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# Extending coherent state transforms to Clifford analysis

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Segal-Bargmann coherent state transforms can be viewed as unitary maps from $L^2$ spaces of functions (or sections of an appropriate line bundle) on a manifold $X$ to spaces of square integrable holomorphic functions (or sections) on $X_{\mathbb{C}}$. It is natural to consider higher dimensional extensions of $X$ based on Clifford algebras as they could be useful in studying quantum systems with internal, discrete, degrees of freedom corresponding to nonzero spins. Notice that the extensions of $X$ based on the Grassmann algebra appear naturally in the study of supersymmetric quantum mechanics. In Clifford analysis, the zero mass Dirac equation provides a natural generalization of the Cauchy-Riemann conditions of complex analysis and leads to monogenic functions. For the simplest but already quite interesting case of $X = \mathbb{R}$, we introduce two extensions of the Segal-Bargmann coherent state transform from $L^2(\mathbb{R}, dx) \otimes \mathbb{R}_m$ to Hilbert spaces of slice monogenic and axial monogenic functions and study their properties. These two transforms are related by the dual Radon transform. Representation theoretic and quantum mechanical aspects of the new representations are studied. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4964448]

## I. INTRODUCTION

Clifford analysis (see Refs. 2 and 9) has been developed to extend the complex analysis of holomorphic functions to Clifford algebra valued functions, satisfying properties generalizing the Cauchy–Riemann conditions.

On the other hand, in quantum physics, Clifford algebra or spinor representation valued functions describe some systems with internal degrees of freedom, such as particles with spin.

Recall that the Segal-Bargmann transform,[1,19,20] for a particle on $\mathbb{R}$, establishes the unitary equivalence of the Schrödinger representation with Hilbert space $L^2(\mathbb{R}, dx)$, with (Fock space-like) representations with Hilbert spaces, $\mathcal{H}L^2(\mathbb{C}, dv)$, of holomorphic functions on the phase space, $\mathbb{R}^2 \cong \mathbb{C}$ which are $L^2$ with respect to a measure $v$. In the Schrödinger representation, the position operator $\hat{x}_{Sch}$ acts diagonally while the momentum operator is $\hat{p}_{Sch} = i\frac{d}{dx}$. On the other hand, on the Segal–Bargmann Hilbert space $\mathcal{H}L^2(\mathbb{C}, dv)$, it is the operator $\hat{x}_{SB} + i\hat{p}_{SB}$ that acts as multiplication by the holomorphic function $x + ip$.

In Ref. 12, Hall has defined coherent state transforms (CSTs) for compact Lie groups $G$ which are analogs of the Segal-Bargmann transform. These CSTs correspond to applying heat kernel evolution, $e^{\frac{\Delta}{2}}$, followed by analytic continuation to the complexification $G_{\mathbb{C}}$ of $G$.[13]

We use the fact that, after applying the heat kernel evolution, the resulting functions are in fact extendable to $\mathbb{R}^{m+1}$ in two natural ways motivated by Clifford analysis. These will lead to two generalizations of the CST, the slice monogenic CST, $U_s$, and the axial monogenic CST, $U_a$, which take values on spaces of $\mathbb{C}_m$-valued functions on $\mathbb{R}^{m+1}$, where $\mathbb{C}_m$ denotes the complex Clifford algebra with $m$ generators. One, $\mathcal{H}_s = \text{Im}\, U_s$, is a subspace of the recently introduced space of square integrable slice monogenic functions,[5] while the other, $\mathcal{H}_a = \text{Im}\, U_a$, is a Hilbert space of, the

more traditional in Clifford analysis, axial monogenic functions.[2,9] We show that the two coherent state transforms are related by the dual Radon transform $\check{R}$,

$$U_a = \check{R} \circ U_s.$$

A possibly interesting alternative way of defining a monogenic CST would be through Fueter's theorem.[11,17,15,16,18] It would be very interesting to relate such a transform with the one studied in the present paper.

As in the case of the usual CST, the aim of these transforms is to describe the quantum states of a particle in $\mathbb{R}$ with internal degrees of freedom parametrized by a Clifford algebra, through slice/axial monogenic functions, thus making available, the powerful analytic machinery of Clifford analysis. In Section V, we show that the operator $\hat{x}_0 + i\hat{p}_0$ has a simple action in both the slice and axial monogenic representations.

