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Integral representations of a class of harmonic functions in the half space

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Abstract

In this article, motivated by the classic Hadamard factorization theorem about an entire function of finite order in the complex plane, we firstly prove that a harmonic function whose positive part satisfies some growth conditions, can be represented by its integral on the boundary of the half space. By using Nevanlinna's representation of harmonic functions and the modified Poisson kernel of the half space, we further prove a representation formula through integration against a certain measure on the boundary hyperplane for harmonic functions not necessarily continuous on the boundary hyperplane whose positive parts satisfy weaker growing conditions than the first question. The result is further generalized by involving a parameter m dealing with the singularity at the infinity.

Keywords: Integral representation; Positive part; Modified Poisson kernel

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1. Introduction

Some fundamental properties of entire functions of finite order and type in the complex plane or analytic functions in the right (upper) half-plane, have been well studied (see [2,10]). In light of results from Complex Analysis, the order of a classic harmonic function with the Poisson integral in the half space of $\mathbb{R}^n$ is 1, if we define the order of harmonic functions in higher-dimensions similarly to that of entire functions. In what follows, $\mathbb{H} = \{x \in \mathbb{R}^n : x = (x', x_n),\ x' \in \mathbb{R}^{n-1},\ x_n > 0\}$ represents the upper half space of $\mathbb{R}^n$.

However, when the order is greater than 1, as far as we know, there has been one paper concerning this higher-dimensional problem: The recent paper [15] establishes an integral representation for harmonic functions in $\mathbb{H}$ with order less than 2, by using Carleman's formula and Nevanlinna's representation [16]. The latter mentioned two formulas in one complex variable were useful in the classical theory of functions of one complex variable. Paper [16] generalized the Carleman's formula for harmonic functions in the half plane to the higher-dimensional half space, and established a Nevanlinna's representation for harmonic functions in the half sphere by using Hörmander's theorem, so they are invaluable tools in the study of harmonic functions in the half space $\mathbb{H}$ as well.

The classic Hadamard factorization theorem of an entire function of finite order [3] and the inner and outer factorization theorem of analytic functions in the Hardy spaces in a half plane [6,9] motivate us to carry out this study on harmonic functions in higher-dimensional spaces as given in the forthcoming two sections. Such a higher-dimensional situation is important, interesting and worthwhile for further investigation. In Section 2 we employ Carleman's formula [16] to give the integral representation of harmonic functions with order less than 3, where integral boundary conditions are assumed in place of growth conditions describing the finite order or type properties for entire functions. We also prove that a harmonic function with a finite order, not necessarily continuous on the boundary hyperplane, has an integral representation involving a measure. We make use of Nevanlinna's representation [16] and the modified Poisson kernel of the half space $\mathbb{H}$ [5]. Integral boundary conditions are used to displace the terminology of finite order as well. In Section 3 we provide proofs of the main results.

2. Preliminaries

The notation and terminology that are used in this article can be found in [4,15].

Recall that $\mathbb{H}$ is the Euclidean half space, we then have the hyperplane $\mathbb{R}^n = \{x \in \mathbb{R}^n : x = (x', x_n), x_n = 0\}$, which will be denoted as $\partial\mathbb{H}$. We identify $\mathbb{R}^n$ with $\mathbb{R}^{n-1} \times \mathbb{R}$ and write $x \in \mathbb{R}^n$ as $x = (x', x_n)$, where $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$. Let θ be the angle between x and $\hat{e}_n$, i.e., $x_n = |x|\cos\theta$, $|x'| = |x|\sin\theta$ $(0 \le \theta < \frac{\pi}{2})$, $x \in \mathbb{H}$. We will write $x = x_1\hat{e}_1 + \cdots + x_{n-1}\hat{e}_{n-1} + x_n\hat{e}_n$, where $\hat{e}_i$ is the ith unit coordinate vector and $\hat{e}_n$ is the normal to $\partial\mathbb{H}$.

For a measurable function u on $\partial\mathbb{H}$, the Poisson integral

$$P[u](x) = \frac{2x_n}{n\omega_n} \int\limits_{\partial\mathbb{H}} \frac{u(y')}{|x - y'|^n} dy' \tag{2.1}$$

will exist and then define a harmonic function in $\mathbb{H}$ if [1,12]

$$\int_{\partial\mathbb{H}} \frac{|u(y')|}{1+|y'|^n} dy' < \infty, \tag{2.2}$$

where $\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}$ is the volume of the unit n-ball. If u is continuous then $P[u](x)$ is the solution of the half space Dirichlet problem.

