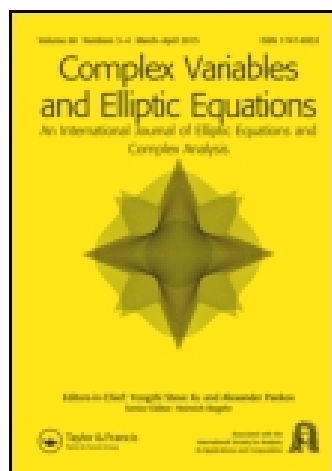


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Estimation of hyperbolically partial derivatives of p -harmonic quasiconformal mappings and its applications

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Estimation of hyperbolically partial derivatives of ρ -harmonic quasiconformal mappings and its applications

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In this paper, for a general metric density ρ , we build a differential inequality to study the hyperbolically partial derivative of ρ -harmonic quasiconformal mappings and generalize a result given by Knežević and Mateljević. As one application, we obtain the hyperbolically $(1/K, K)$ -biLipschitz continuity of such class of mappings. As another application, we generalize the classical Koebe Theorem to this class of mappings and use this result to study its quasihyperbolically biLipschitz continuity.

Keywords: harmonic quasiconformal mappings, hyperbolically biLipschitz continuity, hyperbolically partial derivatives, Koebe theorem, quasihyperbolically Lipschitz

AMS Subject Classifications: Primary: 30C62; Secondly: 58E20, 30C55

1. Introduction

Let Ω and Ω' be two simply connected regions of hyperbolic type in the complex plane \mathbb{C} . For a topology mapping f of Ω onto Ω' , we write $L_f = |f_z| + |f_{\bar{z}}|$, $l_f = |f_z| - |f_{\bar{z}}|$, where $f_z = \frac{1}{2}(f_x - if_y)$ and $f_{\bar{z}} = \frac{1}{2}(f_x + if_y)$. Denote by $\lambda_\Omega(z)|dz|$ the hyperbolic metric on Ω with gaussian curvature -4 . We call $\|\partial f\| = (\lambda_{\Omega'} \circ f / \lambda_\Omega)|f_z|$ and $\|\bar{\partial} f\| = (\lambda_{\Omega'} \circ f / \lambda_\Omega)|f_{\bar{z}}|$ the *hyperbolically partial derivatives* of f . Particularly, if f is a conformal mapping of Ω onto Ω' then $\|\partial f\| = 1$.

Definition 1.1 [1] A topology mapping f of Ω onto Ω' is said to be *K-quasiconformal* (abbreviated by *K-QC*) if it satisfies

- (1) f is ACL in Ω ;
- (2) $L_f \leq Kl_f$, a.e. in Ω , where $K \geq 1$.

Definition 1.2 Let $\rho \in C^\infty(\Omega')$ be a positive metric density on Ω' . A C^2 sense-preserving homeomorphism f of Ω onto Ω' is said to be ρ -harmonic if it satisfies the Euler–Lagrange equation

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$$f_z \bar{z} + 2(\log \rho)_w(f) f_z f_{\bar{z}} = 0, \quad (1.1)$$

where $w = f(z)$.

Particularly, if $\rho(w)|dw|$ is a hyperbolic metric or a positive constant on Ω' then we say that f is *hyperbolically harmonic* or *Euclidean harmonic*, respectively. If $\rho(w) = 1/|w|$ in a domain which does not contain zero, then we call it a *log-harmonic mapping*. If f is ρ -harmonic, then $\rho^2(f) f_z \bar{f}_{\bar{z}} dz^2$ is a holomorphic quadratic differential on Ω . For a survey of harmonic mappings, see [2–4].

Definition 1.3 If a ρ -harmonic mapping f is also K -quasiconformal, then it is called a ρ -harmonic K -quasiconformal mapping.

Lewy [5] showed that the Jacobian determinant of a harmonic homeomorphism between planar domains never vanishes, hence it is a diffeomorphism. Schoen and Yau [6] generalized this result to a harmonic homeomorphism between closed Riemann surfaces of negative curvature. Recently, Martin further proved that a ρ -harmonic quasiconformal mapping is also a diffeomorphism (see Theorem 3 at [7]).

The hyperbolic distance $d_h(z_1, z_2)$ between two points z_1 and z_2 in Ω is defined by $\inf_{\gamma} \int_{\gamma} \lambda_{\Omega}(z) |dz|$, where γ runs through all rectifiable curves in Ω which connect z_1 and z_2 . A mapping f of Ω onto Ω' is said to be *hyperbolically L_1 -Lipschitz* ($L_1 > 0$) if

$$d_h(f(z_1), f(z_2)) \leq L_1 d_h(z_1, z_2), \quad z_1, z_2 \in \Omega.$$

If there also exists a constant $L_2 > 0$ such that

$$L_2 d_h(z_1, z_2) \leq d_h(f(z_1), f(z_2)), \quad z_1, z_2 \in \Omega,$$

then f is said to be *hyperbolically (L_2, L_1) -biLipschitz*. L_1 and L_2 are called the *upper Lipschitz constant* and *lower Lipschitz constant*, correspondingly.

Martio [8] is the first who considered the Euclidean Lipschitz and biLipschitz character of Euclidean harmonic quasiconformal mappings. Under different conditions of the ranges of Euclidean harmonic quasiconformal mappings, recent papers [9–15] studied their Euclidean Lipschitz and biLipschitz character. The study of the Lipschitz (biLipschitz) continuity for certain classes of quasiconformal mappings including the class of ρ -harmonic quasiconformal mappings also arouses interest.

Kalaj and Mateljević [16] obtained

THEOREM A *Let f be a C^2 quasiconformal diffeomorphism from the $C^{1,\alpha}$ Jordan domain Ω onto the $C^{2,\alpha}$ Jordan domain Ω' . If there exists a constant M such that*

$$|\Delta f| \leq M |f_z f_{\bar{z}}|, \quad z \in \Omega,$$

then f has bounded partial derivatives. In particular, it is a Euclidean Lipschitz mapping.

Theorem A implies that if ρ satisfies $|(\rho(w))_w| \leq M\rho(w)$, then a ρ -harmonic mapping f has a bounded partial derivative and the Euclidean Lipschitz character.

Recently, Kalaj and Pavlović [17] showed

THEOREM B *Let $K \geq 1$ be arbitrary and let $g \in C(\overline{D})$ and $\|g\|_\infty := \sup_{z \in D} |g(z)|$. Then there exist constants $N(K)$ and $M(K)$ with $\lim_{K \rightarrow 1} M(K) = 1$ such that: If w is a K -quasiconformal self-mapping of the unit disk D satisfying the partial differential equation*

$$\Delta f = g, \quad g \in C(\overline{D}), \quad f(0) = 0,$$

then for $z_1, z_2 \in D$, it follows

$$\left(\frac{1}{M(K)} - \frac{7\|g\|_\infty}{6} \right) |z_1 - z_2| \leq |w(z_1) - w(z_2)| \leq (M(K) + N(K)\|g\|_\infty) |z_1 - z_2|.$$

Theorem B implies that if a ρ -harmonic quasiconformal mapping f of D onto itself satisfies that the quantity $|(\log \rho)_w \circ f f_z f_{\bar{z}}|$ is bounded and $w(0) = 0$, then f is Euclidean biLipschitz.