## II. PRELIMINARIES

### A. Coherent state transforms (CSTs)

Let $G$ be a compact Lie group with complexification $G_{\mathbb{C}}$. In 1994, Hall[12] introduced a class of unitary integral transforms on $L^2(G, dx)$, where $dx$ is a Haar measure, to spaces of holomorphic functions on $G_{\mathbb{C}}$ which are $L^2$ with respect to an appropriate measure. These are known as coherent state transforms (CSTs) or generalized Segal–Bargmann transforms. These transforms were extended to groups of compact type, which include the case of $G = \mathbb{R}^n$ considered in the present paper, by Driver in Ref. 10. General Lie groups of compact type are direct products of compact Lie groups and $\mathbb{R}^n$, see Corollary 2.2 of Ref. 10. For $G = \mathbb{R}^n$ these transforms coincide with the classical Segal–Bargmann transform.[1,19,20]

We will briefly recall now the case $G = \mathbb{R}$, which we will extend to the context of Clifford analysis in the present paper. The case of arbitrary groups of compact type is very interesting and will be studied in a forthcoming work. Let $\rho_t(x)$ denote the fundamental solution of the heat equation,

$$\frac{\partial}{\partial t} \rho_t = \frac{1}{2} \Delta \rho_t,$$

i.e.,

$$\rho_t(x) = \frac{1}{(2\pi t)^{1/2}} e^{-\frac{x^2}{2t}},$$

where $\Delta$ is the Laplacian for the Euclidian metric and let $\mathcal{H}(\mathbb{C})$ denote the space of entire holomorphic functions on $\mathbb{C}$. The Segal–Bargmann or coherent state transform

$$U : L^2(\mathbb{R}, dx) \longrightarrow \mathcal{H}(\mathbb{C})$$

is defined by

$$U(f)(z) = \int_{\mathbb{R}} \rho_1(z - x) f(x) \, dx =$$
$$= \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-\frac{(z-x)^2}{2}} f(x) \, dx, \tag{2.1}$$

where $\rho_1$ has been analytically continued to $\mathbb{C}$. We see that the transform $U$ in (2.1) factorizes according to the following diagram.

$$\tag{2.2}$$

$\mathcal{A}(\mathbb{R})$ is the space of (complex valued) real analytic functions on $\mathbb{R}$ with unique analytic continuation to entire functions on $\mathbb{C}$, $C$ denotes the analytic continuation from $\mathbb{R}$ to $\mathbb{C}$, and $e^{\frac{\Delta}{2}}(f)$ is the (real

analytic) heat kernel evolution of the function $f \in L^2(\mathbb{R}, dx)$ at time $t = 1$, which is the solution of

$$
\begin{cases}
\dfrac{\partial}{\partial t}\, h_t &= \dfrac{1}{2}\Delta\, h_t, \\
h_0 &= f
\end{cases}
\tag{2.3}
$$

evaluated at time $t = 1$,

$$
e^{\frac{\Delta}{2}}(f) = h_1.
$$

Let $\widetilde{\mathcal{A}}(\mathbb{R}) \subset \mathcal{A}(\mathbb{R})$ denote the image of $L^2(\mathbb{R}, dx)$ by the operator $e^{\frac{\Delta}{2}}$.

**Theorem 2.1 (Segal–Bargmann).** *The transform (2.1)*

$$
\begin{array}{ccc}
 & & \mathcal{H}L^2(\mathbb{C}, \nu\, dx dy) \\
 & \overset{U}{\nearrow} & \uparrow{\scriptstyle C} \\
L^2(\mathbb{R}, dx) & \xrightarrow[\;e^{\frac{\Delta}{2}}\;]{} & \widetilde{\mathcal{A}}(\mathbb{R})
\end{array}
\tag{2.4}
$$

*is a unitary isomorphism, where $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$ and $\nu(y) = e^{-y^2}$.*

## B. Clifford analysis

Clifford analysis has been developed to extend the complex analysis of holomorphic functions to Clifford algebra valued functions, satisfying properties generalizing the Cauchy–Riemann conditions.[2,9] Let us briefly recall from Refs. 5 and 8, some definitions and results from Clifford analysis. Let $\mathbb{R}_m$ denote the real Clifford algebra with $m$ generators, $e_j, j = 1, \dots, m$, identified with the canonical basis of $\mathbb{R}^m \subset \mathbb{R}_m$ and satisfying the relations $e_i e_j + e_j e_i = -2\delta_{ij}$. We have that $\mathbb{R}_m = \oplus_{k=0}^m \mathbb{R}_m^k$, where $\mathbb{R}_m^k$ denotes the space of $k$-vectors, defined by $\mathbb{R}_m^0 = \mathbb{R}$ and $\mathbb{R}_m^k = \operatorname{span}_{\mathbb{R}}\{e_A : A \subset \{1, \dots, m\}, |A| = k\}$. We see that, in particular, $\mathbb{R}^m$ is identified with the space of 1-vectors, $\mathbb{R}^m = \mathbb{R}_m^1$, $\underline{x} = \sum_{j=1}^m x_j e_j$ and $\mathbb{R}^{m+1}$ is identified with the space, $\mathbb{R}_m^{\le 1}$, of paravectors of the form,

$$
\vec{x} = x_0 + \underline{x} = x_0 + \sum_{j=1}^m x_j e_j.
$$

Notice also that $\mathbb{R}_1 \cong \mathbb{C}$ and $\mathbb{R}_2 \cong \mathbb{H}$. The inner product in $\mathbb{R}_m$ is defined by