The recent paper [15] partly employs the methods of [14] to weaken the condition (2.2) into

$$\int_{\partial\mathbb{H}} \frac{u^+(y')}{1+|y'|^{n+1}} dy' < \infty, \tag{2.3}$$

and give the integral representation

$$u = \frac{2x_n}{n\omega_n} \int_{\partial\mathbb{H}} \left(\frac{1}{|x-y'|^n} - \frac{1}{1+|y'|^n} \right) u(y') dy'. \tag{2.4}$$

One of the purposes of this article is to further weaken the boundary condition (2.3). We will first introduce the modified Poisson integral of harmonic functions. Subsequently, we will give the integral expression of harmonic functions by involving a parameter m dealing with singularity at the infinity.

We suppose that a measurable function u on $\partial\mathbb{H}$ satisfies the conditions

$$\int_{\partial\mathbb{H}} \frac{u^+(y')}{1+|y'|^{n+2}} dy' < \infty \tag{2.5}$$

and

$$\int_{\mathbb{H}} \frac{x_n u^+(x)}{1+|x|^{n+1}} dx < \infty. \tag{2.6}$$

By means of Carleman's formula and Nevanlinna's formula [16], with the proof in [15], we can derive the boundary convergence condition and the integral representation of u.

Theorem 2.1. *If a harmonic function $u(x)$ satisfies (2.5) and (2.6), then*

$$\int_{\partial\mathbb{H}} \frac{|u(y')|}{1+|y'|^{n+2}} dy' < \infty; \tag{2.7}$$

and there exist constants c_1 and c_2, such that

$$u(x) = \frac{2x_n}{n\omega_n} \int_{\partial\mathbb{H}} \left(\frac{1}{|x-y'|^n} - \frac{1}{1+|y'|^n} - \frac{|x|}{1+|y'|^{n+1}}\right) u(y') dy' + c_1 x_n |x| + c_2 x_n. \tag{2.8}$$

Remark 2.1. Theorem 2.1 generalizes the results of harmonic functions in [1,5,8,14,15].

Based on the idea of the classic Hadamard theorem of entire functions of finite order and the inner and outer function factorization theorem of analytic functions in Hardy spaces in a half-plane, as well as Riesz representation theorem, we now turn to an interesting connection between integral and measure. When m is an integer, denote by $\mathbb{H}(m)$ the space of functions u that are harmonic in $\mathbb{H}$ and satisfy

$$I = \sup_{0<|\varepsilon|<1} \int_{\partial\mathbb{H}} \frac{u^+(\varepsilon' + y')}{1 + |y'|^{n+m}} dy' < \infty \tag{2.9}$$

and

$$\int_{\mathbb{H}} \frac{x_n u^+(x)}{1 + |x|^{n+m+2}} dx < \infty. \tag{2.10}$$

For $u \in \mathbb{H}(m)$, $u \not\equiv 0$, $u(x)$ is bounded in the half sphere $\overline{B_R^+} = \{x \in \mathbb{H}, |x| = R, x_n > 0, R > 1\}$, thereby, the non-tangential limit of $u(x)$ exists in the cone

$$\Gamma_\alpha(x') = \{x = (x', x_n) \in \mathbb{H}, |x' - y'| < \alpha x_n, y' \in \partial\mathbb{H}, \alpha > 0\},$$

and the non-tangential limit function $u(x')$ is also bounded in $\Gamma_\alpha(x')$ [1].

If $m \geq 0$ is an integer, paper [4] defines the modified Poisson kernel of order m for $x \in \mathbb{H}$ as follows:

$$P_{\mathbb{H}}^m(x, y) = \begin{cases} P_{\mathbb{H}}(x, y), & \text{when } |y| \leq 1, \\ P_{\mathbb{H}}(x, y) - \frac{2x_n}{n\omega_n} \sum_{k=0}^{m-1} \frac{|x|^k}{|y|^{n+k}} C_k^{\frac{n}{2}} \left(\frac{x \cdot y}{|x||y|}\right), & \text{when } |y| > 1. \end{cases}$$

For the coefficients $C_k^{\frac{n}{2}}(t)$ and its properties see [8], pp. 82 and 92. Reference [4] also gives the following estimation:

$$|P_{\mathbb{H}}^m(x, y)| \leq \begin{cases} \dfrac{A|x|^{n+m}}{x_n^{n-1}|y|^{n+m}}, & 1 < |y| \leq 2|x|; \\ \dfrac{Ax_n|x|^m}{x_n^{n-1}|y|^{n+m}}, & |y| > \max\{1, 2|x|\}; \\ \dfrac{2}{\omega_n} \dfrac{1}{x_n^{n-1}}, & |y| \leq 1, \end{cases} \tag{2.11}$$

for $|x| > 1$ and a constant A (as is customary, A will denote a finite, positive constant depending at most on n and m, not necessarily the same on any two occurrences).

