There are many types of continuity results for operators $\mathrm{Op}(a)$. For instance, we have continuity of

$$\mathrm{Op}(a) : C_0^\infty(\Omega, H) \to C^\infty(\Omega, \widetilde{H}),\ \mathcal{W}^s_{\mathrm{comp}}(\Omega, H) \to \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \widetilde{H}),\ s \in \mathbb{R}, \tag{2.21}$$

or, when $a = a(\eta)$ has constant coefficients,

$$A := \mathrm{Op}(a) : \mathcal{W}^s(\mathbb{R}^q, H) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, \widetilde{H}),\ s \in \mathbb{R}. \tag{2.22}$$

Concerning more subtle cases, see, e.g., [21,30]. If a consideration is valid both for classical and general symbols we write subscripts "(cl)".

Let us assume $\Omega = \mathbb{R}^q$ and $a(\eta) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^q_\eta; H, \widetilde{H})$. Then a simple computation shows that

$$\|A\|_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^q, H), \mathcal{W}^{s-\mu}(\mathbb{R}^q, \widetilde{H}))} = \sup_{\eta \in \mathbb{R}^q} \langle \eta \rangle^{-\mu} \|a(\eta)\|_{\mathcal{L}(H, \widetilde{H})}. \tag{2.23}$$

Let $\psi_R(\theta)$ be in $C_0^\infty(\mathbb{R}^q_\theta)$, and $\psi_R(\theta) \equiv 1$ for $|\theta| \leq R/2$, $\psi_R(\theta) \equiv 0$ for $|\theta| \geq R/2$. Setting $a_0(y, y', \eta) := \psi_R(y - y')a(\eta)$ the operator $A_R := \mathrm{Op}(a_0)$ is properly supported. In addition

$$A_R : \mathcal{W}^s(\mathbb{R}^q, H) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, \widetilde{H}) \tag{2.24}$$

is continuous for every $s \in \mathbb{R}$. In fact, let us set

$$k(a_0)(\theta) = \int e^{i\theta\eta} \psi_R(\theta) a(\eta) đ\eta.$$

Then $k(a_0)(y - y')$ is the distributional kernel of A_R, and we can write

$$A_R = A + C_R$$

for $A = \mathrm{Op}(a)$ and

$$C_R u(y) := \iint e^{i(y-y')\eta} (\psi_R(y - y') - 1) a(\eta) u(y') dy' đ\eta.$$

We can write $C_R = \mathrm{Op}(c_R)$ for $c_R(\eta) = \int e^{-i\theta\eta}k(c_R)(\theta)d\theta$ where $k(c_R)(\theta) = \int e^{i\theta\eta}(\psi_R(\theta) - 1)a(\eta)d\eta$. We have $c_R(\eta) \in \mathcal{S}(\mathbb{R}^q_\eta, \mathcal{L}(H, \widetilde{H})) = S^{-\infty}(\mathbb{R}^q; H, \widetilde{H})$. Since $\mathrm{Op}(c_R) : \mathcal{W}^s(\mathbb{R}^q, H) \to \mathcal{W}^{-\infty}(\mathbb{R}^q, \widetilde{H})$ is continuous for every s, and because of the continuity of (2.22) we also obtain the continuity of (2.24).

Note that for $a_R(\eta) := \int e^{-i\theta\eta}k(a_0)(\theta)d\theta \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; H, \widetilde{H})$ we have $A_R = \mathrm{Op}(a_R)$, and also this gives us the continuity of (2.24).

Lemma 2.7 *We have*

$$a_R(\eta) \to a(\eta) \quad \textit{for } R \to \infty$$

in $S^\mu_{(cl)}(\mathbb{R}^q; H, \widetilde{H})$, *and hence*

$$Op(a_R) \to Op(a) \quad \textit{for } R \to \infty$$

in $\mathcal{L}(\mathcal{W}^s(\mathbb{R}^q, H), \mathcal{W}^{s-\mu}(\mathbb{R}^q, \widetilde{H}))$.

This result is known in the context of kernel cut-off operators, cf. [22, Remark 1.1.51].

Proposition 2.8 *Let* $a \in S^\mu(\mathbb{R}^q; H, \widetilde{H})$, *and assume that*

$$a(\eta) : H \to \widetilde{H} \quad \textit{for all } \eta \in \mathbb{R}^q$$

defines isomorphisms, and $a^{-1} \in S^{-\mu}(\mathbb{R}^q; H, \widetilde{H})$. *Then for every* $s \in \mathbb{R}$

(i)

$$Op(a) : \mathcal{W}^s(\mathbb{R}^q, H) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, \widetilde{H})$$

is an isomorphism;

(ii) *there is an* $R_1 > 0$ *such that for all* $R \geq R_1$ *both*

$$Op(a_R) : \mathcal{W}^s(\mathbb{R}^q, H) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, \widetilde{H}) \tag{2.25}$$

and

$$Op(a_R) : \mathcal{W}^s_{comp}(\mathbb{R}^q, H) \to \mathcal{W}^{s-\mu}_{comp}(\mathbb{R}^q, \widetilde{H}) \tag{2.26}$$

are isomorphisms.

Proof (i) follows from $\mathrm{Op}^{-1}(a) = \mathrm{Op}(a^{-1})$.

(ii) is a consequence of (i) together with the convergence $\mathrm{Op}(a_R) \to \mathrm{Op}(a)$ in the space $\mathcal{L}(\mathcal{W}^s(\mathbb{R}^q, H), \mathcal{W}^{s-\mu}(\mathbb{R}^q, \widetilde{H}))$ for $R \to \infty$. In fact, for sufficiently large R the operator (2.25) is an isomorphism, since isomorphisms form an open set in $\mathcal{L}(\mathcal{W}^s(\mathbb{R}^q, H), \mathcal{W}^{s-\mu}(\mathbb{R}^q, \widetilde{H}))$, but $\mathrm{Op}(a_R)$ is properly supported for every $R > 0$ and hence defines a map (2.26) which is obviously bijective when R is sufficiently large. □

For $H = \widetilde{H} = \mathbb{C}$ and trivial group actions, i.e., $\kappa_\delta = \tilde{\kappa}_\delta = \mathrm{id}_{\mathbb{C}}$ for all $\delta \in \mathbb{R}_+$ we recover the scalar symbol spaces $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q)$.

Let $S^\mu_{\mathcal{O}}$ be defined as the set of all $h \in \mathcal{A}(\mathbb{C})$:= the space of entire functions in the complex variable w, such that $h|_{\Gamma_\beta} \in S^\mu_{(\mathrm{cl})}(\Gamma_\beta)$ for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals.

For any $h(r, w) \in C^\infty(\mathbb{R}_+, S^\mu_{\mathcal{O}})$ we set

$$\mathrm{op}^\beta_M(h) = r^\beta \mathrm{op}_M(T^{-\beta}h)r^{-\beta} \tag{2.27}$$

for $(T^{-\beta}h)(r, w) := h(r, w-\beta)$, $\beta \in \mathbb{R}$, where $\mathrm{op}_M(f)u = M^{-1}_{r\to w} f(r, w)(M_{r\to w})$.

Consider an edge-degenerate symbol $p(r, \rho) \in S^\mu_{\mathrm{cl}}(\mathbb{R}_+ \times \mathbb{R})$, i.e., $p(r, \rho) = \tilde{p}(r, r\rho)$ for a $\tilde{p}(r, \tilde{\rho}) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_{+,r} \times \mathbb{R}_{\tilde{\rho}})$. Then a quantisation result, cf. [22, Theorem 3.2.7], tells us that there is an $h(r, w) \in C^\infty(\overline{\mathbb{R}}_+, S^\mu_{\mathcal{O}})$ such that

$$\mathrm{op}^\beta_M(h) = \mathrm{Op}_r(p),$$

modulo an operator with kernel in $C^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$, for every $\beta \in \mathbb{R}$. We then call h a Mellin quantisation of p.

Theorem 2.9 [17,22, Theorem 3.1.27, Remark 3.1.28] *There exists an operator $A \in L^\mu_{cl}(\mathbb{R}_+)$ which induces an isomorphism*

$$A : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}^{s-\mu,\beta}(\mathbb{R}_+)$$

for every $s \in \mathbb{R}$ and prescribed $\gamma, \beta \in \mathbb{R}$ where

$$\iota_\delta A \iota_\delta^{-1} \in C^\infty(\mathbb{R}_{+,\delta}, \mathcal{L}(\mathcal{K}^{s,\gamma}(\mathbb{R}_+), \mathcal{K}^{s-\mu,\beta}(\mathbb{R}_+)))$$

for every $s \in \mathbb{R}$.

Operators A as in Theorem 2.9 can be found in the form $A = g^{\beta-\gamma+\mu}A_1$ for an operator A_1 in the cone algebra on the infinite half-axis, which shifts weights at zero from γ to $\gamma - \mu$, or directly as in [22, Definition 2.4.1],

$$A = r^{\beta-\gamma}\omega \mathrm{op}^\gamma_M(h)\omega' + g(r)^{\beta-\gamma+\mu} r^{-\mu}(1-\omega)\mathrm{Op}_r(p)(1-\omega'') + M + G. \tag{2.28}$$

Here $\omega'' \prec \omega \prec \omega'$ are cut-off functions ($\varphi \prec \varphi'$ means $\varphi' \equiv 1$ on $\mathrm{supp}\,\varphi$), and $g \in C^\infty(\mathbb{R}_+)$ is a function with the properties $g(r) = r$ for $0 < r < \varepsilon_0$, $g(r) = 1$ for $r > \varepsilon_1$ for some $0 < \varepsilon_0 < \varepsilon_1$ where ε_1 is so small that $g(r)^{\beta-\gamma+\mu}r^{-\mu}\omega(r) = r^{\beta-\gamma}\omega(r)$ and $g(r)^{\beta-\gamma+\mu}r^{-\mu}(1-\omega) = r^{-\mu}(1-\omega)$ for large r. The Mellin symbol $h = h(r, w)$ belongs to $C^\infty(\overline{\mathbb{R}}_+, S^\mu_{\mathcal{O}})$, the symbol p is degenerate in the sense $p(r, \rho) = \tilde{p}(r, r\rho)$ for a $\tilde{p}(r, \tilde{\rho}) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_{+,r} \times \mathbb{R}_{\tilde{\rho}})$, and we assume that h is a Mellin quantisation of p. Moreover, M is a smoothing Mellin and G a Green operator in the cone calculus with discrete asymptotics, cf. the terminology of [22].

For any strictly positive function $\tau \mapsto [\tau]$ in $C^\infty(\mathbb{R})$ with $[\tau] = |\tau|$ for $|\tau| \geq C$ for some $C > 0$ we form a function

$$b^\mu(\tau) := [\tau]^\mu \iota_{[\tau]} A \iota_{[\tau]}^{-1} \tag{2.29}$$

which belongs to $C^\infty(\mathbb{R}_\tau, \mathcal{L}(\mathcal{K}^{s,\gamma}(\mathbb{R}_+); \mathcal{K}^{s-\mu,\beta}(\mathbb{R}_+)))$. By virtue of twisted homogeneity

$$b^\mu(\delta\tau) = [\delta\tau]^\mu \iota_{[\delta\tau]} A \iota_{[\delta\tau]}^{-1} = \delta^\mu \iota_\delta b^\mu(\tau) \iota_\delta^{-1}$$

for $\delta \geq 1,\ |\tau| \geq C$, it follows that

$$b^\mu(\tau) \in S^\mu_{\mathrm{cl}}(\mathbb{R}; \mathcal{K}^{s,\gamma}(\mathbb{R}_+); \mathcal{K}^{s-\mu,\beta}(\mathbb{R}_+)), \tag{2.30}$$

cf. Remark 2.6. We will consider below also the double symbol

$$b^\mu(t, t', \tau) := \iota_{[t]} b^\mu(\tau) \iota_{[t']}^{-1} \in S^\mu_{\mathrm{cl}}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}; \mathcal{K}^{s,\gamma}(\mathbb{R}_+), \mathcal{K}^{s-\mu,\beta}(\mathbb{R}_+)). \tag{2.31}$$

We will employ a Mellin generalisation of the spaces (2.14) for a Hilbert space H with group action κ, namely,

$$\mathcal{H}^{s,\gamma}(\mathbb{R}_+, H) = \mathcal{H}^{s,\gamma}(\mathbb{R}_+, H)_\kappa, \tag{2.32}$$

$\gamma \in \mathbb{R}$, defined as the completion of $C_0^\infty(\mathbb{R}_+, H)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+,H)} = \left\{ \int_{\Gamma_{\frac{b+1}{2}-\gamma}} \langle v \rangle^{2s} \|\kappa^{-1}_{\langle v \rangle}(M_{t\to v}u)(v)(\eta)\|^2_H \,đv \right\}^{1/2}, \tag{2.33}$$

for some $b = b(H) \in \mathbb{N}$ which is given together with H. For instance, if $H := \mathcal{K}^{s,\gamma_1}(X^\wedge)$ for some smooth closed manifold X of dimension n we set $b := n + 1$. In our application we will have $H := \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)$ with the group action $\kappa := {}^1\kappa$, i.e., the integration in (2.33) is over $\Gamma_{1-\gamma}$.

Remark 2.10 The map $\iota_\delta : u(t) \mapsto u(\delta t),\ \delta \in \mathbb{R}_+$ fixed, induces an isomorphism

$$\iota_\delta : \mathcal{H}^{s,\gamma}(\mathbb{R}_+, H) \to \mathcal{H}^{s,\gamma}(\mathbb{R}_+, H)$$

for every $s, \gamma \in \mathbb{R}$.

In fact, the replacement of t by δt under the Mellin transform in the expression (2.33) generates a factor δ^v. For $v \in \Gamma_{\frac{b+1}{2}-\gamma}$ this contributes a factor $\delta^{\frac{b+1}{2}-\gamma+i\tau}$ which yields an equivalent norm for every fixed $\delta \in \mathbb{R}_+$.

The operator

$$S_{\gamma-\frac{b}{2}} : C_0^\infty(\mathbb{R}_{+,t}, H) \to C_0^\infty(\mathbb{R}_t, H), \quad u(t) \mapsto e^{-(\frac{b+1}{2}-\gamma)t} u(e^{-t}) \tag{2.34}$$

extends to an isomorphism

$$S_{\gamma-\frac{b}{2}} : \mathcal{H}^{s,\gamma}(\mathbb{R}_+, H) \to \mathcal{W}^s(\mathbb{R}, H) \tag{2.35}$$

for every $s \in \mathbb{R}$.

Let $\mathcal{H}^{s,\gamma}(\mathbb{R}_+, H)_{[c,d]}$ for $0 < c < d$ be the set of all $u \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+, H)$ supported by $[c, d]$. Moreover, let $\mathcal{W}^s(\mathbb{R}, H)_{[c',d']}$ for reals $c' < d'$ be the space of all $u' \in \mathcal{W}^s(\mathbb{R}, H)$ supported by $[c', d']$. The transformation (2.35) induces an isomorphism

$$S_{\gamma-\frac{b}{2}} : \mathcal{H}^{s,\gamma}(\mathbb{R}_+, H)_{[c,d]} \to \mathcal{W}^s(\mathbb{R}, H)_{[c',d']} \tag{2.36}$$

for $c = e^{-c'}$, $d = e^{-d'}$. In fact, the space $C_0^\infty((c', d'), H)$ is dense in $\mathcal{W}^s(\mathbb{R}, H)_{[c',d']}$, similarly as a corresponding property in the case $H = \mathbb{C}$ for the trivial group action. Since (2.34) also induces an isomorphism $S_{\gamma-b/2} : C_0^\infty((c, d), H) \to C_0^\infty((c', d'), H)$ the space $C_0^\infty((c, d), H)$ is dense in $\mathcal{H}^{s,\gamma}(\mathbb{R}_+, H)_{[c,d]}$. Moreover, as a consequence of the invariance of $\mathcal{W}^s_{\text{comp}}(\mathbb{R}, H)$-distributions under diffeomorphisms of $\mathbb{R}$, cf. Remark 2.4 above, and since the multiplication by the exponential factor occurring in $S_{\gamma-\frac{b}{2}}$ transforms that space isomorphically to itself, it follows that $\mathcal{H}^{s,\gamma}(\mathbb{R}_+, H)_{[c,d]} = \mathcal{W}^s(\mathbb{R}, H)_{[c,d]}$ for every $0 < c < d$. This gives us the relation

$$\varphi\mathcal{H}^{s,\gamma}(\mathbb{R}_+, H) = \varphi\mathcal{W}^s(\mathbb{R}, H) \tag{2.37}$$

for every $\varphi \in C_0^\infty(\mathbb{R}_+)$.

Let $t \mapsto [t]$ be a strictly positive smooth function on $\mathbb{R} \ni t$ such that $[t] = 1$ for $|t| \leq 1$ and $[t] = |t|$ for large $|t| \geq c_1$ for some $c_1 > 1$. Define the spaces

$$\mathcal{W}^s_{\text{cone}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa} := \left\{u(t, [t]r) : u(t, \tilde{r}) \in \mathcal{W}^s(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa}\right\}. \tag{2.38}$$

Then $v(t, r) = u(t, \tilde{r})|_{\tilde{r}=[t]r} \in \mathcal{W}^s_{\text{cone}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa}$ is equivalent to

$$\|v(t, r)\|_{\mathcal{W}^s_{\text{cone}}(\mathbb{R},\mathcal{K}^{s,\gamma_1}(\mathbb{R}_{+,r}))_{1_\kappa}} = \|v(t, [t]^{-1}\tilde{r})\|_{\mathcal{W}^s(\mathbb{R}_t,\mathcal{K}^{s,\gamma_1}(\mathbb{R}_{+,\tilde{r}}))_{1_\kappa}} < \infty. \tag{2.39}$$

In applications below the spaces (2.38) will occur only in combination with a factor $1 - \sigma$ for a cut-off function σ on the t half-axis, and the choice of σ is unessential, cf. Lemma 2.16 below. Therefore, it is not necessary here to discuss the influence of the specific function $t \mapsto [t]$ in (2.38) (there is, in fact, no influence). However, in connection with group actions on cone-spaces the difference between t and $[t]$ can be inconvenient. Therefore, on the half-axis $\mathbb{R}_{+,t}$ we define cone-spaces in modified form, compared with (2.38), namely, by

$$\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa} := \left\{u(t, tr) : u(t, \tilde{r}) \in \mathcal{W}^s(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa}\right\}. \tag{2.40}$$

In order to avoid confusion we recall that

$$\mathcal{W}^s(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa} = \mathcal{W}^s(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa}|_{\mathbb{R}_+}.$$

The following observation is motivated by Proposition 2.2.

Proposition 2.11 (i) *The map* $({}^{[2]}\kappa_\delta v)(t,r) := \delta v(\delta t, [t\delta]^{-1}\delta[t]r),\ \delta \in \mathbb{R}_+$, *induces a group action*

$$^{[2]}\kappa_\delta : \mathcal{W}^s_{cone}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa} \to \mathcal{W}^s_{cone}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa}$$

for every $s \in \mathbb{R}$.

(ii) *The map* $({}^2\kappa_\delta v)(t,r) := \delta v(\delta t, r),\ \delta \in \mathbb{R}_+$, *induces a group action*

$$^2\kappa_\delta : \mathcal{W}^s_{cone}(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa} \to \mathcal{W}^s_{cone}(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa}$$

for every $s \in \mathbb{R}$.

Proof (i) According to (2.39) the property $v(t,r) \in \mathcal{W}^s_{\text{cone}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$ means that $v(t,[t]^{-1}\tilde{r}) \in \mathcal{W}^s(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_{+,\tilde{r}}))$. Then, by virtue of Proposition 2.2

$$(\chi_\delta v)(t,[t]^{-1}\tilde{r}) := \delta^{1/2}\,{}^1\kappa_\delta v(\delta t, [\delta t]^{-1}\tilde{r}) = \delta v(\delta t, [\delta t]^{-1}\delta\tilde{r})$$

belongs to $\mathcal{W}^s(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_{+,\tilde{r}}))_{1_\kappa}$. Thus, if we replace $\tilde{r}$ again by $[t]r$, we see that $({}^{[2]}\kappa_\delta v)(t,r) = \delta v(\delta t, [t\delta]^{-1}\delta[t]r)$ belongs to $\mathcal{W}^s_{\text{cone}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$.