$$
\langle u, v \rangle = \Big\langle \sum_A u_A e_A, \sum_B v_B e_B \Big\rangle = \sum_A u_A v_A,
$$

and therefore, $\underline{x}^2 = -|\underline{x}|^2 = -\langle x, x \rangle$. The Dirac operator is defined as

$$
\partial_{\underline{x}} = \sum_{j=1}^m \partial_{x_j} e_j,
$$

and the Cauchy-Riemann operator as

$$
\partial_{\vec{x}} = \partial_{x_0} + \partial_{\underline{x}}.
$$

We have that $\partial_{\underline{x}}^2 = -\sum_{j=1}^m \frac{\partial^2}{\partial x_j^2}$ and $\partial_{\vec{x}}\overline{\partial_{\vec{x}}} = \sum_{j=0}^m \frac{\partial^2}{\partial x_j^2}$.

Recall that a continuously differentiable function $f$ on an open domain $U \subset \mathbb{R}^{m+1}$, with values on $\mathbb{R}_m$ or $\mathbb{C}_m = \mathbb{R}_m \otimes \mathbb{C}$, is called (left) monogenic on $U$ if (see, for example, Refs. 2 and 9)

$$
\partial_{\vec{x}} f(x_0, \underline{x}) = (\partial_{x_0} + \partial_{\underline{x}}) f(x_0, \underline{x}) = 0.
$$

For $m = 1$, monogenic functions on $\mathbb{R}^2$ correspond to holomorphic functions of the complex variable $x_0 + e_1 x_1$.

## III. MONOGENIC EXTENSIONS OF ANALYTIC FUNCTIONS

### A. Slice monogenic extension

Recall from Refs. 5 and 7 that a function $f \; : \; U \subseteq \mathbb{R}^{m+1} \to \mathbb{R}_m$ is slice monogenic if, for any unit vector $\underline{\omega} \in S^{m-1} = \{\underline{x} \in \mathbb{R}_m^1 \; : \; |\underline{x}| = 1\}$, the restrictions $f_{\underline{\omega}}$ of $f$ to the complex planes

$$H_{\underline{\omega}} = \left\{ u + v \, \underline{\omega}, \; u, v \in \mathbb{R} \right\},$$

are holomorphic,

$$(\partial_u + \underline{\omega} \, \partial_v) f_{\underline{\omega}}(u,v) = 0, \; \forall \underline{\omega} \in S^{m-1}. \tag{3.1}$$

Let $\mathcal{SM}(\mathbb{R}^{m+1})$ denote the space of slice monogenic functions on $\mathbb{R}^{m+1}$. From the definition of $\mathcal{A}(\mathbb{R})$ in diagram (2.2) and the Remark 3.4 of Ref. 4 (see also Proposition 2.7 in Ref. 6), one obtains the following.

**Theorem 3.1.** *The slice-monogenic extension map,*

$$M_s \; : \; \mathcal{A}(\mathbb{R}) \otimes \mathbb{R}_m \longrightarrow \mathcal{SM}(\mathbb{R}^{m+1})$$

$$M_s(h)(x_0, \underline{x}) = M_s(\sum_A h_A \, e_A)(x_0, \underline{x}) =$$

$$= \sum_A h_A(x_0 + \underline{x}) \, e_A := \sum_A e^{\underline{x} \, \frac{d}{dx_0}} \, h_A(x_0) \, e_A = \tag{3.2}$$

$$= \sum_A \sum_{k=0}^{\infty} \frac{\underline{x}^k}{k!} \, \frac{d^k h_A}{dx_0^k}(x_0) \, e_A,$$

*is well defined and satisfies $M_s(h)(x_0, 0) = h(x_0), \forall x_0 \in \mathbb{R}$.*

### B. Axial monogenic extension and dual Radon transform

A monogenic function $f(x_0, \underline{x})$ is called axial monogenic (see Ref. 8, p. 322, for the definition of axial monogenic functions of degree $k$) if it is of the form

$$f(x_0, \underline{x}) = \sum_A f_A(x_0, \underline{x}) \, e_A,$$

$$f_A(x_0, \underline{x}) = B_A(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} \, C_A(x_0, |\underline{x}|), \tag{3.3}$$

where $B_A, C_A$ are scalar functions and the functions $f_A$ are monogenic. The monogenicity condition, $\partial_{\vec{x}} f_A = \partial_{x_0} f_A + \partial_{\underline{x}} f_A = 0$, then leads to the Vekua–type system for $B_A, C_A$, generalising the Cauchy-Riemann conditions,

$$\partial_{x_0} B_A - \partial_r C_A = \frac{m-1}{r} \, C_A, \quad \partial_{x_0} C_A + \partial_r B_A = 0, \, r = |\underline{x}|.$$

Let $\mathcal{AM}(\mathbb{R}^{m+1})$ denote the space of axial monogenic functions on $\mathbb{R}^{m+1}$.