Motivated by [14,15], our second aim in this article is to establish the following theorem:

Theorem 2.2. *Suppose* $u(x) \in \mathbb{H}(m)$. *Then*

(1)

$$I = \sup_{0<|\varepsilon|<1} \int_{\partial\mathbb{H}} \frac{|u(\varepsilon' + y')|}{1 + |y'|^{n+m}} dy' < \infty.$$

(2) *There exists a measure* μ *on* $\partial\mathbb{H}$ *such that*

$$\int_{\partial\mathbb{H}} \frac{d|\mu(y')|}{1 + |y'|^{n+m}} < \infty.$$

(3) *There exists a polynomial* $Q(x)$ *of degree* $m - 3$ *such that*

$$u(x) = Q(x) + \int_{\partial\mathbb{H}} P_m(x, y') d\mu(y').$$

Remark 2.2. Theorem 2.2 generalizes the results in [5,14,15].

3. The proofs of the theorems

Proof of Theorem 2.1. Carleman's formula of harmonic functions [15,16] implies

$$\begin{aligned} &\int_{\{x:\,|x|=R,\ x_n>0\}} u^-(x) \frac{n x_n}{R^{n+1}} d\sigma(x) + \\ &\int_{\{x:\,r<|x'|<R,\ x_n=0\}} u^-(x') \left(\frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' = \\ &\int_{\{x:\,|x|=R,\ x_n>0\}} u^+(x) \frac{n x_n}{R^{n+1}} d\sigma(x) + \\ &\int_{\{x:\,r<|x'|<R,\ x_n=0\}} u^+(x') \left(\frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' - \frac{c_1}{r^n} - \frac{c_2}{R^n} \le A u_\rho(R), \end{aligned} \tag{3.1}$$

for $R > 1$, where $u^- = \max\{-u(x), 0\}$ and $u^+ = \max\{u(x), 0\}$ are negative part and positive part of $u(x)$, respectively. $u_1(R) = 1 + \ln(1 + R)$, $u_\rho(R) = 1 + (1 + R)^{\rho-1}$, $\rho \neq 1$. Let

$$\Phi_1(R) := \int_{\{x\in\mathbb{H}:1<|x|<R,x_n=0\}} u^+(x') \left(\frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx'.$$

Then

$$\Phi_1(R) \le \frac{2^n}{2^n-1} \int\limits_{\{x\in\mathbb{H}:1<|x|<2R, x_n=0\}} u^+(x')\left(\frac{1}{|x'|^n} - \frac{1}{(2R)^n}\right)dx' \le Au_\rho(R). \tag{3.2}$$

Taking into account

$$\int_1^\infty \frac{1}{R^{n+2}} \int\limits_{\{x\in\mathbb{H}:1<|x|<R, x_n=0\}} u^-(x')\left(\frac{1}{|x'|^n} - \frac{1}{R^n}\right)dx'dR = \int\limits_{\{x\in\mathbb{H}:|x|>1, x_n=0\}} u^-(x')\int_{|x'|}^\infty \frac{1}{R^{n+2}}\left(\frac{1}{|x'|^n} - \frac{1}{R^n}\right)dRdx',$$

we have

$$\begin{aligned}
&\int_1^\infty \frac{1}{R^{n+2}} \int\limits_{\{x\in\mathbb{H}:1<|x|<R, x_n=0\}} u^-(x')\left(\frac{1}{|x'|^n} - \frac{1}{R^n}\right)dx'dR \\
&\le \int_1^\infty \frac{1}{R^{n+2}} \int\limits_{\{x\in\mathbb{H}:|x|=R, x_n>0\}} u^+(x)\frac{nx_n}{R^{n+1}}d\sigma(x)dR \\
&\quad + \int_1^\infty \frac{1}{R^{n+2}} \int\limits_{\{x\in\mathbb{H}:1<|x|<R, x_n=0\}} u^+(x')\left(\frac{1}{|x'|^n} - \frac{1}{R^n}\right)dx'dR \\
&\quad - \int_1^\infty \frac{1}{R^{n+2}}\left(\frac{c_1}{R^n}+c_2\right)dR < \infty.
\end{aligned}$$