Example 8.1 gives a class of ρ -harmonic quasiconformal mappings with unbounded quantities $|(\rho(w))_w|/\rho(w)$ and $|(\log \rho)_w \circ f f_z f_{\bar{z}}|$ such that their partial derivatives are not bounded, while Example 8.2 shows there indeed exist some ρ -harmonic quasiconformal mappings with unbounded quantities $|(\rho(w))_w|/\rho(w)$ and $|(\log \rho)_w \circ f f_z f_{\bar{z}}|$ satisfying that their partial derivatives are bounded. However, two classes of ρ -harmonic quasiconformal mappings given in Example 8.1 and 8.2 both have bounded hyperbolically partial derivatives and the hyperbolically Lipschitz character. So the study of the Euclidean Lipschitz character of a ρ -harmonic quasiconformal mappings may be very different from the study of a hyperbolically Lipschitz character. In this paper, we aim to study the hyperbolically biLipschitz continuity of ρ -harmonic K -quasiconformal mappings and give their explicit biLipschitz constants.

Wan [18] showed that a hyperbolically harmonic K -quasiconformal mapping of the unit disk onto itself is hyperbolically $(1/K, K)$ -biLipschitz. Without loss of generality, this result can be generalized to a hyperbolically quasiconformal mappings with a simply connected domain and a simply connected range. Hence, a hyperbolically harmonic quasiconformal mapping is always hyperbolically $(1/K, K)$ -biLipschitz.

Knežević and Mateljević [19] showed a Euclidean harmonic K -quasiconformal mappings of the unit disk or the upper half-plane onto itself is hyperbolically $(1/K, K)$ -biLipschitz. For a metric $\rho(w)|dw|$ other than the hyperbolic metric, the composite mapping $\varphi \circ f$ of a ρ -harmonic mapping f and a conformal mapping φ rarely preserves its ρ -harmonicity. Hence, in general, we cannot fix the range of a ρ -harmonic quasiconformal mapping to be the unit disk or the upper half-plane when studying its hyperbolically partial derivatives and its hyperbolically (Euclidean) Lipschitz (biLipschitz) character. Chen and Fang [20] generalized the result given by Knežević and Mateljević to the case of convex ranges.

THEOREM C *If f is a Euclidean harmonic K -quasiconformal mapping of the unit disk D onto a convex domain Ω then the following inequality*

$$\frac{K+1}{2K} \leq \|\partial f\| \leq \frac{K+1}{2}. \quad (1.2)$$

holds for every $z \in D$. Moreover, it is hyperbolically $(1/K, K)$ -biLipschitz.

For Euclidean harmonic quasiconformal mappings with convex ranges, the lower bound $(K+1)/(2K)$ and the upper bound $(K+1)/2$ of the inequality (1.2), and the hyperbolically biLipschitz coefficients $(1/K, K)$ are all sharp (see [20]). In general, for Euclidean harmonic quasiconformal mappings with non-convex ranges, the inequality (1.2) does not hold (see Example 8.4).

A region Ω in the unit disk D is called strongly hyperbolically convex if it has the following property: For any $w_1, w_2, w_3 \in \Omega$, with $w_1 \neq w_2$, the arc $\widehat{w_1 w_2}$ in D of the circle $C(w_1, w_2, 1/\overline{w_3})$ is also contained in Ω . [21] Lately, Chen and Fang [22] showed that a $1/(1-|z|^2)$ -harmonic quasiconformal mapping with a strongly hyperbolically convex range in the unit disk is also hyperbolically $(1/K, K)$ -biLipschitz. We note that a strongly hyperbolically convex domain is not necessary to be Euclidean convex (See Example 8.5).

Above examples show that the hyperbolically biLipschitz continuity of a ρ -harmonic quasiconformal mapping may be closely related to the geometric characterization of its range. Various differential inequalities of hyperbolic metrics are used to characterize domain geometric property (see for example [23]). We naturally pose the following

Question For a general C^∞ metric density ρ on Ω , how to give a proper differential inequality of the hyperbolic metric density λ_Ω on Ω to determine a ρ -harmonic quasiconformal mapping to be hyperbolically biLipschitz?

In order to answer this question, for two metric densities λ_Ω and ρ defined on a given domain Ω , we construct the following differential inequality

$$\frac{|(\log \lambda_\Omega)_{ww} - 2(\log \lambda_\Omega)_w (\log \rho)_w - (\log \rho)_{ww} + 2(\log \rho)_w^2| + |(\log \rho)_{w\bar{w}}|}{(\lambda_\Omega)^2} \leq 1. \quad (1.3)$$

Several explicit examples satisfying the above differential inequality are given in Section 6. For some special metric densities ρ , the above inequality can be simplified to some extent. For example, when choosing ρ to be λ_Ω we have that the left side of (1.3) is just the modulus of the Gaussian curvature of λ_Ω and then the equality at (1.3) holds for any simply connected domain Ω . If ρ is a positive constant, then the inequality (1.3) reduces to the inequality (6.1) which holds for all convex domains by Lemma C. If $\rho = 1/|w|$ then the inequality (1.3) holds for an angular domain with the origin as its vertex. There also exist nondegenerate examples satisfying (1.3), for instance, if $\rho = 1/(1-|w|^2)$ then the inequality (1.3) holds for strongly hyperbolically convex domains in D .

On the basis of the differential inequality (1.3), we give our main result as follows.

THEOREM 1.1 *Let Ω be a simply connected domain of hyperbolic type in the complex plane \mathbb{C} and f a ρ -harmonic K -quasiconformal mapping of the unit disk D onto Ω . If ρ and the hyperbolic metric density λ_Ω of Ω satisfy the inequality (1.3) then the inequality*

$$\|\partial f\| \leq \frac{K+1}{2} \quad (1.4)$$

holds for every $z \in D$. If f also satisfies $\lambda_\Omega \circ f|f_z| \rightarrow +\infty$ as $|z| \rightarrow 1^-$, then

$$\|\partial f\| \geq \frac{K+1}{2K}. \quad (1.5)$$

As an application of Theorem 1.1, we obtain the hyperbolically $(1/K, K)$ -bi-Lipschitz continuity for ρ -harmonic K -quasiconformal mappings under proper conditions.