Axial monogenic functions are determined by their restriction to the real axis, $f(x_0, 0)$. The inverse map of extending (when such an extension exists) a real analytic function $h$ on $\mathbb{R}$ to an axial monogenic function on $\mathbb{R}^{m+1}$ is called generalized axial Cauchy-Kowalevski extension and has been studied by many authors (see, for example, Ref. 8).

Using the dual Radon transform to map slice monogenic functions to monogenic functions as proposed in Ref. 3, we will factorize the axial monogenic extension into the slice monogenic extension followed by the dual Radon transform. Let us first recall the definition of the dual Radon transform. (See, for example, Ref. 14.)

*Definition 3.2. The dual Radon transform of a smooth function $f$ on $\mathbb{R}^{m+1}$ is*

$$\check{R}(f)(x_0, \underline{x}) = \int_{S^{m-1}} f(x_0, \langle \underline{x}, \underline{t} \rangle \underline{t}) \, d\underline{t}. \tag{3.4}$$

It is known from Ref. 3 that $\check{R}$ maps entire slice monogenic functions to entire monogenic functions.

Let us denote a function $f \in \mathcal{A}(\mathbb{R})$ and its analytic continuation to the complex plane $H_{\underline{t}}$ by the same symbol, $f$. The following is a small modification of Theorem 4.2 in Ref. 8.

**Theorem 3.3.** *The axial monogenic or axial Cauchy-Kowalevski extension map*

$$M_a \ : \ \mathcal{A}(\mathbb{R}) \otimes \mathbb{R}_m \longrightarrow \mathcal{AM}(\mathbb{R}^{m+1})$$

$$M_a(h)(x_0, \underline{x}) = M_a(\sum_A h_A e_A)(x_0, \underline{x}) =$$

$$= \sum_A \int_{S^{m-1}} h_A(x_0 + \langle \underline{x}, \underline{t} \rangle \underline{t}) \, d\underline{t} \, e_A, \tag{3.5}$$

*where $d\underline{t}$ denotes the invariant normalized (probability) measure on $S^{m-1}$, is well defined, and satisfies $M_a(h)(x_0, 0) = h(x_0), \forall x_0 \in \mathbb{R} = \mathbb{R}_m^0$.*

*Proof.* From (3.2) and (3.4), we see that the map $M_a$ in (3.5) factorizes to

$$M_a = \check{R} \circ M_s . \tag{3.6}$$

The fact that the image of this map is a subspace of the space of entire monogenic functions on $\mathbb{R}^{m+1}$ is a consequence of theorem A of Ref. 3. We still need to show that the functions $M_a(h)$ are axial monogenic for all $h \in \mathcal{A}(\mathbb{R}) \otimes \mathbb{R}_m$. Notice that the Taylor series of $h$, with center at any point of $\mathbb{R}$, has infinite radius of convergence. Using (3.2), Theorem 3.1, and the fact that for $\underline{\omega} \in S^{m-1}$ one has $\underline{\omega}^{2k} = (-1)^k$, we obtain

$$M_a(h)(x_0, \underline{x}) = M_a(\sum_A h_A \, e_A)(x_0, \underline{x}) =$$

$$= \sum_A \check{R} \circ M_s(h_A)(x_0, \underline{x}) \, e_A = \sum_A \int_{S^{m-1}} \sum_{k=0}^\infty \frac{(\langle \underline{x}, \underline{\omega} \rangle \underline{\omega})^k}{k!} h_A^{(k)}(x_0) d\underline{\omega} \, e_A$$

$$= \sum_A \left( \sum_{j=0}^\infty \int_{S^{m-1}} \frac{(-1)^j}{(2j)!} h_A^{(2j)}(x_0) \langle \underline{x}, \underline{\omega} \rangle^{2j} + \underline{\omega} \frac{(-1)^j}{(2j+1)!} h_A^{(2j+1)}(x_0) \langle \underline{x}, \underline{\omega} \rangle^{2j+1} d\underline{\omega} \right) e_A,$$

and therefore,

$$M_a(h)(x_0, \underline{x}) = \sum_A \left( \sum_{j=0}^\infty \frac{(-1)^j}{(2j)!} h_A^{(2j)}(x_0) C_{m,2j} |\underline{x}|^{2j} + \underline{x} \frac{(-1)^j}{(2j+1)!} h_A^{(2j+1)}(x_0) C_{m,2j+2} |\underline{x}|^{2j} \right) e_A,$$

where

$$C_{m,2j} = \int_0^\pi \sin^{m-1}(\theta) \cos^{2j}(\theta) \, d\theta.$$

This is of the form (3.3) which completes the proof.                                                ∎

We therefore get the following commutative diagram.

$$
\begin{array}{ccc}
 & & \mathcal{AM}(\mathbb{R}^{m+1}) \\
 & \overset{M_a}{\nearrow} & \quad \uparrow \\
\mathcal{A}(\mathbb{R}) \otimes \mathbb{R}_m & & \quad \check{R} \\
 & \underset{M_s}{\searrow} & \quad \downarrow \\
 & & \mathcal{SM}(\mathbb{R}^{m+1})
\end{array}
\tag{3.7}
$$

As an illustration let us consider the axial monogenic extension of plane waves $\varphi_p$, with $\varphi_p(x_0) = e^{ipx_0}$. The axial monogenic extension of $\varphi_p$ follows from Example 2.2.1 and Remark 2.1 of Ref. 8, where the axial monogenic extension of $e^{x_0}$ is given in terms of Bessel functions, by taking $k = 0$ and replacing $\underline{x}$ by $ip\underline{x}$ in the expressions of Example 2.2.1 of Ref. 8.

