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Fefferman-Stein decomposition for Q-spaces and micro-local quantities

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ABSTRACT

In this paper, we study the Fefferman-Stein decomposition of $Q_\alpha(\mathbb{R}^n)$ and give an affirmative answer to an open problem posed by Essén et al. (2000). One of our main methods is to characterize the structure of the predual of $Q_\alpha(\mathbb{R}^n)$ by the micro-local quantities. This result indicates that the norm of the predual space of $Q_\alpha(\mathbb{R}^n)$ depends on the micro-local structure in a self-correlation way.

1. Introduction

In this paper, we give a wavelet characterization of the predual of Q-space $Q_\alpha(\mathbb{R}^n)$ without using a family of Borel measures. By this result, we obtain a Fefferman–Stein type decomposition of $Q_\alpha(\mathbb{R}^n)$. Let R_0 be the unit operator and $R_i, i = 1, \ldots, n$, be the Riesz transforms, respectively. In 1972, in the celebrated paper [5], C. Fefferman and E. M. Stein proved the following result.

Theorem 1 *([18], Theorem B)*. *If $f \in BMO(\mathbb{R}^n)$, then there exist $g_0, \ldots, g_n \in L^\infty(\mathbb{R}^n)$ such that, modulo constants, $f = \sum_{j=0}^n R_i g_i$ and $\sum_{j=0}^n \|g_j\|_{L^\infty} \le C\|f\|_{BMO}$.*

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The importance of the Fefferman–Stein decomposition lays in two aspects. On the one hand, there is a close relation between the $\bar{\partial}$-equation and the Fefferman–Stein decomposition. On the other hand, this decomposition helps understand better the structure of $BMO(\mathbb{R}^n)$ and the distance between $L^\infty(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$. Due to the mentioned two points, the Fefferman–Stein decomposition of $BMO(\mathbb{R}^n)$ has been studied extensively by many researchers since 1970s. We refer the reader to Jones [6,7] and Uchiyama [18] for further information. In the latest decades, the Fefferman–Stein decomposition is also extended to other function spaces, for example, BLO, C^0 and VMO, see [1,14] and the references therein.

As an analogy of $BMO(\mathbb{R}^n)$, Q-spaces own a similar structure and many common properties. It is natural to seek for a Fefferman–Stien type decomposition of Q-spaces. For the Q-spaces on the unit disk, Nicolau–Xiao [12] obtained a decomposition of $Q_p(\partial\mathbb{D})$ similar to the Fefferman–Stein's result for $BMO(\partial\mathbb{D})$ (see [12], Theorem 1.2). On Euclidean space $\mathbb{R}^n$, Essen–Janson–Peng–Xiao [4] introduced $Q_\alpha(\mathbb{R}^n)$ as a generalization of $Q_p(\partial\mathbb{D})$. For $\alpha \in (-\infty, \infty)$, $Q_\alpha(\mathbb{R}^n)$ is defined as the space of all the measurable functions with

$$\sup_I |I|^{2\alpha/n-1} \int_I \int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}} dxdy < \infty, \tag{1.1}$$

where the supremum is taken over all cubes I with the edges parallel to the coordinate. They studied $Q_\alpha(\mathbb{R}^n)$ systemically and listed the Fefferman–Stein decomposition of $Q_\alpha(\mathbb{R}^n)$ as one of the open problems.

Problem 1.1 (*[4, Problem 8.3]*)**.** For $n \geq 2$ and $\alpha \in (0, 1)$. Give a Fefferman–Stein type decomposition for $Q_\alpha(\mathbb{R}^n)$.

In this paper, we will give an affirmative answer to this open problem. Generally speaking, there are two methods to obtain the Fefferman–Stein decomposition of $BMO(\mathbb{R}^n)$. The method of Fefferman–Stein [5] is to split BMO-functions involving an extension theorem based on the Hahn–Banach theorem. In [18], A. Uchiyama gave a constructive proof of Theorem 1. In this paper, using wavelets, we study the micro-local structure of $P^\alpha(\mathbb{R}^n)$, which is the predual of $Q_\alpha(\mathbb{R}^n)$. As an application, we obtain a Fefferman–Stein type decomposition of $Q_\alpha(\mathbb{R}^n)$.

For the Fefferman–Stein decomposition of $Q_\alpha(\mathbb{R}^n)$, the difficulties are two-fold.

(1) For a function f in $P^\alpha(\mathbb{R}^n), 0 < \alpha < n/2$, the higher frequency part and the lower frequency part make different contributions to the norm $\|f\|_{P^\alpha}$. That is to say, each $P^\alpha(\mathbb{R}^n)$ has special micro-local structure. As far as we know, there are little results on such structure of $P^\alpha(\mathbb{R}^n)$.
(2) For any function f, the Riesz transforms may cause a perturbation on all the range of its frequencies. To obtain the Fefferman–Stein type decomposition, we need to control the range of the perturbation.

To overcome the above two difficulties, on the one hand, we analyze the micro-local structure of functions in $P^\alpha(\mathbb{R}^n)$. Such micro-local structure can help get a wavelet characterization of $P^\alpha(\mathbb{R}^n)$ without involving a group of Borel measures. On the other hand, we use the classical Meyer wavelets to control the range of the perturbation.

In Section 2, we will give the definition of wavelet basis $\{\Phi^\epsilon_{j,k}\}_{(\epsilon,j,k)\in\Lambda_n}$. It is well-known that a function g can be written as a sum

$$g(x) = \sum_j g_j(x), \quad \text{where } g_j(x) = \sum_{\epsilon,k} g^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x).$$

Let g be a function in Besov spaces or Triebel–Lizorkin spaces. Roughly speaking, the norms of g can be determined by the $l^p(L^q)$-norm or $L^p(l^q)$-norm of $\{g_j\}$, respectively (see [10,17]). For $g \in P^\alpha(\mathbb{R}^n), 0 < \alpha < \frac{n}{2}$, the situation becomes complicated and we cannot use the above approaches. In Section 3, we introduce the micro-local quantities with levels to study the structure of functions in $P^\alpha(\mathbb{R}^n)$.

Let $g = \sum_{\epsilon,j,k} g^{\epsilon}_{j,k} \Phi^{\epsilon}_{j,k} \in P^{\alpha}(\mathbb{R}^n)$. For any dyadic cube Q, we take the localization of g on Q as

$$g_Q(x) = \sum_{Q_{j,k}\subset Q} g^{\epsilon}_{j,k} \Phi^{\epsilon}_{j,k}(x).$$

Then we restrict the range of frequency by limiting the index j:

$$g_{t,Q}(x) =: \sum_{Q_{j,k}\subset Q: -\log_2 |Q| \le nj \le nt - \log_2 |Q|} g^{\epsilon}_{j,k} \Phi^{\epsilon}_{j,k}(x). \tag{1.2}$$

We obtain three micro-local quantities about $g_{t,Q}$ by using some basic results in analysis. See Section 3 for details.

In Section 4, applying the above micro-local analysis of functions in $P^{\alpha}(\mathbb{R}^n)$, we give a new wavelet characterization of this space. As the predual of $Q_{\alpha}(\mathbb{R}^n)$, $P^{\alpha}(\mathbb{R}^n)$ has been studied by many authors. One method is to define the space $P^{\alpha}(\mathbb{R}^n)$ by a family of Borel measures. See Wu–Xie [20] for $n = 1$ and Yuan–Sickel–Yang [24] for arbitrary spatial dimension. This idea can result in wavelet characterization of the predual space; but the predual space with the induced norm is a pseudo–Banach space. Dafni–Xiao [3] used a method of Hausdorff capacity to study $P^{\alpha}(\mathbb{R}^n)$. L. Peng and Q. Yang defined $P^{\alpha}(\mathbb{R}^n)$ by the atoms (see [13,21]). By these methods, $P^{\alpha}(\mathbb{R}^n)$ are Banach spaces; but these authors did not consider the wavelet characterization of $P^{\alpha}(\mathbb{R}^n)$. However, the micro-local structure of $P^{\alpha}(\mathbb{R}^n)$ has not been considered in the above literatures and we cannot apply them to consider the Fefferman–Stein decomposition.

Compared with the former results of [8,13,21,24], our result has the following advantage. Let $f \in P^{\alpha}(\mathbb{R}^n)$. Our wavelet characterization indicates clearly that different frequencies exert different influences to the P^{α}-norm of f. See Theorems 3.4 and 4.2. To obtain a Fefferman–Stein type decomposition of $Q_{\alpha}(\mathbb{R}^n)$, we need such a wavelet characterization of $P^{\alpha}(\mathbb{R}^n)$.

In Section 5, by the characterization obtained in Section 4 and the properties of the Meyer wavelets and Daubechies wavelets, we characterize $P^{\alpha}(\mathbb{R}^n)$ associated with the Riesz transforms, see Theorem 5.8. Applying this result and the duality between $P^{\alpha}(\mathbb{R}^n)$ and $Q_{\alpha}(\mathbb{R}^n)$, we obtain a Fefferman–Stein type decomposition for $Q_{\alpha}(\mathbb{R}^n)$.

We point out that our definition of $Q_{\alpha}(\mathbb{R}^n)$ is different from the one introduced in [4]. In [4], the scope of α is restricted to $(0, \min\{1, n/2\})$ to make $Q_{\alpha}(\mathbb{R}^n)$ non-trivial, while the scope in our definition can be relaxed to $(0, n/2)$. More importantly, when $\alpha \in (0, \min\{1, n/2\})$, our definition is equivalent to the one in [4]. So the Fefferman–Stein type decomposition obtained in Section 5 gives a positive answer to the open problem proposed in [4].

2. Preliminaries

In this section, we present preliminaries on wavelets, functions and operators which will be used in the sequel.

2.1. Wavelets and classical function spaces

In this paper, we use real-valued tensor product wavelets; which can be the regular Daubechies wavelets or the classical Meyer wavelets. Define the set of n-tuples

$$\{0,1\}^n = \Big\{\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n),\ \epsilon_i = 0 \text{ or } 1, i = 1, 2, \ldots, n\Big\}.$$

Set $E_n = \{0,1\}^n \setminus \{(0,0,\ldots,0)\}$. Let Φ^0 and $\{\Phi^{\epsilon}, \epsilon \in E_n,\}$ be the scale function and the vector-valued wavelet functions, respectively. If Φ^{ϵ} is a Daubechies wavelet, we assume there exist $m > 8n$ and $M \in \mathbb{N}$ such that

(1) $\Phi^{\epsilon} \in C_0^m([-2^M, 2^M]^n)$, $\forall \epsilon \in \{0,1\}^n$;
(2) Φ^{ϵ} has the vanishing moments up to the order $m-1$, $\forall \epsilon \in E_n$.

For further information about wavelets, we refer to [10,19,21].

For $j \in \mathbb{Z}$ and $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$, we denote by $Q_{j,k}$ the dyadic cube $\prod_{s=1}^n [2^{-j}k_s, 2^{-j}(k_s+1)]$ and set $\Omega = \Big\{Q_{j,k}, j \in \mathbb{Z}, k \in \mathbb{Z}^n\Big\}$. Let $\Lambda_n = \Big\{(\epsilon, j, k), \epsilon \in E_n, j \in \mathbb{Z}, k \in \mathbb{Z}^n\Big\}$. For $\epsilon \in \{0,1\}^n$, $j \in \mathbb{Z}, k \in \mathbb{Z}^n$, denote $\Phi^{\epsilon}_{j,k}(x) = 2^{jn/2}\Phi^{\epsilon}(2^j x - k)$. The following result is well-known.

Lemma 2.1 *([10]).* $\Big\{\Phi^{\epsilon}_{j,k},\ (\epsilon, j, k) \in \Lambda_n\Big\}$ *is an orthogonal basis in* $L^2(\mathbb{R}^n)$.