THEOREM 1.2 *Let f be a ρ -harmonic K -quasiconformal mapping of the unit disk onto a simply connected domain Ω . If the pair of metric densities ρ and λ_Ω defined on Ω satisfies the inequality (1.3), then f is hyperbolically K -Lipschitz. If f also satisfies that $\lambda_\Omega|f_z|$ tends to $+\infty$ as $|z|$ tends to 1^- , then f is hyperbolically $(1/K, K)$ -biLipschitz.*

The classical Koebe 1/4-theorem (see Theorem D in Section 5) was generalized to Euclidean harmonic mappings by Clunie and Sheil-Small [24]. Mateljević [25] built its analogue for the class of Euclidean harmonic quasiregular mappings by modulus technique. Chen and Fang [20] generalized the Koebe theorem to the class of Euclidean harmonic K -quasiconformal mappings with convex ranges. As another application of Theorem 1.1, we give a generalization of the Koebe theorem to ρ -harmonic K -quasiconformal mappings (see Theorem 5.1).

The quasihyperbolically Lipschitz (biLipschitz) continuity of Euclidean harmonic (quasiconformal) mappings was also studied in [26,27]. As an application of Theorem 5.1, we show that if the pair of metric densities ρ and λ_Ω defined on Ω satisfies the inequality (1.3) and $\lambda_\Omega|f_z|$ tends to $+\infty$ as $|z|$ tends to 1^- then f is quasihyperbolically $(1/(2K), 4K)$ -biLipschitz (see Corollary 5.1).

This writing of this paper is organized as follows: Section 2 contains preliminary lemmas we need in the following sections. In Section 3, we give the proof of Theorem 1.1. In Section 4, the proof of Theorem 1.2 is given. In Section 5, we first study the generalization of Koebe 1/4 theorem to certain ρ -harmonic K -quasiconformal mappings and then use the result to get the quasihyperbolically biLipschitz continuity of certain ρ -harmonic K -quasiconformal mappings. In Section 6, we give some special density pairs of λ_Ω and ρ satisfying the inequality (1.3). In Section 7, we show that if a ρ -harmonic diffeomorphism satisfies the inequality (1.3) then the lower bound of its hyperbolically partial derivative is $1/2$ and this estimate is optimal (see Theorem 7.1). By constructing a hyperbolically harmonic mapping we show that there does not exist an upper bound for the hyperbolically partial derivative of ρ -harmonic mappings. In Section 8, five auxiliary examples are given.

2. Preliminary lemmas

In order to study the hyperbolically partial derivatives for ρ -harmonic quasiconformal mappings, we need the following lemmas. The well-known Ahlfors-Schwarz lemma [28] says

LEMMA A *If $\rho > 0$ is a C^2 function on the unit disk D and its gaussian curvature K_ρ satisfies $K_\rho \leq -4$ then $\rho \leq \lambda_D$, where λ_D is the hyperbolic metric density of D with gaussian curvature -4 .*

Mateljević [29] gave an inequality of opposite type of the Ahlfors-Schwarz lemma

LEMMA B *If $\rho > 0$ is a C^2 metric density on D for which the gaussian curvature satisfies $K_\rho \geq -4$ and if $\rho(z)$ tends to $+\infty$ when $|z|$ tends to 1^- , then $\lambda_D \leq \rho$.*

It is known that a Euclidean harmonic mapping f defined on a simply connected domain can be represented by two analytic functions h and g with $f = h + \bar{g}$. Hence, such a mapping always belongs to C^∞ . It is natural to ask whether a ρ -harmonic mapping also possesses the same property. In fact, we have

LEMMA 2.1 *Let $f \in C^{n+2}(\Omega)$ be a ρ -harmonic mapping of Ω onto Ω' , where n is a nonnegative integer. If ρ is a $C^{n+2}(\Omega')$ function on Ω' then Δf belongs to $C^{n+1}(\Omega)$.*

Proof After differentiating Δf in z , we obtain the following relation from the definition of ρ -harmonic mappings,

$$\begin{aligned} -f_{z\bar{z}z} &= 2(\log \rho)_{ww} \circ f f_z^2 \bar{f}_{\bar{z}} + 2(\log \rho)_{w\bar{w}} \circ f f_z \bar{f}_{\bar{z}} \bar{f}_{\bar{z}} \\ &\quad + 2(\log \rho)_w \circ f f_{z\bar{z}} \bar{f}_{\bar{z}} + 2(\log \rho)_w \circ f f_z \bar{f}_{\bar{z}\bar{z}}. \end{aligned} \quad (2.1)$$

Similarly, by differentiating Δf in \bar{z} we can also get the following equality

$$\begin{aligned} -f_{z\bar{z}\bar{z}} &= 2(\log \rho)_{ww} \circ f f_z \bar{f}_{\bar{z}}^2 + 2(\log \rho)_{w\bar{w}} \circ f f_z \bar{f}_{\bar{z}} \bar{f}_{\bar{z}} \\ &\quad + 2(\log \rho)_w \circ f f_{z\bar{z}} \bar{f}_{\bar{z}} + 2(\log \rho)_w \circ f f_z \bar{f}_{\bar{z}\bar{z}}. \end{aligned} \quad (2.2)$$

Hence by the assumption that f and ρ are in $C^2(\Omega)$ and $C^2(\Omega')$, respectively, we get that Δf belongs to $C^1(\Omega)$. Relations (2.1) and (2.2) imply that if f and ρ belong to $C^{n+2}(\Omega)$ and $C^{n+2}(\Omega')$ then $\Delta f \in C^{n+1}(\Omega)$ by induction. \square

Let $g = z|z|^5$ then Δg is in C^1 but $g_{z\bar{z}\bar{z}}$ is not continuous at 0. This example shows that $\Delta f \in C^{n+1}(\Omega)$ does not imply that $f \in C^{n+3}(\Omega)$, where n is a nonnegative integer.

The hyperbolic metric of a given planar domain plays an important role in characterizing its corresponding geometric properties. Harmelin [23] proved

LEMMA C *Let Ω be a hyperbolic domain in the complex plane C and $\lambda_\Omega(z)|dz|$ its hyperbolic metric. Then the following three statements are equivalent:*

- (i) Ω is a convex domain;
- (ii)

$$\left| \frac{\partial}{\partial z} \log \lambda_\Omega(z) \right| \leq \lambda_\Omega(z), \quad z \in \Omega; \quad (2.3)$$

- (iii)

$$\lambda_\Omega(z) \left| \frac{\partial^2}{\partial z^2} \lambda_\Omega^{-1} \right| + \left| \frac{\partial}{\partial z} \log \lambda_\Omega(z) \right|^2 \leq \lambda_\Omega(z)^2, \quad z \in \Omega. \quad (2.4)$$

Moreover, by direct verification we have

$$\left| \frac{\partial^2}{\partial z^2} \log \lambda_\Omega(z) \right| \leq \lambda_\Omega(z) \left| \frac{\partial^2}{\partial z^2} \lambda_\Omega^{-1} \right| + \left| \frac{\partial}{\partial z} \log \lambda_\Omega(z) \right|^2, \quad z \in \Omega. \quad (2.5)$$