According to Nevanlinna's formula of harmonic functions in half sphere [16], we know that

$$\begin{aligned}
u(x) = &\int\limits_{\{y\in\mathbb{H}:\ |y|=R,\ y_n>0\}} \frac{R^2-|x|^2}{\omega_n R}\left(\frac{1}{|y-x|^n} - \frac{1}{|y-x^*|^n}\right)u(y)d\sigma(y) \\
&+ \frac{2x_n}{n\omega_n}\int\limits_{\{y\in\mathbb{H}:\ r<|y'|<R,\ y_n=0\}} \left(\frac{1}{|y'-x|^n} - \frac{R^n}{|x|^n}\frac{1}{|y'-\widetilde{x}|^n}\right)u(y')dy',
\end{aligned}$$

for $R>1$, $|x|<R$ and $x_n>0$. Applying Carleman's formula of harmonic functions, we have

$$\begin{aligned}
&\frac{2x_n}{n\omega_n}\int\limits_{\{y\in\mathbb{H}:\ 1<|y'|<R,\ y_n=0\}} \left(\frac{1}{R^n} - \frac{R^n}{|x|^n}\frac{1}{|y'-\widetilde{x}|^n}\right)u(y')dy' \\
&+ \frac{2x_n}{n\omega_n}\int\limits_{\{y\in\mathbb{H}:\ 1<|y'|<R,\ y_n=0\}} \left(\frac{1}{|y'|^n} - \frac{1}{R^n}\right)u(y')dy' = d(R)
\end{aligned}$$

where $d(R)$ is a constant depending on function u and $d(R)$ tending to a constant d as $R \to \infty$. Set

$$L_0(x,R) := \int_{\{y\in\mathbb{H}:\ |y|=R,\ y_n>0\}} \left(\frac{1}{|y-x|^n} - \frac{1}{|y-x^*|^n}\right) u(y)d\sigma(y),$$

$$L_1(x,R) := \frac{R^2-|x|^2}{\omega_n R} L_0(x,R) - \int_{\{x:\ |x|=R,\ x_n>0\}} \frac{n x_n u(x)}{R^{n+1}} d\sigma(x),$$

$$L_2(x,R) := \int_{\{y\in\mathbb{H}:\ |y|=R,\ y_n=0\}} \left(\frac{2x_n}{n\omega_n}\frac{1}{R^n} - \frac{R^n}{|x|^n}|y-\tilde{x}|^n\right) u(y')d\sigma(y'),$$

$$L_3(x,R) := \frac{2x_n}{n\omega_n} \int_{\{y\in\mathbb{H}:\ 1<|y'|<R,\ y_n=0\}} \left(\frac{1}{1+|y'|^n} + \frac{|x|}{1+|y'|^{n+1}} - \frac{R^n}{|x|^n}\frac{1}{|y'-\widetilde{x}|^n}\right) u(y')dy',$$

$$c := d + \frac{2x_n}{n\omega_n} \int_{\{y\in\mathbb{H}:\ 1<|y'|<R,\ y_n=0\}} \left(\frac{1}{1+|y'|^n} - \frac{1}{|y'|^n}\right) u(y')dy'.$$

Write

$$m_+(R) = \frac{n}{R^{n+1}} \int_{\{x:\ |x|=R,\ x_n>0\}} x_n u^+(x)d\sigma(x), \qquad R>0;$$

$$m_-(R) = \frac{n}{R^{n+1}} \int_{\{x:\ |x|=R,\ x_n>0\}} x_n u^-(x)d\sigma(x), \qquad R>0.$$

Since

$$\frac{1}{n}\int_1^\infty m_+(R)dR = \frac{1}{n}\int_1^\infty \int_{\{x:\ |x|=R,\ x_n>0\}} \frac{n x_n u^+(x)}{R^{n+1}} d\sigma(x)dR = \int_{\mathbb{H}} \frac{x_n u^+(x)}{1+|x|^{n+1}}dx < \infty, \tag{3.3}$$

and

$$\frac{1}{n}\int_1^\infty m_-(R)dR < \infty, \tag{3.4}$$

by (3.2), we obtain

$$\begin{aligned}
&\int\limits_{\{x'\in\partial\mathbb{H}:|x'|>1\}} \frac{u^+(x')}{1+|x'|^{n+2}}dx' \\
&= \int_1^\infty \frac{1}{R^{n+2}} \int\limits_{\{x'\in\partial\mathbb{H}:1<|x'|<R\}} \left(\frac{1}{|x'|^n}-\frac{1}{R^n}\right) u^+(x')dx'dR \\
&= \int_1^\infty \frac{\Phi_1(R)}{R^{n+1}}dR < \infty.
\end{aligned} \tag{3.5}$$

By (3.1) and the Fubini theorem, we have

$$\int\limits_{\{x'\in\partial\mathbb{H}:1<|x'|<\infty\}} \frac{u^-(x')}{1+|x'|^{n+2}}dx' < \infty.$$

(2.7) is proved.