Let $f^{\epsilon}_{j,k} = \langle f, \Phi^{\epsilon}_{j,k}\rangle$, $\forall \epsilon \in \{0,1\}^n$ and $k \in \mathbb{Z}^n$. By Lemma 2.1, any L^2-function f has a wavelet decomposition

$$f(x) = \sum_{(\epsilon,j,k)\in\Lambda_n} f^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x).$$

We recall some knowledge on Sobolev spaces and Hardy spaces. For $1 < p < \infty$, we denote by p' the conjugate index of p, that is, $1/p + 1/p' = 1$. For a function space A, we denote by A' the dual space of A. For the Sobolev spaces $W^{r,p}(\mathbb{R}^n), 1 < p < \infty, r \in \mathbb{R}$, it is well-known that $(W^{r,p}(\mathbb{R}^n))' = W^{-r,p'}(\mathbb{R}^n)$ (see [10,17,21] for the details).

Let χ be the characteristic function of the unit cube $[0,1]^n$. We have the following wavelet characterizations of Sobolev spaces and Hardy space, see [10,21,22]:

Proposition 2.2. (i) *Let* $1 < p < \infty$ *and* $|r| < m$. $g = \sum_{(\epsilon,j,k)\in\Lambda_n} g^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k} \in W^{r,p}(\mathbb{R}^n)$ *if and only if*

$$\Big\|\Big(\sum_{(\epsilon,j,k)\in\Lambda_n} 2^{2j(r+n/2)}|g^{\epsilon}_{j,k}|^2\chi(2^j\cdot -k)\Big)^{\frac{1}{2}}\Big\|_{L^p} < \infty.$$

(ii) $g = \sum_{(\epsilon,j,k)\in\Lambda_n} g^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k} \in H^1(\mathbb{R}^n)$ *if and only if*

$$\Big\|\Big(\sum_{(\epsilon,j,k)\in\Lambda_n} 2^{nj}|g^{\epsilon}_{j,k}|^2\chi(2^j\cdot -k)\Big)^{\frac{1}{2}}\Big\|_{L^1} < \infty.$$

2.2. Q-spaces

We know that $Q_{\alpha}(\mathbb{R}^n) = BMO(\mathbb{R}^n)$ for $\alpha < 0$. Further, It is easy to see that the Q-spaces defined in (1.1) are trivial for $\alpha \geq 1$. In fact, for $\alpha \geq 1$ or $\alpha > n/2$, there are only constants in $Q_{\alpha}(\mathbb{R}^n)$ by the definition invoking (1.1).

To get rid of the restriction $\alpha \geq 1$, we introduce a new definition which is non-trivial for $1 \leq \alpha \leq \frac{n}{2}$. For $\alpha \in \mathbb{R}$, denote by

$$f_{\alpha,Q} = |Q|^{-1}\int_Q (-\Delta)^{\alpha/2} f(x)dx$$

the mean value of $(-\Delta)^{\alpha/2} f$ on the cube Q. For $\alpha \in \mathbb{R}$, let

$$B_{\alpha,Q}f = |Q|^{\alpha/n}\left(|Q|^{-1}\int_Q |(-\Delta)^{\alpha/2} f(x) - f_{\alpha,Q}|^2 dx\right)^{1/2}.$$

The Q-spaces $Q_{\alpha}(\mathbb{R}^n)$ and $Q^0_{\alpha}(\mathbb{R}^n)$ are defined as follows.

Definition 2.3. Let $\alpha \in [0, n/2]$.

(i) $Q_\alpha(\mathbb{R}^n)$ is defined as the set of all measurable functions f with

$$\sup_Q B_{\alpha,Q} f < \infty,$$

where the supremum is taken over all cubes Q.

(ii) $Q^0_\alpha(\mathbb{R}^n)$ is defined as the set of all measurable functions $f \in Q_\alpha(\mathbb{R}^n)$ with

$$\begin{cases} \lim\limits_{|Q|\to 0} B_{\alpha,Q} f = 0, \\ \lim\limits_{|Q|\to \infty} B_{\alpha,Q} f = 0, \end{cases}$$

where the supremum and the limit are taken over all cubes Q.

Remark 2.4. If $\alpha = n/2$, $Q_{n/2}(\mathbb{R}^n) = \dot{B}^{n/2,2}_2(\mathbb{R}^n)$. For $1 \le \alpha \le n/2$, the Q-spaces in Definition 2.3 are non-trivial. Further, for other indices α, the corresponding $Q_\alpha(\mathbb{R}^n)$ coincide with those defined in [4]. So $Q_\alpha(\mathbb{R}^n)$, defined in Definition 2.3, is a generalization of Q-spaces defined in (1.1).

For $|\alpha| < m$, $Q \in \Omega$ and $f = \sum_{(\epsilon,j,k)\in\Lambda_n} f^\epsilon_{j,k} \Phi^\epsilon_{j,k}$, let

$$C_{\alpha,Q} f = |Q|^{\alpha/n - 1/2} \Big(\sum_{Q_{j,k}\subset Q} 2^{2j\alpha} |f^\epsilon_{j,k}|^2 \Big)^{1/2}.$$

By (i) of Proposition 2.2, we get the following wavelet characterization of Q-spaces, cf [24]:

Proposition 2.5. *Let* $0 \le \alpha \le n/2$.

(i) $f = \sum_{(\epsilon,j,k)\in\Lambda_n} f^\epsilon_{j,k} \Phi^\epsilon_{j,k} \in Q_\alpha(\mathbb{R}^n)$ *if and only if*

$$\sup_{Q\in\Omega} C_{\alpha,Q} f < \infty.$$

(ii) $f = \sum_{(\epsilon,j,k)\in\Lambda_n} f^\epsilon_{j,k} \Phi^\epsilon_{j,k} \in Q^0_\alpha(\mathbb{R}^n)$ *if and only if*

$$\begin{cases} \sup\limits_{Q\in\Omega} C_{\alpha,Q} f < \infty, \\ \lim\limits_{Q\in\Omega,\, |Q|\to 0} C_{\alpha,Q} f = 0, \\ \lim\limits_{Q\in\Omega,\, |Q|\to \infty} C_{\alpha,Q} f = 0. \end{cases} \tag{2.1}$$

By Propositions 2.2 and 2.5, we may identify a function

$$g = \sum_{(\epsilon,j,k)\in\Lambda_n} g^\epsilon_{j,k} \Phi^\epsilon_{j,k}$$

with the sequence $\{g^\epsilon_{j,k}\}_{(\epsilon,j,k)\in\Lambda_n}$.

2.3. Calderón–Zygmund operators

Now we introduce some preliminaries on Calderón–Zygmund operators, see [10,16]. For $x \neq y$, let $K(\cdot,\cdot)$ be a smooth function such that

$$|\partial_x^\alpha \partial_y^\beta K(x,y)| \leq \frac{C}{|x-y|^{n+|\alpha|+|\beta|}}, \quad \forall |\alpha|+|\beta| \leq N_0, \tag{2.2}$$

where N_0 is a large enough constant.

A linear operator T is said to be a Calderón–Zygmund operator in $CZO(N_0)$ if

(1) T is continuous from $C^1(\mathbb{R}^n)$ to $(C^1(\mathbb{R}^n))'$;
(2) There exists a kernel $K(\cdot,\cdot)$ satisfying (2.2) and for $x \notin \mathrm{supp} f$,

$$Tf(x) = \int K(x,y) f(y) dy;$$

(3) $Tx^\alpha = T^* x^\alpha = 0, \forall \alpha \in \mathbb{N}^n$ and $|\alpha| \leq N_0$.

Remark 2.6. The values of $K(\cdot,\cdot)$ in (2.2) have not been defined for $x = y$. According to Schwartz kernel theorem, the kernel $K(\cdot,\cdot)$ of a linear continuous operator T is only a distribution in $S'(\mathbb{R}^{2n})$.

Let $\{\Phi^\epsilon_{j,k}\}_{(\epsilon,j,k)\in\Lambda_n}$ be a sufficient regular wavelet basis. We denote

$$a^{\epsilon,\epsilon'}_{j,k,j',k'} = \left\langle K(\cdot,\cdot), \Phi^\epsilon_{j,k}(\cdot)\Phi^{\epsilon'}_{j',k'}(\cdot)\right\rangle, \quad (\epsilon,j,k),(\epsilon',j',k') \in \Lambda_n.$$

Lemma 2.7 (*[10]*). (i) *Let $T \in CZO(N_0)$. For all (ϵ,j,k) and $(\epsilon',j',k') \in \Lambda_n$, the coefficients $a^{\epsilon,\epsilon'}_{j,k,j',k'}$ satisfy that*

$$|a^{\epsilon,\epsilon'}_{j,k,j',k'}| \leq C 2^{-|j-j'|(n/2+N_0)} \left(\frac{2^{-j}+2^{-j'}}{2^{-j}+2^{-j'}+|k2^{-j}-k'2^{-j'}|}\right)^{n+N_0}. \tag{2.3}$$

(ii) *If $\{a^{\epsilon,\epsilon'}_{j,k,j',k'}\}_{(\epsilon,j,k),(\epsilon',j',k')\in\Lambda_n}$ satisfies (2.3), then*

$$K(x,y) = \sum_{(\epsilon,j,k),(\epsilon',j',k')\in\Lambda_n} a^{\epsilon,\epsilon'}_{j,k,j',k'} \Phi^\epsilon_{j,k}(x)\Phi^{\epsilon'}_{j',k'}(y)$$

in the sense of distributions. Further, for any $0 < \delta < N_0$, we have $T \in CZO(N_0 - \delta)$.

At the end of this subsection, we list a variant result about the continuity of Calderón–Zygmund operators on Sobolev spaces (see also [11]).

For all $(\epsilon,j,k) \in \Lambda_n$, denote

$$\tilde{g}^\epsilon_{j,k} = \sum_{(\epsilon',j',k')\in\Lambda_n} a^{\epsilon,\epsilon'}_{j,k,j',k'} g^{\epsilon'}_{j',k'}.$$

We have

Lemma 2.8. *Let $|r| < s \leq m$ and $1 < p < \infty$. For (ϵ,j,k), $(\epsilon',j',k') \in \Lambda_n$, if*

$$|a^{\epsilon,\epsilon'}_{j,k,j',k'}| \leq C 2^{-|j-j'|(n/2+s)} \left(\frac{2^{-j}+2^{-j'}}{2^{-j}+2^{-j'}+|k2^{-j}-k'2^{-j'}|}\right)^{n+s},$$

then

$$\int \Big(\sum_{(\epsilon,j,k)\in\Lambda_n} 2^{j(n+2r)} |\tilde{g}^\epsilon_{j,k}|^2 \chi(2^j x - k)\Big)^{p/2} dx \leq C \int \Big(\sum_{(\epsilon,j,k)\in\Lambda_n} 2^{j(n+2r)} |g^\epsilon_{j,k}|^2 \chi(2^j x - k)\Big)^{p/2} dx.$$

2.4. Generalized Hardy spaces

Peng–Yang [13] and Yang [21] used atoms to define the predual of $Q_\alpha(\mathbb{R}^n)$. See also [2] and [23]. Below we introduce the standard atoms, the wavelet atoms and the generalized Hardy spaces related to $Q_\alpha(\mathbb{R}^n)$:

Definition 2.9. Let $0 \le \alpha < n/2$.

(i) A distribution g is an $(\alpha, 2)$-atom on a cube Q if
 (1) $\|(-\Delta)^{-\alpha/2} g\|_{L^2} \le |Q|^{-1/2+\alpha/n}$,
 (2) supp $g \subset Q$,
 (3) $\int x^\beta g(x)dx = 0, \forall |\beta| \le |\alpha|$.
(ii) A distribution f belongs to a Hardy space $P^\alpha(\mathbb{R}^n)$ if $f(x) = \sum_{u\in\mathbb{Z}} \lambda_u g_u(x)$, where $\{\lambda_u\}_{u\in\mathbb{Z}} \in l^1$ and $\{g_u\}$ are $(\alpha, 2)$-atoms.