*Proposition 3.4. The axial monogenic plane waves are given by*

$$M_a(\varphi_p)(x_0, \underline{x}) = \Gamma\left(\frac{m}{2}\right)\left(\frac{2i}{p|\underline{x}|}\right)^{m/2-1}\left(I_{m/2-1}(p|\underline{x}|) + i\frac{\underline{x}}{|\underline{x}|}I_{m/2}(p|\underline{x}|)\right)e^{ipx_0}, \qquad (3.8)$$

*where $I_\alpha$ are the hyperbolic Bessel functions.*

*Proof.* By representing, as in example 2.2.1 of Ref. 8, $M_a(\varphi_p)(x_0)$ in the form

$$M_a(\varphi_p)(x_0, \underline{x}) = \sum_{j=0}^{\infty} c_j \underline{x}^j B_j e^{ipx_0},$$

and expressing the monogenicity of the transform

$$(\partial_{x_0} + \partial_{\underline{x}})\sum_{j=0}^{\infty} c_j \underline{x}^j B_j e^{ipx_0} = 0,$$

we obtain the following recurrence relation for the functions $B_j(x_0)$:

$$B_{j+1}(x_0) = -ipB_j(x_0) - B_j'(x_0), \quad B_0(x_0) = 1.$$

The solution is $B_j(x_0) = (-ip)^j$. Then we see that the transform is obtained by replacing $\underline{x}$ by $ip\underline{x}$ in the expressions of Example 2.2.1 of Ref. 8. ∎

*Remark 3.5.* From Theorem A of Ref. 3, $\check{R} : \mathcal{SM}(\mathbb{R}^{m+1}) \to \mathcal{AM}(\mathbb{R}^{m+1})$ is an injective map. In fact, from Corollary 4.4 of Ref. 3, we see that (non-zero) slice monogenic functions do not belong to Ker $\check{R}$. ◇

*Remark 3.6.* Note that, as in Ref. 8, considering $h \in \mathcal{A}(\mathbb{R}) \otimes \mathbb{C}_m$, one also has

$$M_a(h)(x_0, \underline{x}) = \sum_A \int_{S^{m-1}} h_A(x_0 + i\langle \underline{x}, \underline{t} \rangle)(1 - i\underline{t})\, d\underline{t}\, e_A, \qquad (3.9)$$

which is equivalent to (3.5) and can be readily verified by expansion in power series. ◇

## IV. CLIFFORD EXTENSIONS OF THE CST

The two extensions (3.2) and (3.5) give two natural paths to define coherent state transforms by replacing the vertical arrow of analytic continuation in diagram (2.4).

We refer the reader interested in the representation theoretic and the quantum mechanical meaning of the Hilbert spaces introduced in the present section to Section V.

### A. Slice monogenic coherent state transform (SCST)

The slice monogenic CST is naturally defined by substituting the vertical arrow in diagram (2.4) by the slice monogenic extension (3.2) leading to

$$
\begin{array}{c}
\mathcal{SM}(\mathbb{R}^{m+1}) \otimes \mathbb{C} \\
\xrightarrow{U_s} \qquad \uparrow M_s \\
L^2(\mathbb{R}, dx_0) \otimes \mathbb{C}_m \xhookrightarrow[e^{\frac{\Delta_0}{2}}]{} \widetilde{\mathcal{A}}(\mathbb{R}) \otimes \mathbb{C}_m
\end{array}
\qquad (4.1)
$$

where $\Delta_0 = \frac{d^2}{dx_0^2}$. Notice that even though the plane waves, $\varphi_p(x_0) = e^{ipx_0}$, are not in the Hilbert space $L^2(\mathbb{R}, dx_0)$, they are generalized eigenfunctions of $\Delta_0$ with eigenvalue $-p^2$, and therefore,

$$e^{\frac{\Delta_0}{2}}(\varphi_p)(x_0) = e^{\frac{\Delta_0}{2}}e^{ipx_0} = e^{-\frac{p^2}{2}}e^{ipx_0} = e^{-\frac{p^2}{2}}\varphi_p(x_0). \qquad (4.2)$$

On the other hand, since the plane waves $\varphi_p \in \mathcal{A}(\mathbb{R})$, we can use (3.2) to obtain the following very simple result.