Because

$$\begin{aligned}
L_1(x,R) &\leq \frac{C_1(|x|)}{R^{n+1}} \int\limits_{\{x\in\mathbb{H},|x|=R,x_n>0\}} nx_n u(x)d\sigma(x) \\
&= \frac{C_1(|x|)}{R^n}[m_+(R)+m_-(R)],
\end{aligned}$$

where $C_1(|x|)$ is a positive constant depending on x, there exists an increasing sequence $\{R_n\}$ such that

$$\lim_{R\to\infty} R_n = \infty, \qquad \lim_{R\to\infty} \frac{m_+(R_n)+m_-(R_n)}{R_n^n} = 0,$$

consequently,

$$\lim_{n\to\infty} L_1(x,R) = 0.$$

Similarly, there exists a positive constant $C_2(|x|)$ depending on x, when $R \geq 2|x|+1$, we have

$$|L_2(x,R)| \leq \frac{C_2(|x|)}{R^n} \int\limits_{\{y\in\mathbb{H},|y'|<R,y_n=0\}} \frac{|u(y')|}{1+|y'|^{n+2}}dy',$$

and then

$$\lim_{n\to\infty} L_2(x,R) = 0.$$

$$
\begin{aligned}
L_3(x,R) &= \frac{2x_n}{n\omega_n}\int\limits_{\{y\in\mathbb{H}:\ 1<|y'|<R,\ y_n=0\}} \left(\frac{1}{R^n}-\frac{R^n}{|x|^n}\frac{1}{|y'-\widetilde{x}|^n}\right)u(y')dy' \\
&\quad + \frac{2x_n}{n\omega_n}\int\limits_{\{y\in\mathbb{H}:\ 1<|y'|<R,\ y_n=0\}} \left(\frac{1}{|y'|^n}-\frac{1}{R^n}\right)u(y')dy' \\
&\quad + \frac{2x_n|x|}{n\omega_n}\int\limits_{\{y\in\mathbb{H}:\ 1<|y'|<R,\ y_n=0\}} \frac{u(y')}{|y'|^{n+1}}dy' \\
&\quad + \frac{2x_n}{n\omega_n}\int\limits_{\{y\in\mathbb{H}:\ 1<|y'|<R,\ y_n=0\}} (\frac{1}{1+|y'|^n}-\frac{1}{|y'|^n})u(y')dy' \\
&= d(R)+c_1x_n|x|+c-d,
\end{aligned}
$$

where c is a constant depends on x_n. Therefore,

$$
\begin{aligned}
u(x) &= \frac{2x_n}{n\omega_n}\int\limits_{\{y\in\mathbb{H}:\ 1<|y'|<r,\ y_n=0\}} \left(\frac{1}{|x-y'|^n}-\frac{1}{1+|y'|^n}-\frac{|x|}{1+|y'|^{n+1}}\right)u(y')dy' \\
&\quad + L_1(x,R)+L_2(x,R)+L_3(x,R).
\end{aligned}
$$

Combining the estimates of $L_1(x,R)$, $L_2(x,R)$ and $L_3(x,R)$, the result (2.8) follows. □

Proof of Theorem 2.2. If $u\in\mathbb{H}(m)$, for every $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_n)=(\varepsilon',\varepsilon_n)\in\mathbb{R}^n$, where $\varepsilon'=(\varepsilon_1,\cdots,\varepsilon_{n-1})\in\mathbb{R}^{n-1}$, $0<|\varepsilon'|\le|\varepsilon|<1$, and $|x|<R$ $(R>1)$, applying Nevanlinna's formula of harmonic functions in the half sphere [16] to $u(x+\varepsilon)$, we obtain

$$
\begin{aligned}
u(x+\varepsilon) &= \int\limits_{\{y\in\mathbb{H}:\ |y|=R,\ y_n>0\}} \frac{R^2-|x|^2}{n\omega_nR}\left(\frac{1}{|y-x|^n}-\frac{1}{|y-x^*|^n}\right)u(y+\varepsilon)d\sigma(y) \\
&\quad + \frac{2x_n}{n\omega_n}\int\limits_{\{y\in\mathbb{H}:\ r<|y'|<R,\ y_n=0\}} \left(\frac{1}{|y'-x|^n}-\frac{R^n}{|x|^n}\frac{1}{|y'-\widetilde{x}|^n}\right)u(y'+\varepsilon')dy'. \qquad (3.6)
\end{aligned}
$$