Definition 2.10. Given $0 \le \alpha < n/2$.

(i) A distribution $g = \sum_{\epsilon\in E_n, Q_{j,k}\subset Q} g^\epsilon_{j,k} \Phi^\epsilon_{j,k}$ is a $(\alpha, 2)$-wavelet atom on a dyadic cube Q if

$$\Big(\sum_{(\epsilon,j,k)\in \Lambda_n} 2^{-2j\alpha} |g^\epsilon_{j,k}|^2 \Big)^{1/2} \le |Q|^{\alpha/n-1/2}.$$

(ii) A distribution f belongs to a Hardy space $P^\alpha_w(\mathbb{R}^n)$ if $f(x) = \sum_{u\in\mathbb{Z}} \lambda_u g_u(x)$, where $\{\lambda_u\}_{u\in\mathbb{Z}} \in l^1$ and $\{g_u\}$ are $(\alpha, 2)$-wavelet atoms.

The following results were obtained by Peng–Yang [13] and Yang [21], respectively.

Proposition 2.11. *Let $0 \le \alpha < n/2$.*

(i) $P^\alpha(\mathbb{R}^n) = P^\alpha_w(\mathbb{R}^n)$.
(ii) *Let $T \in CZO(N_0)$. Then T is bounded on $P^\alpha(\mathbb{R}^n)$.*

For $\alpha = n/2$, define $P^{n/2}(\mathbb{R}^n) =: \dot{B}_2^{-n/2,2}(\mathbb{R}^n)$. Applying the same ideas in [13–15,24], we have the following duality relation.

Proposition 2.12. *Let $0 \le \alpha \le n/2$.*

(i) $(P^\alpha(\mathbb{R}^n))' = Q_\alpha(\mathbb{R}^n)$;
(ii) $(Q^0_\alpha(\mathbb{R}^n))' = P^\alpha(\mathbb{R}^n)$.

3. Micro-local quantities for $P^\alpha(\mathbb{R}^n)$

For the spaces $P^\alpha(\mathbb{R}^n)$, we can see that $P^0(\mathbb{R}^n) = H^1(\mathbb{R}^n)$ and $P^{\frac{n}{2}}(\mathbb{R}^n) = \dot{B}_2^{-\frac{n}{2},2}(\mathbb{R}^n)$. It is well known that the norms of $P^0(\mathbb{R}^n)$ and $P^{\frac{n}{2}}(\mathbb{R}^n)$ depend only on the $L^p(l^2)$-norms of function series $\{f_j = Q_j f\}_{j\in\mathbb{Z}}$ for $p = 1$ and $p = 2$, respectively. For the case $0 < \alpha < \frac{n}{2}$, the situation is complicated. In this section, we use wavelets to analyze the micro-local structure of $P^\alpha(\mathbb{R}^n)$. First, we present a theorem on conditional maximum value in Section 3.1. Then we consider the micro-local quantities in Section 3.2.

3.1. Conditional maximal value for non-negative sequence

For $u \in \mathbb{N}$, denote

$$\begin{cases} \Lambda_{u,n} = & \{0, 1, \ldots, 2^u - 1\}^n; \\ G_{u,n} = & \{(\epsilon, s, v), \epsilon \in E_n, 0 \leq s \leq u, v \in \Lambda_{s,n}\}. \end{cases}$$

Definition 3.1. For $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, $t \in \mathbb{N}$, let $\tilde{g}^t_{j,k} = \{g^\epsilon_{j+s,2^s k+u}\}_{(\epsilon,s,u)\in G_{t,n}}$ be a sequence. We call $\tilde{g}^t_{j,k} = \{g^\epsilon_{j+s,2^s k+u}\}_{(\epsilon,s,u)\in G_{t,n}}$ a non-negative sequence if $\tilde{g}^t_{j,k}$ satisfies

$$\forall\ (\epsilon, s, u) \in G_{t,n}, \quad g^\epsilon_{j+s,2^s k+u} \geq 0. \tag{3.1}$$

For a non-negative sequence $\tilde{g}^t_{j,k}$, we find the maximum value of the following quantities:

$$\tau_{f^t_{j,k},\tilde{g}^t_{j,k}} = \sum_{(\epsilon,s,u)\in G_{t,n}} f^t_{j,k}\tilde{g}^t_{j,k}, \tag{3.2}$$

where the non-negative sequence $f^t_{j,k} = \{f^\epsilon_{j+s,2^s k+u}\}_{(\epsilon,s,u)\in G_{t,n}}$ satisfies the following $\sum_{0\leq s\leq t} 2^{ns}$ restricted conditions

$$\begin{cases} 2^{n(j+t)} \sum\limits_{\epsilon\in E_n} (f^\epsilon_{j+t,2^t k+u})^2 & \leq 1, \quad \forall u \in \Lambda_{t,n}; \\ 2^{n(j+t-1)} \sum\limits_{(\epsilon,s,v)\in G_{1,n}} 2^{2s\alpha}(f^\epsilon_{j+t-1+s,2^s(2^{t-1}k+u)+v})^2 & \leq 1, \quad \forall u \in \Lambda_{t-1,n}; \\ 2^{n(j+t-2)} \sum\limits_{(\epsilon,s,v)\in G_{2,n}} 2^{2s\alpha}(f^\epsilon_{j+t-2+s,2^s(2^{t-2}k+u)+v})^2 & \leq 1, \quad \forall u \in \Lambda_{t-2,n}; \\ \cdots & \leq 1, \quad \cdots; \\ 2^{nj} \sum\limits_{(\epsilon,s,v)\in G_{t,n}} 2^{2s\alpha}(f^\epsilon_{j+s,2^s k+v})^2 & \leq 1 \quad . \end{cases} \tag{3.3}$$

There exist $(2^n - 1)\sum_{0\leq s\leq t} 2^{ns}$ elements in $G_{t,n}$. We can see that $f^t_{j,k}$ is a sequence, where the number of nonnegative terms is at most $(2^n - 1)\sum_{0\leq s\leq t} 2^{ns}$.

Definition 3.2. For $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, $t \in \mathbb{N}$, we say $f^t_{j,k} = \{f^\epsilon_{j+s,2^s k+u}\}_{(\epsilon,s,u)\in G_{t,n}} \in F^t_{j,k}$ if $f^t_{j,k}$ is a non-negative sequence satisfying (3.3).

We have

Theorem 3.3. *Let $0 \leq \alpha < n/2$ and $t \geq 0$. For any non-negative sequence $\tilde{g}^t_{j,k} = \{g^\epsilon_{j+s,2^s k+u}\}_{(\epsilon,s,u)\in G_{t,n}}$, there exists at least one sequence $\bar{f}^t_{j,k} = \{\bar{f}^\epsilon_{j+s,2^s k+u}\}_{(\epsilon,s,u)\in G_{t,n}} \in F^t_{j,k}$ such that*

$$\tau_{\bar{f}^t_{j,k},\tilde{g}^t_{j,k}} = \max_{f^t_{j,k}\in F^t_{j,k}} \tau_{f^t_{j,k},\tilde{g}^t_{j,k}}.$$

Proof. The $(2^n - 1)\sum_{0\leq s\leq t} 2^{ns}$ variables $\{f^\epsilon_{j+s,2^s k+u}\}_{(\epsilon,s,u)\in G_{t,n}}$ of the sequence $f^t_{j,k}$ are restricted in a closed domain, so the conclusion is obvious. □

3.2. Micro-local quantities in $P^\alpha(\mathbb{R}^n)$

From Proposition 2.12, we know that $(Q^0_\alpha(\mathbb{R}^n))' = P^\alpha(\mathbb{R}^n)$. To prove a function $g \in P^\alpha(\mathbb{R}^n)$, we only need to consider $\sup\langle f,\ g\rangle$, where the supremum is taken over all $f \in Q^0_\alpha$ with $\|f\|_{Q^0_\alpha} \leq 1$. However, by this method, we cannot know the micro-local structure of g in details.

To avoid this difficulty, we introduce a new method. Let

$$g(x)=\sum_{(\epsilon,j,k)\in\Lambda_n} g^{\epsilon}_{j,k}\,\Phi^{\epsilon}_{j,k}(x).$$

We localize g by restricting its wavelet coefficients $g^{\epsilon}_{j,k}$ such that $Q_{j,k}\subset Q$. Then we limit the range of frequencies and analyze its micro-local information. For this purpose, we analyze the function $g_{t,Q}$ defined in (1.2). For such a $g_{t,Q}$, the number of (ϵ,j,k) such that $g^{\epsilon}_{j,k}\neq 0$ is at most $(2^n-1)\sum_{0\leq s\leq t}2^{ns}$. We study micro-local functions $g_{t,Q}$ in $P^{\alpha}(\mathbb{R}^n)$ and obtain three kinds of micro-local quantities.

For all $t,j\in\mathbb{Z},k\in\mathbb{Z}^n$ and $t\geq 0$, we consider the series

$$g^t_{j,k}=\Big\{g^{\epsilon}_{j+s,2^sk+v},\epsilon\in E_n,0\leq s\leq t,v\in\Lambda_{s,n}\Big\}.$$

Denote

$$g^t_{j,k}(x)=\sum_{(\epsilon,s,u)\in G_{t,n}} g^{\epsilon}_{j+s,2^sk+u}\,\Phi^{\epsilon}_{j+s,2^sk+u}(x). \tag{3.4}$$

Since the correspondence between the sequences $g^t_{j,k}$ and the function $g^t_{j,k}(x)$ is one-to-one, in the notation-wise, we sometimes do not distinguish them.

For simplicity, we suppose that our functions are real-valued. Let

$$\begin{cases} f(x)=\sum_{(\epsilon,j,k)\in\Lambda_n} f^{\epsilon}_{j,k}\,\Phi^{\epsilon}_{j,k}(x); \\ g(x)=\sum_{(\epsilon,j,k)\in\Lambda_n} g^{\epsilon}_{j,k}\,\Phi^{\epsilon}_{j,k}(x). \end{cases}$$

If $\langle f,g\rangle$ and $\sum_{(\epsilon,j,k)\in\Lambda_n} f^{\epsilon}_{j,k}g^{\epsilon}_{j,k}$ are well defined, then we have

$$\tau_{f,g}=:\langle f,\ g\rangle=\sum_{(\epsilon,j,k)\in\Lambda_n} f^{\epsilon}_{j,k}g^{\epsilon}_{j,k}. \tag{3.5}$$

To compute $\max_{\|f\|_{Q^0_{\alpha}}\leq 1}\tau_{f,g^t_{j,k}}$, according to (3.5), we can restrict f to the function

$$f^t_{j,k}=\sum_{(\epsilon,s,u)\in G_{t,n}} f^{\epsilon}_{j+s,2^sk+u}\,\Phi^{\epsilon}_{j+s,2^sk+u}$$

with $\|f^t_{j,k}\|_{Q^0_{\alpha}}\leq 1$. The number of (ϵ,j,k) such that $f^{\epsilon}_{j,k}\neq 0$ is at most $(2^n-1)\sum_{0\leq s\leq t}2^{ns}$. Applying (3.5), we transfer the problem to finding out the supremum under an infinite number of constraint conditions to a maximal value problem on $\sum_{s=0}^{t}2^{ns}$ restricted conditions on the series of quantities $\{f^{\epsilon}_{j+s,2^sk+u}\}_{(\epsilon,s,u)\in G_{t,n}}$.

Based on Theorem 3.3, we begin to consider the micro-local quantities of $g^t_{j,k}$ in $P^{\alpha}(\mathbb{R}^n)$.