(ii) The property $v(t,r) \in \mathcal{W}^s_{\text{cone}}(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$ means that $v(t,t^{-1}\tilde{r}) \in \mathcal{W}^s(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_{+,\tilde{r}}))$. Similarly as in Proposition 2.2 we form

$$(\chi_\delta v)(t,t^{-1}\tilde{r}) := \delta^{1/2}\,{}^1\kappa_\delta v(\delta t, (\delta t)^{-1}\tilde{r}) = \delta v(\delta t, (\delta t)^{-1}\delta\tilde{r})$$

which belongs to $\mathcal{W}^s(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_{+,\tilde{r}}))_{1_\kappa}$. Thus, replacing $\tilde{r}$ by tr, it follows that $({}^2\kappa_\delta v)(t,r) = \delta v(\delta t, (t\delta)^{-1}\delta tr) = \delta v(\delta t, r)$ belongs to $\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$. □

Observe that for $s = \gamma_1 = 0$ we have

$$\mathcal{W}^0_{\text{cone}}(\mathbb{R}, \mathcal{K}^{0,0}(\mathbb{R}_+))_{1_\kappa} := [t]^{-1/2}L^2(\mathbb{R}\times\mathbb{R}_+). \tag{2.41}$$

In fact, since the group ${}^1\kappa$ is unitary in $\mathcal{K}^{0,0}(\mathbb{R}_+) = L^2(\mathbb{R}_+)$ we have

$$\mathcal{W}^0(\mathbb{R}, \mathcal{K}^{0,0}(\mathbb{R}_+))_{1_\kappa} = L^2(\mathbb{R}\times\mathbb{R}_+).$$

Thus, $v(t,[t]r) \in \mathcal{W}^0_{\text{cone}}(\mathbb{R}, \mathcal{K}^{0,0}(\mathbb{R}_+))_{1_\kappa}$ means that the function $v(t,\tilde{r})$ in the notation of (2.38) belongs to $L^2(\mathbb{R}_t \times \mathbb{R}_{+,\tilde{r}})$. This is equivalent to

$$v(t,[t]r) \in [t]^{-1/2}L^2(\mathbb{R}_t \times \mathbb{R}_{+,r}),$$

i.e., $\iint |[t]^{1/2}v(t,[t]r)|^2 dr dt = \iint |[t]^{1/2}v(t,\tilde{r})|^2 [t]^{-1} d\tilde{r} dt = \|v(t,\tilde{r})\|^2_{L^2(\mathbb{R}_t\times\mathbb{R}_{+,\tilde{r}})}$.

Remark 2.12 The spaces $\mathcal{W}^s(\mathbb{R}^q, H)_\kappa$ have been widely studied in connection with operators on a manifold with edge, first in [20], and then in numerous papers and monographs, see, in particular, [6,21]. For corner singularities of order ≥ 2 the involved spaces H may depend on the edge variable y. This effect plays a role also in [27]. There is no functional analytic investigation for such a situation in general. Even for the spaces (2.38) the influence of the edge variable t is nontrivial. However, there are specific operator-valued symbols, also studied in [17], cf. the consideration after Theorem 2.9, which can be applied to such spaces, cf. the proof of Proposition 2.14 below.

Consider the space

$$\mathcal{K}^{\infty,\infty;\infty}(\mathbb{R}_+) := \bigcap_{s,\gamma,e\in\mathbb{R}} \mathcal{K}^{s,\gamma;e}(\mathbb{R}_+),$$

cf. the formula (2.8), which is dense in $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ for every s, γ. The operator ι : $u(t,r) \mapsto u(t,[t]r)$ induces an isomorphism

$$\iota : C_0^\infty(\mathbb{R}, \mathcal{K}^{\infty,\infty;\infty}(\mathbb{R}_+)) \to C_0^\infty(\mathbb{R}, \mathcal{K}^{\infty,\infty;\infty}(\mathbb{R}_+)).$$

Remark 2.13 The space $C_0^\infty(\mathbb{R}, \mathcal{K}^{\infty,\infty;\infty}(\mathbb{R}_+))$ is dense both in $\mathcal{W}^s(\mathbb{R}, \mathcal{K}^{s,\gamma}(\mathbb{R}_+))_{1_\kappa}$ and $\mathcal{W}^s_{\text{cone}}(\mathbb{R}, \mathcal{K}^{s,\gamma}(\mathbb{R}_+))_{1_\kappa}$ for every $s, \gamma \in \mathbb{R}$.

Proposition 2.14 *We have*

$$\mathcal{W}^s_{cone}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa} \subset \mathcal{W}^s_{loc}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa}, \tag{2.42}$$

and for every $\varphi \in C^\infty(\mathbb{R})$

$$\varphi\mathcal{W}^s_{cone}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa} = \varphi\mathcal{W}^s(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa}. \tag{2.43}$$

Moreover, the space $\mathcal{W}^s_{cone}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa}$ *is independent of the choice of the function* $t \to [t]$, $s \in \mathbb{R}$.

Proof For abbreviation in this proof we drop subcripts $^1\kappa$. Let us set $(\iota_{[t]}u)(t,r) := u(t,[t]r)$. Then, by definition, we have isomorphisms

$$\iota_{[t]} : \mathcal{W}^s(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)) \to \mathcal{W}^s_{\text{cone}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$$

and

$$\iota_{[t]} : \mathcal{W}^0(\mathbb{R}, \mathcal{K}^{0,0}(\mathbb{R}_+)) \to \mathcal{W}^0_{\text{cone}}(\mathbb{R}, \mathcal{K}^{0,0}(\mathbb{R}_+)) = [t]^{-1/2}L^2(\mathbb{R} \times \mathbb{R}_+).$$

Next we employ a consequence of Theorem 2.9, namely, the existence of a symbol (2.29), now for $\mu = s$ denoted by

$$b(\tau) \in S^s_{\text{cl}}(\mathbb{R}_\tau; \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+), \mathcal{K}^{0,0}(\mathbb{R}_+)),$$

taking values in a modification of the cone algebra on the infinite half-axis $\overline{\mathbb{R}}_+$, interpreted as a manifold with conical singularity at $r = 0$ and conical exit for $r \to \infty$, such that

$$b(\tau) : \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+) \to \mathcal{K}^{0,0}(\mathbb{R}_+)$$

is a family of isomorphisms, $b^{-1}(\tau) \in S^{-s}_{\mathrm{cl}}(\mathbb{R}_\tau; \mathcal{K}^{0,0}(\mathbb{R}_+), \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$, and

$$B := \mathrm{Op}_t(b) : \mathcal{W}^s(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)) \to \mathcal{W}^0(\mathbb{R}, \mathcal{K}^{0,0}(\mathbb{R}_+))$$

is an isomorphism. Then also

$$\iota_{[t]} B \iota_{[t]}^{-1} : \mathcal{W}^s_{\mathrm{cone}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)) \to [t]^{-1/2} L^2(\mathbb{R} \times \mathbb{R}_+) = [t]^{-1/2} \mathcal{W}^0(\mathbb{R}, \mathcal{K}^{0,0}(\mathbb{R}_+))$$

is an isomorphism. The inverse is of the form

$$\iota_{[t]} B^{-1} \iota_{[t]}^{-1} = \iota_{[t]} \mathrm{Op}_t(b^{-1}) \iota_{[t]}^{-1} = \mathrm{Op}_t(l).$$

for a double symbol $l(t, t', \tau) \in S^{-s}_{\mathrm{cl}}(\mathbb{R}_t \times \mathbb{R}_{t'} \times \mathbb{R}_\tau; \mathcal{K}^{0,0}(\mathbb{R}_+), \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$. Clearly in this computation we interpret the t-variable on the right of $\mathrm{Op}_t(\cdot)$ as t'. The operator

$$\mathrm{Op}_t(l) : \mathcal{W}^0_{\mathrm{comp}}(\mathbb{R}, \mathcal{K}^{0,0}(\mathbb{R}_+)) \to \mathcal{W}^s_{\mathrm{loc}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$$

is known to be continuous by a general result of the pseudo-differential calculus with operator-valued symbols and twisted symbolic estimates, cf. the second relation of (2.21). Then also

$$\mathrm{Op}_t(l) : [t]^{-1/2} \mathcal{W}^0_{\mathrm{comp}}(\mathbb{R}, \mathcal{K}^{0,0}(\mathbb{R}_+)) \to \mathcal{W}^s_{\mathrm{loc}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$$

is continuous, since $[t]^{-1/2} \mathcal{W}^0_{\mathrm{comp}}(\mathbb{R}, \mathcal{K}^{0,0}(\mathbb{R}_+)) \subseteq \mathcal{W}^0_{\mathrm{comp}}(\mathbb{R}, \mathcal{K}^{0,0}(\mathbb{R}_+))$. But we know that $\mathrm{Op}_t(l)$ extends to $[t]^{-1/2} \mathcal{W}^0(\mathbb{R}, \mathcal{K}^{0,0}(\mathbb{R}_+)) = \mathcal{W}^0_{\mathrm{cone}}(\mathbb{R}, \mathcal{K}^{0,0}(\mathbb{R}_+))$, and still maps to $\mathcal{W}^s_{\mathrm{loc}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$. Since the image is equal to $\mathcal{W}^s_{\mathrm{cone}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$ the relation (2.42) is proved. The property (2.43) is a refinement. For

$$\mathcal{W}^s_{\mathrm{cone,comp}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)) := \{u(t, r) \in \mathcal{W}^s_{\mathrm{cone}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)) : u(t, r) = 0 \text{ for } t \notin K \text{ for some } K \Subset \mathbb{R}_+\} \tag{2.44}$$

it suffices to show

$$\mathcal{W}^s_{\mathrm{cone,comp}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)) = \mathcal{W}^s_{\mathrm{comp}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)). \tag{2.45}$$

Because of $\mathcal{W}^0_{\mathrm{cone}}(\mathbb{R}, \mathcal{K}^{0,0}(\mathbb{R}_+)) = [t]^{-1/2} L^2(\mathbb{R} \times \mathbb{R}_+)$ and $\mathcal{W}^0(\mathbb{R}, \mathcal{K}^{0,0}(\mathbb{R}_+)) = L^2(\mathbb{R} \times \mathbb{R}_+)$ the relation (2.45) is true for $s = 0,\ \gamma_1 = 0$. Moreover, we have

$$\mathcal{W}^s_{\mathrm{cone,comp}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)) = \iota_{[t]} \mathcal{W}^s_{\mathrm{comp}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)).$$

In fact, $u \in \mathcal{W}^s_{\mathrm{comp}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$ implies $\iota_{[t]}u \in \mathcal{W}^s_{\mathrm{cone,comp}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$. Conversely, $v \in \mathcal{W}^s_{\mathrm{cone,comp}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$ gives rise to $\iota^{-1}_{[t]}v \in \mathcal{W}^s_{\mathrm{comp}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$. We now apply elements of the proof of Proposition 2.8 (ii). We form the symbol b_R and obtain the properly supported operator $\mathrm{Op}(b_R)$ which gives rise to an isomorphism

$$\mathrm{Op}(b_R) : \mathcal{W}^s_{\mathrm{comp}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)) \to \mathcal{W}^0_{\mathrm{comp}}(\mathbb{R}, \mathcal{K}^{0,0}(\mathbb{R}_+)).$$

Also $\iota_{[t]}\mathrm{Op}(b_R)\iota^{-1}_{[t]}$ induces an isomorphism

$$\iota_{[t]}\mathrm{Op}(b_R)\iota^{-1}_{[t]} : \mathcal{W}^s_{\mathrm{cone,comp}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)) \to \mathcal{W}^0_{\mathrm{cone,comp}}(\mathbb{R}, \mathcal{K}^{0,0}(\mathbb{R}_+)), \tag{2.46}$$

since $\mathcal{W}^s_{\mathrm{cone,comp}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)) = \iota_{[t]}\mathcal{W}^s_{\mathrm{comp}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$. Moreover, the operator $\iota_{[t]}\mathrm{Op}(b_R)\iota^{-1}_{[t]}$ is properly supported and defines an isomorphism

$$\iota_{[t]}\mathrm{Op}(b_R)\iota^{-1}_{[t]} : \mathcal{W}^s_{\mathrm{comp}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)) \to \mathcal{W}^0_{\mathrm{comp}}(\mathbb{R}, \mathcal{K}^{0,0}(\mathbb{R}_+)). \tag{2.47}$$

Because of (2.45) for $s = \gamma_1 = 0$ the spaces in the preimages of (2.46) and (2.47) coincide, and hence we obtain the relation (2.45) in general. We immediately obtain the relation (2.43) and also the independence of the cone-spaces of the choice of the function $t \to [t]$. □

Recall that

$$\mathcal{W}^s(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa} := \mathcal{W}^s(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa}|_{\mathbb{R}_+}. \tag{2.48}$$

Moreover, let

$$\mathcal{W}^s_{\mathrm{cone}}(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa} := \mathcal{W}^s_{\mathrm{cone}}(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa}|_{\mathbb{R}_+}. \tag{2.49}$$

Definition 2.15 For $\gamma_1, \gamma_2 \in \mathbb{R}$ we define

(i)

$$\begin{aligned}\mathcal{K}^{s,\gamma_2,\gamma_1}(\mathbb{R}_+ \times \mathbb{R}_+) &:= [\sigma]\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa} \\ &\quad + [1-\sigma]\mathcal{W}^s_{\mathrm{cone}}(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa}\end{aligned} \tag{2.50}$$

for a cut-off function σ on the t half-axis, cf. (2.32) for $\gamma = \gamma_2$, $H = \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)$, $\kappa = {}^1\kappa$, and formula (2.49);

(ii) for the interval $I := \{0 \le r \le 1\}$ we set

$$\mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge) := [\omega_0]\mathcal{K}^{s,\gamma_2,\gamma_1}(\mathbb{R}_+ \times \mathbb{R}_+) + \vartheta^*[\omega_1]\mathcal{K}^{s,\gamma_2,\gamma_1}(\mathbb{R}_+ \times \mathbb{R}_+), \tag{2.51}$$

for cut-off functions ω_0, ω_1 on the r half-axis such that $\{\omega_0, \vartheta^*\omega_1\}$ form a partition of unity on I, cf. notation (2.18).

Lemma 2.16 *The spaces in* (2.50) *are independent of the choice of* σ. *Those in* (2.51) *are independent of the involved partition of unity* $\{\omega_0, \vartheta^*\omega_1\}$ *on* I.

Proof In the proof we drop again subscripts ${}^1\kappa$.
(i) Let σ^1 and σ^2 be two cut-off functions on the t half-axis. Then

$$\begin{aligned}\sigma^1\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+,\mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)) + (1-\sigma^1)\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+,\mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))\\ = \sigma^2\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+,\mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)) + (1-\sigma^2)\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+,\mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))\\ + (\sigma^1-\sigma^2)\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+,\mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)) + (\sigma^2-\sigma^1)\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+,\mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)).\end{aligned}$$

Similarly as before the interpretation of the latter relations is that we talk about the spaces consisting of the sets of sums of elements in the involved spaces, e.g.,

$$\sigma^2 u_1 + (1-\sigma^2)u_2 + (\sigma^1-\sigma^2)u_3 + (\sigma^2-\sigma^1)u_4$$

for arbitrary $u_1, u_3 \in \mathcal{H}^{s,\gamma_2}(\mathbb{R}_+,\mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$ and $u_2, u_4 \in \mathcal{W}^s_{\text{cone}}(\mathbb{R}_+,\mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$. From (2.37), i.e.,

$$\varphi\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+,\mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)) = \varphi\mathcal{W}^s(\mathbb{R}_+,\mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$$

and

$$\varphi\mathcal{W}^s(\mathbb{R}_+,\mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)) = \varphi\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+,\mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$$

for every $\varphi \in C_0^\infty(\mathbb{R}_+)$ we obtain

$$\varphi\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+,\mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)) = \varphi\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+,\mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$$

for every $\varphi \in C_0^\infty(\mathbb{R}_+)$. This shows that $\mathcal{K}^{s,\gamma_2,\gamma_1}(\mathbb{R}_+\times\mathbb{R}_+)$ is independent of the choice of σ.
(ii) Let us first recall some tools on the spaces

$$\mathcal{W}^s(\mathbb{R}^q,\mathcal{K}^{s,\gamma}(X^\wedge)), \tag{2.52}$$

for $s,\gamma\in\mathbb{R}$, and a smooth compact manifold X, $n = \dim X$, cf. (2.12). These spaces belong to the edge pseudo-differential calculus for an edge of dimension q. Despite of the anisotropic description of (2.52) we have the relation

$$H^s_{\text{comp}}(X^\wedge\times\mathbb{R}^q) \subset \mathcal{W}^s(\mathbb{R}^q,\mathcal{K}^{s,\gamma}(X^\wedge)) \subset H^s_{\text{loc}}(X^\wedge\times\mathbb{R}^q)$$

for every $s,\gamma\in\mathbb{R}$, cf. [22, Proposition 3.1.21]. This property relies on the estimate

$$c_1\|u\|_{H^s(\mathbb{R}^{1+n+q})} \le \|u\|_{\mathcal{W}^s(\mathbb{R}^q,\mathcal{K}^{s,\gamma}((S^n)^\wedge))} \le c_2\|u\|_{H^s(\mathbb{R}^{1+n+q})} \tag{2.53}$$

for all $u \in C_0^\infty(\mathbb{R}^q, C_0^\infty(\mathbb{R}^{1+n})^R)$ for every $R > 0$, for constants $c_i = c_i(R) > 0$, $i = 1,2$, with S^n being the unit sphere in $\mathbb{R}^{1+n}$. Here $C_0^\infty(\mathbb{R}^{1+n})^R$ means the subspace of all $u \in C_0^\infty(\mathbb{R}^{1+n}\setminus\{0\})$ supported by $\{\tilde{x}\in\mathbb{R}^{1+n} : |\tilde{x}| \ge R\}$. We apply this to the spaces $\mathcal{W}^s(\mathbb{R},\mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$ which are a special case of (2.52) for $q = 1$ and $X^\wedge = \mathbb{R}_+$, i.e., $n = 0$, and t now plays the role of the edge variable y. From (2.38) and (2.49), (2.48) we see that the elements $u(t, r)$ of $\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$ are characterised by the property $u(t, [t]^{-1}r) \in \mathcal{W}^s(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$. As a consequence of (2.53) we have the relations

$$(1-\sigma)\mathcal{W}^s(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_{+,\tilde{r}}))|_{\tilde{r}>R} = (1-\sigma)H^s(\mathbb{R}\times\mathbb{R}_{+,\tilde{r}})|_{\tilde{r}>R} \tag{2.54}$$

and

$$\begin{aligned}(1-\sigma)\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_{+,r}))|_{tr>R} &= (1-\sigma)\mathcal{W}^s(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_{+,tr}))|_{tr>R} \\ &= (1-\sigma)H^s(\mathbb{R}\times\mathbb{R}_{+,tr})|_{tr>R}. \end{aligned} \tag{2.55}$$

Now if we have two cut-off functions ω_0^1 and ω_0^2, then the spaces

$$\omega_0^i(1-\sigma)\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_{+,r}))|_{tr>R}$$

for $i = 1, 2$ differ from each other by

$$\varphi_0(1-\sigma)\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_{+,r}))|_{tr>R}$$

for a $\varphi_0 \in C_0^\infty(\mathbb{R}_{+,r})$. Translated into the variables $(t, \tilde{r})$ the change of the spaces is caused by the change from $\omega_0^1(\tilde{r}/t)$ to $\omega_0^2(\tilde{r}/t)$. By virtue of (2.54) we are far from $t = 0$, and $\varphi_0(\tilde{r}/t)(1-\sigma(t))$ cuts out standard Sobolev spaces $H^s(\mathbb{R}\times\mathbb{R}_{+,\tilde{r}})$ in a region of $\mathbb{R}^2_{t,\tilde{r}}$ which is conical for large t. So the nature of the spaces close to $r = 0$ on the interval I is not changed under changing the cut-offs in r. Close to $r = 1$ we have a similar effect, but since the involved cut-off functions ω_0^i and $\vartheta^*\omega_1^i$ form a partition of unity both for $i = 1$ and $i = 2$, the change of the spaces near $r = 0$ caused by replacing ω_0^1 by ω_0^2 is compensated with the opposite sign by the change from ω_1^1 to ω_1^2 near $r = 1$. That means the space $(1-\sigma)\mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge)$ remains unchanged under changing the partition of unity on I.

It remains to show that $\sigma\mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge)$ is independent of the chosen partition of unity on I. Although there is an additional weight γ_2 the arguments are a little easier. We apply the isomorphism

$$S_{\gamma_2-\frac{1}{2}} : \mathcal{H}^{s,\gamma_2}(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)) \to \mathcal{W}^s(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)),$$

cf. (2.34), for $H = \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)$ and $b = 1$. Then $\omega_0^1\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$ is transformed to $\omega_0^1\mathcal{W}^s(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$. This space differs from $\omega_0^2\mathcal{W}^s(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$ by $\varphi_0\mathcal{W}^s(\mathbb{R}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)) = \varphi_0 H^s(\mathbb{R}\times\mathbb{R}_+)$. In a similar manner we can argue for the change from ω_1^1 to ω_1^2, and the change over I is with the opposite sign, when we change the partitions of unity $\{\omega_0^i, \vartheta^*\omega_0^i\}$ from $i = 1$ to $i = 2$. At the same time we see that the spaces $\sigma\mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge)$ remain unchanged. □

Proposition 2.17 *The space $\mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge)$ is a Hilbert space with group action ${}^2\kappa = \{{}^2\kappa_\delta\}_{\delta\in\mathbb{R}_+}$,*

$$({}^2\kappa_\delta u)(t) := \delta u(\delta t),\ \delta\in\mathbb{R}_+. \tag{2.56}$$

Proof From Definition 2.15 we see that $\mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge)$ is a sum of two spaces, namely,

$$[\omega_0]\big\{[\sigma]\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+,\mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))+[1-\sigma]\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+,\mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))\big\} \tag{2.57}$$

and an analogous space referring to $r=1$. They are of the same structure; so we consider (2.57). The change from t to δt acts in the cut-off function σ and in the remaining (t,r)-variables. Because of Lemma 2.16 the change of σ preserves functions within the space (2.57). Therefore, we may focus on the other (t,r). Here it suffices to apply Remark 2.10 and Proposition 2.11. □

Remark 2.18 Let $\varphi\in C_0^\infty(\operatorname{int} I)$ and σ a cut-off function on the t half-axis.