*Lemma 4.1.  The slice monogenic plane waves are given by*

$$M_s(\varphi_p)(x_0) = M_s(e^{ipx_0}) = e^{ip\vec{x}} = \left(\cosh(p|\underline{x}|) + i\frac{x}{|\underline{x}|}\sinh(p|\underline{x}|)\right)e^{ipx_0}. \tag{4.3}$$

*Proof.* From (3.2) we obtain

$$M_s(\varphi_p)(x_0) = e^{ipx_0}\sum_{k=0}^{\infty}\frac{(ip\underline{x})^k}{k!} = \left(\cosh(p|\underline{x}|) + i\frac{x}{|\underline{x}|}\sinh(p|\underline{x}|)\right)e^{ipx_0}.$$

∎

*Proposition 4.2.  Let $f \in L^2(\mathbb{R}, dx_0)$ and*

$$f(x_0) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{ipx_0}\tilde{f}(p)\,dp.$$

*We have*

$$U_s(f)(x_0,\underline{x}) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-\frac{p^2}{2}}e^{ip\vec{x}}\tilde{f}(p)\,dp = \tag{4.4}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-\frac{p^2}{2}}e^{ipx_0}\cosh(p|\underline{x}|)\tilde{f}(p)\,dp + i\frac{x}{|\underline{x}|}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-\frac{p^2}{2}}e^{ipx_0}\sinh(p|\underline{x}|)\tilde{f}(p)\,dp.$$

*Proof.*  This result follows from Lemma 4.1, (3.2), and (4.2).                                    ∎

Consider the standard inner product on $\mathbb{C}_m$. Our main result in this section is the following.

**Theorem 4.3.**  *The SCST, $U_s$ in Diagram (4.1), is unitary onto its image for the measure $d\nu_m$ on $\mathbb{R}^{m+1}$ given by*

$$d\nu_m = \frac{2}{\sqrt{\pi}}\frac{1}{Vol(S^{m-1})}\frac{e^{-|\underline{x}|^2}}{|\underline{x}|^{m-1}}\,dx_0d\underline{x},$$

*where $Vol(S^{m-1})$ denotes the volume of the unit radius sphere in $\mathbb{R}^m$, i.e., the map $U_s$ in the diagram*

$$\begin{array}{ccc} & & \mathcal{H}_s \\ & {}^{\displaystyle U_s}\nearrow & \uparrow {\scriptstyle M_s} \\ L^2(\mathbb{R}, dx_0)\otimes\mathbb{C}_m & \xrightarrow[e^{\frac{\Delta_0}{2}}]{} & \widetilde{\mathcal{A}}(\mathbb{R})\otimes\mathbb{C}_m \end{array} \tag{4.5}$$

*is a unitary isomorphism, where $\mathcal{H}_s = U_s(L^2(\mathbb{R}, dx_0)\otimes\mathbb{C}_m) \subset \mathcal{SM}L^2(\mathbb{R}^{m+1}, d\nu_m)$.*

*Proof.* Let $\mathcal{S}(\mathbb{R})$ be the space of Schwarz functions on $\mathbb{R}$. For $f, h \in \mathcal{S}(\mathbb{R})\otimes\mathbb{R}_m$, with $f = \sum_A f_A e_A$, $h = \sum_A h_A e_A$, we have

$$\langle U_s(f), U_s(h)\rangle = \frac{2}{\sqrt{\pi}}\frac{1}{Vol(S^{m-1})}\sum_A\int_{\mathbb{R}\times\mathbb{R}^m}[e^{2i\underline{x}p}]_0\,e^{-p^2}\tilde{f}_A(p)\overline{\tilde{h}_A}(p)\frac{e^{-|\underline{x}|^2}}{|\underline{x}|^{m-1}}d^m x\,dp =$$

$$= \frac{2}{\sqrt{\pi}}\frac{1}{Vol(S^{m-1})}\sum_A\int_{\mathbb{R}}e^{-p^2}\tilde{f}_A(p)\overline{\tilde{h}_A}(p)\left(\int_{\mathbb{R}^m}\cosh(2|\underline{x}|p)\frac{e^{-|\underline{x}|^2}}{|\underline{x}|^{m-1}}\,d^m x\right)dp =$$

$$= \frac{2}{\sqrt{\pi}}\frac{1}{Vol(S^{m-1})}\sum_A\int_{\mathbb{R}}e^{-p^2}\tilde{f}_A(p)\overline{\tilde{h}_A}(p)\left(\int_0^{\infty}\cosh(2up)e^{-u^2}\,du\right)dp =$$

$$= \sum_A\int_{\mathbb{R}}\tilde{f}_A(p)\overline{\tilde{h}_A}(p)\,dp = \langle f, h\rangle.$$

From the density of $\mathcal{S}(\mathbb{R})\otimes\mathbb{C}$ in $L^2(\mathbb{R})$, we conclude that $U_s$ is unitary onto its image.                    ∎