Theorem 3.4. *Suppose that $0<\alpha<n/2$ and $t\geq 0$. Let $g^t_{j,k}$ be the function defined by (3.4) and $\|g^t_{j,k}\|_{P^{\alpha}}>0$.*

(i) *There exists a function*

$$Sf^t_{j,k}=\sum_{(\epsilon,s,u)\in G_{t,n}} S^t_j f^{\epsilon}_{j+s,2^sk+u}\,\Phi^{\epsilon}_{j+s,2^sk+u}$$

with $\|S^t_jf^t_{j,k}\|_{Q^0_{\alpha}}\leq 1$ such that

$$\max_{\|f\|_{Q^0_{\alpha}}\leq 1}\tau_{f,g^t_{j,k}}=\sum_{(\epsilon,s,u)\in G_{t,n}} S^t_j f^{\epsilon}_{j+s,2^sk+u}\cdot g^t_{j+s,2^sk+u}.$$

(ii) *There exists a positive number* $P^t_j g^t_{j,k}$ *which is defined by the absolute values of the wavelet coefficient of* $g^t_{j,k}$ *such that*

$$P^t_j g^t_{j,k} = \|g^t_{j,k}\|_{P^\alpha} = \max_{\|f\|_{Q^0_\alpha}\le 1} \tau_{f,g^t_{j,k}} = \tau_{Sf^t_{j,k},g^t_{j,k}}.$$

(iii) *There exists a sequence* $\{Q^t_j g^\epsilon_{j,k}\}_{\epsilon\in E_n}$ *such that* $\sum_{\epsilon\in E_n} Q^t_j g^\epsilon_{j,k}\Phi^\epsilon_{j,k}$ *has the same norm in* $P^\alpha(\mathbb{R}^n)$ *as* $g^t_{j,k}$ *does.*

Proof. For $g^t_{j,k} = \{g^\epsilon_{j+s,2^sk+u}\}_{(\epsilon,s,u)\in G_{t,n}}$, set $\tilde{g}^t_{j,k} = \{|g^\epsilon_{j+s,2^sk+u}|\}_{(\epsilon,s,u)\in G_{t,n}}$. Denote

$$G^{t,j,k}_g = \Big\{(\epsilon,s,u)\in G_{t,n}, g^\epsilon_{j+s,2^sk+u}\ne 0\Big\}.$$

For $f^t_{j,k}(x) = \sum_{(\epsilon,s,u)\in G_{t,n}} f^\epsilon_{j+s,2^sk+u}\Phi^\epsilon_{j+s,2^sk+u}(x)$, define

$$f^{\epsilon,g}_{j+s,2^sk+u} = \begin{cases} |f^\epsilon_{j+s,2^sk+u}|\cdot|g^\epsilon_{j+s,2^sk+u}|^{-1}\overline{g^\epsilon_{j+s,2^sk+u}}, & (\epsilon,s,u)\in G_{t,n};\\ 0, & (\epsilon,s,u)\notin G_{t,n}.\end{cases}$$

We denote by $F^{t,j,k}_g$ the set

$$\Big\{f^t_{j,k}:\ f^t_{j,k}(x) = \sum_{(\epsilon,s,u)\in G_{t,n}} f^{\epsilon,g}_{j+s,2^sk+u}\Phi^\epsilon_{j+s,2^sk+u}(x) \text{ and } \|f^t_{j,k}\|_{Q^0_\alpha}\le 1\Big\}.$$

By (ii) of Proposition 2.5, we have

$$\max_{\|f^t_{j,k}\|_{Q^0_\alpha}\le 1} \tau_{\tilde{f}^t_{j,k},g^t_{j,k}} = \max_{f^t_{j,k}\in F^{t,j,k}_g} \tau_{f^t_{j,k},g^t_{j,k}} = \max_{\tilde{f}^t_{j,k}\in F^{t,j,k}_g} \tau_{\tilde{f}^t_{j,k},\tilde{g}^t_{j,k}}. \tag{3.6}$$

By Proposition 2.5, the condition $\|\tilde{f}^t_{j,k}\|_{Q^0_\alpha}\le 1$ is equivalent to (3.3). Further, for fixed $\tilde{g}^t_{j,k}$, due to (3.5), if $(\epsilon,s,u)\in G_{t,n}$ and $(\epsilon,s,u)\notin G^{t,j,k}_g$, then the coefficients $f^\epsilon_{j+s,2^sk+u}$ make no contribution to $\tau_{\tilde{f}^t_{j,k},\tilde{g}^t_{j,k}}$. Hence we get

$$\max_{\tilde{f}^t_{j,k}\in F^{t,j,k}_g} \tau_{\tilde{f}^t_{j,k},\tilde{g}^t_{j,k}} = \max_{\tilde{f}^t_{j,k}\in F^t_{j,k}} \tau_{\tilde{f}^t_{j,k},\tilde{g}^t_{j,k}}.$$

According to Theorem 3.3, there exists at least one sequence

$$\bar{f}^t_{j,k} = \{\tilde{f}^\epsilon_{j+s,2^sk+u}\}_{(\epsilon,s,u)\in G_{t,n}} \in F^t_{j,k}$$

such that

$$\tau_{\bar{f}^t_{j,k},\tilde{g}^t_{j,k}} = \max_{f^t_{j,k}\in F^t_{j,k}} \tau_{f^t_{j,k},\tilde{g}^t_{j,k}}. \tag{3.7}$$

Let $Sf^t_{j,k}(x) = \sum_{(\epsilon,s,u)\in G_{t,n}} S^t_j f^\epsilon_{j+s,2^sk+u}\Phi^\epsilon_{j+s,2^sk+u}(x)$, where

$$S^t_j f^\epsilon_{j+s,2^sk+u} = \begin{cases} \tilde{f}^\epsilon_{j+s,2^sk+u}|g^\epsilon_{j+s,2^sk+u}|^{-1}\overline{g^\epsilon_{j+s,2^sk+u}}, & \forall(\epsilon,s,u)\in G_{t,n};\\ 0, & \forall(\epsilon,s,u)\notin G_{t,n}.\end{cases}$$

According to (3.5) and (3.6), $Sf^t_{j,k}$ satisfies (i).

Let $P^t_j g^t_{j,k} = \tau_{\bar{f}^t_{j,k},\tilde{g}^t_{j,k}}$. According to the last equality in (3.6), $P^t_j g^t_{j,k}$ is defined by the absolute values of the wavelet coefficients of $g^t_{j,k}$. According to (3.5)–(3.7), $P^t_j g^t_{j,k}$ satisfies (ii).

Denote

$$Q_j^t g_{j,k}^\epsilon = \begin{cases} 2^{(j-1)n/2} P_j^t g_{j,k}^t, & \text{if } \sum_{\epsilon\in E_n} |g_{j,k}^\epsilon| = 0; \\ 2^{n/2j} P_j^t g_{j,k}^t \Big(\sum_{\epsilon\in E_n} |g_{j,k}^\epsilon|^2\Big)^{-1/2} g_{j,k}^\epsilon, & \text{if } \sum_{\epsilon\in E_n} |g_{j,k}^\epsilon| \neq 0. \end{cases}$$

Applying (ii) of Proposition 2.5 again, we know that $\{Q_j^t g_{j,k}^\epsilon\}_{\epsilon\in E_n}$ satisfies the condition (iii). □

Remark 3.5. For $\alpha = 0$ and $\alpha = n/2$, if we deal with $P_j^t g_{j,k}^t$ in a similar way, then:

(i) For $\alpha = 0$, according to the wavelet characterization of $H^1(\mathbb{R}^n)$ in [10], $P_j^t g_{j,k}^t$ is equivalent with

$$\Big\| \Big(\sum_{(\epsilon,s,u)\in G_{t,n}} 2^{n(j+s)} |g_{j+s,2^s k+u}^\epsilon|^2 \chi(2^{j+s}\cdot -2^s k - u)\Big)^{1/2} \Big\|_{L^1}.$$

(ii) For $\alpha = n/2$, $P_j^t g_{j,k}^t$ can be written as $\Big(\sum_{(\epsilon,s,u)\in G_{t,n}} 2^{-nj} |g_{j,k}^\epsilon|^2\Big)^{1/2}$.

However, for $0 < \alpha < n/2$, $P_j^t g_{j,k}^t$ cannot be expressed in an explicit way. Luckily, the three parts

$$\begin{cases} \{Q_j^t g_{j,k}^\epsilon\}_{\epsilon\in E_n}, \\ Sf_{j,k}^t = \sum_{(\epsilon,s,u)\in G_{t,n}} S_j^t f_{j+s,2^s k+u}^\epsilon \Phi_{j+s,2^s k+u}^\epsilon, \\ P_j^t g_{j,k}^t \end{cases} \tag{3.8}$$

indicate the micro-local characters in both the frequency structure and the local structure.

In the rest of this paper, the quantities defined by (3.8) will be used repeatedly. Micro-local quantities reveal the global information of functions in $P^\alpha(\mathbb{R}^n)$. In Section 4, this idea will be used to get the wavelet characterization of $P^\alpha(\mathbb{R}^n)$ by a group of L^1-functions defined by the absolute values of wavelet coefficients. Such wavelet characterization does not involve the action of a group of Borel measures.

4. Wavelet characterization of $P^\alpha(\mathbb{R}^n)$

For $s \in \mathbb{Z}$ and $N \in \mathbb{N}$, let

$$\Omega^{s,N} = \Big\{ Q \in \Omega : 2^{-sn} \le |Q| \le 2^{(N-s)n} \Big\}.$$

For $0 \le t \le N, m \in \mathbb{Z}^n, Q = Q_{s-N,m}$, define

$$\Omega_{s,t,Q} = \Omega_{s,m}^{t,N} = \Big\{ Q' \in \Omega : 2^{-sn} \le |Q'| \le 2^{(t-s)n}, Q' \subset Q_{s-N,m} \Big\}.$$

We can see that $\Omega^{s,N} = \bigcup_{m\in\mathbb{Z}^n} \Omega_{s,m}^{N,N}$. For $s \in \mathbb{Z}$ and $N \in \mathbb{N}$, we define

$$g_{s-N,m}^N(x) = \sum_{Q_{j,k}\in\Omega_{s,m}^{N,N}} g_{j,k}^\epsilon \Phi_{j,k}^\epsilon(x) \tag{4.1}$$

and

$$g_{s,N}(x) = \sum_{m\in\mathbb{Z}^n} g_{s-N,m}^N(x). \tag{4.2}$$

For $g = \sum_{(\epsilon,j,k)\in\Lambda_n} g_{j,k}^\epsilon \Phi_{j,k}^\epsilon \in P^\alpha(\mathbb{R}^n)$, let $\Lambda_n^g = \Big\{ (\epsilon,j,k) \in \Lambda_n : g_{j,k}^\epsilon \neq 0 \Big\}$.

Let $\{f^{\epsilon,g}_{j,k}\}$ be the sequences such that

$$f^{\epsilon,g}_{j,k} = \begin{cases} |f^{\epsilon,g}_{j,k}|\,|g^{\epsilon}_{j,k}|^{-1}\overline{g^{\epsilon}_{j,k}}, & (\epsilon,j,k)\in \Lambda^g_n;\\ 0, & (\epsilon,j,k)\notin \Lambda^g_n.\end{cases}$$

We denote by $Q^{0,g}_{\alpha}$ the set

$$\Big\{ f:\ f(x) = \sum_{(\epsilon,j,k)\in\Lambda_n} f^{\epsilon,g}_{j,k}\Phi^{\epsilon}_{j,k}(x) \text{ and } \|f\|_{Q^0_\alpha}\le 1\Big\}.$$

By (3.5), we have

$$\sup_{\|f\|_{Q^0_\alpha}\le 1}\tau_{f,g} = \sup_{f\in Q^{0,g}_\alpha}\tau_{f,g}. \tag{4.3}$$

We prove first an approximation lemma for $P^\alpha(\mathbb{R}^n)$.

Lemma 4.1. *For $g=\sum_{(\epsilon,j,k)\in\Lambda_n} g^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}\in P^\alpha(\mathbb{R}^n)$, let*

$$\tilde g_{s,N}(x) = \sum_{|m|\le 2^n} g^N_{s-N,m}(x).$$

For arbitrary $\delta>0$, there exist s and N such that $\|g-\tilde g_{s,N}\|_{P^\alpha}\le\delta$.