(i) Let us identify the interval I with a closed interval I_1 on the unit circle S^1 via a fixed diffeomorphism $\iota: I\to I_1\subset S^1\backslash\{2\pi\}$; in the following notation we suppress ι again. For every $s,\gamma_2,\gamma_1\in\mathbb{R}$ we have a continuous embedding

$$\sigma\varphi\mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge)\hookrightarrow\mathcal{H}^{s,\gamma_2}((S^1)^\wedge).$$

(ii) There are continuous embeddings

$$\sigma\mathcal{K}^{s',\gamma_2',\gamma_1'}(I^\wedge)\hookrightarrow\sigma\mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge)$$

for $s'\geq s,\gamma_2'\geq\gamma_2,\gamma_1'\geq\gamma_1$ that are compact for $s'>s,\gamma_2'>\gamma_2,\gamma_1'>\gamma_1$.

(iii) The space

$$\mathcal{K}^{s,\gamma_2,\gamma_1;e}(I^\wedge):=\langle t\rangle^{-e}\mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge),\ \ e\in\mathbb{R},$$

is a Hilbert space with group action ${}^2\kappa$, and we have continuous embeddings

$$\mathcal{K}^{s',\gamma_2',\gamma_1';e'}(I^\wedge)\hookrightarrow\mathcal{K}^{s,\gamma_2,\gamma_1;e}(I^\wedge)$$

for $s'\geq s,\gamma_2'\geq\gamma_2,\gamma_1'\geq\gamma_1,e'\geq e$, that are compact for $s'>s,\gamma_2'>\gamma_2,\gamma_1'>\gamma_1,e'>e$.

Proposition 2.17 allows us to form edge spaces

$$\mathcal{W}^s(\mathbb{R}^d,\mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge))_{{}^2\kappa} \tag{2.58}$$

based on the corner spaces in Definition 2.15 (ii). Those play a role as local models of weighted corner spaces. Moreover, let M be a compact manifold with second order corner $Z=\sigma_2(M)$, cf. Sect. 2.1. Then

$$H^{s,\gamma_2,\gamma_1}(M) \tag{2.59}$$

is defined as the subspace of all $u\in H^{s,\gamma_1}_{[\text{loc})}(M\backslash Z)$ such that for any singular chart $\chi: V|_D\to I^\Delta\times\mathbb{R}^d$ associated with a chart $D\to\mathbb{R}^d$ on Z and $\chi_{\text{st}}:=\chi|_{V|_D\backslash Z}: V|_D\backslash Z\to\mathbb{R}_+\times I\times\mathbb{R}^d$ we have $(\chi_{\text{st}}^{-1})^*\sigma u\in\mathcal{W}^s(\mathbb{R}^d,\mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge))_{{}^2\kappa}$. Here σ is any element of $C^\infty(\mathbb{M})$ of the form $\chi_{\text{st}}^*\sigma_0$ for some cut-off function on the t-half-axis.

2.3 Iterated Asymptotics and Corner Green Operators

Asymptotics of distributions $u \in \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ as $r \to 0$ will be expressed in terms of singular functions of the form

$$\omega(r) r^{-p} \log^k r$$

for $p \in \mathbb{C}$, $k \in \mathbb{N}$, and some cut-off function ω on the half-axis. A sequence

$$P := \{(p_j, m_j)\}_{j \in \mathbb{J}} \subset \mathbb{C} \times \mathbb{N}, \tag{2.60}$$

$\mathbb{J} \subseteq \mathbb{N}$, is called a (discrete) asymptotic type if $\pi_{\mathbb{C}} P := \{p_j\}_{j \in \mathbb{J}}$ is either finite or $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$. We say that P is associated with the weight data (γ, Θ) for a weight $\gamma \in \mathbb{R}$ and a weight interval $\Theta := (\vartheta, 0]$, $-\infty \le \vartheta < 0$, if

$$\pi_{\mathbb{C}} P \subset \{1/2 - \gamma + \vartheta < \operatorname{Re} w < 1/2 - \gamma\}.$$

In future, for convenience, we assume that P satisfies the shadow condition, i.e., $p \in \pi_{\mathbb{C}} P$ implies $p - l \in \pi_{\mathbb{C}} P$ for all $l \in \mathbb{N}$ such that $1/2 - \gamma + \vartheta < \operatorname{Re} p - l$. If P is associated with (γ, Θ) and Θ finite, then

$$\mathcal{E}_P^{\gamma}(\mathbb{R}_+) := \left\{ u = \omega(r) \sum_{j \in \mathbb{J}} \sum_{k=0}^{m_j} c_{jk} r^{-p_j} \log^k r : c_{jk} \in \mathbb{C}, 0 \le k \le m_j, j \in \mathbb{J} \right\} \tag{2.61}$$

for a fixed cut-off function ω is a finite-dimensional subspace of $\mathcal{K}^{\infty,\gamma}(\mathbb{R}_+)$. The coefficients c_{jk} are uniquely determined by u. We set

$$\mathcal{K}_P^{s,\gamma}(\mathbb{R}_+) := \mathcal{E}_P^{\gamma}(\mathbb{R}_+) + \mathcal{K}_{\Theta}^{s,\gamma}(\mathbb{R}_+) \tag{2.62}$$

for

$$\mathcal{K}_{\Theta}^{s,\gamma}(\mathbb{R}_+) := \varprojlim_{\varepsilon > 0} \mathcal{K}^{s,\gamma-\vartheta-\varepsilon}(\mathbb{R}_+). \tag{2.63}$$

The space (2.63) is Fréchet, and also (2.62) as a direct sum of Fréchet spaces. In the case of infinite Θ we define $\mathcal{K}_P^{s,\gamma}(\mathbb{R}_+) := \varprojlim_{n \in \mathbb{N}} \{\mathcal{E}_{P_n}^{\gamma}(\mathbb{R}_+) + \mathcal{K}_{\Theta_n}^{s,\gamma}(\mathbb{R}_+)\}$ for $\Theta_n := (-(n+1), 0]$ and $P_n := \{(p, m) \in P : 1/2 - \gamma - (n+1) < \operatorname{Re} p < 1/2 - \gamma\}$. Since asymptotics only refer to $r \to 0$ it makes sense also to form

$$\mathcal{H}_P^{s,\gamma}(\mathbb{R}_+) := \omega \mathcal{K}_P^{s,\gamma}(\mathbb{R}_+) + (1 - \omega)\mathcal{H}^{s,\gamma}(\mathbb{R}_+).$$

Analogously as (2.17) we define

$$H_P^{s,\gamma}(I) := [\omega_0]\mathcal{H}_P^{s,\gamma}(\mathbb{R}_+) + \vartheta^*[\omega_1]\mathcal{H}_P^{s,\gamma}(\mathbb{R}_+) \ \text{ for } \ s, \gamma \in \mathbb{R}, \tag{2.64}$$

for asymptotic types

$$P \ \text{ associated with } \ (\gamma, \Theta). \tag{2.65}$$

Recall that

$$\vartheta : {}^1\overline{\mathbb{R}}_- \to \overline{\mathbb{R}}_+ \ \text{ for } \ {}^1\overline{\mathbb{R}}_- = \{r \in \mathbb{R} : -\infty < r \leq 1\}, \tag{2.66}$$

is defined by $\vartheta(r) = -r + 1$, cf. (2.18), and ω_0, ω_1 are cut-off functions on the half-axis such that $\omega_0(r) + \omega_1(-r+1) = 1$ on the interval I. Moreover, let

$$\mathcal{H}_P^{s,\gamma_2,\gamma_1}(I^\wedge) := [\omega_0]\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+, \mathcal{K}_P^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa} + \vartheta^*[\omega_1]\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+, \mathcal{K}_P^{s,\gamma_1}(\mathbb{R}_+))_{1_\kappa}. \tag{2.67}$$

In addition for finite $\Lambda = (\lambda, 0]$, we set

$$\mathcal{H}_{\Lambda,P}^{s,\gamma_2,\gamma_1}(I^\wedge) := [\sigma] \varprojlim_{\varepsilon>0} \mathcal{H}_P^{s,\gamma_2-\lambda-\varepsilon,\gamma_1}(I^\wedge) + [1-\sigma]\mathcal{H}_P^{s,\gamma_2,\gamma_1}(I^\wedge) \tag{2.68}$$

for some cut off function σ on the t half-axis. Recall that we could admit different weights at the end points of I and different asymptotic types P. This generalisation is simple and left to the reader.

In order to define functions with iterated asymptotics for $r \to 0$ and $t \to 0$ we also consider singular functions in t-direction

$$\sigma(t)t^{-q} \log^l t$$

for $q \in \mathbb{C}$, $l \in \mathbb{N}$, and some cut-off function σ on the t half-axis. Let

$$Q := \{(q_i, n_i)\}_{i \in \mathbb{I}} \subset \mathbb{C} \times \mathbb{N}, \tag{2.69}$$

$\mathbb{I} \subseteq \mathbb{N}$, be a (discrete) asymptotic type with respect to t, associated with the weight data (β, Λ) for a weight $\beta \in \mathbb{R}$ and $\Lambda := (\lambda, 0]$, $-\infty \leq \lambda < 0$, i.e.,

$$\pi_{\mathbb{C}} Q \subset \{1/2 - \beta + \lambda < \operatorname{Re} v < 1/2 - \beta\}.$$

From now on, for convenience, we set $\mathbb{I} = \{0, 1, \ldots, N\}$ for some $N \in \mathbb{N} \cup \{\infty\}$. If Q is associated with γ_2 and finite Λ, i.e., finite N, and P as in (2.64) we set

$$\mathcal{F}_{Q,P}^{\gamma_2,\gamma_1}(I^\wedge) := \left\{ f = \sigma(t) \sum_{i=0}^{N} \sum_{l=0}^{n_i} c_{il} t^{-q_i} \log^l t : c_{il} \in \mathcal{H}_P^{\infty,\gamma_1}(I), 0 \leq l \leq m_i, i \in \mathbb{I} \right\} \tag{2.70}$$

for some fixed cut-off function σ in t. Similarly as in (2.61) the coefficients c_{il} are uniquely determind by f.

Moreover, let

$$\mathcal{H}_{Q,P}^{s,\gamma_2,\gamma_1}(I^\wedge) := \mathcal{F}_{Q,P}^{\gamma_2,\gamma_1}(I^\wedge) + [1-\sigma]\mathcal{H}_P^{s,\gamma_2,\gamma_1}(I^\wedge), \tag{2.71}$$

cf. notation (2.67).

Definition 2.19 We set

(i)

$$\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}_P(\mathbb{R}_{+,r}))_{{}^1\kappa} := \left\{ v(t, tr) : v(t, \tilde{r}) \in \mathcal{W}^s(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}_P(\mathbb{R}_{+,\tilde{r}}))_{{}^1\kappa} \right\},$$

$({}^1\kappa_\delta u)(\tilde{r}) = \delta^{1/2} u(\delta \tilde{r})$, and

$$\mathcal{W}^{s,\gamma_1}_{\text{cone},P}(I^\wedge) := [\omega_0]\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}_P(\mathbb{R}_+))_{{}^1\kappa} + \vartheta^*[\omega_1]\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}_P(\mathbb{R}_+))_{{}^1\kappa};$$

(ii)

$$\mathcal{K}^{s,\gamma_2,\gamma_1}_{Q,P}(I^\wedge) := [\sigma]\mathcal{H}^{s,\gamma_2,\gamma_1}_{Q,P}(I^\wedge) + [1-\sigma]\mathcal{W}^{s,\gamma_1}_{\text{cone},P}(I^\wedge); \tag{2.72}$$

(iii) $\mathcal{K}^{s,\gamma_2,\gamma_1;e}_{Q,P}(I^\wedge) := [\sigma]\mathcal{K}^{s,\gamma_2,\gamma_1}_{Q,P}(I^\wedge) + [1-\sigma]\langle t\rangle^{-e}\mathcal{W}^{s,\gamma_1}_{\text{cone},P}(I^\wedge),\ e \in \mathbb{R}.$

A Fréchet space, written as a projective limit of Hilbert spaces $E = \varprojlim_{j\in\mathbb{N}} E^j$ with continuous embeddings $E^{j+1} \hookrightarrow E^j$ for all j, is said to be endowed with a group action $\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ if κ is a group action in E^0, cf. Sect. 2.2, and $\kappa|_{E^j}$ a group action in E^j for every j.

Proposition 2.20 *The spaces in Definition 2.19 are Fréchet in a natural way, and the group actions of Propositions 2.11(ii) and 2.17 restrict to group actions in those spaces.*

Proof Let us first consider the spaces in Definition 2.19 (i) for P associated with the weight data (γ_1, Θ), Θ finite. The case of $\Theta = (-\infty, 0]$ can be easily reduced to finite Θ by passing to a projective limit. This step is left to the reader. It is known that we can write

$$\mathcal{K}^{s,\gamma_1}_P(\mathbb{R}_+) = \mathcal{K}^{s,\gamma_1}_\Theta(\mathbb{R}_+) + \mathcal{K}^{\infty,\gamma_1}_P(\mathbb{R}_+)$$

as a non-direct sum of Fréchet spaces, where $\mathcal{K}^{s,\gamma_1}_\Theta(\mathbb{R}_+)$ is Fréchet as a projective limit (2.63) of Hilbert spaces $\mathcal{K}^{s,\gamma_1-\vartheta-\varepsilon_l}(\mathbb{R}_+)$, for any $0 < \varepsilon_l$, $l \in \mathbb{N}$, tending to 0 as $l \to \infty$. We can choose ε_l in such a way that $1/2 + \vartheta + \varepsilon_l < 1/2$ and $\pi_\mathbb{C} P \cap \Gamma_{1/2-\gamma_1+\vartheta+\varepsilon_l} = \emptyset$ for all $l \in \mathbb{N}$. Then

$$E^l := \mathcal{K}^{s,\gamma_1-\vartheta-\varepsilon_l}(\mathbb{R}_+) + \mathcal{E}^{\gamma_1}_{P_l}(\mathbb{R}_+)$$

for $P_l := \{(p, m) \in P : 1/2 - \gamma_1 + \vartheta + \varepsilon_l < \operatorname{Re} p\}$ is a Hilbert space with group action ${}^1\kappa$. Thus

$$\mathcal{K}^{s,\gamma_1}_P(\mathbb{R}_+) = \varprojlim_{l\in\mathbb{N}} E^l \tag{2.73}$$

is a Fréchet space with group action. This gives us

$$\mathcal{W}^s(\mathbb{R}_{+,t}, \mathcal{K}^{s,\gamma_1}_P(\mathbb{R}_+)) = \varprojlim_{l\in\mathbb{N}} \mathcal{W}^s(\mathbb{R}_t, E^l)|_{\mathbb{R}_{+,t}}.$$

Setting

$$\mathcal{W}^s_{\text{cone}}(\mathbb{R}_{+,t}, E^l) := \{v(t, tr) : v(t, \tilde{r}) \in \mathcal{W}^s(\mathbb{R}_{+,t}, E^l_{\tilde{r}})\}$$

for $\mathcal{W}^s(\mathbb{R}_{+,t}, E^l_{\tilde{r}}) := \mathcal{W}^s(\mathbb{R}_t, E^l_{\tilde{r}})|_{\mathbb{R}_{+,t}}$, with $E^l_{\tilde{r}}$ being the space of functions in E^l in the variable $\tilde{r}$, it follows that

$$\mathcal{W}^s_{\text{cone}}(\mathbb{R}_{+,t}, \mathcal{K}^{s,\gamma_1}_P(\mathbb{R}_{+,r})) = \varprojlim_{l\in\mathbb{N}} \mathcal{W}^s_{\text{cone}}(\mathbb{R}_{+,t}, E^l).$$

Now we can proceed in a similar manner as in the proof of Proposition 2.11 (ii). From Proposition 2.2 we obtain a group action $\{\chi_\delta\}_{\delta\in\mathbb{R}_+}$ in $\mathcal{W}^s(\mathbb{R}_t, E^l_{\tilde{r}})_{^1\kappa}$ which induces a group action in $\mathcal{W}^s(\mathbb{R}_{+,t}, E^l_{\tilde{r}})_{^1\kappa}$, cf. Remark 2.3; here we use for the moment subscript $^1\kappa$ which is involved in χ.

The property $v(t, r) \in \mathcal{W}^s_{\text{cone}}(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}_P(\mathbb{R}_+))_{^1\kappa}$ means that

$$v(t, t^{-1}\tilde{r}) \in \mathcal{W}^s(\mathbb{R}_+, E^l_{\tilde{r}})_{^1\kappa}$$

for every $l \in \mathbb{N}$. Similarly as in Proposition 2.2 we form

$$(\chi_\delta v)(t, t^{-1}\tilde{r}) := \delta^{1/2}\, {}^1\kappa_\delta v(\delta t, (\delta t)^{-1}\tilde{r}) = \delta v(\delta t, (\delta t)^{-1}\delta\tilde{r})$$

which belongs to $\mathcal{W}^s(\mathbb{R}_+, E^l_{\tilde{r}})_{^1\kappa}$. Thus, replacing $\tilde{r}$ by tr, we see that $({}^2\kappa_\delta v)(t, r) = \delta v(\delta t, (t\delta)^{-1}\delta tr) = \delta v(\delta t, r)$ belongs to $\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+, E^l)_{^1\kappa}$. Since this is true for every l we obtain $({}^2\kappa_\delta v)(t, r) \in \mathcal{W}^s_{\text{cone}}(\mathbb{R}_+, \mathcal{K}^{s,\gamma_1}_P(\mathbb{R}_{+,r}))_{^1\kappa}$. It is now evident that ${}^2\kappa = \{{}^2\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ is also a group action on the Fréchet space $\mathcal{W}^{s,\gamma_1}_{\text{cone},P}(I^\wedge)$.

We now turn to (ii). Let us set $E := \mathcal{K}^{s,\gamma_1}_P(\mathbb{R}_+)$, endowed with the group action ${}^1\kappa$, and

$$\mathcal{H}^{s,\gamma_2}_\Lambda(\mathbb{R}_+, E) := [\sigma] \varprojlim_{\varepsilon>0} \mathcal{H}^{s,\gamma_2-\lambda-\varepsilon}(\mathbb{R}_+, E) + [1-\sigma]\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+, E),$$

cf. the notation (2.32) for the Fréchet space E rather than H and formula (2.68). Then we have

$$\mathcal{K}^{s,\gamma_2,\gamma_1}_{Q,P}(I^\wedge) = \mathcal{F}^{\gamma_2,\gamma_1}_{Q,P}(I^\wedge) + \mathcal{K}^{s,\gamma_2,\gamma_1}_{\Lambda,P}(I^\wedge)$$

for

$$\begin{aligned}\mathcal{K}^{s,\gamma_2,\gamma_1}_{\Lambda,P}(I^\wedge) &= [\sigma]\{[\omega_0]\mathcal{H}^{s,\gamma_2}_\Lambda(\mathbb{R}_+, E) + \vartheta^*[\omega_1]\mathcal{H}^{s,\gamma_2}_\Lambda(\mathbb{R}_+, E)\}\\ &\quad + [1-\sigma]\{[\omega_0]\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+, E) + \vartheta^*[\omega_1]\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+, E)\}\\ &= [\omega_0]\{[\sigma]\mathcal{H}^{s,\gamma_2}_\Lambda(\mathbb{R}_+, E) + [1-\sigma]\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+, E)\}\\ &\quad + \vartheta^*[\omega_1]\{[\sigma]\mathcal{H}^{s,\gamma_2}_\Lambda(\mathbb{R}_+, E) + [1-\sigma]\mathcal{W}^s_{\text{cone}}(\mathbb{R}_+, E)\}. \end{aligned} \tag{2.74}$$

The action of ${}^2\kappa$ on the space (2.74) can be verified in a similar manner as in Proposition 2.17. It remains to look at the effect of the group action on $\mathcal{F}_{Q,P}^{\gamma_2,\gamma_1}(I^\wedge)$. But here it is clear that we do not leave the space up to a remainder in $\mathcal{K}_{\Lambda,P}^{s,\gamma_2,\gamma_1}(I^\wedge)$. □

It follows that there are edge spaces modelled on $\mathcal{K}_{Q,P}^{s,\gamma_2,\gamma_1}(I^\wedge)$, namely, analogously as (2.58),

$$\mathcal{W}^s(\mathbb{R}^d, \mathcal{K}_{Q,P}^{s,\gamma_2,\gamma_1}(I^\wedge))_{{}^2\kappa} \tag{2.75}$$

for any pair of asymptotic types Q and P, associated with the weight data (γ_2, Λ) and (γ_1, Θ), respectively. Moreover, for an open set $\Omega \subseteq \mathbb{R}^d$ we have comp/loc-spaces,

$$\mathcal{W}^s_{\text{comp}}(\Omega, \mathcal{K}_{Q,P}^{s,\gamma_2,\gamma_1}(I^\wedge))_{{}^2\kappa}, \ \mathcal{W}^s_{\text{loc}}(\Omega, \mathcal{K}_{Q,P}^{s,\gamma_2,\gamma_1}(I^\wedge))_{{}^2\kappa}.$$

Another topic of this subsection are Green symbols and Green operators of different kind. Recall from the classical theory of elliptic boundary value problems that there appear Green's functions. For instance, in the case of the Dirichlet problem for the Poisson equation in a smooth bounded domain, Green's function (regarded as an operator) solves the inhomogeneous equation for vanishing boundary conditions. Pseudo-differential boundary value problems also employ such operators. In Boutet de Monvel's calculus for operators with the transmission property at the boundary, cf. [1], these operators contain parts with a symbolic structure locally along the boundary, with specific operator-valued symbols, in this case referring to Taylor asymptotics $\{(-j, 0)\}_{j\in\mathbb{N}}$ in normal direction. Also the edge pseudo-differential calculus, developed in [20] as well as diverse corner theories, cf. [10,23,29], contains adapted variants of Green symbols and associated operators. In the present article we intend to establish such a concept on corner manifolds M in the sense of Sect. 2.1.