A conformal map f of D into D is said to be strongly hyperbolically convex if $f(D)$ is a strongly hyperbolically convex domain. Curz and Mejía [21] showed that if it contains the origin then it is Euclidean convex, and if it is symmetric about the origin, then it is strongly hyperbolically convex if and only if it is spherically convex. Moreover,

LEMMA D If g is strongly hyperbolically convex, then for every $\zeta \in D$

$$\frac{1}{2}(1 - |\zeta|^2)^2 |S_g| + \frac{1}{4} \left| \left[2\bar{\zeta} - (1 - |\zeta|^2) \frac{g''}{g'} \right] - \frac{2\bar{g}g'(1 - |\zeta|^2)}{1 - |g|^2} \right|^2 + \frac{(1 - |\zeta|^2)^2 |g'|^2}{(1 - |g|^2)^2} \leq 1.$$

Using Lemma D, Chen and Fang [22] showed

LEMMA E Let Ω be a strongly hyperbolically convex domain in the unit disk D and $\lambda_\Omega(w)|dw|$ its hyperbolic metric of Gaussian curvature -4 . If $\rho_\Omega(w) = 1/(1 - |w|^2)$ then

$$\frac{|(\log \lambda_\Omega)_{ww} - 2(\log \lambda_\Omega)_w (\log \rho_\Omega)_w - [(\log \rho_\Omega)_{ww} - 2(\log \rho_\Omega)_w^2]| + |(\log \rho_\Omega)_{w\bar{w}}|}{(\lambda_\Omega)^2} \leq 1.$$

In order to study the generalization of the Koebe theorem to ρ -harmonic quasiconformal mappings. We also need the following Lemma F given by Harmelin [23] (See also [30]).

LEMMA F Let Ω be a simply connected domain of hyperbolic type in the complex plane C . Then the following inequality holds for every z in Ω :

$$\frac{1}{4} \leq \lambda_\Omega(z) \delta_\Omega(z) \leq 1, \quad (2.6)$$

and the equality of the upper bound holds if and only if Ω is a disk with centre z , the equality of the lower bound holds if and only if Ω is a slit plane.

Moreover, if Ω is a convex domain of C then for every z in Ω it follows

$$\frac{1}{2} \leq \lambda_\Omega(z) \delta_\Omega(z) \leq 1, \quad (2.7)$$

and the equality of the lower bound holds if and only if Ω is a half-plane.

3. Proof of Theorem 1.1

Proof Let f be a ρ -harmonic K -quasiconformal mapping of D onto Ω and $k = (K - 1)/(K + 1)$. Suppose that $\sigma(z) = (1 - k)\lambda_\Omega(f(z))|f_z|$, $z \in D$, where $\lambda_\Omega(w)|dw|$ is the hyperbolic metric of Ω . By the fact that a ρ -harmonic quasiconformal mapping is diffeomorphism [7], we have that $\sigma(z) > 0$. Then it follows

$$(\Delta \log \sigma)(z) = 4[(\log \lambda_\Omega \circ f)_{z\bar{z}}(z) + (\log |f_z|)_{z\bar{z}}]. \quad (3.1)$$

By the chain rule,[1] we get

$$\begin{aligned} 4(\log \lambda_\Omega \circ f)_{z\bar{z}}(z) = 4 \left\{ ((\log \lambda_\Omega)_{w\bar{w}} \circ f)(|f_z|^2 + |f_{\bar{z}}|^2) \right. \\ \left. + 2\Re[(\log \lambda_\Omega)_{w\bar{w}} \circ f]f_z f_{\bar{z}} + 2\Re[(\log \lambda_\Omega)_w \circ f]f_{z\bar{z}} \right\}. \end{aligned} \quad (3.2)$$

From the assumption that f is ρ -harmonic, it follows

$$f_{z\bar{z}} + 2(\log \rho)_w \circ f f_z f_{\bar{z}} = 0. \quad (3.3)$$

Thus, using (3.2) and (3.3) we obtain

$$4(\log \lambda_\Omega \circ f)_{z\bar{z}}(z) = 4\left\{((\log \lambda_\Omega)_{w\bar{w}} \circ f)(|f_z|^2 + |f_{\bar{z}}|^2) + 2\Re[(\log \lambda_\Omega)_{ww} \circ f - 2(\log \lambda_\Omega)_w \circ f(\log \rho)_w \circ f]f_z f_{\bar{z}}\right\}. \quad (3.4)$$

By differentiating (3.3) in z we have

$$f_{z\bar{z}z} = -2(\log \rho)_{ww} \circ f f_z^2 f_{\bar{z}} - 2(\log \rho)_{w\bar{w}} \circ f f_z f_{\bar{z}} \bar{f}_z \\ - 2(\log \rho)_w \circ f f_{z\bar{z}} f_{\bar{z}} - 2(\log \rho)_w \circ f f_z f_{z\bar{z}}.$$

Hence by Lemma 2.1 we get that $f_{z\bar{z}z} = f_{z\bar{z}\bar{z}}$. Thus, it follows that

$$f_{z\bar{z}z} f_z - f_{z\bar{z}} f_{z\bar{z}} \\ = -2(\log \rho)_{ww} \circ f f_{\bar{z}} f_z^3 - 2(\log \rho)_{w\bar{w}} \circ f |f_{\bar{z}}|^2 f_z^2 + 4[(\log \rho)_w \circ f]^2 f_{\bar{z}} f_z^3.$$

Hence, we obtain

$$4(\log |f_z|)_{z\bar{z}} = 4\Re \frac{f_{z\bar{z}z} f_z - f_{z\bar{z}} f_{z\bar{z}}}{f_z^2} \\ = -4\Re\{2(\log \rho)_{w\bar{w}} \circ f |f_{\bar{z}}|^2 + 2(\log \rho)_{ww} \circ f f_z f_{\bar{z}} \\ - 4[(\log \rho)_w \circ f]^2 f_z f_{\bar{z}}\}. \quad (3.5)$$

Combining (3.1) with (3.4) and (3.5), we get

$$(\Delta \log \sigma)(z) = 4((\log \lambda_\Omega)_{w\bar{w}} \circ f)(|f_z|^2 + |f_{\bar{z}}|^2) \\ - 4\Re\{2(\log \rho)_{w\bar{w}} \circ f |f_{\bar{z}}|^2 + [2(\log \rho)_{ww} \circ f - 4((\log \rho)_w \circ f)^2 \\ + 4(\log \lambda_\Omega)_w \circ f(\log \rho)_w \circ f - 2(\log \lambda_\Omega)_{ww} \circ f]f_z f_{\bar{z}}\}. \quad (3.6)$$