Proof. For any $0<\delta<\|g\|_{P^\alpha}/8$, according to Proposition 2.11, there exists $\{\lambda_u\}_{u\in\mathbb{N}_+}\in l^1$ and a group of $(\alpha,2)$-wavelet atoms $\{a_u\}$ such that $g(x)=\sum_{u\in\mathbb{N}_+}\lambda_u a_u(x)$ and

$$\Big|\sum_{u\in\mathbb{N}}|\lambda_u| - \|g\|_{P^\alpha}\Big| \le \delta/8.$$

Further there exists an integer $N_\delta>0$ such that

$$\sum_{u>N_\delta}|\lambda_u|\le\delta/8. \tag{4.4}$$

Now, for $u=1,\dots,N_\delta$, we consider the atoms

$$a_u(x) = \sum_{(\epsilon,j,k)\in\Lambda_n,Q_{j,k}\subset Q_u} a^{\epsilon,u}_{j,k}\Phi^{\epsilon}_{j,k}(x).$$

Since

$$\Big(\sum_{(\epsilon,j,k)\in\Lambda_n,Q_{j,k}\subset Q_u} 2^{-2j\alpha}|a^{\epsilon,u}_{j,k}|^2\Big)^{1/2} \le |Q_u|^{\alpha/n-1/2},$$

there exists an integer $\tilde N_\delta>0$ such that

$$\Big(\sum_{(\epsilon,j,k)\in\Lambda_n,Q_{j,k}\subset Q_u,j>\tilde N_\delta} 2^{-2j\alpha}|a^{\epsilon,u}_{j,k}|^2\Big)^{1/2} \le \frac{\delta}{16\|g\|_{P^\alpha}}|Q_u|^{\alpha/n-1/2}. \tag{4.5}$$

Since $1\le u\le N_\delta$, there exists an integer $j_\delta\in\mathbb{Z}$ such that

$$\bigcup_{1\le u\le N_\delta} Q_u \subset \bigcup_{|m|\le 2^n} Q_{j_\delta,m}. \tag{4.6}$$

For $u = 1, \dots, N_\delta$, let $b_u(x) = \sum_{(\epsilon,j,k)\in\Lambda_n, Q_{j,k}\subset Q_u, j\leq \tilde{N}_\delta} a^{\epsilon,u}_{j,k}\Phi^\epsilon_{j,k}(x)$. According to (4.4) and (4.5), we know that

$$\begin{aligned} &\Big\|\sum_{u>N_\delta}\lambda_u a_u\Big\|_{P^\alpha} \leq \delta/8;\\ &\Big\|\sum_{1\leq u\leq N_\delta}\lambda_u(a_u-b_u)\Big\|_{P^\alpha} \leq \delta\sum_{1\leq u\leq N_\delta}|\lambda_u|/16\|g\|_{P^\alpha} \leq \delta/8. \end{aligned} \tag{4.7}$$

Let

$$g_\delta(x) = \sum_{u>N_\delta}\lambda_u a_u + \sum_{1\leq u\leq N_\delta}\lambda_u(a_u-b_u) = \sum_{(\epsilon,j,k)\in\Lambda_n} g^{\epsilon,\delta}_{j,k}\Phi^\epsilon_{j,k}(x).$$

Then $\|g_\delta\|_{P^\alpha} \leq \delta/4$. Let

$$g_{1,\delta}(x) = \sum_{(\epsilon,j,k)\in\Lambda_n, j\geq N_\delta, Q_{j,k}\subset \bigcup\limits_{|m|\leq 2^n} Q_{j_\delta,m}} g^{\epsilon,\delta}_{j,k}\Phi^\epsilon_{j,k}(x)$$

and $g_{2,\delta}(x) = g_\delta(x) - g_{1,\delta}(x)$. According to (4.3), we have $\|g_{1,\delta}\|_{P^\alpha} \leq \delta/4$ and $\|g_{2,\delta}\|_{P^\alpha} \leq \delta/4$. Take $s = \tilde{N}_\delta$ and $N = s - j_\delta$. Let

$$\tilde{g}_{s,N}(x) = g_{1,\delta}(x) + \sum_{1\leq u\leq N_\delta}\lambda_u b_u(x).$$

According to the above construction process, $\tilde{g}_{s,N}$ satisfies the condition of Lemma 4.1. □

Given $0 \leq t \leq N$, $m \in \mathbb{Z}^n$ and $Q = Q_{s-N,m}$. If $t = 0$, we denote

$$g^{\epsilon,s,t,N}_{j,k} = \begin{cases} 0, & j > s;\\ g^\epsilon_{j,k}, & j = s. \end{cases}$$

Let $t \geq 1$. For $Q^t_j g^\epsilon_{j,k}$ defined in Theorem 3.4, we denote

$$g^{\epsilon,s,t,N}_{j,k} = \begin{cases} 0, & j > s-t;\\ Q^t_j g^\epsilon_{j,k}, & j = s-t;\\ g^\epsilon_{j,k}, & j < s-t. \end{cases}$$

Let $g_{s,t,N}(x) = \sum_{\epsilon,j,k} g^{\epsilon,s,t,N}_{j,k}\Phi^\epsilon_{j,k}(x)$. We define

$$P_{s,t,N}g(x) = \Big(\sum_{\epsilon, Q_{j,k}\in\Omega^{s,N}, j\leq s-t} 2^{nj}|g^{\epsilon,s,t,N}_{j,k}|^2\chi(2^jx-k)\Big)^{1/2}$$

and

$$Q_{s,t,N}g = \Big\|\Big(\sum_{\epsilon, Q_{j,k}\in\Omega^{s,N}, j=s-t} 2^{jn}|g^{\epsilon,s,t,N}_{j,k}|^2\chi(2^j\cdot -k)\Big)^{1/2}\Big\|_{L^1}.$$

For $t = N$, we have

$$Q_{s,N,N}g^N_{s-N,m} = \|P_{s,N,N}g^N_{s-N,m}\|_{L^1}. \tag{4.8}$$

Now we prove a wavelet characterization without involving Borel measures.

Theorem 4.2. *If* $0 < \alpha < n/2$, *then*

$$P^\alpha(\mathbb{R}^n) = \Big\{g : \sup_{s\in\mathbb{Z}, N\in\mathbb{N}}\min_{0\leq t\leq N}\|P_{s,t,N}g\|_{L^1} < \infty\Big\}.$$

Proof. According to (4.3) and Lemma 4.1, $\forall \delta > 0$, there exists $\tau_\delta > 0$ such that for $s > \tau_\delta, N \geq 2s$, we have

$$\|g_{s,N} - g\|_{P^\alpha} + \sum_{|m|>2^n} \|g^N_{s-N,m}\|_{P^\alpha} \leq \delta \tag{4.9}$$

and

$$8^{-n} \max_{|m|\leq 2^n} \|g^N_{s-N,m}\|_{P^\alpha} - \delta \leq \|g_{s,N}\|_{P^\alpha} \leq \sum_{|m|\leq 2^n} \|g^N_{s-N,m}\|_{P^\alpha} + \delta, \tag{4.10}$$

where $g_{s,N}$ and $g^N_{s-N,m}$ are defined by (4.1) and (4.2).

By (4.8) and Theorem 3.4, we have

$$\|g^N_{s-N,m}\|_{P^\alpha} = Q_{s,N,N} g^N_{s-N,m} = \|P_{s,N,N} g^N_{s-N,m}\|_{L^1}. \tag{4.11}$$

Furthermore, we have

$$\|g_{s,t,N}\|_{P^\alpha} \leq \|g_{s,t,N}\|_{H^1} = \|P_{s,t,N} g\|_{L^1}. \tag{4.12}$$

According to (4.9)–(4.12), the proof of Theorem 4.2 is complete. □

5. A Fefferman–Stein type decomposition of the Q-spaces

In this section, by Theorem 4.2, we give a Fefferman–Stein type decomposition of $Q_\alpha(\mathbb{R}^n)$. In [4], the authors proved that $Q_\alpha(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$. Equivalently, we can obtain

Proposition 5.1. *If* $0 < \alpha < n/2$, $H^1(\mathbb{R}^n) \subset P^\alpha(\mathbb{R}^n)$.

Further, for the proof of Theorem 5.7 below, we need some special properties of the Daubechies wavelets. Except for Theorem 5.7, we use the classical Meyer wavelets throughout Sections 5 and 6. The support of the Fourier transform of the classical Meyer wavelet in [10] satisfies the following conditions

$$\begin{cases} \text{supp } \widehat{\Phi^0} \subset [-4\pi/3, 4\pi/3]; \\ \text{supp } \widehat{\Phi^1} \subset [-8\pi/3, 8\pi/3] \setminus (-2\pi/3, 2\pi/3). \end{cases} \tag{5.1}$$

For tensor product Meyer wavelets satisfying (5.1), $\forall (\epsilon, j, k), (\epsilon', j', k') \in \Lambda_n$ and $|j - j'| \geq 2$, we have

$$\langle R_i \Phi^\epsilon_{j,k}, \ \Phi^{\epsilon'}_{j',k'} \rangle = 0, \quad \forall i = 1, \ldots, n. \tag{5.2}$$

5.1. *Adapted L^1 and L^∞ spaces*

For $g(x) = \sum_{(\epsilon,j,k)\in\Lambda_n} g^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x)$ and $j \in \mathbb{Z}$, denote

$$Q_j g(x) = \sum_{\epsilon\in E_n, k\in\mathbb{Z}^n} g^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x). \tag{5.3}$$

For $s \in \mathbb{Z}$ and $N \in \mathbb{N}$, we set

$$P_{s,N} g(x) = \sum_{\epsilon, s-N\leq j\leq s, k} g^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x). \tag{5.4}$$

For all integers $t = 0, \ldots, N$, denote

$$T^1_{s,t,N} g(x) = \sum_{\epsilon, s-t\leq j\leq s, k} g^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x)$$

and

$$T^2_{s,t,N}g(x)=\sum_{\epsilon,s-N\le j<s-t,k} g^\epsilon_{j,k}\,\Phi^\epsilon_{j,k}(x).$$

By Theorem 4.2, we introduce spaces $\tilde{P}^\alpha(\mathbb{R}^n)$.