Write

$$
\begin{aligned}
m^{\pm}(R,\varepsilon) &= \frac{R^2-|x|^2}{\omega_nR}\int\limits_{\{y\in\mathbb{H}:\ |y|=R,\ y_n>0\}} \left(\frac{1}{|y-x|^n}-\frac{1}{|y-x^*|^n}\right)u^{\pm}(y+\varepsilon)d\sigma(y); \\
n^{\pm}(R,\varepsilon') &= \frac{2x_n}{n\omega_n}\int\limits_{\{y\in\mathbb{H}:\ r<|y'|<R,\ y_n=0\}} \left(\frac{1}{|y'-x|^n}-\frac{R^n}{|x|^n}\frac{1}{|y'-\widetilde{x}|^n}\right)u^{\pm}(y'+\varepsilon')dy'.
\end{aligned}
$$

If $R>1$, (3.6) becomes

$$
m^-(R,\varepsilon)+n^-(R,\varepsilon')=m^+(R,\varepsilon)+n^+(R,\varepsilon')-u(x+\varepsilon), \qquad (3.7)
$$

and

$$\begin{aligned}
\int_2^{+\infty} \frac{m^+(R,\varepsilon)}{R^{m+1}} dR &= \int_{\{y\in\mathbb{H}:\ |y|=R,\ y_n>0\}} \frac{R^2-|x|^2}{n\omega_n R}\left(\frac{1}{|y-x|^n}-\frac{1}{|y-x^*|^n}\right)u^+(y+\varepsilon)d\sigma(y)\\
&\le A \int_{\{y\in\mathbb{H}:\ |y|=R,\ y_n>0\}} \frac{y_n u^+(y+\varepsilon)}{R^{n+m}} d\sigma(y)\\
&\le A \int_{\{y\in\mathbb{H}:\ |y|=R,\ y_n>0\}} \frac{(y_n+\varepsilon_n)u^+(y+\varepsilon)}{[(y'+\varepsilon')^2+(y_n+\varepsilon_n)^2]^{\frac{n+m}{2}}} d\sigma(y) < AI.
\end{aligned}$$

Hence

$$\sup_{0<|\varepsilon|<1} \int_2^{+\infty} \frac{m^+(R,\varepsilon)}{R^{m+1}} dR \le AI < \infty, \tag{3.8}$$

and

$$\sup_{0<|\varepsilon|<1} \liminf_{R\to\infty} m_+(R,\varepsilon) = 0.$$

By

$$\begin{aligned}
n^+(R,\varepsilon') &= \frac{2x_n}{n\omega_n} \int_{\{y\in\mathbb{H}:\ r<|y'|<R,\ y_n=0\}} \left(\frac{1}{|y'-x|^n} - \frac{R^n}{|x|^n}\frac{1}{|y'-\widetilde{x}|^n}\right)u^+(y'+\varepsilon')dy'\\
&\le \int_{\{y\in\mathbb{H}:\ r<|y'|<R,\ y_n=0\}} \frac{2x_n}{n\omega_n}\frac{u^+(y'+\varepsilon')}{|y|^n \sin^n\varphi} dy'\\
&\le \frac{2x_n}{n\omega_n} \int_{\{y\in\mathbb{H}:\ r<|y'|<R,\ y_n=0\}} \frac{u^+(y'+\varepsilon')}{|y'|^n} dy',
\end{aligned}$$

we have

$$\int_2^{+\infty} \frac{n^+(R,\varepsilon')}{R^{m+1}} dR \le \frac{2x_n}{n\omega_n} \int_{\{y'\in\partial\mathbb{H}:\ r<|y'|<R\}} \frac{u^+(y'+\varepsilon')}{|y'|^{n+m+1}} dy' < \infty. \tag{3.9}$$

It is evident that

$$\sup_{0<|\varepsilon'|<1} \int_2^{+\infty} \frac{-u(y'+\varepsilon')}{R^{m+1}} dR = \frac{1}{2^{m+1}} \sup_{0<|\varepsilon|<1} [-u(y'+\varepsilon')] < \infty. \tag{3.10}$$

(3.7)–(3.10) imply

$$\sup_{0<|\varepsilon'|<1} \int_2^{+\infty} \frac{n^-(R,\varepsilon')}{R^{m+1}} dR < \infty, \tag{3.11}$$

and

$$\sup_{0<|\varepsilon'|<1} \int_2^{+\infty} \int_{\partial\mathbb{H}} \frac{y_n u^-(y'+\varepsilon')}{R^{m+n}} dy' dR \leq \sup_{0<|\varepsilon|<1} \int_2^{+\infty} \frac{m^-(R,\varepsilon)}{R^{m+n}} dR < \infty.$$