Then by (3.6) we obtain the Gaussian curvature K_σ of σ satisfies that

$$K_\sigma = -\frac{\Delta \log \sigma}{\sigma^2} \\ = -\frac{4}{(1-k)^2} \left\{ \frac{\Delta \log \lambda_\Omega}{4(\lambda_\Omega)^2} \circ f \frac{|f_z|^2 + |f_{\bar{z}}|^2}{|f_z|^2} - 2\Re \left[\frac{(\log \rho)_{w\bar{w}}}{(\lambda_\Omega)^2} \circ f \frac{|f_{\bar{z}}|^2}{|f_z|^2} \right. \right. \\ \left. \left. + \frac{(\log \rho)_{ww} - 2(\log \rho)_w^2 + 2(\log \lambda_\Omega)_w(\log \rho)_w - (\log \lambda_\Omega)_{ww}}{\lambda_\Omega^2} \circ f \frac{f_z f_{\bar{z}}}{|f_z|^2} \right] \right\}. \quad (3.7)$$

From the relation (3.7) and the fact that $|f_{\bar{z}}| \leq |f_z|$ we have

$$K_\sigma \leq -\frac{4}{(1-k)^2} \left\{ (1 + |\mu|^2) \right. \\ \left. - 2 \frac{|(\log \lambda_\Omega)_{ww} - 2(\log \lambda_\Omega)_w(\log \rho)_w - [(\log \rho)_{ww} - 2(\log \rho)_w^2]| + |(\log \rho)_{w\bar{w}}|}{(\lambda_\Omega)^2} |\mu| \right\},$$

where $|\mu| = |f_{\bar{z}}|/|f_z|$. If λ_Ω and ρ satisfy (1.3), then we obtain

$$K_\sigma \leq -\frac{4}{(1-k)^2} (1 + |\mu|^2 - 2|\mu|) = -4 \frac{(1 - |\mu|)^2}{(1-k)^2}.$$

By the assumption that f is also K -quasiconformal, we have $|\mu| \leq k$, then we get

$$K_\sigma \leq -4.$$

Thus, $\sigma(z)$ is a ultrahyperbolic metric in the unit disk D . So by the Lemma A we get $\sigma \leq \lambda_D$, that is,

$$||\partial f|| = \frac{\lambda_\Omega \circ f}{\lambda_D} |f_z| \leq \frac{K+1}{2}.$$

Next we will prove the second inequality of Theorem 1.1. Let $\eta = (1+k)\lambda_\Omega \circ f|f_z|$. Similarly, by (3.5) and (3.6) it follows

$$\begin{aligned} K_\eta &= -\frac{\Delta \log \eta}{\eta^2} \\ &= -\frac{4}{(1+k)^2} \left\{ \frac{\Delta \log \lambda_\Omega}{4(\lambda_\Omega)^2} \circ f \frac{|f_z|^2 + |f_{\bar{z}}|^2}{|f_z|^2} - 2\Re \left[\frac{(\log \rho)_{w\bar{w}}}{(\lambda_\Omega)^2} \circ f \frac{|f_{\bar{z}}|^2}{|f_z|^2} \right. \right. \\ &\quad \left. \left. + \frac{(\log \rho)_{ww} - 2((\log \rho)_w)^2 + 2(\log \lambda_\Omega)_w(\log \rho)_w - (\log \lambda_\Omega)_{ww}}{\lambda_\Omega^2} \circ f \frac{f_z f_{\bar{z}}}{|f_z|^2} \right] \right\}. \end{aligned}$$

Hence, the above relation and the fact that $|f_{\bar{z}}| \leq |f_z|$ implies that

$$\begin{aligned} K_\eta &\geq -\frac{4}{(1+k)^2} \left\{ (1+|\mu|^2) \right. \\ &\quad \left. + 2 \frac{|(\log \lambda_\Omega)_{ww} - 2(\log \lambda_\Omega)_w(\log \rho)_w - [(\log \rho)_{ww} - 2(\log \rho)_w^2]| + |(\log \rho)_{w\bar{w}}|}{(\lambda_\Omega)^2} |\mu| \right\}. \end{aligned}$$

If λ_Ω and ρ satisfy (1.3), then we obtain

$$K_\eta \geq -\frac{4}{(1+k)^2} (1+|\mu|^2 + 2|\mu|) = -4 \frac{(1+|\mu|)^2}{(1+k)^2}.$$

Since f is K -quasiconformal we have $|\mu| \leq k$. Thus, we get

$$K_\eta \geq -4.$$

By the assumption that $\lambda_\Omega \circ f|f_z| \rightarrow +\infty$ as $|z| \rightarrow 1^-$, it follows from Lemma B $\eta \geq \lambda_D$, that is,

$$||\partial f|| = \frac{\lambda_\Omega \circ f}{\lambda_D} |f_z| \geq \frac{K+1}{2K}.$$

The proof of Theorem 1.1 is complete. \square

4. Proof of Theorem 1.2

Proof Let γ be the hyperbolic geodesic between z_1 and z_2 , where z_1 and z_2 are two arbitrary points in D . Then it follows

$$\int_{f(\gamma)} \lambda_\Omega(w) |dw| \leq \int_\gamma \lambda_\Omega(f(z)) L_f(z) |dz| \leq \frac{2K}{K+1} \int_\gamma \frac{\lambda_\Omega(f(z)) |f_z(z)|}{\lambda_D(z)} \lambda_D(z) |dz|,$$

where $w = f(z)$. By the inequality of (1.4) and the fact that $d_h(f(z_1), f(z_2)) \leq \int_{f(\gamma)} \lambda_\Omega(w) |dw|$, we obtain from the above inequality that

$$d_h(f(z_1), f(z_2)) \leq K \int_\gamma \lambda_D(z) |dz| = K d_h(z_1, z_2).$$

Hence, f is hyperbolically K -Lipschitz.

Let $f(\gamma) \subset \Omega$ be the hyperbolic geodesic connected $f(z_1)$ with $f(z_2)$. By the assumption that $\lambda_\Omega|f_z|$ tends to $+\infty$ as $|z| \rightarrow 1^-$, we have that the inequality (1.5) also holds. Hence, we also have

$$d_h(f(z_1), f(z_2)) = \int_{f(\gamma)} \lambda_\Omega(w) |dw| \geq \frac{1}{K} \int_\gamma \lambda_D(z) |dz| \geq \frac{1}{K} d_h(z_1, z_2),$$

where $w = f(z)$. Thus, f is hyperbolically $(1/K, K)$ -biLipschitz. The proof of Theorem 1.2 is complete. \square

5. Koebe theorem for ρ -harmonic quasiconformal mappings

In this section, we want to generalize the classical Koebe theorem to the class of ρ -harmonic quasiconformal mappings. Let $\delta_\Omega(z)$ denote the distance from $z \in \Omega$ to the boundary of Ω . The well-known Koebe theorem is as follows

THEOREM D [31] *If $f(z)$ is a conformal mapping of D onto a simply connected domain Ω then*

$$\frac{1}{4}(1 - |z|^2)|f'(z)| \leq \delta_\Omega(f(z)) \leq (1 - |z|^2)|f'(z)|.$$

Particularly, if f satisfies that $f(0) = 0$ and $f'(0) = 1$ then

$$\frac{1}{4} \leq \delta_\Omega(0) \leq 1.$$

Next we will give another application of Theorem 1.1 to obtain an analogue of the Koebe theorem for ρ -harmonic quasiconformal mappings.