Definition 5.2. Let $\alpha\in[0,n/2)$. We say that $g\in\tilde{P}^\alpha(\mathbb{R}^n)$ if

$$\sup_{s\in\mathbb{Z},N\in\mathbb{N}}\inf_{0\le t\le N}(\|T^1_{s,t,N}g\|_{P^\alpha}+\|T^2_{s,t,N}g\|_{H^1})<\infty.$$

The space $\tilde{P}^\alpha(\mathbb{R}^n)$ is not really new. In fact,

Theorem 5.3. (i) *If $\alpha=0$, then $P^0(\mathbb{R}^n)=\tilde{P}^0(\mathbb{R}^n)=H^1(\mathbb{R}^n)$.*
(ii) *If $0<\alpha<n/2$, then $P^\alpha(\mathbb{R}^n)=\tilde{P}^\alpha(\mathbb{R}^n)$.*

Proof. $P^0(\mathbb{R}^n)=H^1(\mathbb{R}^n)$ is known, so (i) is evident. Now we consider the cases $0<\alpha<n/2$. If $g\in P^\alpha(\mathbb{R}^n)$, then $\|P_{s,N}g\|_{P^\alpha}\le\|g\|_{P^\alpha}$. Further

$$\inf_{0\le t\le N}(\|T^1_{s,t,N}g\|_{P^\alpha}+\|T^2_{s,t,N}g\|_{H^1})\le\|T^1_{s,N,N}g\|_{P^\alpha}=\|P_{s,N}g\|_{P^\alpha}.$$

Hence

$$\sup_{s\in\mathbb{Z},N\in\mathbb{N}}\inf_{0\le t\le N}(\|T^1_{s,t,N}g\|_{P^\alpha}+\|T^2_{s,t,N}g\|_{H^1})\le\|g\|_{P^\alpha}.$$

Conversely, if $g\in\tilde{P}^\alpha(\mathbb{R}^n)$, we have

$$\begin{aligned}\|P_{s,N}g\|_{P^\alpha}&\le\|T^1_{s,t,N}g\|_{P^\alpha}+\|T^2_{s,t,N}g\|_{P^\alpha}\\&\le\|T^1_{s,t,N}g\|_{P^\alpha}+\|T^2_{s,t,N}g\|_{H^1}.\end{aligned}$$

Hence

$$\|P_{s,N}g\|_{P^\alpha}\le\inf_{0\le t\le N}(\|T^1_{s,t,N}g\|_{P^\alpha}+\|T^2_{s,t,N}g\|_{H^1}).$$

According to (4.9), $g\in P^\alpha(\mathbb{R}^n)$. □

For $g\in P^\alpha(\mathbb{R}^n)$, we can deduce from Theorem 5.3 that the P^α-norm of g is equivalent to

$$\sup_{s\in\mathbb{Z},N\in\mathbb{N}}\inf_{0\le t\le N}(\|T^1_{s,t,N}g\|_{P^\alpha}+\|T^2_{s,t,N}g\|_{H^1}),$$

which implies that for $0<\alpha<n/2$, the high-frequency part $T^1_{s,t,N}g$ and the low-frequency part $T^2_{s,t,N}g$ make different contributions to the norm. Now we use such property to construct $L^{1,\alpha}(\mathbb{R}^n)$ and $L^{\infty,\alpha}(\mathbb{R}^n)$ which will be adapted to the Fefferman–Stein type decomposition of $Q_\alpha(\mathbb{R}^n)$.

Let $f(x)=\sum_{(\epsilon,j,k)\in\Lambda_n} f^\epsilon_{j,k}\,\Phi^\epsilon_{j,k}(x)$. For $s,t,N\in\mathbb{Z}$ and $0\le t\le N$, we denote

$$\begin{aligned}P_{s,N}f(x)&=\sum_{\epsilon,s-N\le j\le s,k} f^\epsilon_{j,k}\,\Phi^\epsilon_{j,k}(x),\\S^1_{s,t,N}f(x)&=\sum_{\epsilon,s-t\le j\le s,k} f^\epsilon_{j,k}\,\Phi^\epsilon_{j,k}(x),\\S^2_{s,t,N}f(x)&=\sum_{\epsilon,s-N\le j<s-t,k} f^\epsilon_{j,k}\,\Phi^\epsilon_{j,k}(x).\end{aligned}$$

The spaces $L^{1,\alpha}(\mathbb{R}^n)$ and $L^{\infty,\alpha}(\mathbb{R}^n)$ are defined as follows.

Definition 5.4. Let $f(x)=\sum_{(\epsilon,j,k)\in\Lambda_n} f^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x)$ and $g(x)=\sum_{(\epsilon,j,k)\in\Lambda_n} g^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x)$.

(i) We say that $g\in L^{1,\alpha}(\mathbb{R}^n)$ if

$$\sup_{s\in\mathbb{Z},N\in\mathbb{N}}\min_{0\le t\le N}\Big(\|T^1_{s,t,N}g\|_{P^\alpha}+\|T^2_{s,t,N}g\|_{L^1}\Big)<\infty.$$

(ii) We say that $f\in L^{\infty,\alpha}(\mathbb{R}^n)$ if

$$\sup_{s\in\mathbb{Z},N\in\mathbb{N}}\sup_{0\le t\le N}\Big(\|S^1_{s,t,N}f\|_{Q_\alpha}+\|S^2_{s,t,N}f\|_{L^\infty}\Big)<\infty.$$

By Proposition 5.1 and Theorem 5.3, we have

Theorem 5.5. *Given* $0\le\alpha<n/2$.

(i) $P^\alpha(\mathbb{R}^n)\subset L^{1,\alpha}(\mathbb{R}^n)$;
(ii) $L^{\infty,\alpha}(\mathbb{R}^n)=Q_\alpha(\mathbb{R}^n)\bigcap L^\infty(\mathbb{R}^n)$;
(iii) $(L^{1,\alpha}(\mathbb{R}^n))'=L^{\infty,\alpha}(\mathbb{R}^n)$.

Remark 5.6. For the case $\alpha=0$, we have:

(i) $P^0(\mathbb{R}^n)=H^1(\mathbb{R}^n)$ and $Q_0(\mathbb{R}^n)=BMO(\mathbb{R}^n)$;
(ii) $L^{1,0}(\mathbb{R}^n)=L^1(\mathbb{R}^n)$ and $L^{\infty,0}(\mathbb{R}^n)=L^\infty(\mathbb{R}^n)$.

Now, we use the Daubechies wavelets to prove that $L^{\infty,\alpha}(\mathbb{R}^n)\subsetneq Q_\alpha(\mathbb{R}^n)$. We know that there exist some integer M and a Daubechies scale function $\Phi^0\in C_0^{n+2}([-2^M,2^M]^n)$ satisfying

$$C_D=\int\frac{-y_1}{|y|^{n+1}}\Phi^0(y-2^{M+1}e)dy<0,\quad e=(1,1,\dots,1). \tag{5.5}$$

Theorem 5.7. *Let* $\Phi(x)=\Phi^0(x-2^{M+1}e)$ *and let* f *be defined as*

$$f(x)=\sum_{j\in2\mathbb{N}}\Phi(2^jx). \tag{5.6}$$

If $0\le\alpha<n/2$, *then* $f\in L^{\infty,\alpha}(\mathbb{R}^n)$ *and* $R_1(f)\notin L^\infty(\mathbb{R}^n)$, *that is,* $L^{\infty,\alpha}(\mathbb{R}^n)\subsetneq Q_\alpha(\mathbb{R}^n)$.

Proof. For $j,j'\in2\mathbb{N}$ with $j\neq j'$, the supports of $\Phi(2^j\cdot)$ and $\Phi(2^{j'}\cdot)$ are disjoint. Hence the above f in (5.6) belongs to $L^\infty(\mathbb{R}^n)$. The same reasoning gives $\sum_{j\in\mathbb{N},2j>j'}\Phi(2^{2j}\cdot)\in L^\infty(\mathbb{R}^n)$ for any $j'\in\mathbb{N}$.

Now we compute the wavelet coefficients of f in (5.6). For $(\epsilon',j',k')\in\Lambda_n$, let $f^{\epsilon'}_{j',k'}=\langle f,\ \Phi^{\epsilon'}_{j',k'}\rangle$. We divide the proof into two cases: $j'<0$ and $j'\ge0$.

For $j'<0$, since supp $f\subset[-3\cdot2^M,3\cdot2^M]^n$, we know that $f^{\epsilon'}_{j',k'}=0$ for $|k'|>2^{2M+5}$. For $|k'|\le2^{2M+5}$, we have

$$|f^{\epsilon'}_{j',k'}|\le C2^{nj'/2}\int|f(x)|dx\le C2^{nj'/2}.$$

For $j' \geq 0$, by orthogonality of the wavelets, we have

$$f^{\epsilon'}_{j',k'} = \left\langle f,\ \Phi^{\epsilon'}_{j',k'} \right\rangle = \Big\langle \sum_{j\in\mathbb{N},2j>j'} \Phi(2^{2j}\cdot),\ \Phi^{\epsilon'}_{j',k'} \Big\rangle.$$

By the same reasoning, for the case $j' \geq 0$, we know that if $|k'| > 2^{2M+5}$, then $f^{\epsilon'}_{j',k'} = 0$. Since $\sum_{j\in\mathbb{N},2j>j'} \Phi(2^{2j}\cdot) \in L^\infty$, if $|k'| \leq 2^{2M+5}$, we have

$$|f^{\epsilon'}_{j',k'}| \leq C \int |\Phi^{\epsilon'}_{j',k'}(x)|dx \leq C2^{-nj'/2}.$$

By the above estimates and (i) of Proposition 2.5, we conclude that $f \in Q_\alpha(\mathbb{R}^n)$, that is, $f \in Q_\alpha(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Since $\Phi^0 \in C_0^{n+2}([-2^M, 2^M]^n)$, we know that

$$\Phi = \Phi^0(\cdot - 2^{M+1}e) \in C_0^{n+2}([2^M, 3\cdot 2^M]^n).$$

Further, if $|x| \leq 2^{M-1}$ and $y \in [2^M, 3\cdot 2^M]^n$, then $|x-y| > 2^{M-1}$. Hence $R_1\Phi$ is smooth in the ball $\{x:\ |x| \leq 2^{M-1}\}$.

Applying (5.5), there exists a positive $\delta > 0$ such that for $|x| < \delta$, $R_1\Phi(x) < C_D/2 < 0$. By a dilation, we can see that $R_1\Phi(2^{2j}x) < C_D/2 < 0$ for $2^{2j}|x| < \delta$. Hence $R_1 f \notin L^\infty(\mathbb{R}^n)$. □

5.2. Fefferman–Stein decomposition of $Q_\alpha(\mathbb{R}^n)$

Fefferman–Stein [5] used the Riesz transformations and the L^1 norm to characterize Hardy space $H^1(\mathbb{R}^n)$:

Theorem 2. *$g \in H^1(\mathbb{R}^n)$ if and only if*

$$C_{Riesz}(g) = \|g\|_{L^1(\mathbb{R}^n)} + \sum_{i=1}^{n} \|R_i g\|_{L^1(\mathbb{R}^n)} < \infty.$$

Theorem 2 gives rise to the Fefferman–Stein decomposition of $BMO(\mathbb{R}^n)$. The following theorem extends Theorem 2 to $P^\alpha(\mathbb{R}^n)$. If $\alpha = 0$, Theorem 5.8 becomes Theorem 2, so we omit the proof of this case. The proof for the cases $0 < \alpha < n/2$ is rather long. So we only state this result here and postpone the proof to Section 6. For $0 \leq \alpha < n/2$ and a function g, denote

$$C_{\alpha,Riesz}(g) = \|g\|_{L^{1,\alpha}} + \sup_{s\in\mathbb{Z},N\in\mathbb{N}} \min_{0\leq t\leq N} \sum_{i=1}^{n} \{\|T^1_{s,t,N} R_i g\|_{P_\alpha} + \|T^2_{s,t,N} R_i g\|_{L^1}\}.$$

Theorem 5.8. *If $0 \leq \alpha < n/2$, then $g \in P^\alpha(\mathbb{R}^n)$ if and only if*

$$C_{\alpha,Riesz}(g) < \infty. \tag{5.7}$$

If Theorem 5.8 holds, by Theorem 5.7, we could obtain a Fefferman–Stein type decomposition of $Q_\alpha(\mathbb{R}^n)$ using Fefferman–Stein's skill in [5]. This result solves Problem 1.1 (Problem 8.3 in [4]).

Theorem 5.9. *If $0 \leq \alpha < n/2$, then $f \in Q_\alpha(\mathbb{R}^n)$ if and only if $f(x) = \sum_{0\leq i\leq n} R_i f_i(x)$, where $f_i \in Q_\alpha(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$.*

Proof. By the continuity of the Calderón–Zygmund operators on the Q-spaces, we know that if $f_i \in Q_\alpha(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$, then

$$\sum_{0\le i\le n} R_i f_i \in Q_\alpha(\mathbb{R}^n).$$

Now we prove the converse result. Let

$$B = \Big\{(g_0, g_1, \dots, g_n) : g_i \in L^{1,\alpha}(\mathbb{R}^n), i = 0, \dots, n\Big\}.$$

The norm of B is defined as

$$\|(g_0, g_1, \dots, g_n)\|_B = \sum_{i=0}^{n} \|g_i\|_{L^{1,\alpha}}.$$

We define

$$S = \Big\{(g_0, g_1, \dots, g_n) \in B : g_i = R_i g_0, i = 0, 1, \dots, n\Big\}.$$

S is a closed subset of B. By Theorem 5.8, The mapping

$$g_0 \to (g_0, R_1 g_0, \dots, R_n g_0)$$

defines a norm preserving map from $P^\alpha(\mathbb{R}^n)$ to S. Hence the set of continuous linear functionals on $P^\alpha(\mathbb{R}^n)$ is equivalent to the set of continuous linear functionals on S. The continuous linear functionals on S can extend to a continuous linear functionals on B preserving the same norm. We know that the dual space of $L^{1,\alpha}(\mathbb{R}^n) \oplus \cdots \oplus L^{1,\alpha}(\mathbb{R}^n)$ is $L^{\infty,\alpha}(\mathbb{R}^n) \oplus \cdots \oplus L^{\infty,\alpha}(\mathbb{R}^n)$.