*Remark 4.4.* For each complex plane $H_{\underline{\omega}} := \{u + v\underline{\omega} : u, v \in \mathbb{R}\}$ and for $f \in L^2(\mathbb{R}, dx) \otimes \mathbb{C}_m$, $f = \sum_A f_A e_A$, the map $f \mapsto U_s(f)\big|_{H_{\underline{\omega}}}$ coincides, for each component $f_A$ of $f$, with the Segal–Bargmann transform, which is surjective to $\mathcal{H}L^2(H_{\underline{\omega}}, d\nu)$ and unitary for the measure $d\nu = e^{-v^2} du \, dv$ on $H_{\underline{\omega}}$. ◇

## B. Axial monogenic coherent state transform (ACST)

The axial monogenic CST is also naturally defined as the heat kernel evolution followed by the axial Cauchy-Kowalevski extension

$$U_a = M_a \circ e^{\frac{\Delta_0}{2}},$$

i.e., by substituting the vertical arrow in diagram (2.2) by the axial monogenic extension (3.5).

$$
\begin{array}{ccc}
 & & \mathcal{A}\mathcal{M}(\mathbb{R}^{m+1}) \otimes \mathbb{C} \\
 & \nearrow^{U_a} & \uparrow M_a \\
L^2(\mathbb{R}, dx_0) \otimes \mathbb{C}_m & \xrightarrow[e^{\frac{\Delta_0}{2}}]{} & \mathcal{A}(\mathbb{R}) \otimes \mathbb{C}_m
\end{array}
\qquad (4.6)
$$

The following is an easy consequence of Theorem 4.3, (3.6), and Remark 3.5.

**Theorem 4.5.** *Let $\mathcal{H}_a \subset \mathcal{A}\mathcal{M}(\mathbb{R}^{m+1}) \otimes \mathbb{C}$ denote the image of $L^2(\mathbb{R}, dx_0) \otimes \mathbb{C}_m$ under $U_a$. The restriction of the dual Radon transform to $\mathcal{H}_s$ defines an isomorphism to $\mathcal{H}_a$.*

*The diagram*

$$
\begin{array}{ccc}
 & & \mathcal{H}_a \\
 & U_a \nearrow \quad \nearrow M_a & \uparrow \\
L^2(\mathbb{R}, dx_0) \otimes \mathbb{C}_m \xrightarrow{e^{\frac{\Delta_0}{2}}} \widetilde{\mathcal{A}}(\mathbb{R}) \otimes \mathbb{C}_m & & \check{R} \\
 & U_s \searrow \quad \searrow M_s & \downarrow \\
 & & \mathcal{H}_s
\end{array}
\qquad (4.7)
$$

*is commutative and its exterior arrows are unitary isomorphisms for the inner product on $\mathcal{H}_a$ given by $\langle \cdot, \cdot \rangle_{\mathcal{H}_a}$,*

$$\langle F, G \rangle_{\mathcal{H}_a} = \int_{\mathbb{R}^{m+1}} (\check{R})^{-1}(F)(\check{R})^{-1}(G) d\nu_m, \qquad (4.8)$$

*where $d\nu_m$ was defined in Theorem 4.3.*

*Proof.* The injectivity of $\check{R}_{|\mathcal{H}_s}$ follows from Remark 3.5. From (3.6), we conclude that $U_a = \check{R} \circ U_s$ which implies the surjectivity of $\check{R}_{|\mathcal{H}_s} : \mathcal{H}_s \longrightarrow \mathcal{H}_a$. Then, the inner product (4.8) is well defined, the diagram (4.7) is commutative, and the exterior arrows are unitary isomorphisms.  ∎

*Remark 4.6.* As mentioned in the Introduction, a possibly interesting alternative way of defining a monogenic CST would be by replacing the dual Radon transform in (3.7) and in diagram (4.7) by the Fueter mapping, $\Delta^{\frac{m-1}{2}}$, where $\Delta = \sum_{j=0}^{m} \frac{\partial^2}{\partial x_j^2}$ (see Refs. 11, 17, 15, 16, and 18). Notice however that the map $\Delta^{\frac{m-1}{2}} \circ M_s$ does not correspond to a monogenic extension of analytic functions of one variable as the restriction to the real line does not give back the original functions. It leads nevertheless to an interesting transform and it would be very interesting to relate it with $U_a$. ◇

## V. REPRESENTATION THEORETIC AND QUANTUM MECHANICAL INTERPRETATION

Recall that the Schrödinger representation in quantum mechanics is the representation for which the position operator $\hat{x}_0$ acts by multiplication on $L^2(\mathbb{R}, dx_0)$. The momentum operator is then

given by

$$\hat{p}_0 = i\frac{d}{dx_0}.$$

The CST from Section II A intertwines the Schrödinger representation with the Segal-Bargmann representation, on which the operator $\hat{x}_0 + i\hat{p}_0$ acts as the operator of multiplication by the holomorphic function $x_0 + ip_0$ (see Theorem 6.3 of Ref. 13),

$$\left(U \circ (\hat{x}_0 + i\hat{p}_0) \circ U^{-1}\right)(f)(x_0, p_0) = (x_0 + ip_0)f(x_0, p_0). \tag{5.1}$$

We will prove now the analogous result that the slice monogenic CST intertwines the Schrödinger representation with the representation on which $\hat{x}_0 + i\hat{p}_0$ acts as the operator of left multiplication by the slice monogenic function $x_0 + \underline{x}$.