THEOREM 5.1 *Let f be a ρ -harmonic K -quasiconformal mapping of the unit disk D onto a simply connected domain $\Omega \subset \mathbb{C}$. If the pair of metric densities ρ and λ_Ω defined on Ω satisfies the inequality (1.3) and $\lambda_\Omega|f_z|$ tends to $+\infty$ as $|z|$ tends to 1^- then*

$$\frac{1}{4K} \frac{L_f(z)}{\lambda_D(z)} \leq \delta_\Omega(f(z)) \leq K \frac{l_f(z)}{\lambda_D(z)}. \quad (5.1)$$

Particularly, if Ω is also convex then

$$\frac{1}{2K} \frac{L_f(z)}{\lambda_D(z)} \leq \delta_H(f(z)) \leq K \frac{l_f(z)}{\lambda_D(z)}. \quad (5.2)$$

If Ω is the upper half-plane H then

$$\frac{1}{2K} \frac{L_f(z)}{\lambda_D(z)} \leq \delta_H(f(z)) \leq \frac{K}{2} \frac{l_f(z)}{\lambda_D(z)}, \quad (5.3)$$

and the inequality is sharp for every $z \in D$.

Proof Using the K -quasiconformality of f and the inequality of (1.4), we have

$$\frac{\lambda_{\Omega}(f(z))}{\lambda_D(z)} L_f(z) \leq \frac{2K}{K+1} \frac{\lambda_{\Omega}(f(z))}{\lambda_D(z)} |f_z| \leq K. \quad (5.4)$$

By the left inequality (2.6) of Lemma F, it follows from (5.4) that

$$\frac{1}{4K} \frac{L_f(z)}{\lambda_D(z)} \leq \delta_{\Omega}(f(z)).$$

Using the K -quasiconformality of f and the inequality of (1.5), we have

$$\frac{\lambda_{\Omega}(f(z))}{\lambda_D(z)} l_f(z) \geq \frac{2}{K+1} \frac{\lambda_{\Omega}(f(z))}{\lambda_D(z)} |f_z| \geq \frac{1}{K}. \quad (5.5)$$

By the right inequality (2.6) of Lemma F, we obtain from (5.4) that

$$\delta_{\Omega}(f(z)) \leq K \frac{l_f(z)}{\lambda_D(z)}.$$

Particularly, if Ω is also convex then from the inequality (2.7) at Lemma F we similarly have that

$$\frac{1}{2K} \frac{L_f(z)}{\lambda_D(z)} \leq \delta_H(f(z)) \leq K \frac{l_f(z)}{\lambda_D(z)}.$$

If Ω is the upper half-plane H then it is clear that $\lambda_H(z)\delta_H(z) = 1/2$. Hence, from (1.4) and (1.5) we get that

$$\frac{1}{2K} \frac{L_f(z)}{\lambda_D(z)} \leq \delta_H(f(z)) \leq \frac{K}{2} \frac{l_f(z)}{\lambda_D(z)}.$$

Let $F = -2\Im z/|1 - z|^2 + i(1 - |z|^2)/(K|1 - z|^2)$, $z \in D$. Then F is a Euclidean harmonic K -quasiconformal mapping of D onto H . We have

$$\lambda_D(z) = \frac{1}{1 - |z|^2}, \quad L_F(z) = \frac{2}{|1 - z|^2}, \quad \delta_H(F(z)) = \Im F(z) = \frac{1 - |z|^2}{K|1 - z|^2}.$$

Thus,

$$\frac{L_F(z)}{2K\lambda_D(z)} = \frac{1 - |z|^2}{K|1 - z|^2} = \delta_H(F(z)).$$

Hence, the left inequality of (5.3) is sharp for every $z \in D$.

Similarly, Let $G = -2\Im z/|1 - z|^2 + i(K(1 - |z|^2))/(|1 - z|^2)$, $z \in D$. Hence, G is a Euclidean harmonic K -quasiconformal mapping of D onto H . One can easily verify that the equality of the right inequality of (5.3) holds for the mapping G . Thus, the right inequality of (5.3) is sharp for every $z \in D$. The proof of Theorem 5.1 is complete. \square

Let $\eta_{\Omega} = 1/\delta_{\Omega}$. We call η_{Ω} the quasihyperbolic metric density of a domain Ω . The quasihyperbolic metric was first introduced by Gehring and Palak [32]. The quasihyperbolic metric $\eta_D|dz|$ of the unit disk is $1/(1 - |z|)|dz|$. As an application of the generalized Koebe theorem of harmonic quasiconformal mappings, we next give the quasihyperbolically biLipschitz continuity of a class of harmonic quasiconformal mappings.

COROLLARY 5.1 *Let f be a ρ -harmonic K -quasiconformal mapping of the unit disk D onto a simply connected domain $\Omega \subset \mathbb{C}$. If the pair of metric densities ρ and λ_{Ω} defined*

on Ω satisfies the inequality (1.3) and $\lambda_\Omega |f_z|$ tends to $+\infty$ as $|z|$ tends to 1^- then f is quasihyperbolically $(1/(2K), 4K)$ -biLipschitz.

Proof By the inequality (5.1) at Theorem 5.1, we have

$$\frac{1}{K} \frac{\lambda_D(z)}{l_f(z)} \leq \eta_\Omega(f(z)) \leq 4K \frac{\lambda_D(z)}{L_f}.$$

By the fact that $\eta_D(z)/2 \leq \lambda_D(z) \leq \eta_D(z)$ and the above inequality, it follows

$$\frac{1}{2K} \frac{\eta_D(z)}{l_f} \leq \eta_\Omega(f(z)) \leq 4K \frac{\eta_D(z)}{L_f}. \quad (5.6)$$

Using the relation that $l_f |dz| \leq |df| \leq L_f |dz|$, we get the quasihyperbolically $(1/(2K), 4K)$ -biLipschitz continuity of f by the same method as the proof of Theorem 1.2.

6. Several pairs of densities λ_Ω and ρ satisfying the inequality (1.3)

In this section we will give several pairs of densities λ_Ω and ρ satisfying the inequality (1.3) and obtain some classes of ρ -harmonic K -quasiconformal mappings with hyperbolically $(1/K, K)$ -biLipschitz continuity.