For $f \in Q_\alpha(\mathbb{R}^n)$, f defines a continuous linear functional l on $P^\alpha(\mathbb{R}^n)$ and also on S. Hence there exist $\tilde{f}_i \in L^{\infty,\alpha}(\mathbb{R}^n), i = 0, 1, \dots, n$, such that for any $g_0 \in P^\alpha(\mathbb{R}^n)$,

$$\begin{aligned} l(f) &= \int_{\mathbb{R}^n} f(x) g_0(x) dx \\ &= \int_{\mathbb{R}^n} \tilde{f}_0(x) g_0(x) dx + \sum_{i=1}^{n} \int_{\mathbb{R}^n} \tilde{f}_i(x) R_i g_0(x) dx \\ &= \int_{\mathbb{R}^n} \tilde{f}_0(x) g_0(x) dx - \sum_{i=1}^{n} \int_{\mathbb{R}^n} R_i(\tilde{f}_i)(x) g_0(x) dx. \end{aligned}$$

Hence $f(x) = \tilde{f}_0(x) - \sum_{i=1}^n R_i(\tilde{f}_i)(x)$. □

Triebel–Lizorkin spaces $\dot{F}^{0,q}_\infty$ are introduced in [17], Besov–Morrey spaces and Triebel–Lizorkin–Morrey spaces are introduced in [24]. These spaces play an important role in harmonic analysis and non-linear problems, see [9] etc. Since Fefferman–Stein decomposition of $BMO(\mathbb{R}^n)$ plays an important role in harmonic analysis, we propose the following open problems:

Remark 5.10. (1) In dimension n, how to give a Fefferman–Stein type decomposition for Triebel–Lizorkin spaces $\dot{F}^{0,q}_\infty$?

(2) More generally, for other Besov–Morrey spaces or Triebel–Lizorkin–Morrey spaces, whether there is also a Fefferman–Stein type decomposition?

6. The proof of Theorem 5.8

At first, we prove that

$$g \in P^\alpha(\mathbb{R}^n) \Longrightarrow g \text{ satisfies } (5.7). \tag{6.1}$$

By (ii) of Proposition 2.11, the fact $g \in P^{\alpha}(\mathbb{R}^n)$ implies that

$$R_i g \in P^{\alpha}(\mathbb{R}^n), \quad i = 1, \dots, n.$$

By (i) of Theorem 5.5, we obtain that g satisfies (5.7).

The proof of the converse of (6.1) is cumbersomeness and we will complete it in Section 6.2. Next, as a preliminary, we give the following lemma.

6.1. A lemma

Lemma 6.1. *For $g(x) = \sum_{(\epsilon,j,k)\in\Lambda_n} g^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x)$ and arbitrary $j \in \mathbb{Z}$, denote $g_j(x) = \sum_{\epsilon\in E_n, k\in\mathbb{Z}^n} g^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x)$ and denote $\tilde{g}_j(x) = \sum_{j'\leq j} g_{j'}(x)$. For $0 < \alpha < \frac{n}{2}$, we have*

(i) $\|g_j\|_{H^1} \leq C\|g\|_{L^1}$.
(ii) $\max\Big\{\|\tilde{g}_j\|_{P^{\alpha}},\ \|g-\tilde{g}_j\|_{P^{\alpha}}\Big\} \leq \|g\|_{P^{\alpha}} \leq \|\tilde{g}_j\|_{P^{\alpha}} + \|g-\tilde{g}_j\|_{P^{\alpha}}$.
(iii) $\|g_j\|_{P^{\alpha}} \leq C\|g\|_{L^{1,\alpha}}$.

Proof. (i) Applying (ii) of Proposition 2.2 and the orthogonality of $\{\Phi^{\epsilon}_{j,k}\}$, we have

$$\begin{aligned}\|g_j\|_{H^1} &\leq C\Big\|\Big(\sum_{\epsilon\in E_n, k\in\mathbb{Z}^n} 2^{nj}|\langle g_j, \Phi^{\epsilon}_{j,k}\rangle|^2\chi(2^j\cdot -k)\Big)^{1/2}\Big\|_{L^1}\\ &\leq C\sum_{\epsilon\in E_n}\Big\|\sum_{k\in\mathbb{Z}^n} 2^{n/2j}|\langle g, \Phi^{\epsilon}_{j,k}\rangle|\chi(2^j\cdot -k)\Big\|_{L^1}\\ &\leq C\|g\|_{L^1}.\end{aligned}$$

(ii) $P^{\alpha}(\mathbb{R}^n)$ is a Banach space, hence we have

$$\|g\|_{P^{\alpha}} \leq \|\tilde{g}_j\|_{P^{\alpha}} + \|g-\tilde{g}_j\|_{P^{\alpha}}.$$

To prove the first inequality of (ii), denote

$$G_g = \Big\{(\epsilon, j, k) \in \Lambda_n, g^{\epsilon}_{j,k} \neq 0\Big\}.$$

For $f(x) = \sum_{(\epsilon,j,k)\in\Lambda_n} f^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x)$, define

$$f^{\epsilon,g}_{j,k} = \begin{cases} |f^{\epsilon}_{j,k}|\cdot|g^{\epsilon}_{j,k}|^{-1}\overline{g^{\epsilon}_{j,k}}, & (\epsilon,j,k)\in G_g;\\ 0, & (\epsilon,j,k)\notin G_g.\end{cases}$$

We denote by F_g the set

$$\Big\{f:\ f(x) = \sum_{(\epsilon,j,k)\in G_g} f^{\epsilon,g}_{j,k}\Phi^{\epsilon}_{j,k}(x) \text{ and } \|f\|_{Q^0_{\alpha}} \leq 1\Big\}.$$

Define

$$\underline{f}(x) = \sum_{(\epsilon,j,k)\in\Lambda_n} |f^{\epsilon}_{j,k}|\Phi^{\epsilon}_{j,k}(x)$$

and

$$\underline{g}(x) = \sum_{(\epsilon,j,k)\in\Lambda_n} |g^{\epsilon}_{j,k}|\Phi^{\epsilon}_{j,k}(x).$$

By (ii) of Proposition 2.5, we have

$$\sup_{\|f\|_{Q^0_\alpha}\leq 1} \tau_{f,g} = \sup_{f\in F_g} \tau_{f,g} = \sup_{\underline{f}\in F_g} \tau_{\underline{f},\underline{g}} = \sup_{\|\underline{f}\|_{Q^0_\alpha}\leq 1} \tau_{\underline{f},\underline{g}}. \tag{6.2}$$

Hence we have

$$\max\Big\{\|\tilde{g}_j\|_{P^\alpha},\ \|g-\tilde{g}_j\|_{P^\alpha}\Big\} \leq \|g\|_{P^\alpha}.$$

(iii) By the definition of the $L^{1,\alpha}$-norm of g, for $s\in\mathbb{Z}$, $N\in\mathbb{N}$ with $s-N\leq j\leq s$, there exists j_0 such that $0\leq j_0\leq N$ and

$$\Big\|\sum_{s-j_0\leq j'\leq s} g_{j'}\Big\|_{P^\alpha} + \Big\|\sum_{s-N\leq j'<s-j_0} g_{j'}\Big\|_{L^1} \leq \|g\|_{L^{1,\alpha}}.$$

If $j<s-j_0$, we apply (i) to get the desired assertion. If $j\geq s-j_0$, we apply (ii) to get the desired assertion. □

6.2. The proof of the converse part

For the proof of the converse of (6.1), it is sufficient to prove that $\forall s_1\in\mathbb{Z}$, $N_1\geq 1$ and $g_{s_1,N_1}(x) = P_{s_1,N_1}g(x)$ defined in (5.4), we have

$$\|g_{s_1,N_1}\|_{P^\alpha} \leq CC_{\alpha,Riesz}(g_{s_1,N_1}). \tag{6.3}$$

Owing to (5.2), there exists $\{g^{\epsilon,i}_{j,k}\}_{(\epsilon,j,k)\in\Lambda_n}$ such that for $i=1,2,\ldots,n$,

$$R_i g_{s_1,N_1}(x) = \sum_{(\epsilon,j,k)\in\Lambda_n, s_1-N_1-1\leq j\leq s_1+1} g^{\epsilon,i}_{j,k}\Phi^\epsilon_{j,k}(x). \tag{6.4}$$

Due to (6.4), to estimate the $L^{1,\alpha}$-norm of $R_i g_{s_1,N_1}, i=0,1,\ldots,n$, it is sufficient to consider $s=s_1+1$ and $N=N_1+2$. For such s and N, there exist $t^0_{s,N}$ and $t^1_{s,N}$ such that

$$\|T^1_{s,t^0_{s,N},N}g_{s_1,N_1}\|_{P^\alpha} + \|T^2_{s,t^0_{s,N},N}g_{s_1,N_1}\|_{L^1} = \min_{0\leq t\leq N}\Big(\|T^1_{s,t,N}g_{s_1,N_1}\|_{P^\alpha} + \|T^2_{s,t,N}g_{s_1,N_1}\|_{L^1}\Big); \tag{6.5}$$

$$\begin{aligned}&\sum_{1\leq i\leq n}\{\|T^1_{s,t^1_{s,N},N}R_i g_{s_1,N_1}\|_{P^\alpha} + \|T^2_{s,t^1_{s,N},N}R_i g_{s_1,N_1}\|_{L^1}\}\\ &= \min_{0\leq t\leq N}\sum_{1\leq i\leq n}\{\|R_i T^1_{s,t,N}g_{s_1,N_1}\|_{P^\alpha} + \|R_i T^2_{s,t,N}g_{s_1,N_1}\|_{L^1}\}.\end{aligned} \tag{6.6}$$

We divide the proof into three cases.