The following definition concerns symbols referring to the edge Z, cf. the notation in Sect. 2.1. Therefore, variables and covariables will now be denoted by z and ζ, respectively, with z varying in $\mathbb{R}^d$. Then U means an open set in $\mathbb{R}^b$ for some $b \in \mathbb{N}\backslash\{0\}$, and we employ the notation (2.2).

Green symbols refer to formal adjoints in $\mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge)$, $\mathcal{K}^{s,\gamma_1}(\mathbb{R}_+)$, etc., with respect to the scalar products of spaces of smoothness and weight zero, cf. (2.10) for the case over $\mathbb{R}_+$. Concerning $I^\wedge$ we employ the identification

$$\mathcal{K}^{0,0,0}(I^\wedge) = t^{-1/2}L^2(\mathbb{R}_+ \times I) = t^{-1/2}L^2(\mathbb{R}_+, L^2(I)).$$

The operator ${}^2\kappa_\delta$ given by $({}^2\kappa_\delta u)(t) = \delta u(\delta t)$, is unitary in $\mathcal{K}^{0,0}(I^\wedge)$, and ${}^1\kappa_\delta$ given by $({}^1\kappa_\delta u')(t) = \delta^{1/2}u'(\delta t)$, is unitary in $\mathcal{K}^{0,0}(\mathbb{R}_+)$ for every $\delta \in \mathbb{R}_+$. Analogously as (2.10) we have sesquilinear pairings

$$\mathcal{K}^{s,\gamma_2,\gamma_1;e}(I^\wedge) \times \mathcal{K}^{-s,-\gamma_2,-\gamma_1;-e}(I^\wedge) \to \mathbb{C} \tag{2.76}$$

for every $s, \gamma_2, \gamma_1, e \in \mathbb{R}$.

Definition 2.21 Let $U \subseteq \mathbb{R}^p$ be an open set and $\mu \in \mathbb{R}$.

(i) An $I^\wedge$-Green symbol $g(z, \zeta)$ of order $\mu \in \mathbb{R}$ is a

$$g(z,\zeta) \in \bigcap_{s,e\in\mathbb{R}} S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^d; \mathcal{K}^{s,\gamma_2,\gamma_1;e}(I^\wedge), \mathcal{K}^{\infty,\gamma_2-\mu,\gamma_1-\mu;\infty}_{P_2,P_1}(I^\wedge))$$

such that

$$g^*(z,\zeta) \in \bigcap_{s,e\in\mathbb{R}} S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^d; \mathcal{K}^{s,-\gamma_2+\mu,-\gamma_1+\mu;e}(I^\wedge), \mathcal{K}^{\infty,-\gamma_2,-\gamma_1;\infty}_{Q_2,Q_1}(I^\wedge))$$

for certain g-dependent asymptotic types $P_j,\ Q_j,\ j = 1, 2$.

(ii) An $(I^\wedge, \partial^0 I^\wedge)$-Green symbol $g(z, \zeta)$ of order $\mu \in \mathbb{R}$ is a

$$g(z,\zeta) \in \bigcap_{s,e\in\mathbb{R}} S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^d; \mathcal{K}^{s,\gamma_2,\gamma_1;e}(I^\wedge), \mathcal{K}^{\infty,\gamma_2-\mu;\infty}_{P_2^0}(\partial^0 I^\wedge))$$

such that

$$g^*(z,\zeta) \in \bigcap_{s,e\in\mathbb{R}} S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^d; \mathcal{K}^{s,-\gamma_2+\mu;e}(\partial^0 I^\wedge), \mathcal{K}^{\infty,-\gamma_2,-\gamma_1;\infty}_{Q_2,Q_1}(I^\wedge))$$

for certain g-dependent asymptotic types $P_2^0,\ Q_j,\ j = 1, 2$.

(iii) A $(\partial^0 I^\wedge, \partial^0 I^\wedge)$-Green symbol $g(y, \eta)$ of order $\mu \in \mathbb{R}$ is a

$$g(z,\zeta) \in \bigcap_{s,e\in\mathbb{R}} S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^d; \mathcal{K}^{s,\gamma_2;e}(\partial^0 I^\wedge), \mathcal{K}^{\infty,\gamma_2-\mu;\infty}_{P_2^0}(\partial^0 I^\wedge))$$

such that

$$g^*(z,\zeta) \in \bigcap_{s,e\in\mathbb{R}} S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^d; \mathcal{K}^{s,-\gamma_2+\mu;e}(\partial^0 I^\wedge), \mathcal{K}^{\infty,-\gamma_2;\infty}_{Q_2^0}(\partial^0 I^\wedge))$$

for certain g-dependent asymptotic types P_2^0, Q_2^0.

(iv) In a similar manner we define

$$(\partial^0 I^\wedge, I^\wedge)\text{-},\ (I^\wedge, \partial^1 I^\wedge)\text{-},\ (\partial^1 I^\wedge, \partial^1 I^\wedge)\text{-},\ (\partial^m I^\wedge, \partial^n I^\wedge)\text{-},$$

etc., Green symbols, the latter for $m, n = 0, 1$, and $m \neq n$.

There are more types of Green symbols, e.g., trace and potential symbols for the edge Z, but we drop the details, since we mainly focus here on $I^\wedge$-Green symbols. However, in order to give an impression on the full symbolic information, we already observe that corner symbols in (z, ζ) take values in continuous operators

$$a(z,\zeta):\begin{array}{c}\mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge)\\ \oplus\\ \mathcal{K}^{s,\gamma_1}(\partial^0 I^\wedge)\\ \oplus\\ \mathcal{K}^{s,\gamma_1}(\partial^1 I^\wedge)\\ \oplus\\ \mathbb{C}\end{array}\rightarrow\begin{array}{c}\mathcal{K}^{s-\mu,\gamma_2-\mu,\gamma_1-\mu}(I^\wedge)\\ \oplus\\ \mathcal{K}^{s-\mu,\gamma_1-\mu}(\partial^0 I^\wedge)\\ \oplus\\ \mathcal{K}^{s-\mu,\gamma_1-\mu}(\partial^1 I^\wedge)\\ \oplus\\ \mathbb{C}\end{array}, \tag{2.77}$$

or between corresponding subspaces with asymptotics and decay for $t\to\infty$. The expression (2.77) contains some simplification concerning smoothness, orders and weights that may depend on the respective entries of the block matrix a. In addition, in applications to mixed elliptic corner problems, similarly as [4], in the edge case, it makes sense to admit vector-valued spaces, for instance, $\mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge,\mathbb{C}^l)$, $\mathcal{K}^{s-\mu,\gamma_2-\mu,\gamma_1-\mu}(I^\wedge,\mathbb{C}^m)$, etc. However, for the generalities of the corner pseudo-differential calculus it suffices to consider spaces of scalar functions. In any case the shape of block matrices (2.77) shows the kind of entries which are not yet formulated in Definition 2.21, namely, those referring to $\mathbb{C}$. Of course, they are part of the calculus as well. For instance, writing $a(z,\zeta)=(a(z,\zeta)_{kl})_{k,l=1,\ldots,4}$, the component $a_{14}(z,\zeta)$ takes values in $\mathcal{L}(\mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge),\mathbb{C})$ and has the meaning of a trace symbol with respect to the edge $U\ni z$, while $a_{41}(z,\zeta)$ takes values in $\mathcal{L}(\mathbb{C},\mathcal{K}^{\infty,\gamma_2-\mu,\gamma_1-\mu;\infty}_{P_2,P_1}(I^\wedge))$ and has the meaning of a potential symbol. Both refer to $I^\wedge$. Similarly we have trace and potential symbols with respect to the edge $U\ni z$, referring to $\partial^i I^\wedge$, $i=0,1$. The lower right corner $a_{44}(z,\zeta)$ is a matrix of classical scalar symbols.

Let us fix notation for the symbol spaces in Definition 2.21. By

$$\mathcal{R}^\mu_G(U\times\mathbb{R}^d,\boldsymbol{g})_{(I^\wedge,I^\wedge)} \tag{2.78}$$

for

$$\boldsymbol{g}:=(\boldsymbol{g}_2,\boldsymbol{g}_1),\ \boldsymbol{g}_i:=(\gamma_i,\gamma_i-\mu,\Theta_i),\ i=1,2, \tag{2.79}$$

we denote the space of all Green symbols, defined by Definition 2.21 (i). Similarly we have the operator spaces

$$\mathcal{R}^\mu_G(U\times\mathbb{R}^d,\boldsymbol{g})_{(I^\wedge,\partial^0 I^\wedge)},\ \mathcal{R}^\mu_G(U\times\mathbb{R}^d,\boldsymbol{g}_2)_{(\partial^0 I^\wedge,\partial^0 I^\wedge)}, \tag{2.80}$$

etc., with obvious meaning of notation.

The properties of Green symbol spaces in the present context are to some extent analogous to those in the edge calculus of singularity order 1, see, for instance, [11,22], or [4]. Therefore, we content ourselves on the case of upper left corners of the indicated block matrices.

Theorem 2.22 *Let $g_j(z,\zeta)\in\mathcal{R}^{\mu-j}_G(U\times\mathbb{R}^d,\boldsymbol{g})_{(I^\wedge,I^\wedge)}$, $j\in\mathbb{N}$ be an arbitrary sequence of Green symbols where the involved asymptotic types are independent of j. Then there is an asymptotic sum $g(z,\zeta)\sim\sum_{j=0}^\infty g_j(z,\zeta)$, $g(z,\zeta)\in\mathcal{R}^\mu_G(U\times\mathbb{R}^d,\boldsymbol{g})_{(I^\wedge,I^\wedge)}$, unique modulo $\mathcal{R}^{-\infty}_G(U\times\mathbb{R}^d,\boldsymbol{g})_{(I^\wedge,I^\wedge)}$, which means that for every $N\in\mathbb{N}$ we have*

$$g(z,\zeta)-\sum_{j=0}^{N}g_j(z,\zeta)\in\mathcal{R}_G^{\mu-(N+1)}(U\times\mathbb{R}^d,\boldsymbol{g})_{(I^\wedge,I^\wedge)}.$$

Proof The proof employs the following fact. If H is a Hilbert space with group action and $E=\varprojlim_{k\in\mathbb{N}}E^k$ a Fréchet space with another group action, then a sequence of symbols in $g_j\in S_{\mathrm{cl}}^{\mu-j}(U\times\mathbb{R}^d;H,E):=\varprojlim_{k\in\mathbb{N}}S_{\mathrm{cl}}^{\mu-j}(U\times\mathbb{R}^d;H,E^k)$ has an asymptotic sum. In the present case, if the involved asymptotic types are the same for all j, the spaces E^k are independent of j. To be more precise, the involved asymptotic types are contained in a larger fixed asymptotic type for all j. For the formal adjoints the argument is similar. □

We apply Green symbols in the case $U:=\Omega\times\Omega$, $\Omega\subseteq\mathbb{R}^d$ open, denote the variables by $(z,z')\in\Omega\times\Omega$, and form associated operators $\mathrm{Op}(g)$,

$$\mathrm{Op}(g)u(z):=\iint e^{i(z-z')\zeta}g(z,z',\zeta)u(z')dz'đ\zeta, \tag{2.81}$$

first for functions $u\in C_0^\infty(\Omega,\mathcal{K}^{\infty,\gamma_2,\gamma_1}(I^\wedge))$. In addition we define smoothing Green operators C associated with the weight data (2.79) in terms of mapping properties. Such an operator is asked to induce continuous maps

$$C:\mathcal{W}^s_{\mathrm{comp}}(\Omega,\mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge))\to\mathcal{W}^\infty_{\mathrm{loc}}(\Omega,\mathcal{K}^{\infty,\gamma_2-\mu,\gamma_1-\mu}_{P_2,P_1}(I^\wedge)),$$
$$C^*:\mathcal{W}^s_{\mathrm{comp}}(\Omega,\mathcal{K}^{s,-\gamma_2+\mu,-\gamma_1+\mu}(I^\wedge))\to\mathcal{W}^\infty_{\mathrm{loc}}(\Omega,\mathcal{K}^{\infty,-\gamma_2,-\gamma_1}_{Q_2,Q_1}(I^\wedge)),$$

for all $s\in\mathbb{R}$, and corresponding C-dependent asymptotic types $P_j,\ Q_j,\ j=1,2$, where C^* is the formal adjoint of C with respect to the $\mathcal{W}^0_{\mathrm{comp}}(\Omega,\mathcal{K}^{0,0,0}(I^\wedge))$-scalar product. Green operators on an open set $\Omega\subseteq\mathbb{R}^d$, of order $\mu\in\mathbb{R}$, associated with the weight data (2.79) are defined as sums $G:=\mathrm{Op}(g)+C$ for a Green symbol g and a smoothing Green operator.

For $g(z,z',\zeta)$ in (2.81) we find a left symbol $g_{\mathrm{L}}(z,\zeta)\in\mathcal{R}_G^\mu(U\times\mathbb{R}^d,\boldsymbol{g})_{(I^\wedge,I^\wedge)}$ such that $\mathrm{Op}(g)-\mathrm{Op}(g_{\mathrm{L}})$ is a smoothing Green operator. The proof is similar to the case of classical scalar pseudo-differential operators. Starting from (2.81) it suffices to pass to $g_{\mathrm{L}}(z,\zeta)\sim\sum_{\alpha\in\mathbb{N}^d}1/\alpha!(\partial_{z'}^\alpha D_\zeta^\alpha g)|_{z'=z}(z,\zeta)$, where the assumptions of Theorem 2.22 for the asymptotic summation are satisfied. If a Green operator is written $G:=\mathrm{Op}(g)+C$ for a $g(z,\zeta)\in\mathcal{R}_G^{\mu-(N+1)}(\Omega\times\mathbb{R}^d,\boldsymbol{g})_{(I^\wedge,I^\wedge)}$, $\Omega\subseteq\mathbb{R}^d$ open, and a smoothing Green operator C, we set

$$\sigma_1(G)(z,\zeta):=g_{(\mu)}(z,\zeta) \tag{2.82}$$

where $g_{(\mu)}(z,\zeta)$ is the homogeneous principal part of g as a classical symbol of order μ. Incidentally, instead of $g_{(\mu)}(z,\zeta)$ we also write $\sigma_1(g)(z,\zeta)$.

Proposition 2.23 *Every Green operator G can be written in the form $G=G_0+C$ for a properly supported Green operator G_0 and a smoothing Green operator*

Proof Write $g(z, z', \zeta)$ in (2.81) as $\psi(z, z')g(z, z', \zeta) + (1 - \psi(z, z'))g(z, z', \zeta)$ for a function $\psi(z, z') \in C^\infty(\Omega \times \Omega)$ with proper support (i.e., every strip $A \times \Omega$ and $\Omega \times B$ for arbitrary $A, B \Subset \Omega$ intersects $\operatorname{supp} \psi$ in a compact set) such that $\operatorname{supp} \psi$ contains $\operatorname{diag}(\Omega \times \Omega)$ in its open interior. Then $G_0 = \operatorname{Op}(\psi g)$ is properly supported. Applying the asymptotic expansion that turns $(1 - \psi)g$ to a left symbol we easily see that $C = \operatorname{Op}((1 - \psi)g)$ is a smoothing Green operator. □

Theorem 2.24 *Let $G := Op(g) + C$ be a Green operator on $\Omega \subseteq \mathbb{R}^d$, of order μ, associated with the weight data* (2.79)*. Then G induces continuous operators*

$$G : \mathcal{W}^s_{comp}(\Omega, \mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge)) \to \mathcal{W}^{s-\mu}_{loc}(\Omega, \mathcal{K}^{\infty,\gamma_2-\mu,\gamma_1-\mu}_{P_2,P_1}(I^\wedge)),$$

for all $s \in \mathbb{R}$, for asymptotic types P_2, P_1, independent of s. If G is properly supported we can write loc or comp or comp on both sides.

Proof The proof is a direct consequence of the second part of formula (2.21). □

Theorem 2.25 *Let $G := Op(g) + C$ and $L := Op(l) + D$ be Green operators with symbols $g(z, \zeta) \in \mathcal{R}^\mu_G(\Omega \times \mathbb{R}^d, \boldsymbol{g})_{(I^\wedge, I^\wedge)}$ and $l(z, \zeta) \in \mathcal{R}^\nu_G(\Omega \times \mathbb{R}^d, \boldsymbol{c})_{(I^\wedge, I^\wedge)}$, for $\mu, \nu \in \mathbb{R}$, and corresponding smoothing Green operators C and D, respectively. We realise G, L as continuous operators*

$$G : \mathcal{W}^s_{comp}(\Omega, \mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge)) \to \mathcal{W}^{s-\mu}_{loc}(\Omega, \mathcal{K}^{s-\mu,\gamma_2-\mu,\gamma_1-\mu}(I^\wedge)),$$
$$L : \mathcal{W}^{s-\mu}_{comp}(\Omega, \mathcal{K}^{s-\mu,\gamma_2-\mu,\gamma_1-\mu}(I^\wedge)) \to \mathcal{W}^{s-\nu}_{loc}(\Omega, \mathcal{K}^{s-\mu-\nu,\gamma_2-\mu-\nu,\gamma_1-\mu-\nu}(I^\wedge)),$$

assuming an obvious compatibility of weights in the involved data $\boldsymbol{g}$, $\boldsymbol{c}$. Moreover, we assume that B or G is properly supported, such that B or G operate both in comp and loc-spaces. Then the composition LG is a Green operator, i.e., of the form

$$LG = Op(f) + B$$

for some $f(z, \zeta) \in \mathcal{R}^{\mu+\nu}_G(\Omega \times \mathbb{R}^d, \boldsymbol{b})_{(I^\wedge, I^\wedge)}$ with weight data $\boldsymbol{b} = \boldsymbol{l} \circ \boldsymbol{g} := (\gamma_i, \gamma_i - (\mu + \nu), \Theta_i)_{i=1,2}$, and a smoothing Green operator B, where

$$\sigma_1(LG)(z, \zeta) = \sigma_1(L)(z, \zeta)\sigma_1(G)(z, \zeta). \tag{2.83}$$

Proof The proof follows in an analogous manner as in the scalar calculus of pseudo-differential operators. □

3 Mellin Operators

3.1 Mellin Operators of First Singularity Order

Pseudo-differential operators based on the Mellin transform will appear in this paper in different variants. In this subsection we briefly recall the shape of Mellin operators that are known from the cone and edge calculus, i.e., of singularitiy order 1. We also formulate Green operator-valued Mellin symbols on the interval I. Those will contribute to the corner pseudo-differential calculus over $I^\wedge$. In the simplest case we have

$$\mathrm{op}_M^\gamma(f)u(t) := \int_{\Gamma_{1/2-\gamma}} \int_{\mathbb{R}_+} (t/t')^{-v} f(v)u(t')dt'/t' đ v, \quad đ v = (2\pi i)^{-1} dv, \quad (3.1)$$

for a symbol $f(v) \in S^\mu(\Gamma_{1/2-\gamma})$, cf. also (2.27). In this notation $\tau = \mathrm{Im}\, v$ plays the role of the covariable. The expression (3.1) is interpreted as a Mellin oscillatory integral, first for $u \in C_0^\infty(\mathbb{R}_+)$ and then extended to more general distribution spaces, e.g., $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$. We apply here Mellin operators in numerous variants, e.g., with symbols

$$f(t, t', v) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, S^\mu(\Gamma_{1/2-\gamma}))$$

with variable coefficients, or taking values in several operator classes, analogously as those with twisted symbolic estimates, cf. the terminology in Sect. 2.3. We first consider operators (3.1) where the symbol f extends to the complex v-plane as a meromorphic function.

By $\mathcal{A}(G)$, $G \subseteq \mathbb{C}$ open, we denote the space of all holomorphic functions in G. Similarly as (2.60) we consider sequences

$$S := \{(s_l, n_l)\}_{l \in \mathbb{L}} \subset \mathbb{C} \times \mathbb{N} \quad (3.2)$$

for an index set $\mathbb{L} \subseteq \mathbb{Z}$, and we assume that $\pi_{\mathbb{C}} S := \{s_l\}_{l \in \mathbb{L}}$ intersects every strip $\{c \leq \mathrm{Re}\, v \leq c'\}$ in a finite set. We call S a Mellin asymptotic type. Then

$$M_S^{-\infty}$$

denotes the set of all $f(v) \in \mathcal{A}(\mathbb{C}_v \setminus \pi_{\mathbb{C}} S)$ that are meromorphic with poles at the points s_l of multiplicity $n_l + 1$ and such that for any $\pi_{\mathbb{C}} S$-excision function χ (i.e., $\chi \in C^\infty(\mathbb{C})$, $\chi(v) = 0$ for $\mathrm{dist}\,(\pi_{\mathbb{C}} S, v) < \varepsilon_0$, $\chi = 1$ for $\mathrm{dist}\,(\pi_{\mathbb{C}} S, v) > \varepsilon_1$, for some $0 < \varepsilon_0 < \varepsilon_1$)

$$\chi f|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta)$$

for every real β, uniformly in compact β-intervals.