Wan [18] considered the case that ρ is a hyperbolic metric density and showed that a hyperbolically harmonic K -quasiconformal mapping of the unit disk onto itself is always hyperbolically $(1/K, K)$ -biLipschitz. In fact, if ρ is chosen to be the hyperbolic metric density λ_Ω , then for an arbitrary simply connected domain Ω the inequality (1.3) reduces to be

$$(\log \lambda_\Omega)_{w\bar{w}} = (\lambda_\Omega)^2.$$

The above equality always holds by the definition of the Gaussian curvature of a hyperbolic metric. In this case, Chen [33] showed that the biLipschitz constants can be improved to be $(2/(K+1), \sqrt{K})$, and its hyperbolically Jacobian $\lambda_\Omega^2(f)/\lambda_D^2(|f_z|^2 - |f_{\bar{z}}|^2)$ satisfies that

$$\frac{1}{K} \leq \frac{\lambda_\Omega^2(f)}{\lambda_D^2} (|f_z|^2 - |f_{\bar{z}}|^2) \leq K.$$

Knezević and Mateljević [19] showed a Euclidean harmonic K -quasiconformal mapping is hyperbolically $(1/K, K)$ -biLipschitz when its range is the unit disk or the upper half-plane. If ρ is a positive constant, then the inequality (1.3) becomes

$$|(\log \lambda_\Omega)_{ww}| \leq (\lambda_\Omega)^2. \quad (6.1)$$

Using Lemma C, Chen and Fang [20] generalized the result obtained by Knezević and Mateljević to the class of Euclidean harmonic K -quasiconformal mappings with convex ranges and showed that the convex assumption is optimal (see Example 8.4).

If $\rho = 1/|\varphi|$, where φ is an analytic function without zeros in Ω , then the inequality (1.3) can be simplified as

$$\left| (\log \lambda_\Omega)_{ww} + \frac{\varphi'}{\varphi} (\log \lambda_\Omega)_w + \frac{1}{2} \frac{\varphi''}{\varphi} \right| \leq \lambda_\Omega^2.$$

Particularly, let Ω be an angular domain with the origin of the complex plane C as its vertex and $\varphi = w$. Chen [34] showed that

$$(\log \lambda_\Omega)_{ww} + \frac{(\log \lambda_\Omega)_w}{w} + \frac{\bar{w}}{w} (\lambda_\Omega)^2 = 0.$$

This implies that the inequality (1.3) also holds. So, the hyperbolically Lipschitz continuity of a log-harmonic quasiconformal mapping of the unit disk onto an angular range was built.

Curz and Mejía [21] introduced a strongly hyperbolically convex domain and characterized its geometric property by the Schwarzian derivative of a strongly hyperbolically convex function. Chen and Fang [22] built a differential inequality determined by the hyperbolic metric for a strongly hyperbolically convex domain in the unit disk (See Lemma E). Using this inequality, we showed that a $1/(1 - |z|^2)$ -harmonic quasiconformal mapping of the unit disk onto a strongly hyperbolically convex domain in the unit disk is also hyperbolically $(1/K, K)$ -biLipschitz.

7. Hyperbolically partial derivatives for ρ -harmonic diffeomorphisms

Since a hyperbolic metric is a conformal invariant, we have that the hyperbolically partial derivative of any conformal mapping f of D onto a domain Ω satisfies that

$$||\partial f|| = \frac{\lambda_\Omega \circ f}{\lambda_D} |f'| = 1.$$

Let $f(z) = x + i/a \sinh(ay)$, where $z = x + iy$ and $a > 0$. Then f is a hyperbolically harmonic diffeomorphism of the upper half-plane H onto itself. Moreover, we have that

$$||\partial f|| = \frac{\lambda_H \circ f}{\lambda_H} |f_z| = \frac{ay(e^{ay} + 1)}{2(e^{ay} - 1)} \rightarrow \infty$$

as y tends to infinity. This example shows that there does not exist an upper bound for the hyperbolically partial derivative of a ρ -harmonic diffeomorphism.

However, for a hyperbolically harmonic diffeomorphism f of the unit disk onto Ω , it follows that

$$||\partial f|| = \frac{\lambda_\Omega \circ f}{\lambda_D} |f_z| \geq 1.$$

In fact, for a general ρ -harmonic diffeomorphism of the unit disk onto a simply connected domain Ω , we have

THEOREM 7.1 *Let Ω be a simply connected domain of hyperbolic type in the complex plane C . Assume that f is a ρ -harmonic diffeomorphism of the unit disk D onto Ω satisfying $\lambda_\Omega \circ f |f_z| \rightarrow +\infty$ as $|z| \rightarrow 1^-$. If ρ and the hyperbolic metric density λ_Ω of Ω satisfy the inequality (1.3), then the inequality*

$$||\partial f|| = \frac{\lambda_\Omega \circ f}{\lambda_D} |f_z| \geq \frac{1}{2}.$$

Proof Assume that f is a ρ -harmonic diffeomorphism of the unit disk D onto Ω . Let $\varpi = 2\lambda_\Omega(f)|f_z|$. By direct verification, the Gaussian curvature of ϖ can be expressed by

$$\begin{aligned} K_\varpi &= -\frac{\Delta \log \varpi}{\varpi^2} \\ &= -\left\{ \frac{\Delta \log \lambda_\Omega}{4(\lambda_\Omega)^2} \circ f \frac{|f_z|^2 + |f_{\bar{z}}|^2}{|f_z|^2} - 2\Re \left[\frac{(\log \rho)_{w\bar{w}}}{(\lambda_\Omega)^2} \circ f \frac{|f_{\bar{z}}|^2}{|f_z|^2} \right. \right. \\ &\quad \left. \left. + \frac{(\log \rho)_{ww} - 2((\log \rho)_w)^2 + 2(\log \lambda_\Omega)_w(\log \rho)_w - (\log \lambda_\Omega)_{ww}}{\lambda_\Omega^2} \circ f \frac{f_z f_{\bar{z}}}{|f_z|^2} \right] \right\}. \end{aligned}$$

Let $\mu = f_{\bar{z}}/f_z$. Then the above relation implies that

$$\begin{aligned} K_\varpi &\geq -\left\{ (1 + |\mu|^2) \right. \\ &\quad \left. + 2 \frac{|(\log \lambda_\Omega)_{ww} - 2(\log \lambda_\Omega)_w(\log \rho)_w - [(\log \rho)_{ww} - 2(\log \rho)_w^2]| + |(\log \rho)_{w\bar{w}}|}{(\lambda_\Omega)^2} |\mu| \right\}. \end{aligned}$$

By the assumption that λ_Ω and ρ satisfy (1.3), then we obtain

$$K_\varpi \geq -(1 + |\mu|^2 + 2|\mu|) \geq -4.$$

Since $\lambda_\Omega \circ f|f_z| \rightarrow +\infty$ as $|z| \rightarrow 1^-$, it follows from Lemma B

$$\varpi \geq \lambda_D,$$

that is,

$$\|\partial f\| = \frac{\lambda_\Omega \circ f}{\lambda_D} |f_z| \geq \frac{1}{2}.$$

The proof of Theorem 7.1 is complete. \square

Let $f = x + e^{-y} \sin x + iy$ be defined on H . Then f is a Euclidean harmonic mapping of H onto itself. Example 8.3 says that it is (K, K') -quasiconformal but not quasiconformal (see [35] for the definition of (K, K') -quasiconformal mappings). After some calculations, it follows that

$$\|\partial f\| = \frac{\lambda_H \circ f}{\lambda_H} |f_z| = \frac{1}{2}(2 + e^{-y} \cos x) > \frac{1}{2}$$

and

$$\|\partial f\| = \frac{\lambda_H \circ f}{\lambda_H} |f_z| = \frac{1}{2}(2 + e^{-y} \cos x) \rightarrow \frac{1}{2}$$

as $x = 2k\pi + \pi$ and y tends to zero. Let φ be a conformal mapping of the unit disk D onto H . Using the fact that $\|\partial(f \circ \varphi)\| = \|\partial f\|$ and the ρ -harmonicity of $f \circ \varphi$ is the same as f , [20] we show that the lower bound $1/2$ of $\|\partial f\|$ is optimal.