*Proposition 5.1. The observable $x_0 + ip_0$ is represented in the slice monogenic representation by the operator of multiplication by the slice monogenic function $x_0 + \underline{x}$, i.e.,*

$$\left(U_s \circ (\hat{x}_0 + i\hat{p}_0) \circ U_s^{-1}\right)(f)(x_0, \underline{x}) = (x_0 + \underline{x})f(x_0, \underline{x}), \ f \in \mathcal{H}_s. \tag{5.2}$$

*Proof.* We have $U_s = M_s \circ e^{\frac{\Delta_0}{2}}$. From the injectivity of the slice monogenic extension map $M_s$, (5.2) is equivalent to

$$\left(e^{\frac{\Delta_0}{2}} \circ (x_0 - \frac{d}{dx_0}) \circ e^{-\frac{\Delta_0}{2}}\right)(f)(x_0) = x_0 f(x_0).$$

This follows from Theorem 6.3 of Ref. 13.                                                    ∎

For the axial monogenic coherent state transform defining the axial monogenic representation, on the other hand, we have a more complicated representation of $x_0 + ip_0$ involving the Cauchy-Kowalevski extension of the polynomials $\underline{x}^j, j \in \mathbb{N}_0$.

Recall, from Theorem 2.2.1 of Ref. 9, that the Cauchy-Kowalevski extension of $\underline{x}^j$ is given by the polynomial $X_0^{(j)}(x_0, \underline{x})$, such that $X_0^{(j)}(0, \underline{x}) = \underline{x}^j$, where

$$X_0^{(j)}(x_0, \underline{x}) = CK(\underline{x}^j) = \mu_0^j |x|^j \left(C_j^{(m-1)/2}\left(\frac{x_0}{|x|}\right) + \frac{m-1}{m+j-1}C_{j-1}^{(m+1)/2}\left(\frac{x_0}{|x|}\right)\frac{\underline{x}}{|x|}\right),$$

with

$$\mu_0^{2j} = (-1)^j (C_{2j}^{(m-1)/2}(0))^{-1}, \ \mu_0^{2j+1} = (-1)^j \frac{m+2j}{m-1}(C_{2j}^{(m+1)/2}(0))^{-1}$$

and the Gegenbauer polynomials

$$C_j^\nu(y) = \sum_{i=0}^{[j/2]} \frac{(-1)^i(\nu)_{j-i}}{i!(j-2i)!}(2y)^{j-2i},$$

where $(\nu)_j = \nu(\nu+1)\cdots(\nu+j-1)$.

*Proposition 5.2. Let $f \in \mathcal{H}_a$ be given by*

$$f(x_0, \underline{x}) = \sum_{i=0}^\infty X_0^{(i)}(x_0, \underline{x})f_i. \tag{5.3}$$

*The observable $x_0 + ip_0$ is represented in the axial monogenic representation by the following operator:*

$$\left(U_a \circ (\hat{x}_0 + i\hat{p}_0) \circ U_a^{-1}\right)(f)(x_0, \underline{x}) = \sum_{i=0}^\infty \left(\frac{2i+1}{2i+m}X_0^{(2i+1)}(x_0, \underline{x})f_{2i} + X_0^{(2i+2)}(x_0, \underline{x})f_{2i+1}\right). \tag{5.4}$$

*Proof.* From Theorem 3.4 of Ref. 3, any entire axial monogenic function has an expansion of the form (5.3). On the other hand, from equations (22) and (23) of Ref. 3, we obtain

$$\left(\check{R} \circ (x_0 + \underline{x}) \circ \check{R}^{-1}\right)\left(X_0^{2j}\right) = \frac{2j+1}{2j+m}\, X_0^{2j+1},$$

$$\left(\check{R} \circ (x_0 + \underline{x}) \circ \check{R}^{-1}\right)\left(X_0^{2j+1}\right) = X_0^{2j+2}, \qquad j \in \mathbb{N}_0.$$

These identities together with Proposition 5.1 and the fact that $U_a = \check{R} \circ U_s$ prove (5.4). ∎

*Remark 5.3.* On the axial monogenic representation, one does not expect to have operators of multiplication by nontrivial functions as the product of monogenic functions is in general not monogenic. The axial monogenic representation of $x_0 + ip_0$ given by (5.4) is in a sense the closest one can get to such an operator as it maps the monogenic polynomial of order $k$, $X_0^k = CK(\underline{x}^k)$, to a scalar times the monogenic polynomial of order $k+1$, $X_0^{k+1}$. ◇

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