Let us now turn to smoothing Mellin symbols of the corner calculus. First we formulate such symbols for $\partial^i I^\wedge \cong \mathbb{R}_+$, $i = 0, 1$. Those are well-known in the calculus of boundary value problems without the transmission property at the boundary, here in the framework of the edge calculus over the half space $\overline{\mathbb{R}}_{+,t} \times \mathbb{R}_z^d$ where the boundary $\mathbb{R}_z^d$ is interpreted as an edge. We fix any strictly positive function $\zeta \to [\zeta]$ in $C^\infty(\mathbb{R}^d)$ with the property $[\zeta] = |\zeta|$ for $|\zeta| \geq c$ for some $c > 0$. Moreover, we choose arbitrary cut-off functions σ, σ' on the t half-axis. For any function $\varphi(t)$ we set

$$\varphi_\zeta(t) := \varphi(t[\zeta]).$$

Now a smoothing Mellin edge symbol in $(z, \zeta) \in U \times \mathbb{R}^d$ for open $U \subseteq \mathbb{R}^b$ is of the form

$$m(z, \zeta) := t^{-\mu} \sigma_\zeta \sum_{j=0}^{k} t^j \sum_{|\alpha| \leq j} \mathrm{op}_{M_t}^{\gamma_{2,j\alpha}} (f_{j\alpha})(z) \zeta^\alpha \sigma'_\zeta \tag{3.3}$$

for $f_{j\alpha}(z, v) \in C^\infty(U, M_{S_{j\alpha}}^{-\infty})$, where $S_{j\alpha}$ are Mellin asymptotic types and $\gamma_{2,j\alpha} \in \mathbb{R}$ weights such that

$$\gamma_2 - j \leq \gamma_{2,j\alpha} \leq \gamma_2, \quad \pi_{\mathbb{C}} S_{j\alpha} \cap \Gamma_{1/2-\gamma_{2,j\alpha}} = \emptyset$$

for all j, α. The meaning of $k \in \mathbb{N}$ in the sum (3.3) is that whenever we talk about families of such Mellin operators we assume that

$$\Lambda := (-(k+1), 0]$$

is the weight interval in asymptotics on the t half-axis for $t \to 0$. Recall that we have

$$m(z, \zeta) \in S_{\mathrm{cl}}^\mu (U \times \mathbb{R}^d; \mathcal{K}^{s,\gamma_2}(\mathbb{R}_+), \mathcal{K}^{\infty,\gamma_2-\mu}(\mathbb{R}_+)) \tag{3.4}$$

and

$$m(z, \zeta) \in S_{\mathrm{cl}}^\mu (U \times \mathbb{R}^d; \mathcal{K}_{P_2}^{s,\gamma_2}(\mathbb{R}_+), \mathcal{K}_{Q_2}^{\infty,\gamma_2-\mu}(\mathbb{R}_+)) \tag{3.5}$$

for every $s \in \mathbb{R}$ and every asymptotic type P_2 for some resulting Q_2; clearly P_2 and Q_2 refer to asymptotics for $t \to 0$. Identifying $\mathbb{R}_+$ with $\partial^0 I^\wedge$ by

$$\mathcal{R}_{M+G}^\mu (U \times \mathbb{R}^d, \boldsymbol{g}_2)_{(\partial^0 I^\wedge, \partial^0 I^\wedge)} \quad \text{for} \quad \boldsymbol{g}_2 = (\gamma_2, \gamma_2 - \mu, \Lambda) \tag{3.6}$$

we denote the set of all $(m + g)(z, \zeta)$ for arbitrary $m(z, \zeta)$ of the form (3.3) and $g(z, \zeta) \in \mathcal{R}_{M+G}^\mu (U \times \mathbb{R}^d, \boldsymbol{g}_2)_{(\partial^0 I^\wedge, \partial^0 I^\wedge)}$.

3.2 Mellin Operators of Second Singularity Order

Another kind of smoothing Mellin symbols is based on Green operators, referring to the interval I with two conical end points. According to the general terminology of the cone pseudo-differential calculus by

$$L_G(I, \boldsymbol{g}_1) \tag{3.7}$$

for weight data $\boldsymbol{g}_1 := (\gamma_1, \gamma_1 - \mu, \Theta)$, with a weight interval Θ as in (2.64) and a weight $\gamma_1 \in \mathbb{R}$, we denote the space of all $G \in \bigcap_{s \in \mathbb{R}} \mathcal{L}(H^{s,\gamma_1}(I), H^{s-\mu,\gamma_1-\mu}(I))$, cf. the spaces (2.17), that induce continuous operators

$$G : H^{s,\gamma_1}(I) \to H_P^{s-\mu,\gamma_1-\mu}(I)$$

and

$$G^* : H^{-s+\mu,-\gamma_1+\mu}(I) \to H_Q^{-s,-\gamma_1}(I)$$

for all $s \in \mathbb{R}$ and G-dependent asymptotic types P and Q, see the notation (2.65).

If we fix P and Q we obtain a subspace $L_G(I, \boldsymbol{g}_1)_{P,Q} \subset L_G(I, \boldsymbol{g}_1)$ which is Fréchet in a natural way. Now let us fix a Mellin asymptotic type T as in (3.2), and let

$$M_T^{-\infty}(I, \boldsymbol{g}_1)_{P,Q}$$

be the set of all

$$f(v) \in \mathcal{A}(\mathbb{C}\backslash\pi_{\mathbb{C}}T, L_G(I, \boldsymbol{g}_1)_{P,Q})$$

such that f is meromorphic with poles at the points s_l of multiplicity $n_l + 1$ and such that for any $\pi_{\mathbb{C}}T$-excision function χ we have

$$\chi f|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta, L_G(I, \boldsymbol{g}_1)_{P,Q})$$

for every real β, uniformly in compact β-intervals. In addition we require that the Laurent coefficients of $f(v)$ at the powers $(v - s_l)^{-(k+1)}$, $0 \le k \le n_l$, are of finite rank.

Set

$$M_T^{-\infty}(I, \boldsymbol{g}_1) := \bigcup_{P,Q} M_T^{-\infty}(I, \boldsymbol{g}_1)_{P,Q}$$

where the union is taken over all asymptotic types P and Q, associated with the weight data involved in the definition of $L_G(I, \boldsymbol{g}_1)_{P,Q}$.

In the corner calculus of boundary value problems we have Mellin operator families of a similar structure as (3.3), namely,

$$m(z, \zeta) := t^{-\mu}\sigma_\zeta \sum_{j=0}^{k} t^j \sum_{|\alpha|\le j} \mathrm{op}_{M_t}^{\gamma_{2,j\alpha}-1/2}(f_{j\alpha})(z)\zeta^\alpha\sigma'_\zeta \tag{3.8}$$

for $f_{j\alpha}(z, v) \in C^\infty(U, M_{T_{j\alpha}}^{-\infty}(I, \boldsymbol{g}_1))$, where $T_{j\alpha}$ are Mellin asymptotic types and $\gamma_{2,j\alpha} \in \mathbb{R}$ weights such that

$$\gamma_2 - j \le \gamma_{2,j\alpha} \le \gamma_2, \quad \pi_{\mathbb{C}}T_{j\alpha} \cap \Gamma_{1-\gamma_{2,j\alpha}} = \emptyset$$

for all j, α.

Proposition 3.1 *The family of operators* (3.8) *defines elements*

$$m(z, \zeta) \in S_{cl}^\mu(U \times \mathbb{R}^d; \mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge), \mathcal{K}^{\infty,\gamma_2-\mu,\gamma_1-\mu}(I^\wedge)) \tag{3.9}$$

and

$$m(z,\zeta)\in S^{\mu}_{cl}(U\times\mathbb{R}^d;\mathcal{K}^{s,\gamma_2,\gamma_1}_{P_2,P_1}(I^\wedge),\mathcal{K}^{\infty,\gamma_2-\mu,\gamma_1-\mu}_{Q_2,Q_1}(I^\wedge)) \tag{3.10}$$

for every $s\in\mathbb{R}$ *and every pair of asymptotic types* P_2,P_1 *for some resulting* Q_2,Q_1.

Proof Let us write (3.8) in the form

$$m(z,\zeta):=t^{-\mu}\sigma_\zeta\sum_{j=0}^{k}t^j\sum_{l=0}^{j}\sum_{j-|\alpha|=l}\mathrm{op}^{\gamma_2,j\alpha-1/2}_{M_t}(f_{j\alpha})(z)\zeta^\alpha\sigma'_\zeta.$$

Then, for

$$m_{\mu-l}(z,\zeta):=t^{-\mu}\sigma_\zeta\sum_{j=0}^{k}t^j\sum_{j-|\alpha|=l}\mathrm{op}^{\gamma_2,j\alpha-1/2}_{M_t}(f_{j\alpha})(z)\zeta^\alpha\sigma'_\zeta \tag{3.11}$$

we have $m(z,\zeta)=\sum_{l=0}^{k}m_{\mu-l}(z,\zeta)$ and

$$m_{\mu-l}(z,\delta\zeta)=\delta^{\mu-l}\,{}^2\kappa_\delta m_{\mu-l}(z,\zeta)\,({}^2\kappa_\delta)^{-1}$$

for all $\delta\geq 1$, $|\zeta|\geq C$, for some $C>0$. Because of Remark 2.6 it remains to observe that $m(z,\zeta)$ is a smooth function with values in

$$\mathcal{L}(\mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge),\mathcal{K}^{\infty,\gamma_2-\mu,\gamma_1-\mu}(I^\wedge))\quad\text{and}\quad\mathcal{L}(\mathcal{K}^{s,\gamma_2,\gamma_1}_{P_2,P_1}(I^\wedge),\mathcal{K}^{\infty,\gamma_2-\mu,\gamma_1-\mu}_{Q_2,Q_1}(I^\wedge)),$$

respectively, for all $s\in\mathbb{R}$. □

Definition 3.2 By

$$\mathcal{R}^{\mu}_{M+G}(U\times\mathbb{R}^d,\boldsymbol{g})_{(I^\wedge,I^\wedge)}$$

for $\mu\in\mathbb{R}$ and weight data $\boldsymbol{g}:=(\boldsymbol{g}_2,\boldsymbol{g}_1)$ for $\boldsymbol{g}_i=(\gamma_i,\gamma_i-\mu,(-(k+1),0])$, $i=1,2$, we denote the set of all operator families

$$(m+g)(z,\zeta) \tag{3.12}$$

for $m(z,\zeta)$ as in (3.8) and $g(z,\zeta)\in\mathcal{R}^{\mu}_{G}(U\times\mathbb{R}^d,\boldsymbol{g})_{(I^\wedge,I^\wedge)}$.

4 Corner-Degenerate Operators

4.1 Corner Symbols and Quantisations

Let $\mathrm{Diff}^\mu(X)$ for a smooth manifold X be the space of all differential operators on X of order $\mu\in\mathbb{N}$ with smooth coefficients in local coordinates. Moreover, if B is a manifold with edge Y, cf. the notation in Sect. 2.1, by $\mathrm{Diff}^{\mu}_{\deg}(B)$ we denote the space of all $A \in \mathrm{Diff}^{\mu}(s_0(B))$ that are locally near Y in the variables $(r, x, y) \in \mathbb{R}_+ \times X \times \mathbb{R}^q$ for $q > 0$, cf. the formula (2.5), of the form

$$A = r^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j\alpha}(r, y)(-r\partial r)^j (rD_y)^{\alpha} \tag{4.1}$$

for coefficients $a_{j\alpha} \in C^{\infty}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X))$. For $q = 0$ the manifold B has conical singularities. In this case, instead of (4.1) we assume

$$A = r^{-\mu} \sum_{j=0}^{\mu} a_j(r)(-r\partial r)^j \tag{4.2}$$

for coefficients $a_j \in C^{\infty}(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j}(X))$. The base X of the local cone close to the conical point $s_1(B)$ may have different connected components. Those can be interpreted as several conical singularities of B. If we want to distinguish them we ask the local form (4.2) close to the different conical points $\{c_0, c_1, \ldots\} = s_0(B)$ with respect to the individual base manifolds X_l that depend on the corresponding c_l. In particular, for $B := I = \{r \in \mathbb{R} : 0 \le r \le 1\}$ we have two different conical points $r = 0$ and $r = 1$, and the respective cone bases are of dimension 0. In this case the operators in

$$\mathrm{Diff}^{\mu}_{\mathrm{deg}}(I) \tag{4.3}$$

are characterised by scalar coefficients $\{a^i_j\}_{j=0,\ldots,\mu}$, for $i = 0$ or $i = 1$, according to $r = 0$ or $r = 1$.

Let $M \in \mathfrak{M}_2$ be a stratified space as in Sect. 2.1. Then $\mathrm{Diff}^{\mu}_{\mathrm{deg}}(M)$ is defined as the space of all $A \in \mathrm{Diff}^{\mu}(s_0(M))$ belonging to $\mathrm{Diff}^{\mu}_{\mathrm{deg}}(M \setminus s_2(M))$ that are locally near $Z = s_2(M)$ and $r = 0$ in the variables $(t, r, z) \in \mathbb{R}_+ \times I \times \mathbb{R}^d$, cf. (4.1), of the form

$$A = r^{-\mu} t^{-\mu} \sum_{j+|\alpha|+l+|\beta|\le\mu} a_{j\alpha l\beta}(r, y, t, z)(-r\partial r)^j (rD_y)^{\alpha} (-rt\partial t)^l (rtD_z)^{\beta} \tag{4.4}$$

for coefficients $a_{j\alpha l\beta} \in C^{\infty}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q \times \overline{\mathbb{R}}_+ \times \mathbb{R}^d)$. A similar representation is assumed locally near Z and $r = 1$, the end point of I. Instead of (4.4) for $r = 0$ and $r = 1$ we can equivalently assume

$$A = t^{-\mu} \sum_{k+|\delta|\le\mu} c_{k\delta}(t, z)(-t\partial t)^k (tD_z)^{\delta} \tag{4.5}$$

for coefficients $c_{k\delta}(t, z) \in C^{\infty}(\overline{\mathbb{R}}_+ \times \mathbb{R}^d, \mathrm{Diff}^{\mu-(k+|\delta|)}_{\mathrm{deg}}(I))$.

Definition 4.1 Let $\Theta_1 := (-(k_1+1), 0]$, $k_1 \in \mathbb{N}$.

(i) Let $R^{\mu}_{\mathrm{edge},G}(\overline{\mathbb{R}}_+ \times U \times \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}}, \boldsymbol{g}_1)$ for $U \subseteq \mathbb{R}^b$ open, $\boldsymbol{g}_1 := (\gamma_1, \gamma_1 - \mu, \Theta_1)$, be the space of all

$$g_{\mathrm{edge}}(t, z, \tilde{\tau}, \tilde{\zeta}) \in \bigcap_{s,e \in \mathbb{R}} S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times U \times \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}}; \mathcal{K}^{s,\gamma_1;e}(\mathbb{R}_+), \mathcal{K}^{\infty,\gamma_1-\mu;\infty}_{P_1}(\mathbb{R}_+))$$

such that

$$g^*_{\mathrm{edge}}(t, z, \tilde{\tau}, \tilde{\zeta}) \in \bigcap_{s,e \in \mathbb{R}} S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times U \times \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}}; \mathcal{K}^{s,-\gamma_1+\mu;e}(\mathbb{R}_+), \mathcal{K}^{\infty,-\gamma_1;\infty}_{Q_1}(\mathbb{R}_+))$$

for certain g_{edge}-dependent asymptotic types P_1, Q_1, associated with $(\gamma_1 - \mu, \Theta_1)$ and $(-\gamma_1, \Theta_1)$, respectively.

(ii) Let $R^{\mu}_{\mathrm{edge},M+G}(\overline{\mathbb{R}}_+ \times U \times \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}}, \boldsymbol{g}_1)$ for U and $\boldsymbol{g}_1$ as in (i) be the space of all operator families

$$(m_{\mathrm{edge}} + g_{\mathrm{edge}})(t, z, \tilde{\tau}, \tilde{\zeta})$$

for $g_{\mathrm{edge}}(t, z, \tilde{\tau}, \tilde{\zeta}) \in R^{\mu}_{\mathrm{edge},G}(\overline{\mathbb{R}}_+ \times U \times \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}}, \boldsymbol{g}_1)$ and for cut-off functions ω, ω' on the r half-axis

$$m_{\mathrm{edge}}(t, z, \tilde{\tau}, \tilde{\zeta}) := r^{-\mu} \omega_{\tilde{\tau},\tilde{\zeta}} \sum_{j=0}^{k_1} r^j \sum_{|\alpha| \le j} \mathrm{op}^{\gamma_{1,j\alpha}}_{M_r}(f_{j\alpha})(t, z)(\tilde{\tau}, \tilde{\zeta})^{\alpha} \omega'_{\tilde{\tau},\tilde{\zeta}}. \tag{4.6}$$

Here $f_{j\alpha}(t, z) \in C^{\infty}(\overline{\mathbb{R}}_+ \times U, M^{-\infty}_{R_{j\alpha}})$ for Mellin asymptotic types $R_{j\alpha}$ referring to the $\mathbb{C}_w$-plane and weights $\gamma_{j\alpha} \in \mathbb{R}$ such that

$$\gamma_1 - j \le \gamma_{1,j\alpha} \le \gamma_1, \pi_{\mathbb{C}} R_{j\alpha} \cap \Gamma_{1/2-\gamma_{1,j\alpha}} = \emptyset.$$

(iii) By $C^{\infty}(\overline{\mathbb{R}}_+ \times U, L^{\mu}_{\mathrm{edge},M+G}(I, \boldsymbol{g}; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}}))$ for $\boldsymbol{g} = (\boldsymbol{g}_0, \boldsymbol{g}_1)$ we denote the space of all operator functions of the form

$$\begin{aligned}\tilde{b}_{\mathrm{edge},M+G}(t, z, \tilde{\tau}, \tilde{\zeta}) &:= \omega_0 (m_{\mathrm{edge},0} + g_{\mathrm{edge},0})(t, z, \tilde{\tau}, \tilde{\zeta}) \omega'_0 \\ &\quad + \vartheta_*^{-1} \omega_1 (m_{\mathrm{edge},1} + g_{\mathrm{edge},1})(t, z, \tilde{\tau}, \tilde{\zeta}) \omega'_1 \end{aligned} \tag{4.7}$$

for symbols $(m+g)_{\mathrm{edge},i}(t, z, \tilde{\tau}, \tilde{\zeta}) \in R^{\mu}_{\mathrm{edge},M+G}(\overline{\mathbb{R}}_+ \times U \times \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}}, \boldsymbol{g}_1)$ introduced in (ii). Here $\omega_i \prec \omega'_i$, $i = 0, 1$, are cut-off functions on the r half-axis such that $\omega_0(r) + \omega_1(-r+1) = 1$ on the interval I, and ϑ_*^{-1} is the push forward belonging to the inverse of (2.66).

Let us now consider symbols

$$p_{i,\text{loc}}(t,r,z,\tilde{\tau},\rho,\tilde{\zeta}) = \tilde{\tilde{p}}_{i,\text{loc}}(t,r,z,r\tilde{\tau},r\rho,r\tilde{\zeta})$$

for $\tilde{\tilde{p}}_{i,\text{loc}}(t,r,z,\tilde{\tilde{\tau}},\tilde{\rho},\tilde{\tilde{\zeta}}) \in S^{\mu}_{\text{cl}}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times U \times \mathbb{R}^{2+d}_{\tilde{\tilde{\tau}},\tilde{\rho},\tilde{\tilde{\zeta}}})$, $i = 0, 1$. Via Mellin quantisation in r-direction with $\tilde{\tilde{p}}_{i,\text{loc}}(t,r,z,\tilde{\tilde{\tau}},r\rho,\tilde{\tilde{\zeta}})$ we associate an

$$\tilde{\tilde{h}}_{i,\text{loc}}(t,r,z,\tilde{\tilde{\tau}},w,\tilde{\tilde{\zeta}}) \in S^{\mu}_{\mathcal{O}_w}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times U \times \mathbb{R}^{1+d}_{\tilde{\tilde{\tau}},\tilde{\tilde{\zeta}}})$$

such that for $h_{i,\text{loc}}(t,r,z,\tilde{\tau},w,\tilde{\zeta}) := \tilde{\tilde{h}}_{i,\text{loc}}(t,r,z,r\tilde{\tau},w,r\tilde{\zeta})$ we have

$$\text{Op}_r(p_{i,\text{loc}})(t,z,\tilde{\tau},\tilde{\zeta}) = \text{op}^{\beta}_{M_r}(h_{i,\text{loc}})(t,z,\tilde{\tau},\tilde{\zeta})$$

modulo $C^\infty(\overline{\mathbb{R}}_+ \times U, L^{-\infty}(\mathbb{R}_+; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}}))$, for every $\beta \in \mathbb{R}$.