If $R > 1$, we have

$$\begin{aligned} n^-(R,\varepsilon') &\geq \frac{2x_n}{n\omega_n} \int_{\{y\in\mathbb{H}:\ 1<|y'|<\frac{R}{2},\ y_n=0\}} \left(\frac{1}{|y'-x|^n} - \frac{R^n}{|x|^n} \frac{1}{|y'-\widetilde{x}|^n} \right) u^-(y'+\varepsilon') dy' \\ &\geq \int_{\{y\in\mathbb{H}:\ 1<|y'|<\frac{R}{2},\ y_n=0\}} \left(\frac{1}{|y'|^n+1} - \frac{R^n}{1+|y'|^n} \right) u^-(y'+\varepsilon') dy' \\ &\geq \int_{\{y\in\mathbb{H}:\ 1<|y'|<\frac{R}{2},\ y_n=0\}} \frac{u^-(y'+\varepsilon')}{1+|y'|^n} dy'. \end{aligned} \tag{3.12}$$

By (3.10), (3.12) and the Fubini theorem, we see that

$$\begin{aligned} &\sup_{0<|\varepsilon'|<1} \int_2^{+\infty} \int_1^{\frac{R}{2}} \frac{u^-(y'+\varepsilon')}{1+|y'|^n} dy' dR = \\ &\sup_{0<|\varepsilon'|<1} \int_1^{+\infty} \frac{u^-(y'+\varepsilon')}{(1+|y'|^n)|y'|^{n+m-1}} dy' \leq \\ &\sup_{0<|\varepsilon'|<1} \int_2^{+\infty} \frac{n^-(R,\varepsilon')}{R^{n+m}} dR < \infty. \end{aligned} \tag{3.13}$$

By (2.9) and (3.13),

$$I = \sup_{0<|\varepsilon'|<1} \int_{\{y\in\mathbb{H}:\ r<|y'|<R,\ y_n=0\}} \frac{|u(y'+\varepsilon')|}{|y'|^{n+m}} dy' < \infty.$$

(1) is proved.

Let

$$u_m(x+\varepsilon') = \int_{\partial\mathbb{H}} P_{\mathbb{H}}^m(x,y')u(y'+\varepsilon')dy'.$$

Suppose $\chi_{\overline{B_R^+}}(y')$ is the characteristic function of $\overline{B_R^+} = B_R \cap \mathbb{H}$, we fix a boundary point $a' = (a_1, a_2, \cdots, a_{n-1}) \in \partial\mathbb{H}$ and choose a large $T > |a'| + 1$, then $u_m(x+\varepsilon')$ may be written as

$$\begin{aligned} u_m(x+\varepsilon') = & \int_{|y'|\le 2T} P_{\mathbb{H}}(x,y')u(y'+\varepsilon')dy' \\ & - \frac{2x_n}{n\omega_n} \int_{1\le|y'|\le 2T} \sum_{k=0}^{m-1} \frac{|x|^k}{|y|^{n+k}} C_k^{n/2}\left(\frac{x\cdot y}{|x||y|}\right) u(y'+\varepsilon')dy' \\ & + \int_{|y'|\ge 2T} P_{\mathbb{H}}^m(x,y')u(y'+\varepsilon')dy' \\ = & X_{\varepsilon'}(x) - Y_{\varepsilon'}(x) + Z_{\varepsilon'}(x). \end{aligned}$$

The function $X_{\varepsilon'}(x)$ is harmonic in $\overline{B_R^+}$ and is the Poisson integral of $u(y'+\varepsilon')\chi_{\overline{B_R^+}}(y')$, with $X_{\varepsilon'}(x) = u(y'+\varepsilon')$, $Y_{\varepsilon'}(x)$ is a harmonic polynomial multiplied by x_n with $Y_{\varepsilon'}(y') = 0$, $y' \in \overline{B_R^+}(y')$, $Z_{\varepsilon'}(x)$ is harmonic in $\overline{B_R^+}$ with $Z_{\varepsilon'}(y') = 0$, $|y'| \le T$. Hence $u_m(x,\varepsilon')$ is harmonic in $\overline{\mathbb{H}}$ for any $T > 2$ and

$$\lim_{x_n\to 0} u_m(x+\varepsilon') = u(x'+\varepsilon'), \ \ x' \in \partial\mathbb{H}.$$