8. Auxiliary examples

Example 8.1 Suppose that $f = z|z|^{K-1}$, $K > 1$. Then f is a ρ -harmonic K -quasiconformal mapping of the upper half-plane H onto itself, here $\rho(w) = 1/|w|$.

Moreover,

$$\lim_{|z| \rightarrow \infty} |f_z| = \lim_{|z| \rightarrow \infty} \frac{K+1}{2} |z|^{K-1} = +\infty,$$

$$\lim_{z \rightarrow \infty} |f(z)|/|z| = \lim_{z \rightarrow \infty} |z|^{K-1} = \infty, \quad \|\partial f\| = (K+1)/2,$$

and

$$|(\log \rho(w))_w| = |1/(2w)| \rightarrow \infty, \quad w \rightarrow 0.$$

If $K > 2$, then

$$2|(\log \rho)_w \circ f f_z f_{\bar{z}}| = \frac{K^2 - 1}{4} |z|^{K-2} \rightarrow \infty,$$

as z tends to the infinity point.

Example 8.2 Let $f(z) = \frac{-2|z|^2 + i(z - \bar{z}) + iK(z + \bar{z})}{(z - \bar{z}) - K(z + \bar{z}) + 2i}$. Then f is a ρ -harmonic K -quasiconformal mapping of the unit disk D onto itself satisfying that $|(\rho)_w|/\rho(w)$ and $|(\log \rho)_w \circ f f_z f_{\bar{z}}|$ are unbounded, where $\rho(w)|dw| = 2/|w + i|^2 |dw|$. However, f has bounded (hyperbolically) partial derivatives and then it is hyperbolically biLipschitz and Euclidean biLipschitz.

Proof In fact we can write $f = \psi \circ A_K \circ \varphi$, where

$$\varphi(z) = \frac{z - i}{z + i}, \quad A_K(\zeta) = u + iKv, \quad \zeta = u + iv, \quad \psi(\xi) = i \frac{1 + \xi}{1 - \xi}.$$

Hence, f is a Teichmüller mapping of the unit disk onto itself. By Example 2.2 in [36], it follows that f is ρ -harmonic with $\rho(w) = 2/|w + i|^2$.

Set $z = x + iy$. We have

$$|\rho(w)_w|/\rho(w) = \frac{2}{|w + i|}, \quad |f_z| = \frac{(K+1)[x^2 + (y+1)^2]}{2[K^2x^2 + (y+1)^2]},$$

and

$$2|(\log \rho)_w \circ f f_z f_{\bar{z}}| = \frac{4(K+1)|\bar{z} - i| |(z - \bar{z}) - K(z + \bar{z}) + 2i + (K+1)(\bar{z} - i)|}{|(z - \bar{z}) - K(z + \bar{z}) + 2i|^3}$$

Thus, when $K > 1$ it follows

$$2|(\log \rho)_w \circ f f_z f_{\bar{z}}| = \frac{K^2 - 1}{2(1 + y)} \rightarrow \infty,$$

as z tends to the point $-i$ along the imaginary axis.

Similarly, we have

$$|(\rho(w))_w|/|\rho(w)| \rightarrow \infty, \quad w \rightarrow -i.$$

By the Schwarz-Pick Lemma [1] we obtain the equality

$$\frac{\lambda_D(f(z))}{\lambda_D(z)} |f_z| = \frac{\lambda_H(A_K(\varphi(z)))}{\lambda_D(z)} |(A_K(\varphi(z)))_z|.$$

Thus, by Theorem C

$$\frac{K+1}{2K} \leq \frac{\lambda_H(A_K(\varphi(z)))}{\lambda_D(z)} |(A_K(\varphi(z)))_z| \leq \frac{K+1}{2}.$$

Hence, f has a bounded hyperbolically partial derivative and then it is $(1/K, K)$ -hyperbolically biLipschitz.

Moreover, since

$$\frac{K+1}{2K^2} \leq |f_z| \leq \frac{K+1}{2},$$

f also has a bounded partial derivative. Then it is Euclidean biLipschitz.

Example 8.3 [37] Let $f(z) = x + e^{-y} \sin x + iy$, where $y > 0$. Then f is $(2, 1)$ -QC mapping of the upper half-plane onto itself but it is not K -QC for any $K \geq 1$.

Example 8.4 [20] Let $\Omega = C - [0, +\infty)$ and λ_Ω be its hyperbolic metric density with the Gaussian curvature -4 . Then

$$\lambda_\Omega(z) |dz| = \frac{i}{2(\sqrt{z} - \overline{\sqrt{z}})|\sqrt{z}|} |dz|.$$

Suppose that $A_K = Kx + iy$, $K > 1$. Then A_K is a Euclidean harmonic K -quasiconformal mapping of Ω onto itself and satisfies the inequality

$$||\partial A_K|| = \frac{\lambda_\Omega \circ A_K}{\lambda_\Omega} |(A_K)_z| > \frac{K+1}{2} \quad (8.1)$$

when $x > 0$, $y > 0$.

If $B_K = (1/K)x + iy$ then B_K is a Euclidean harmonic K -quasiconformal mapping of Ω onto itself and satisfies the inequality

$$||\partial B_K|| = \frac{\lambda_\Omega \circ B_K}{\lambda_\Omega} |(B_K)_z| < \left(\frac{K+1}{2K} \right)$$

when $x > 0$, $y \neq 0$.

Example 8.5 Let the boundary of a region $\Omega \subset D$ be the curves $\Gamma_1 = \{z | z = 1/2 + 1/2e^{i\theta}, \theta \in [0, \pi]\}$ and $\Gamma_2 = \{z | z = 1/2 - i/2 + \sqrt{2}/2e^{i\theta}, \theta \in [\pi/4, 3\pi/4]\}$. Then Γ_1 has hyperbolic curvature 2 and Γ_2 has hyperbolic curvature $\sqrt{2}$. Hence, Ω is a crescent-shaped domain. Thus, Ω is strongly hyperbolically convex but not Euclidean convex (see page 1409 of [21]).

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