Let us now form

$$\begin{aligned}\tilde{a}_{\text{edge}}(t,z,\tilde{\tau},\tilde{\zeta}) := \omega_0 r^{-\mu}\{&\omega_{\tilde{\tau},\tilde{\zeta}}\text{op}^{\gamma_{1,0}}_{M_r}(h_{0,\text{loc}})(t,z,\tilde{\tau},\tilde{\zeta})\omega'_{\tilde{\tau},\tilde{\zeta}}\\ &+ (1-\omega_{\tilde{\tau},\tilde{\zeta}})\text{Op}_r(p_{0,\text{loc}})(t,z,\tilde{\tau},\tilde{\zeta})(1-\omega''_{\tilde{\tau},\tilde{\zeta}})\}\omega'_0\\ &+ \vartheta_*^{-1}\omega_1 r^{-\mu}\{\omega_{\tilde{\tau},\tilde{\zeta}}\text{op}^{\gamma_{1,1}}_{M_r}(h_{1,\text{loc}})(t,z,\tilde{\tau},\tilde{\zeta})\omega'_{\tilde{\tau},\tilde{\zeta}}\\ &+ (1-\omega_{\tilde{\tau},\tilde{\zeta}})\text{Op}_r(p_{1,\text{loc}})(t,z,\tilde{\tau},\tilde{\zeta})(1-\omega''_{\tilde{\tau},\tilde{\zeta}})\}\omega'_1,\end{aligned} \tag{4.8}$$

$\omega_{\tilde{\tau},\tilde{\zeta}}(r) := \omega(r|\tilde{\tau},\tilde{\zeta}|)$. Since the final results are independent of the choice of the cut-off functions $\omega'' \prec \omega \prec \omega'$ on the r half-axis we take the same both for $i = 0$ and $i = 1$.

Let

$$C^\infty(\overline{\mathbb{R}}_+ \times U, L^{\mu}(I, \boldsymbol{g}_1; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}})) \tag{4.9}$$

be the set of all operator functions

$$\tilde{p}(t,z,\tilde{\tau},\tilde{\zeta}) := \tilde{a}_{\text{edge}}(t,z,\tilde{\tau},\tilde{\zeta}) + \tilde{b}_{\text{edge,M+G}}(t,z,\tilde{\tau},\tilde{\zeta}) + \tilde{c}_{\text{edge}}(t,z,\tilde{\tau},\tilde{\zeta}), \tag{4.10}$$

where $\tilde{a}_{\text{edge}}(t,z,\tilde{\tau},\tilde{\zeta})$ and $\tilde{b}_{\text{edge,M+G}}(t,z,\tilde{\tau},\tilde{\zeta})$ are given by (4.8) and (4.7), respectively, while $\tilde{c}_{\text{edge}}(t,z,\tilde{\tau},\tilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times U, \mathcal{S}(\mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}}, L_G(I, \boldsymbol{g}_1)))$, cf. formula (3.7). In an analogous manner we define

$$C^\infty(\overline{\mathbb{R}}_+ \times U, L^{\mu}(I, \boldsymbol{g}_1; \mathbb{R}^{d}_{\tilde{\zeta}}))$$

by simply omitting everywhere the variable $\tilde{\tau}$ and

$$C^\infty(\overline{\mathbb{R}}_+ \times U, L^{\mu}(I, \boldsymbol{g}_1; \Gamma_\beta \times \mathbb{R}^{d}_{\tilde{\zeta}}))$$

by replacing $\tilde{\tau}$ in (4.9) by $\operatorname{Im} v$ for $v \in \Gamma_\beta$.

Now let

$$C^\infty(\overline{\mathbb{R}}_+ \times U, M^\mu_{\mathcal{O}_v}(I, \boldsymbol{g}_1; \mathbb{R}^d_{\tilde{\zeta}})) \tag{4.11}$$

be the space of all

$$\tilde{h}(t, z, v, \tilde{\zeta}) \in \mathcal{A}(\mathbb{C}_v, C^\infty(\overline{\mathbb{R}}_+ \times U, L^\mu(I, \boldsymbol{g}_1; \mathbb{R}^d_{\tilde{\zeta}})))$$

such that

$$\tilde{h}(t, z, \beta + i\tau, \tilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times U, L^\mu(I, \boldsymbol{g}_1; \Gamma_\beta \times \mathbb{R}^d_{\tilde{\zeta}}))$$

for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals. We employ the following Mellin quantisation result:

Theorem 4.2 *For every*

$$p(t, z, \tau, \zeta) := \tilde{p}(t, z, t\tau, t\zeta)$$

$\tilde{p}(t, z, \tilde{\tau}, \tilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times U, L^\mu(I, \boldsymbol{g}_1; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}}))$, *there exists an* $\tilde{h}(t, z, v, \tilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times U, M^\mu_{\mathcal{O}_v}(I, \boldsymbol{g}_1; \mathbb{R}^d_{\tilde{\zeta}}))$ *such that for*

$$h(t, z, v, \zeta) = \tilde{h}(t, z, v, t\zeta)$$

we have

$$op^\beta_{M_t}(h)(z, \zeta) = Op_t(p)(z, \zeta)$$

modulo $C^\infty(U, L^{-\infty}(\mathbb{R}_+ \times I, \boldsymbol{g}_1; \mathbb{R}^d_\zeta))$, *for every* $\beta \in \mathbb{R}$.

Theorems of that kind have been first established in connection with cone and edge pseudo-differential algebras, cf. [22, Theorem 2.3.7]. There are many variants and alternative proofs, see, in particular, [18] in the framework of boundary value problems with the transmission property at the boundary, Krainer [13] in connection with parabolic operators, or the iterative constructions for higher singularities in [9], [3]. For purposes below we form

$$p_0(t, z, \tau, \zeta) := \tilde{p}(0, z, t\tau, t\zeta),\ h_0(t, z, v, \zeta) := \tilde{h}(0, z, v, t\zeta).$$

Then, similarly as in Theorem 4.2 we have

$$\mathrm{op}^\beta_{M_t}(h_0)(z, \zeta) = \mathrm{Op}_t(p_0)(z, \zeta)$$

modulo $C^\infty(U, L^{-\infty}(\mathbb{R}_+ \times I, \boldsymbol{g}_1; \mathbb{R}^d_\zeta))$, for every $\beta \in \mathbb{R}$.

Definition 4.3 The space

$$\mathcal{R}^{\mu}(U \times \mathbb{R}^d, \boldsymbol{g}), \tag{4.12}$$

for $\mu \in \mathbb{R}$ and $\boldsymbol{g} = (\boldsymbol{g}_2, \boldsymbol{g}_1)$, $\boldsymbol{g}_i = (\gamma_i, \gamma_i - \mu, \Theta_i)$, $\Theta_i = (-(k_i + 1), 0]$, $i = 1, 2$, is defined as the set of all operator families

$$\begin{aligned} a(z, \zeta) := \boldsymbol{\sigma} t^{-\mu} \{\sigma_\zeta \mathrm{op}_{M_t}^{\gamma_2 - 1/2}(h)(z, \zeta)\sigma'_\zeta + (1 - \sigma_\zeta)\mathrm{Op}_t(p)(z, \zeta)(1 - \sigma''_\zeta)\}\boldsymbol{\sigma}' \\ + \boldsymbol{\varphi}\mathrm{Op}_t(p_{\mathrm{int}})(z, \zeta)\boldsymbol{\varphi}' + (m + g)(z, \zeta) \end{aligned} \tag{4.13}$$

for arbitrary p, h as in Theorem 4.2, $(m+g)(z, \zeta) \in \mathcal{R}^{\mu}_{M+G}(U \times \mathbb{R}^d, \boldsymbol{g})$, cf. Definition 3.2, $p_{\mathrm{int}}(t, z, \tau, \zeta) \in C^\infty(\mathbb{R}_+ \times U, L^\mu(I, \boldsymbol{g}_1; \mathbb{R}^{1+d}_{\tau,\zeta}))$, cut-off functions $\sigma'' \prec \sigma \prec \sigma'$, $\boldsymbol{\sigma}, \boldsymbol{\sigma}'$ on the t half-axis, and $\boldsymbol{\varphi}, \boldsymbol{\varphi}' \in C_0^\infty(\mathbb{R}_{+,t})$.

If $U = \Omega_2 \times \Omega_2$, $\Omega_2 \subseteq \mathbb{R}^d$ open, we write $(z, z') \in \Omega_2 \times \Omega_2$ rather than z. For $a(z, \zeta) \in \mathcal{R}^{\mu}(\Omega_2 \times \mathbb{R}^d, \boldsymbol{g})$ we set

$$\begin{aligned} \sigma_2(a)(z, \zeta) := t^{-\mu}\big\{\sigma_{|\zeta|}\mathrm{op}_{M_t}^{\gamma_2 - 1/2}(h_0)(z, \zeta)\sigma'_{|\zeta|} \\ + (1 - \sigma_{|\zeta|})\mathrm{Op}_t(p_0)(z, \zeta)(1 - \sigma''_{|\zeta|})\big\} + \sigma_2(m + g)(z, \zeta), \end{aligned} \tag{4.14}$$

$(z, \zeta) \in \Omega_2 \times (\mathbb{R}^d \setminus \{0\})$ for $\sigma_{|\zeta|}(t) := \sigma(t|\zeta|)$, etc., and $\sigma_2(m + g)(z, \zeta) := (m + g)_{(\mu)}(z, \zeta)$, with (μ) indicating the $({}^2\kappa)$-twisted homogeneous principal component of order μ of the corresponding classical symbol.

The operator families $a(z, \zeta) \in \mathcal{R}^{\mu}(U \times \mathbb{R}^d, \boldsymbol{g})$ contain information from the calculus of pseudo-differential operators on $I^\wedge$, interpreted as a (non-compact) manifold with edge $s_1(I^\wedge) = \partial^0 I^\wedge \cup \partial^1 I^\wedge$, cf. notation (2.2). Assume for the moment that $a = a(\zeta)$ is independent of z; the z-dependent case is straightforward and tacitly included below.

It is convenient for the moment to refer to a general manifold B with edge $s_1(B) = Y$, main stratum $s_0(B) = B \setminus Y$, where B is locally near $s_1(B)$ modelled on $X^\Delta \times \Omega_1$ for a smooth closed manifold X, $n = \dim X$, and open $\Omega_1 \subseteq \mathbb{R}^q$, corresponding to a chart on Y, $q = \dim Y$. A special case is $B = I^\wedge$, $Y = s_1(I^\wedge)$, $s_0(I^\wedge) = \mathbb{R}_+ \times (0, 1)$, where X is a single point. The well-known parameter-dependent edge calulus (edge algebra) contains edge-degenerate pseudo-differential operators, together with smoothing edge Mellin and Green operators. It is furnished by spaces

$$L^\mu(B, \boldsymbol{g}_1; \mathbb{R}^d_\zeta) \subseteq L^\mu_{\mathrm{cl}}(s_0(B); \mathbb{R}^d_\zeta) \tag{4.15}$$

of ζ-dependent classical pseudo-differential operators over $s_0(B) = B \setminus Y$, associated with the weight data $\boldsymbol{g}_1 = (\gamma_1, \gamma_1 - \mu, \Theta_1)$, cf. Dorschfeldt [5], or [2,3,11,12]. Notation has been changed and unified during the development of the past decade, in order to make the calculus iterative for increasing orders of singularities. In the present article we freely use notation and results of [27].

A $W(\zeta) \in L^\mu(B, \boldsymbol{g}_1; \mathbb{R}^d_\zeta)$ has a parameter-dependent homogeneous principal symbol of order μ

$$\sigma_0(W)(x, \xi, \zeta), \tag{4.16}$$

determined by $W(\zeta)$ regarded as an element of $L^{\mu}_{\mathrm{cl}}(s_0(B); \mathbb{R}^d_{\zeta})$, cf. (4.15). Here (x, ξ) means variables and covariables in $T^*(s_0(B))$, and (4.16) is homogeneous in $(\xi, \zeta) \neq 0$ of order μ. Moreover, let $\Omega_1 \subseteq \mathbb{R}^q$ for $q := \dim Y$ be an open set, belonging to a chart on $s_1(B) = Y$, with variables and covariables (y, η) on $\Omega_1 \times \mathbb{R}^q = T^*(\Omega_1)$. Then $W(\zeta) \in L^{\mu}(B, \boldsymbol{g}_1; \mathbb{R}^d_{\zeta})$ is locally near $s_1(B)$ modulo a local smoothing parameter-dependent edge operator of the form

$$\mathrm{Op}_y({}^1a)(\zeta) \tag{4.17}$$

for an ${}^1a(y, \eta, \zeta)$ belonging to a space of edge amplitude functions (for simplicity, left symbols)

$${}^1\mathcal{R}^{\mu}(\Omega_1 \times \mathbb{R}^{q+d}_{\eta,\zeta}, \boldsymbol{g}_1) \tag{4.18}$$

which is of a similar structure as (4.12). More precisely, first we have an analogue of Theorem 4.2, namely,

Theorem 4.4 *For every*

$${}^1p(r, y, \rho, \eta, \zeta) := {}^1\tilde{p}(r, y, r\rho, r\eta, r\zeta)$$

${}^1\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta}, \tilde{\zeta}) \in C^{\infty}(\overline{\mathbb{R}}_{+,r} \times \Omega_1, L^{\mu}_{cl}(X; \mathbb{R}^{q+d}_{\tilde{\rho},\tilde{\eta},\tilde{\zeta}}))$, *there exists an* ${}^1\tilde{h}(r, y, w, \tilde{\eta}, \tilde{\zeta}) \in C^{\infty}(\overline{\mathbb{R}}_{+} \times \Omega_1, M^{\mu}_{\mathcal{O}_w}(X; \mathbb{R}^{d}_{\tilde{\eta},\tilde{\zeta}}))$ *such that for*

$${}^1h(r, y, w, \eta, \zeta) := {}^1\tilde{h}(r, y, w, r\eta, r\zeta)$$

we have

$$op^{\beta}_{M_r}({}^1h)(y, \eta, \zeta) = Op_r({}^1p)(y, \eta, \zeta)$$

modulo $C^{\infty}(\Omega_1, L^{-\infty}(\mathbb{R}_+ \times X; \mathbb{R}^{q+d}_{\eta,\zeta}))$, *for every* $\beta \in \mathbb{R}$.

Setting

$${}^1p_0(r, y, \rho, \eta, \zeta) := {}^1\tilde{p}(0, y, r\rho, r\eta, r\zeta),\ {}^1h_0(r, y, w, \eta, \zeta) := {}^1\tilde{h}(0, y, w, r\eta, r\zeta),$$

we also have

$$\mathrm{op}^{\beta}_{M_r}({}^1h_0)(y, \eta, \zeta) = \mathrm{Op}_r({}^1p_0)(y, \eta, \zeta)$$

modulo $C^{\infty}(\Omega_1, L^{-\infty}(\mathbb{R}_+ \times X; \mathbb{R}^{q+d}_{\eta,\zeta}))$, for every $\beta \in \mathbb{R}$.

Other ingredients of (4.18) are spaces

$${}^1\mathcal{R}^{\mu}_{G}(\Omega_1 \times \mathbb{R}^{q+d}_{\eta,\zeta}, \boldsymbol{g}_1) \text{ and } {}^1\mathcal{R}^{\mu}_{M+G}(\Omega_1 \times \mathbb{R}^{q+d}_{\eta,\zeta}, \boldsymbol{g}_1)$$

of Green and smoothing Mellin plus Green edge symbols, respectively, cf. [22, Definitions 3.3.6, 3.3.14]. They are of a similar structure as those in Definition 3.2. The formal difference is that I is replaced by X, and the smoothing Mellin symbols $f(y, w)$ belong to $C^\infty(\Omega_1, M_S^{-\infty}(X))$ for a Mellin asymptotic type $S = \{(s_l, n_l)\}_{l\in\mathbb{L}}$, cf. formula (3.2), where $M_S^{-\infty}(X)$ consists of the set of all meromorphic functions with values in $L^{-\infty}(X) \cong C^\infty(X\times X)$ with poles at the points s_l of multiplicity n_l+1, and finite rank Laurent coefficients at $(w - s_l)^{-(k+1)}$, $0 \le k \le n_l$, and $\chi f|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta, L^{-\infty}(X))$ for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals.

Then (4.18) is the space of families of operators

$$
\begin{aligned}
{}^1a(y, \eta, \zeta) := \boldsymbol{\omega} r^{-\mu}\{\omega_{\eta,\zeta}\mathrm{op}_{M_r}^{\gamma_1-n/2}({}^1h)(y, \eta, \zeta)\omega'_{\eta,\zeta} + (1 - \omega_{\eta,\zeta})\mathrm{Op}_r({}^1p)(y, \eta, \zeta)\\
(1 - \omega''_{\eta,\zeta})\}\boldsymbol{\omega}' + \boldsymbol{\psi}\mathrm{Op}_r({}^1p_{\mathrm{int}})(y, \eta, \zeta)\boldsymbol{\psi}' + ({}^1m + {}^1g)(y, \eta, \zeta) \qquad (4.19)
\end{aligned}
$$

for arbitrary ${}^1p$, ${}^1h$ as in Theorem 4.4, $({}^1m + {}^1g)(y, \eta, \zeta) \in {}^1\mathcal{R}^\mu_{M+G}(\Omega_1 \times \mathbb{R}^{q+d}, \boldsymbol{g}_1)$, moreover, ${}^1p_{\mathrm{int}}(r, y, \rho, \eta, \zeta) \in C^\infty(\mathbb{R}_+ \times \Omega_1, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q+d}_{\rho,\eta,\zeta}))$, cut-off functions $\omega'' \prec \omega \prec \omega'$, $\boldsymbol{\omega}, \boldsymbol{\omega}'$ on the r half-axis, and $\boldsymbol{\psi}, \boldsymbol{\psi}' \in C_0^\infty(\mathbb{R}_{+,r})$.

For ${}^1a(y, y, \zeta) \in {}^1\mathcal{R}^\mu(\Omega_1 \times \mathbb{R}^{q+d}_{\eta,\zeta}, \boldsymbol{g}_1)$ we set

$$
\begin{aligned}
\sigma_1({}^1a)(y, \eta, \zeta) := r^{-\mu}\big\{&\omega_{|\eta,\zeta|}\mathrm{op}_{M_r}^{\gamma_1-n/2}({}^1h_0)(y, \eta, \zeta)\omega'_{|\eta,\zeta|}\\
&+(1 - \omega_{|\eta,\zeta|})\mathrm{Op}_r({}^1p_0)(y, \eta, \zeta)(1 - \omega''_{|\eta,\zeta|})\big\}\\
&+\sigma_2({}^1m + {}^1g)(y, \eta, \zeta), \qquad (4.20)
\end{aligned}
$$

$(y, \eta, \zeta) \in \Omega_1 \times (\mathbb{R}^{q+d}\setminus\{0\})$ for $\omega_{|\eta,\zeta|}(r) := \omega(r|\eta, \zeta|)$, etc., and $\sigma_2({}^1m + {}^1g)(y, \eta, \zeta) := ({}^1m + {}^1g)_{(\mu)}(y, \eta, \zeta)$, with (μ) indicating the $({}^1\kappa)$-twisted homogeneous principal component of order μ of the corresponding classical symbol.

A parameter-dependent operator $W(\zeta) \in L^\mu(B, \boldsymbol{g}_1; \mathbb{R}^d)$ then has a parameter-dependent homogeneous principal edge symbol $\sigma_1(W)$ of order μ, locally near $s_1(B)$ determined by (4.17), and we set

$$
\sigma_1(W)(y, \eta, \zeta) = \sigma_1({}^1a)(y, \eta, \zeta). \qquad (4.21)
$$

Together with (4.16) we have the principal symbolic hierarchy

$$
\sigma(W) = (\sigma_0(W), \sigma_1(W)) \qquad (4.22)
$$

of operators W in the edge calculus.

Remark 4.5 Note that the specific choice of the functions $\omega'' \prec \omega \prec \omega'$, $\boldsymbol{\omega}, \boldsymbol{\omega}'$ on the r half-axis, and $\boldsymbol{\psi}, \boldsymbol{\psi}' \in C_0^\infty(\mathbb{R}_{+,r})$ is not essential. Remainders under changing these functions remain in (4.18). In particular, if we assume $\boldsymbol{\omega} \succ \boldsymbol{\psi}$, $\boldsymbol{\omega}' \succ \boldsymbol{\psi}'$ the summand in (4.19) with the factors $\boldsymbol{\psi}, \boldsymbol{\psi}'$ can be integrated in the one with the factors $\boldsymbol{\omega}, \boldsymbol{\omega}'$, modulo a flat Green remainder (flat means trivial asymptotic types), though ${}^1p_{\mathrm{int}}(r, y, \rho, \eta, \zeta)$ is not edge-degenerate. Without loss of generality we could assume the latter contribution to be edge-degenerate, but since this term is localised off $r = 0$ both versions are equivalent modulo a flat Green term.