Denoted by $C[1+|y'|^{n+m}]$ the space of all continuous function $G(x)$ on $\mathbb{H}$ for which

$$\lim_{|y'|\to\pm\infty} |G(y')|(1+|y'|^{n+m}) = 0,$$

define the norm

$$\|G\| = \sup_{y'\in\partial\mathbb{H}} |G(y')|(1+|y'|^{n+m}),$$

so $C[1+|y'|^{n+m}]$ is a Banach space and

$$P_{\mathbb{H}}^m(x,y') \in C[1+|y'|^{n+m}].$$

Let $\delta_k = (\delta_1^{(k)}, \cdots, \delta_{n-1}^{(k)}, 0)$, $|\delta_k| \searrow 0$. Define the linear functional on $C[1+|y'|^{n+m}]$

$$\Lambda_k[G(y')] = \int_{\partial\mathbb{H}} G(y')u(y'+\delta_k)dy',$$

and

$$|\Lambda_n[G(y')]| \le [\sup_{y'\in\partial\mathbb{H}} |G(y')|(1+|y'|^{n+m})] \cdot \int_{\partial\mathbb{H}} \frac{|u(y'+\delta_k)|}{|y'|^{n+m}} dy'$$

$$\le \| G \| \sup_{0<|\delta_k|<1} \int_{\partial\mathbb{H}} \frac{|u(y'+\delta_k)|}{1+|y'|^{n+m}} dy';$$

$$\| \Lambda_n \| \le \sup_{0<|\delta_k|<1} \int_{\partial\mathbb{H}} \frac{|u(y'+\delta_k)|}{1+|y'|^{n+m}} dy',$$

so Λ_k is a bounded linear functional on $C[1+|y'|^{n+m}]$, and we can construct a subsequence of $u(\varepsilon'_k + y')$, $\varepsilon'_k = (\varepsilon_1^{(k)}, \cdots, \varepsilon_{n-1}^{(k)}, 0)$, such that [11]

$$\Lambda(G) = \lim_{k\to\infty} \Lambda_k(G) = \lim_{k\to\infty} \int_{\partial\mathbb{H}} G(y')u(y'+\varepsilon'_k)dy', \ G(y') \in C[1+|y'|^{n+m}], \tag{3.14}$$

and

$$\| \Lambda \| \le \sup_{0<|\varepsilon'_k|<1} \int_{\partial\mathbb{H}} \frac{|u(\varepsilon'_k + y')|}{1+|y'|^{n+m}} dy',$$

Λ is a bounded linear functional on $C[1+|y'|^{n+m}]$. By the Riesz representation theorem, there exists a measure μ on $\partial\mathbb{H}$ such that

$$\Lambda(G) = \int_{\partial\mathbb{H}} G(y')d\mu(y'), G(y') \in C[1+|y'|^{n+m}], \tag{3.15}$$

$$\int_{\partial\mathbb{H}} \frac{d|\mu|(y')}{1+|y'|^{n+m}} = \lim_{k\to\infty} \int_{\partial\mathbb{H}} \frac{|u(y'+\varepsilon'_k)|}{1+|y'|^{n+m}} dt \le \sup_{0<|\varepsilon'_k|<1} \int_{\partial\mathbb{H}} \frac{|u(\varepsilon'_k+y')|}{1+|y'|^{n+m}} dy' < \infty.$$

Thus (2) holds.

Let

$$h_{\varepsilon'}(x) = u(x+\varepsilon') - u_m(x+\varepsilon').$$

By the Schwarz reflection principle [11], p. 68 and [7], p. 28, there exists a harmonic function $h_{\varepsilon'}(x*)$ in $\mathbb{R}^n$, such that

$$h_{\varepsilon'}(x^*) = -h_{\varepsilon'}(x) = -[u(x+\varepsilon') - u_m(x+\varepsilon')], \ \ x \in \mathbb{H},$$

$x^* = (x', -x_n)$ is the reflection point of x with respect to $\partial\mathbb{H}$, $h(y') \equiv 0$, $y' \in \partial\mathbb{H}$. By the spherical harmonic expansion theorem [11], p. 100, Theorem 2.1 in [13], p. 139, orthogonality of spherical harmonics [13], p. 141, and the proof of Theorem [4], p. 58, we know that

$$h_{\varepsilon'}(x) = \sum_{k=0}^{m+1} P_k(x + \varepsilon')$$

is a harmonic polynomial $Q_{\varepsilon'}(x)$ of degree not greater than $m + 1$ which vanishes on the boundary $\partial\mathbb{H}$ such that

$$u(x, \varepsilon') = \sum_{k=0}^{m+1} P_k(x + \varepsilon') + u_m(x, \varepsilon') = Q_{\varepsilon'}(x) + u_m(x, \varepsilon'), \quad x \in \mathbb{H}, \tag{3.16}$$

in which $P_k(x + \varepsilon')(k = 0, 1, \cdots)$ are homogeneous harmonic polynomials of degree k.

Let $\varepsilon \to 0$ in (3.16), it follows that

$$u(x) = Q(x) + u_m(x). \qquad \square$$

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