In particular, we may assume $\boldsymbol{\omega} \succ \omega_\eta$, $\boldsymbol{\omega}' \succ \omega'_\eta$ for all η. Thus

$$\boldsymbol{\omega}\omega_\eta = \omega_\eta, \ \boldsymbol{\omega}'\omega'_\eta = \omega'_\eta.$$

Let us now recall the following important relations. For every $s \in \mathbb{R}$ we have

$$\mathcal{R}^\mu(\Omega_1 \times \mathbb{R}^{q+d}, \boldsymbol{g}) \subset S^\mu(U \times \mathbb{R}^{q+d}; H, \widetilde{H}) \tag{4.23}$$

for the pair of spaces

$$H := \mathcal{K}^{s,\gamma_1}(X^\wedge), \ \widetilde{H} := \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^\wedge) \ \text{ or } \ H := \mathcal{K}^{s,\gamma_1}_{P_1}(X^\wedge), \ \widetilde{H} := \mathcal{K}^{s-\mu,\gamma_1-\mu}_{Q_1}(X^\wedge) \tag{4.24}$$

for asymptotic types P_1, associated with the weight data (γ_1, Θ_1) and some resulting Q_1, associated with $(\gamma_1 - \mu, \Theta_1)$. For references below we sketch here the main arguments, cf. also the constructions in [19]. Let us ignore elements $({}^1m + {}^1g)(y, \eta, \zeta) \in {}^1\mathcal{R}^\mu_{M+G}(\Omega_1 \times \mathbb{R}^{q+d}_{\eta,\zeta}, \boldsymbol{g}_1)$ of (4.18) which are even classical symbols with more specific properties. It suffices to consider symbols $a = a(y, \eta)$ since dimensions of variables and covariables are independent, and changing notation we may drop ζ. The dependence on the variable y does not cause any specific difficulty; so we drop it. Moreover, it is convenient first to assume that ${}^1\tilde{p}(r, \tilde{\rho}, \tilde{\eta})$ and ${}^1\tilde{h}(r, w, \tilde{\eta})$ are independent of r; the general case is treated by applying a tensor product argument, cf. details below. Thus, taking into account Remark 4.5 and setting

$${}^1p_0(r, \rho, \eta) := {}^1\tilde{p}(0, r\rho, r\eta), \ {}^1h_0(r, w, \eta) := {}^1\tilde{h}(0, w, r\eta) \tag{4.25}$$

it remains

$${}^1a(\eta) := {}^1b(\eta) + {}^1e(\eta) \tag{4.26}$$

for

$${}^1b(\eta) := r^{-\mu}\omega_\eta \mathrm{op}_{M_r}^{\gamma_1 - n/2}({}^1h_0)(\eta)\omega'_\eta, \tag{4.27}$$

$${}^1e(\eta) := \boldsymbol{\omega}\,{}^1f(\eta)\boldsymbol{\omega}' \ \text{ for } \ {}^1f(\eta) = r^{-\mu}(1 - \omega_\eta)\mathrm{Op}_r({}^1p_0)(\eta)(1 - \omega''_\eta). \tag{4.28}$$

Now we have ${}^1b(\eta) \in C^\infty(\mathbb{R}^q, \mathcal{L}(H, \tilde{H}))$ for the spaces in (4.24). The spaces with asymptotics in the second pair are written as projective limits of Hilbert spaces

$$\varprojlim_{m\in\mathbb{N}} H^m, \quad \varprojlim_{l\in\mathbb{N}} \widetilde{H}^l \tag{4.29}$$

for Hilbert subspaces

$$\cdots H^{m+1} \hookrightarrow H^m \hookrightarrow \cdots \hookrightarrow H^0 = \mathcal{K}^{s,\gamma_1}(X^\wedge)$$

and

$$\cdots \widetilde{H}^{l+1} \hookrightarrow \widetilde{H}^l \hookrightarrow \cdots \hookrightarrow \widetilde{H}^0 = \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^\wedge)$$

with group action ${}^1\kappa$, cf. also formula (2.73). In this case $A \in \mathcal{L}(H, \tilde{H})$ means the existence of a function $\mathrm{r} : \mathbb{N} \to \mathbb{N}$ such that $A \in \mathcal{L}(H^{\mathrm{r}(l)}, \tilde{H}^l)$ for all $l \in \mathbb{N}$, while

$$S^{\mu}_{(\mathrm{cl})}(\Omega_1 \times \mathbb{R}^q; H, \widetilde{H}) := \bigcup_{\mathrm{r}} \bigcap_{l \in \mathbb{N}} S^{\mu}_{(\mathrm{cl})}(\Omega_1 \times \mathbb{R}^q; H^{\mathrm{r}(l)}, \tilde{H}^l) \tag{4.30}$$

where the union in (4.30) is taken over all mappings $\mathrm{r} : \mathbb{N} \to \mathbb{N}$. Remark 2.6 is valid both for pairs of Hilbert and Fréchet spaces with group action. In our case we can apply this to the function (4.27) which belongs to $C^\infty(\mathbb{R}^q, \mathcal{L}(H^{\mathrm{r}(l)}, \tilde{H}^l))$ for a suitable $\mathrm{r} : \mathbb{N} \to \mathbb{N}$. Without loss of generality we can assume $\mathrm{r}(0) = 0$. Because of

$${}^1b(\delta\eta) = \delta^{\mu}\, {}^1\kappa_\delta\, {}^1b(\eta)({}^1\kappa_\delta)^{-1} \tag{4.31}$$

for all $\delta \geq 1$ and $|\eta| \geq const$ for some constant > 0 the assumptions of Remark 2.6 are satisfied, and we obtain the desired symbol property for ${}^1b(\eta)$. In addition the map

$$M^{\mu}_{\mathcal{O}_w}(X; \mathbb{R}^q_{\tilde{\eta}})) \to S^{\mu}_{\mathrm{cl}}(\Omega_1 \times \mathbb{R}^q; H^{\mathrm{r}(l)}, \tilde{H}^l),\ {}^1\tilde{h}(0, w, \tilde{\eta}) \mapsto {}^1b(\delta\eta)$$

for the indicated r is continuous.

The arguments for (4.28) are as follows. First note that

$${}^1e(\eta) \in C^\infty(\mathbb{R}^q, \mathcal{L}(H, \widetilde{H})). \tag{4.32}$$

Then, for any excision function $\chi(\eta)$ we write

$${}^1e(\eta) = c(\eta) + d(\eta)$$

for $c(\eta) := (1 - \chi(\eta))\, {}^1e(\eta)$, $d(\eta) := \chi(\eta)\, {}^1e(\eta)$. Since $c(\eta)$ is of compact support in η it follows together with (4.32) that $c(\eta) \in S^{-\infty}(\mathbb{R}^q; H, \widetilde{H})$. Moreover, we have

$$d(\eta) = \boldsymbol{\omega}\chi(\eta)\, {}^1f(\eta)\boldsymbol{\omega}'. \tag{4.33}$$

Since the operators of multiplication by $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ both belong to $S^0(\mathbb{R}^q; H, H)$ and $S_0(\mathbb{R}^q; \widetilde{H}, \widetilde{H})$, it remains to observe the relation

$$\chi(\eta)^1 f(\eta) = \chi(\eta) r^{-\mu}(1 - \omega_\eta)\mathrm{Op}_r({}^1p_0)(\eta)(1 - \omega''_\eta) \in S^{\mu}(\mathbb{R}^q; H, \widetilde{H}) \tag{4.34}$$

and the continuity of

$$L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^q_{\tilde{\rho},\tilde{\eta}})) \to S^{\mu}(\mathbb{R}^q; H, \widetilde{H}),\ {}^1\tilde{p}(0, \tilde{\rho}, \tilde{\eta}) \mapsto f(\eta). \tag{4.35}$$

Remark 4.6 Let $C^\infty_{[0,R]}(\overline{\mathbb{R}}_+)$ be the subspace of all $\varphi \in C^\infty(\overline{\mathbb{R}}_+)$ supported by $[0, R]$ for some $R > 0$, the operator $\mathcal{M}_\varphi$ of multiplication by $\varphi \in C^\infty_{[0,R]}(\overline{\mathbb{R}}_+)$ belongs to $S^0(\mathbb{R}^q; H, H)$ and $S_0(\mathbb{R}^q; \widetilde{H}, \widetilde{H})$, and the corresponding operators

$$C^{\infty}_{[0,R]}(\overline{\mathbb{R}}_+) \to S^0(\mathbb{R}^q; H, H),\ \varphi \mapsto \mathcal{M}_\varphi,$$

are continuous. Analogous relations are true with respect to $\widetilde{H}$.

Now for r-dependent ${}^1\tilde{p}(r, \tilde{\rho}, \tilde{\eta})$ and ${}^1\tilde{h}(r, w, \tilde{\eta})$ there is a tensor product argument. The abstract background is that the elements of the projective tensor product $E \hat{\otimes}_\pi F$ of Fréchet spaces E and F can be written as a convergent sum

$$\sum_{j=0}^{\infty} \lambda_j e_j \otimes f_j \tag{4.36}$$

for $\lambda_j \in \mathbb{C}$, $\sum_{j=0}^{\infty} |\lambda_j| < \infty$ and $e_j \in E$, $f_j \in F$, tending to 0 in the respective spaces as $j \to \infty$. In the present case this can be applied to $E := C^{\infty}_{[0,R]}(\overline{\mathbb{R}}_+)$ (the subspace of all $\varphi \in C^\infty(\overline{\mathbb{R}}_+)$ supported by $[0, R]$ for some $R > 0$) and $F = L^\mu_{\mathrm{cl}}(X; \mathbb{R}^q_{\tilde{\rho},\tilde{\eta}})$, i.e.,

$${}^1\tilde{p}(r, \tilde{\rho}, \tilde{\eta}) \in C^{\infty}_{[0,R]}(\overline{\mathbb{R}}_+, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^q_{\tilde{\rho},\tilde{\eta}})) = C^{\infty}_{[0,R]}(\overline{\mathbb{R}}_+) \hat{\otimes}_\pi L^\mu_{\mathrm{cl}}(X; \mathbb{R}^q_{\tilde{\rho},\tilde{\eta}})$$

or $F = M^\mu_{\mathcal{O}_w}(X; \mathbb{R}^q_{\tilde{\eta}})$ and

$${}^1\tilde{h}(r, w, \tilde{\eta}) \in C^{\infty}_{[0,R]}(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}_w}(X; \mathbb{R}^q_{\tilde{\eta}})) = C^{\infty}_{[0,R]}(\overline{\mathbb{R}}_+) \hat{\otimes}_\pi M^\mu_{\mathcal{O}_w}(X; \mathbb{R}^q_{\tilde{\eta}}).$$

Proposition 4.7 *We have*

$$\mathcal{R}^\mu(U \times \mathbb{R}^d, \boldsymbol{g}) \subset S^\mu(U \times \mathbb{R}^d; H, \widetilde{H}) \tag{4.37}$$

for the pair of spaces

$$H := \mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge),\ \widetilde{H} := \mathcal{K}^{s-\mu,\gamma_2-\mu,\gamma_1-\mu}(I^\wedge)$$

as well as

$$H := \mathcal{K}^{s,\gamma_2,\gamma_1}_{P_2,P_1}(I^\wedge),\ \widetilde{H} := \mathcal{K}^{s-\mu,\gamma_2-\mu,\gamma_1-\mu}_{Q_2,Q_1}(I^\wedge)$$

for every $s \in \mathbb{R}$ and asymptotic types P_2, P_1, for some resulting Q_2, Q_1.

Proof Throughout this proof we assume that the operator functions (4.13) are independent of z. The general case is straightforward and left to the reader. By notation we have $(m + g)(\zeta) \in \mathcal{R}^\mu_{M+G}(\mathbb{R}^d_\zeta, \boldsymbol{g})$. By virtue Definition 2.21 (i) the Green summand $g(\zeta)$ is as claimed. Moreover, Proposition 3.1 tells us that also $m(\zeta)$ is as desired, even a classical symbol.

Applying an analogue of Remark 4.5 to elements (4.13) of (4.12) we may ignore the summands with factors $\boldsymbol{\varphi}$, $\boldsymbol{\varphi}'$ completely. In addition without loss of generality we may assume $\boldsymbol{\sigma} \succ \sigma_\zeta$, $\boldsymbol{\sigma}' \succ \sigma'_\zeta$ for all ζ. Thus $\boldsymbol{\sigma}\sigma_\zeta = \sigma_\zeta$, $\boldsymbol{\sigma}'\sigma'_\zeta = \sigma'_\zeta$, and it remains to look at

$$a(\zeta) := b(\zeta) + e(\zeta) \tag{4.38}$$

for

$$b(\zeta) := t^{-\mu}\sigma_\zeta \mathrm{op}_{M_t}^{\gamma_2-1/2}(h)(\zeta)\sigma'_\zeta, \quad e(\zeta) := \boldsymbol{\sigma} f(\zeta)\boldsymbol{\sigma}', \tag{4.39}$$

and

$$f(\zeta) := t^{-\mu}(1-\sigma_\zeta)\mathrm{Op}_t(p)(\zeta)(1-\sigma''_\zeta). \tag{4.40}$$

We have $b(\eta) \in C^\infty(\mathbb{R}^d, \mathcal{L}(H, \tilde{H}))$ for the spaces in (4.37). The spaces with asymptotics in the second pair are written as projective limits of Hilbert spaces analogously as (4.29) for Hilbert subspaces $\cdots H^{m+1} \hookrightarrow H^m \hookrightarrow \cdots \hookrightarrow H^0 = \mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge)$ and $\cdots \widetilde{H}^{l+1} \hookrightarrow \widetilde{H}^l \hookrightarrow \cdots \hookrightarrow \widetilde{H}^0 = \mathcal{K}^{s-\mu,\gamma_2-\mu,\gamma_1-\mu}(I^\wedge)$ with group action ${}^2\kappa$, cf. also Proposition 2.20. We have

$$a(\zeta) \in C^\infty(\mathbb{R}^d, \mathcal{L}(H, \tilde{H})). \tag{4.41}$$

This can be concluded from $b(\zeta),\ e(\zeta) \in C^\infty(\mathbb{R}^d, \mathcal{L}(H, \tilde{H}))$, cf. (4.39). The desired symbol property of $b(\zeta)$ follows from a tensor product argument, combined with Remarks 2.6 and 4.6 which also holds for the spaces in (4.37). More precisely, we may assume $\tilde{h}(t, v, \tilde{\zeta}) \in C^\infty_{[0,R]}(\overline{\mathbb{R}}_+)\hat{\otimes}_\pi M^\mu_{\mathcal{O}_v}(I, \boldsymbol{g}_1; \mathbb{R}^d_{\tilde{\zeta}})$ for a sufficiently large $R > 0$, i.e., we can write

$$\tilde{h}(t, v, \tilde{\zeta}) = \sum_{j=0}^\infty \lambda_j \varphi_j(t)\tilde{h}_j(v, \tilde{\zeta})$$

for $\lambda_j \in \mathbb{C}$, $\sum_{j=0}^\infty |\lambda_j| < \infty$, $\varphi_j \in C^\infty_{[0,R]}(\overline{\mathbb{R}}_+)$ and $\tilde{h}_j(v, \tilde{\zeta}) \in M^\mu_{\mathcal{O}_v}(I, \boldsymbol{g}_1; \mathbb{R}^d_{\tilde{\zeta}})$, tending to 0 in the respective spaces as $j \to \infty$. This gives us

$$b(\zeta) = \sum_{j=0}^\infty \lambda_j \mathcal{M}_{\varphi_j} b_j(\zeta) \tag{4.42}$$

for

$$b_j(\zeta) = t^{-\mu}\sigma_\zeta \mathrm{op}_{M_t}^{\gamma_2-1/2}(h_j)(\zeta)\sigma'_\zeta, \quad h_j(v, \zeta) = \tilde{h}_j(v, t\zeta).$$

Because of

$$b_j(\delta\zeta) = \delta^\mu\, {}^2\kappa_\delta\, b_j(\zeta)({}^2\kappa_\delta)^{-1} \tag{4.43}$$

for all $\delta \geq 1$ and $|\zeta| \geq c$ for some $c > 0$, the assumptions of Remark 2.6 are satisfied, and we see that $b_j(\zeta)$ is a classical symbol, tending to zero as $j \to \infty$. Thus (4.42) converges in the claimed symbol space. In order to treat $e(\zeta)$ we choose an excision function $\chi(\zeta)$ in $\mathbb{R}^d$ and write

$$e(\zeta) = c(\zeta) + d(\zeta) \tag{4.44}$$

for $c(\zeta) = \boldsymbol{\sigma}(1-\chi(\zeta))f(\zeta)\boldsymbol{\sigma}',\ d(\zeta) = \boldsymbol{\sigma}\chi(\zeta)f(\zeta)\boldsymbol{\sigma}'$. Since $c(\zeta) \in C^\infty(\mathbb{R}^d, \mathcal{L}(H, \tilde{H}))$ is of compact support in ζ it follows that $c(\zeta) \in S^{-\infty}(\mathbb{R}^d; H, \widetilde{H})$ which is contained in the desired symbol space. Moreover, we may assume

$$\tilde{p}(t, \tilde{\tau}, \tilde{\zeta}) \in C^{\infty}_{[0,R]}(\overline{\mathbb{R}}_+)\hat{\otimes}_{\pi} L^{\mu}(I, \boldsymbol{g}_1; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}})$$

for a sufficiently large $R > 0$, i.e., we can write

$$\tilde{p}(t, \tilde{\tau}, \tilde{\zeta}) = \sum_{j=0}^{\infty} \lambda_j \varphi_j(t) \tilde{p}_j(\tilde{\tau}, \tilde{\zeta})$$

for $\lambda_j \in \mathbb{C}$, $\sum_{j=0}^{\infty} |\lambda_j| < \infty$, $\varphi_j \in C^{\infty}_{[0,R]}(\overline{\mathbb{R}}_+)$ and $\tilde{p}_j(\tilde{\tau}, \tilde{\zeta}) \in L^{\mu}(I, \boldsymbol{g}_1; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}})$, tending to 0 in the respective spaces as $j \to \infty$. This gives us

$$d(\zeta) = \sum_{j=0}^{\infty} \lambda_j \mathcal{M}_{\varphi_j} d_j(\zeta) \tag{4.45}$$

for

$$d_j(\zeta) = \boldsymbol{\sigma} \chi(\zeta) f_j(\zeta) \boldsymbol{\sigma}',$$
$$f_j(\zeta) = t^{-\mu}(1 - \sigma_{\zeta})\mathrm{Op}_t(p_j)(\zeta)(1 - \sigma'_{\zeta}), \quad p_j(t, \tau, \zeta) = \tilde{p}_j(t\tau, t\zeta).$$

A computation based on oscillatory integrals yields that

$$L^{\mu}(I, \boldsymbol{g}_1; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}}) \to S^{\mu}(\mathbb{R}^d; H, \widetilde{H}), \ \tilde{p}_j(\tilde{\tau}, \tilde{\zeta}) \mapsto d_j(\zeta),$$

is continuous, and hence (4.45) converges in $S^{\mu}(\mathbb{R}^d; H, \widetilde{H})$. □

The elements $a(z, \zeta) \in \mathcal{R}^{\mu}(U \times \mathbb{R}^d, \boldsymbol{g})$ are particular families of parameter-dependent edge operators

$$a(z, \zeta) \in C^{\infty}(\Omega_2, L^{\mu}(I^{\wedge}, \boldsymbol{g}_1; \mathbb{R}^d)).$$

As such they have the symbols $\sigma_0(\cdot)$ and $\sigma_1(\cdot)$, smoothly depending on $z \in \Omega_2$, namely,

$$\sigma_0(a)(t, r, z, \tau, \rho, \zeta),$$

cf. (4.16), where x is replaced by $(t, r) \in \mathbb{R} \times (0, 1)$ and ξ by $(\tau, \rho) \in \mathbb{R}^2$, and

$$\sigma_1(a)(t, z, \tau, \zeta),$$

cf. the formula (4.21), with (y, η) being replaced by (t, τ). Together with (4.14) this gives us the principal symbolic hierarchy in $\mathcal{R}^{\mu}(\Omega_2 \times \mathbb{R}^d, \boldsymbol{g}) \ni a(z, \zeta)$, namely,

$$\sigma(a) = (\sigma_0(a), \sigma_1(a), \sigma_2(a)). \tag{4.46}$$

Setting

$$\mathcal{R}^{\mu-1}(\Omega_2 \times \mathbb{R}^d, \boldsymbol{g}) \ni a(z,\zeta) := \{a \in \mathcal{R}^{\mu}(\Omega_2 \times \mathbb{R}^d, \boldsymbol{g}) : \sigma(a) = 0\}$$

we obtain a subspace of elements which have a triple of principal symbols of order $\mu - 1$, namely,

$$\sigma^{\mu-1}(a) = (\sigma_0^{\mu-1}(a), \sigma_1^{\mu-1}(a), \sigma_2^{\mu-1}(a)).$$

Successively we obtain subspaces

$$\mathcal{R}^{\mu-(N+1)}(\Omega_2 \times \mathbb{R}^d, \boldsymbol{g}) \subset \mathcal{R}^{\mu}(\Omega_2 \times \mathbb{R}^d, \boldsymbol{g}),\ N \in \mathbb{N},$$

the weight data of which are independent of N. Analogously as in the edge calculus we have the following result on asymptotic summation.

Theorem 4.8 *For every sequence $a_j(z,\zeta) \in \mathcal{R}^{\mu-j}(\Omega_2 \times \mathbb{R}^d, \boldsymbol{g}),\ j \in \mathbb{N}$, where the weight intervals contained in $\boldsymbol{g}$ are finite and the asymptotic types of the involved Green symbols independent of j, there is an asymptotic sum*

$$a(z,\zeta) \sim \sum_{j=0}^{\infty} a_j(z,\zeta),$$

$a(z,\zeta) \in \mathcal{R}^{\mu}(\Omega_2 \times \mathbb{R}^d, \boldsymbol{g})$, unique modulo $\mathcal{R}_G^{-\infty}(\Omega_2 \times \mathbb{R}^d, \boldsymbol{g})_{(I^\wedge, I^\wedge)}$, i.e., for every $N \in \mathbb{N}$ we have

$$a(z,\zeta) - \sum_{j=0}^{N} a_j(z,\zeta) \in \mathcal{R}^{\mu-(N+1)}(\Omega_2 \times \mathbb{R}^d, \boldsymbol{g}).$$

The main ideas of the proof are similar to that of a corresponding result on asymptotic summation of edge symbols. So we drop the proof here.

4.2 Corner Boundary Value Problems

We now study the operators of the corner calculus, locally generated by symbols $a(z,\zeta)$ in the sense of Definition 4.3.

Theorem 4.9 *$a \in \mathcal{R}^{\nu}(\Omega_2 \times \mathbb{R}^d, \boldsymbol{g})$, $b \in \mathcal{R}^{\mu}(\Omega_2 \times \mathbb{R}^d, \boldsymbol{h})$ for $\boldsymbol{g} = (\boldsymbol{g}_2, \boldsymbol{g}_1)$, $\boldsymbol{h} = (\boldsymbol{h}_2, \boldsymbol{h}_1)$,*

$$\boldsymbol{g}_i = (\gamma_i - \nu, \gamma_i - \nu - \mu, \Theta_i),\ \boldsymbol{h}_i = (\gamma_i, \gamma_i - \nu, \Theta_i),\ \Theta_i = (-(k_i + 1), 0],\ k_i \in \mathbb{N},$$

implies $ab \in \mathcal{R}^{\mu+\nu}(\Omega_2 \times \mathbb{R}^d, \boldsymbol{g} \circ \boldsymbol{h})$ for

$$\boldsymbol{g} \circ \boldsymbol{h} = (\boldsymbol{g}_i \circ \boldsymbol{h}_i)_{i=0,1},\ \boldsymbol{g}_i \circ \boldsymbol{h}_i = (\gamma_i, \gamma_i - \nu - \mu, \Theta_i),\ \ i = 1, 2,$$

and we have

$$\sigma_i(ab) = \sigma_i(a)\sigma_i(b), \quad i = 0, 1, 2.$$

Proof The result employs the known composition behaviour of operators in the edge calculus, i.e., the fact that ab also contains the pointwise composition between the values of operator-valued symbols in weighted spaces, controlled as in Proposition 4.7, namely,

$$a \in L^{\nu}(I^{\wedge}, \boldsymbol{g}_1),\ b \in L^{\mu}(I^{\wedge}, \boldsymbol{h}_1) \Rightarrow ab \in L^{\mu+\nu}(I^{\wedge}, \boldsymbol{g}_1 \circ \boldsymbol{h}_1).$$

In addition, similarly as in the composition of symbols in the edge calculus for singularity order 1, cf. [8], we can refer to a quantisation only based on holomorphic symbols as obtained for singularity order 2 in the article [27]. This gives us the composition in the corner symbol spaces themselves. □

Remark 4.10 Let $a \in \mathcal{R}^{\mu}(\Omega_2 \times \mathbb{R}^d, \boldsymbol{g})$ for $\boldsymbol{g} = (\boldsymbol{g}_i)_{i=1,2},\ \boldsymbol{g}_i = (\gamma_i, \gamma_i - \mu, \Theta_i)$. Then for the (z, ζ) wise formal adjoint with respect to the $\mathcal{K}^{0,0,0}(I^{\wedge})$-scalar product we have $a^* \in \mathcal{R}^{\mu}(\Omega_2 \times \mathbb{R}^d, \boldsymbol{g}^*)$ for $\boldsymbol{g}^* = (\boldsymbol{g}_i^*)_{i=1,2},\ \boldsymbol{g}_i^* = (-\gamma_i + \mu, -\gamma_i, \Theta_i)$.

Let M be a stratified space as at the beginning of Sect. 2.1. We now assume that M is compact. Recall that close to $Z = s_2(M)$ the space M is modelled on $I^{\Delta} \times \mathbb{R}^d$. Moreover, $M \backslash Z$ is a non-compact manifold of dimension $2+d$ with boundary $\partial(M\backslash Z)$ of dimension $1 + d$ for $d \geq 1$. We treat $M\backslash Z$ as a manifold with smooth edge, since our operators will not have the transmission property at the boundary. On $M\backslash Z$ we have the well-known edge operator spaces

$$L^{\mu}(M\backslash Z, \boldsymbol{g}_1) \quad \text{for} \quad \boldsymbol{g}_1 = (\gamma_1, \gamma_1 - \mu, \Theta_1)$$

and weighted edge spaces

$$H^{s,\gamma_1}_{[\text{loc})}(M\backslash Z) \subset H^s_{\text{loc}}(\text{int}\,(M\backslash Z)), \tag{4.47}$$

locally near $\partial(M\backslash Z)$ modelled on

$$\mathcal{W}^s(\mathbb{R}^{1+d}, \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+))$$

where $\mathbb{R}_+$ is the inner normal of the boundary $\partial(M\backslash Z)$ in $M\backslash Z$. Moreover, we have subspaces

$$H^{s,\gamma_1}_{[\text{loc})P_1}(M\backslash Z), \tag{4.48}$$

locally described by

$$\mathcal{W}^s(\mathbb{R}^{1+d}, \mathcal{K}^{s,\gamma_1}_{P_1}(\mathbb{R}_+)).$$

Finally on $M\backslash Z$ locally near Z in the splitting ov variables

$$(t, r, z) \in \mathbb{R}_+ \times I \times \mathbb{R}^d$$

we have the spaces

$$H^{s,\gamma_2,\gamma_1}(\mathbb{R}_+ \times I \times \mathbb{R}^d) := \mathcal{W}^s(\mathbb{R}^d, \mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge)) \tag{4.49}$$

and subspaces with asymptotics

$$H^{s,\gamma_2,\gamma_1}_{P_2,P_1}(\mathbb{R}_+ \times I \times \mathbb{R}^d) := \mathcal{W}^s(\mathbb{R}^d, \mathcal{K}^{s,\gamma_2,\gamma_1}_{P_2,P_1}(I^\wedge)). \tag{4.50}$$

By gluing together (4.47) and (4.49) via charts and a subordinate partition of unity we obtain weighted spaces

$$H^{s,\gamma_2,\gamma_1}(M) \tag{4.51}$$

over M. In a similar manner we obtain weighted spaces with asymptotics

$$H^{s,\gamma_2,\gamma_1}_{P_2,P_1}(M) \tag{4.52}$$

by gluing together (4.48) and (4.50), cf. formula (2.59).

By

$$L^{-\infty}(M, \boldsymbol{g})$$

for $\boldsymbol{g} = (\boldsymbol{g}_2, \boldsymbol{g}_1)$ as in Definition 4.3 we denote the space of all continuous $C : H^{s,\gamma_2,\gamma_1}(M) \to H^{\infty,\gamma_2-\mu,\gamma_1-\mu}(M)$, $s \in \mathbb{R}$, that induce continuous operators

$$C : H^{s,\gamma_2,\gamma_1}(M) \to H^{\infty,\gamma_2-\mu,\gamma_1-\mu}_{P_2,P_1}(M),$$
$$C^* : H^{s,-\gamma_2+\mu,-\gamma_1+\mu}(M) \to H^{\infty,-\gamma_2,-\gamma_1}_{Q_2,Q_1}(M),$$

$s \in \mathbb{R}$, for C-dependent asymptotic types P_i and Q_i, associated with the weight data $(\gamma_i - \mu, \Theta_i)$ and $(-\gamma_i, \Theta_i)$, respectively. Here C^* is the formal adjoint of C with respect to the non-degenerate sesquilinear pairings

$$H^{s,\gamma_2,\gamma_1}(M) \times H^{-s,-\gamma_2,-\gamma_1}(M) \to \mathbb{C},$$

based on the $H^{0,0,0}(M)$-scalar product.

Definition 4.11 The space of corner operators

$$L^\mu(M, \boldsymbol{g})$$

for $\mu \in \mathbb{R}$ and $\boldsymbol{g} = (\boldsymbol{g}_2, \boldsymbol{g}_1)$ is defined as the set of all $A \in L^\mu(M \setminus Z, \boldsymbol{g}_1)$ which are modulo $L^{-\infty}(M, \boldsymbol{g})$ locally near Z of the form $\mathrm{Op}_z(a)$ for some $a \in \mathcal{R}^\mu(\Omega_2 \times \mathbb{R}^d, \boldsymbol{g})$, where $\Omega_2 \subseteq \mathbb{R}^d$ corresponds to a chart on Z.

The elements of $L^\mu(M, \boldsymbol{g})$ represent boundary value problems on M, more precisely, upper left corners of operator block matrices, analogously as (2.77).

Theorem 4.12 *An operator* $A \in L^\mu(M, \boldsymbol{g})$ *for* $\mu \in \mathbb{R}$ *and* $\boldsymbol{g} = (\boldsymbol{g}_2, \boldsymbol{g}_1)$ *induces continuous operators*

$$A : H^{s,\gamma_2,\gamma_1}(M) \to H^{s-\mu,\gamma_2-\mu,\gamma_1-\mu}(M),$$
$$A : H^{s,\gamma_2,\gamma_1}_{P_2,P_1}(M) \to H^{s-\mu,\gamma_2-\mu,\gamma_1-\mu}_{Q_2,Q_1}(M),$$

for every $s \in \mathbb{R}$ *and arbitrary asymptotic types* P_i*, associated with* (γ_i, Θ_i) *and resulting* Q_i*, associated with* $(\gamma_i - \mu, \Theta_i)$*, depending on* P_i *and the operator* A*.*

Proof The results are a direct consequence of the local continuity of operators off $s_2(M)$ as edge operators and of Proposition 4.7 combined with relation (2.21) and its analogue for Fréchet spaces with group action. □

The inclusions

$$L^\mu(M, \boldsymbol{g}) \subset L^\mu_{\mathrm{cl}}(s_0(M)), \quad L^\mu(M, \boldsymbol{g}) \subset L^\mu(M \backslash Z, \boldsymbol{g}_1) \tag{4.53}$$

show that an operator $A \in L^\mu(M, \boldsymbol{g})$ has the (standard) homogeneous principal symbol $\sigma_0(A)$ as a classical pseudo-differential operator over the smooth manifold $s_0(M)$ and the (twisted) homogeneous principal symbol $\sigma_1(A)$ as an operator in the edge calculus over the manifold $M\backslash Z$ with smooth edge, in this case with boundary $s_1(M) = \partial(M\backslash Z)$. Locally near $s_1(M)$ in variables and covariables $(y, \eta) \in \Omega_1 \times (\mathbb{R}\backslash\{0\})$ for an open set $\Omega_1 \subseteq \mathbb{R}$, representing a chart on $s_1(M)$, the symbol $\sigma_1(A)$ is a family of continuous operators

$$\sigma_1(A)(y, \eta) : \mathcal{K}^{s,\gamma_1}(\mathbb{R}_+) \to \mathcal{K}^{s-\mu,\gamma_1-\mu}(\mathbb{R}_+),$$

continuous for all $s \in \mathbb{R}$ and twisted homogeneous of order μ, namely,

$$\sigma_1(A)(y, \delta\eta) = \delta^\mu \, {}^1\kappa_\delta \sigma_1(A)(y, \eta)({}^1\kappa_\delta)^{-1}$$

for all $\delta \in \mathbb{R}_+$.

Moreover, locally near $s_2(M) = Z$ in variables and covariables $(z, \zeta) \in \Omega_2 \times (\mathbb{R}^d\backslash\{0\})$ for an open set $\Omega_2 \subseteq \mathbb{R}^d$, representing a chart on $s_2(M)$, the symbol $\sigma_2(A)$ is a family of continuous operators

$$\sigma_2(A)(z, \zeta) : \mathcal{K}^{s,\gamma_2,\gamma_1}(I^\wedge) \to \mathcal{K}^{s-\mu,\gamma_2-\mu,\gamma_1-\mu}(I^\wedge),$$

continuous for all $s \in \mathbb{R}$ and twisted homogeneous of order μ, in this case,

$$\sigma_2(A)(z, \delta\zeta) = \delta^\mu \, {}^2\kappa_\delta \sigma_2(A)(z, \zeta)({}^2\kappa_\delta)^{-1}$$

for all $\delta \in \mathbb{R}_+$.

Theorem 4.13 *Let* $A \in L^\nu(M, \boldsymbol{g}),\ B \in L^\mu(M, \boldsymbol{h})$ *for* $\boldsymbol{g}, \boldsymbol{h}$ *as in Theorem 4.9. Then we have*

$$AB \in L^{\mu+\nu}(M, \boldsymbol{g} \circ \boldsymbol{h}), \tag{4.54}$$

and

$$\sigma_i(AB) = \sigma_i(A)\sigma_i(B), \quad i = 0, 1, 2. \tag{4.55}$$

Proof The composition AB is well-defined in the sense of continuous operators between corresponding weighted corner spaces, cf. the first assertion of Theorem 4.12. By virtue of (4.53) this corresponds to compositions both of classical pseudo-differential operators over $s_0(M)$ and edge operators over $M \setminus Z$. Since the principal symbols $\sigma_i(\cdot)$ for $i = 0, 1$ refer to (4.53), and because of the known composition behaviour in the corresponding operator spaces, including the symbolic rules (4.55) for $i = 0, 1$, it remains to show the relation (4.54) and (4.55) for $i = 2$.

It suffices to characterise local compositions of the kind

$$\varphi \mathrm{Op}_z(a)\varphi_0 \mathrm{Op}_z(b)\varphi' \tag{4.56}$$

for symbols $a(z,\zeta) \in \mathcal{R}^\nu(\Omega_2 \times \mathbb{R}^d, \boldsymbol{g}), b(z,\zeta) \in \mathcal{R}^\mu(\Omega_2 \times \mathbb{R}^d, \boldsymbol{h})$, for functions $\varphi, \varphi_0, \varphi' \in C_0^\infty(\Omega_2)$, where Ω_2 corresponds to a chart on $Z = s_2(M)$. In order to localise expressions after treating (4.56) once again in a compact subset of Ω_2, instead of (4.56) we can write $\tilde{\varphi}\varphi \mathrm{Op}_z(a)\varphi_0 \mathrm{Op}_z(b)\varphi'\tilde{\varphi}'$ for functions $\tilde{\varphi} \succ \varphi$, $\tilde{\varphi}' \succ \varphi'$ in $C_0^\infty(\Omega_2)$. We have $\varphi_0 b \in \mathcal{R}^\mu(\Omega_2 \times \mathbb{R}^d, \boldsymbol{h})$, and the Leibniz product $c := a\#(\varphi_0 b) \sim \sum_{\alpha \in \mathbb{N}^d} 1/\alpha! \partial_\zeta^\alpha a D_z^\alpha(\varphi_0 b)$ can be carried out in $\mathcal{R}^{\mu+\nu}(\Omega_2 \times \mathbb{R}^d, \boldsymbol{g} \circ \boldsymbol{h})$, cf. Theorem 4.8. By using the right behaviour of the symbol classes in Definition 4.3 under pointwise formal adjoints we obtain that (4.56) is equal to $\tilde{\varphi}\mathrm{Op}_z(c)\tilde{\varphi}'$ modulo a smoothing operator localised in $I^\wedge \times \Omega_2$. We easily see also the symbolic rule (4.55) for $i = 2$. □

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References

1. Boutet de Monvel, L.: Boundary problems for pseudo-differential operators. Acta Math. **126**, 11–51 (1971)
2. Calvo, D., Schulze, B.-W.: Edge symbolic structures of second generation. Math. Nachr. **282**, 348–367 (2009)
3. Chang, D.-C., Habal, N., Schulze, B.-W.: Quantisation on a manifold with singular edge, NCTS Preprints in Mathematics 2013-1-002, Taiwan, 2013. J. Pseudo-Differ Oper Appl. **4**(3), 317–343 (2013)
4. Chang, D.-C., Habal, N., Schulze, B.-W.: The edge algebra structure of the Zaremba problem, NCTS Preprints in Mathematics 2013–6-002, Taiwan. J. Pseudo-Differ. Oper. Appl. **5**(2014), 69–155 (2014). doi:10.1007/s11868-013-0088-7
5. Dorschfeldt, Ch.: Algebras of Pseudo-Differential Operators Near Edge and Corner Singularities, Math. Res. 102, Akademie, Berlin (1998)
6. Egorov, V., Schulze, B.-W.: Pseudo-Differential Operators, Singularities, Applications, Operator Theory: Advances and Applications, vol. 93. Birkhäuser, Basel (1997)
7. Eskin, G.I.: Boundary value problems for elliptic pseudodifferential equations, Translations of Nauka, Moskva, 1973, Mathematical Monographs, American Mathematical Society vol. 52. Providence, Rhode Island (1980)

8. Gil, J.B., Schulze, B.-W., Seiler, J.: Cone pseudodifferential operators in the edge symbolic calculus. Osaka J. Math. **37**, 221–260 (2000)
9. Habal, N., Schulze, B.-W.: Mellin quantisation in corner operators, operator theory, advances and applications. In: Karlovich, Y.I., et al. (eds.), "Operator Theory, Pseudo-Differential Equations, and Mathematical Physics", The Vladimir Rabinovich Anniversary Volume, vol. 228, pp. 151–172. Birkhäuser, Basel (2013)
10. Habal, N., Schulze, B.-W.: Holomorphic corner symbols. J. Pseudo-Differ. Oper. Appl. **2**(4), 419–465 (2011)
11. Harutjunjan, G., Schulze, B.-W.: Elliptic Mixed, Transmission and Singular Crack Problems. European Mathematical Society, Zürich (2008)
12. Kapanadze, D., Schulze, B.-W.: Crack Theory and Edge Singularities. Kluwer Academic Publisher, Dordrecht (2003)
13. Krainer, T.: On the inverse of parabolic systems of partial differential equations of general form in an infinite space-time cylinder, operator theory, advances and applications 138. In: Albeverio, S., Demuth, M., Schrohe, E., Schulze, B.-W. (eds.) Adv. in Partial Differential Equations "Parabolicity, Volterra Calculus, and Conical Singularities", pp. 93–278. Birkhäuser, Basel (2002)
14. Rempel, S., Schulze, B.-W.: Index Theory of Elliptic Boundary Problems. Akademie, Berlin (1982)
15. Rempel, S., Schulze, B.-W.: Parametrices and boundary symbolic calculus for elliptic boundary problems without transmission property. Math. Nachr. **105**, 45–149 (1982)
16. Rempel, S., Schulze, B.-W.: Complete Mellin and Green symbolic calculus in spaces with conormal asymptotics. Ann. Glob. Anal. Geom. **4**(2), 137–224 (1986)
17. Rungrottheera, W., Schulze, B.-W.: Weighted spaces on corner manifolds. Complex Var. Elliptic Equ. **59**(12):1706–1738 (2014). http://www.tandfonline.com/doi/full/10.1080/17476933.2013.876416
18. Schrohe, E., Schulze, B.-W.: Boundary value problems in Boutet de Monvel's calculus for manifolds with conical singularities II. Ib: Proceedings of Adv. in Partial Differential Equations "Boundary Value Problems, Schrödinger Operators, Deformation Quantization", pp. 70–205. Akademie Verlag, Berlin (1995)
19. Schrohe, E., Schulze, B.-W.: Edge-degenerate boundary value problems on cones. In: Proceedings of Evolution Equations and their Applications in Physical and Life Sciences, Bad Herrenalb, Karlsruhe (2000)
20. Schulze, B.-W.: Pseudo-differential operators on manifolds with edges, Teubner-Texte zur Mathematik, Symp. "Partial Differential Equations", Holzhau 1988, 112, Leipzig, pp. 259–287 (1989)
21. Schulze, B.-W.: Pseudo-Differential Operators on Manifolds with Singularities. North-Holland, Amsterdam (1991)
22. Schulze, B.-W.: Boundary Value Problems and Singular Pseudo-Differential Operators. Wiley, Chichester (1998)
23. Schulze, B.-W.: Operators with symbol hierarchies and iterated asymptotics. Publ. RIMS Kyoto Univ. **38**(4), 735–802 (2002)
24. Schulze, B.-W.: The iterative structure of the corner calculus, operator theory, advances and applications. In: Rodino, L., et al. (eds.) Pseudo-Differential Operators: Analysis, Application and Computations, vol. 213, pp. 79–103. Birkhäuser, Basel (2011)
25. Schulze, B.-W., Seiler, J.: The edge algebra structure of boundary value problems. Ann. Glob. Anal. Geom. **22**, 197–265 (2002)
26. Schulze, B.-W., Tepoyan, L.: The singular functions of branching edge asymptotics, operator theory, advances and applications. In: Molahajloo, S., Pilipovič, S., Toft, J., Wong, M.W., (eds.) Pseudo-Differential Operators, Generalized Functions and Asymptotics, vol. 231, pp. 27–53. Birkhäuser, Basel (2013). arXiv:1202.0387v2 [math.AP]
27. Schulze, B.-W., Wei, Y.: The Mellin-edge quantisation for corner operators. Complex Anal. Oper. Theory **8**(4), 803–841 (2014). doi:10.1007/s11785-013-0289-3
28. Schulze, B.-W., Volpato, A.: Branching asymptotics on manifolds with edge, J. Pseudo-Differ. Oper. Appl. **1** (2010), 433–493 (2010). arXiv:1004.0332 [math.DG]
29. Schulze, B.-W., Wong, M.W.: Mellin and Green operators of the corner calculus. J. Pseudo-Differ. Oper. Appl. **2**(4), 467–507 (2011)
30. Seiler, J.: Continuity of edge and corner pseudo-differential operators. Math. Nachr. **205**, 163–182 (1999)