Case 1: $t^0_{s,N} = t^1_{s,N}$. Let Q_j be the projection operators defined by (5.3). We divide the function g_{s_1,N_1} into two functions

$$g_{s_1,N_1}(x) = g^1_{s,N}(x) + g^2_{s,N}(x),$$

where

$$g^1_{s,N}(x) = \sum_{j\geq s-t^0_{s,N}} Q_j g_{s_1,N_1}(x)$$

and

$$g^2_{s,N}(x) = \sum_{j<s-t^0_{s,N}} Q_j g_{s_1,N_1}(x).$$

By (6.5), we have $g^2_{s,N} \in L^1(\mathbb{R}^n)$. By Lemma 6.1,

$$Q_{s-t^0_{s,N}-1} g_{s_1,N_1} + Q_{s-t^0_{s,N}-2} g_{s_1,N_1} \in H^1(\mathbb{R}^n) \tag{6.7}$$

and

$$g^2_{s,N} - \left(Q_{s-t^0_{s,N}-1} g_{s_1,N_1} + Q_{s-t^0_{s,N}-2} g_{s_1,N_1}\right) \in L^1(\mathbb{R}^n). \tag{6.8}$$

Further, for $i = 1, \ldots, n$, we have

$$\begin{aligned} T^2_{s,t^1_{s,N},N} R_i g_{s_1,N_1}(x) &= T^2_{s,t^1_{s,N},N} R_i \Big[g^2_{s,N}(x) + Q_{s-t^0_{s,N}} g_{s_1,N_1}(x)\Big] \\ &= T^2_{s,t^1_{s,N},N} R_i \Big[g^2_{s,N}(x) - \Big(Q_{s-t^0_{s,N}-1} g_{s_1,N_1}(x) + Q_{s-t^0_{s,N}-2} g_{s_1,N_1}(x)\Big) \\ &\quad + \Big(Q_{s-t^0_{s,N}-1} g_{s_1,N_1}(x) + Q_{s-t^0_{s,N}-2} g_{s_1,N_1}(x)\Big) + Q_{s-t^0_{s,N}} g_{s_1,N_1}(x)\Big] \\ &= R_i \Big[g^2_{s,N}(x) - \Big(Q_{s-t^0_{s,N}-1} g_{s_1,N_1}(x) + Q_{s-t^0_{s,N}-2} g_{s_1,N_1}(x)\Big)\Big] \\ &\quad + T^2_{s,t^1_{s,N},N} R_i \Big[Q_{s-t^0_{s,N}-1} g_{s_1,N_1}(x) + Q_{s-t^0_{s,N}-2} g_{s_1,N_1}(x)\Big] \\ &\quad + T^2_{s,t^1_{s,N},N} R_i Q_{s-t^0_{s,N}} g_{s_1,N_1}(x). \end{aligned}$$

Hence, by (6.7), for $i = 1, \ldots, n$,

$$II^i =: R_i \Big[g^2_{s,N} - \Big(Q_{s-t^0_{s,N}-1} g_{s_1,N_1} + Q_{s-t^0_{s,N}-2} g_{s_1,N_1}\Big)\Big] + T^2_{s,t^1_{s,N},N} R_i Q_{s-t^0_{s,N}} g_{s_1,N_1} \in L^1(\mathbb{R}^n).$$

By (5.2), there exists $\{\tau^{\epsilon,i}_{j,k}\}_{(\epsilon,j,k)\in\Lambda_n}$ such that

$$\begin{aligned} I^i(x) &=: R_i \Big[g^2_{s,N}(x) - \Big(Q_{s-t^0_{s,N}-1} g_{s_1,N_1}(x) + Q_{s-t^0_{s,N}-2} g_{s_1,N_1}(x)\Big)\Big] \\ &= \sum_{(\epsilon,j,k)\in\Lambda_n, j\le s-t^0_{s,N}-2} \tau^{\epsilon,i}_{j,k} \Phi^\epsilon_{j,k}(x) \end{aligned}$$

and

$$R_i Q_{s-t^0_{s,N}} g_{s_1,N_1}(x) = \sum_{(\epsilon,j,k)\in\Lambda_n, s-t^0_{s,N}-1\le j\le s-t^0_{s,N}+1} \tau^{\epsilon,i}_{j,k} \Phi^\epsilon_{j,k}(x).$$

For arbitrary L^∞ function

$$h(x) = \sum_{(\epsilon,j,k)\in\Lambda_n} h^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x)$$

and $j_0 \in \mathbb{Z}$, denote the operator

$$P_{j_0} h(x) = \sum_{k\in\mathbb{Z}^n} \langle h(x), \Phi^0_{j_0,k}(x)\rangle \Phi^0_{j_0,k}(x).$$

We can see that $P_{j_0} h \in L^\infty(\mathbb{R}^n)$. In fact, by the fact

$$|\langle h, \ \Phi^0_{j_0,k}\rangle| \le C 2^{-nj_0/2},$$

we can get

$$\begin{aligned} |P_{j_0} h(x)| &\le C \sum_{k\in\mathbb{Z}^n} 2^{-nj_0/2} |\Phi^0_{j_0,k}(x)| \\ &\le C \sum_{k\in\mathbb{Z}^n} |\Phi^0(2^{j_0}x - k)| \\ &\le C. \end{aligned}$$

Let

$$h_0(x) =: P_{s-t^0_{s,N}-2}h(x) = \sum_{(\epsilon,j,k)\in\Lambda_n, j\leq s-t^0_{s,N}-2} h^\epsilon_{j,k}\,\Phi^\epsilon_{j,k}(x).$$

Hence $h_0 \in L^\infty(\mathbb{R}^n)$. Further, because $II^i \in L^1(\mathbb{R}^n)$,

$$\begin{aligned} |\langle I^i, h\rangle| &= |\langle I^i, h_0\rangle| \\ &= |\langle II^i, h_0\rangle| \\ &\leq \|II^i\|_{L^1}\|h_0\|_\infty. \end{aligned}$$

The last estimate implies that for $i = 1, \ldots, n$, the functions

$$I^i = R_i\Big[g^2_{s,N} - \Big(Q_{s-t^0_{s,N}-1}g_{s_1,N_1} + Q_{s-t^0_{s,N}-2}g_{s_1,N_1}\Big)\Big] \in L^1(\mathbb{R}^n).$$

This fact and (6.12) imply that

$$g^2_{s,N} - \Big(Q_{s-t^0_{s,N}-1}g_{s_1,N_1} + Q_{s-t^0_{s,N}-2}g_{s_1,N_1}\Big) \in H^1(\mathbb{R}^n).$$

By (6.7), we get $g^2_{s,N} \in H^1(\mathbb{R}^n)$. Further, we have $g^1_{s,N} \in P^\alpha(\mathbb{R}^n)$. Applying (6.5), we get $g_{s_1,N_1} \in P^\alpha(\mathbb{R}^n)$.

Case 2: $t^0_{s,N} > t^1_{s,N}$. For this case, we decompose g_{s_1,N_1} as

$$g_{s_1,N_1}(x) = g^1_{s,N}(x) + g^2_{s,N}(x) + g^3_{s,N}(x),$$

where

$$g^1_{s,N}(x) = \sum_{j\geq s-t^1_{s,N}} Q_j g_{s_1,N_1}(x),$$
$$g^2_{s,N}(x) = \sum_{s-t^0_{s,N}\leq j<s-t^1_{s,N}} Q_j g_{s_1,N_1}(x)$$

and

$$g^3_{s,N}(x) = \sum_{j<s-t^0_{s,N}} Q_j g_{s_1,N_1}(x).$$

We know that

$$T^2_{s,t^1_{s,N},N}R_i g_{s_1,N_1}(x) = T^2_{s,t^1_{s,N},N}R_i\Big[g^3_{s,N}(x) + g^2_{s,N}(x) + Q_{s-t^1_{s,N}}g_{s_1,N_1}(x)\Big].$$

Then for $h(x) = \sum_{\epsilon,s-N\leq j<s-t^0_{s,N},k} h^\epsilon_{j,k}\,\Phi^\epsilon_{j,k}(x)$ with $\|h\|_{L^\infty} \leq 1$, we know that

$$\langle T^2_{s,t^1_{s,N},N}R_i g_{s_1,N_1},\ h\rangle = \langle R_i g^3_{s,N},\ h\rangle. \tag{6.9}$$

By (6.5), $g^3_{s,N} \in L^1(\mathbb{R}^n)$. This fact implies that

$$Q_{s-t^0_{s,N}-1}g_{s_1,N_1} + Q_{s-t^0_{s,N}-2}g_{s_1,N_1} \in H^1(\mathbb{R}^n). \tag{6.10}$$

Owing to (6.9) and (6.10), for $i = 0, \ldots, n$, we have

$$R_i\Big[g^3_{s,N} - \Big(Q_{s-t^0_{s,N}-1}g_{s_1,N_1} + Q_{s-t^0_{s,N}-2}g_{s_1,N_1}\Big)\Big] \in L^1(\mathbb{R}^n).$$

Hence we obtain

$$g^3_{s,N} - \Big(Q_{s-t^0_{s,N}-1}g_{s_1,N_1} + Q_{s-t^0_{s,N}-2}g_{s_1,N_1}\Big) \in H^1(\mathbb{R}^n).$$

So we have $g^3_{s,N} \in H^1(\mathbb{R}^n)$. Since

$$g^1_{s,N}(x) + g^2_{s,N} \in P^\alpha(\mathbb{R}^n),$$

we have $g_{s_1,N_1} \in P^\alpha(\mathbb{R}^n)$.

Case 3: $t^0_{s,N} < t^1_{s,N}$. We decompose g_{s_1,N_1} into three functions

$$g_{s_1,N_1}(x) = g^1_{s,N}(x) + g^2_{s,N}(x) + g^3_{s,N}(x),$$

where

$$g^1_{s,N}(x) = \sum_{j \ge s - t^0_{s,N}} Q_j g_{s_1,N_1}(x),$$
$$g^2_{s,N}(x) = \sum_{s-t^1_{s,N} \le j < s - t^0_{s,N}} Q_j g_{s_1,N_1}(x)$$

and

$$g^3_{s,N}(x) = \sum_{j < s - t^1_{s,N}} Q_j g_{s_1,N_1}(x).$$

For $i = 1, \ldots, n$, we know that

$$\begin{aligned} T^2_{s,t^1_{s,N},N} R_i g_{s_1,N_1}(x) &= T^2_{s,t^1_{s,N},N} R_i \Big[g^3_{s,N}(x) + Q_{s-t^1_{s,N}} g_{s_1,N_1}(x) \Big] \\ &= R_i \Big[g^3_{s,N}(x) - \Big(Q_{s-t^1_{s,N}-1} g_{s_1,N_1}(x) + Q_{s-t^1_{s,N}-2} g_{s_1,N_1}(x) \Big) \Big] \\ &\quad + T^2_{s,t^1_{s,N},N} R_i \Big[Q_{s-t^1_{s,N}-1} g_{s_1,N_1}(x) + Q_{s-t^1_{s,N}-2} g_{s_1,N_1}(x) \Big] \\ &\quad + T^2_{s,t^1_{s,N},N} R_i Q_{s-t^1_{s,N}} g_{s_1,N_1}(x). \end{aligned}$$

Define the function h_i, $i = 1, 2, 3, 4$, as

$$h_1(x) = \sum_{\epsilon, s-N \le j < s - t^1_{s,N} - 2, k} h^{\epsilon,1}_{j,k} \Phi^\epsilon_{j,k}(x),$$
$$h_2(x) = \sum_{\epsilon, j = s - t^1_{s,N} - 2, k} h^{\epsilon,2}_{j,k} \Phi^\epsilon_{j,k}(x),$$
$$h_3(x) = \sum_{\epsilon, j = s - t^1_{s,N} - 1, k} h^{\epsilon,3}_{j,k} \Phi^\epsilon_{j,k}(x),$$
$$h_4(x) = \sum_{\epsilon, j = s - t^1_{s,N}, k} h^{\epsilon,4}_{j,k} \Phi^\epsilon_{j,k}(x),$$

where the sequences $\{h^{\epsilon,i}_{j,k}\}$, $i = 1, 2, 3, 4$, are four arbitrary sequences satisfying $\|h_i\|_{L^\infty} \le 1$. We consider

$$\int T^2_{s,t^1_{s,N},N} R_i g_{s_1,N_1}(x) h_i(x) dx.$$

By (6.6) and the definition of $t^1_{s,N}$, we have

$$g^3_{s,N}(x) + g^2_{s,N} \in L^1(\mathbb{R}^n). \tag{6.11}$$

Hence

$$g^2_{s,N} - \Big(Q_{s-t^1_{s,N}-1} g_{s_1,N_1} + Q_{s-t^1_{s,N}-2} g_{s_1,N_1} \Big) \in L^1 \tag{6.12}$$

and

$$Q_{s-t^1_{s,N}-i}g_{s_1,N_1} \in H^1(\mathbb{R}^n), \quad i = 0, 1, 2. \tag{6.13}$$

Similar to Case 1, by (5.2), the fact that

$$T^2_{s,t^1_{s,N},N} R_i g_{s_1,N_1} \in L^1(\mathbb{R}^n), \quad i = 1, \dots, n,$$

implies that for $i = 1, \dots, n$,

$$R_i\Big[g^3_{s,N} - \Big(Q_{s-t^1_{s,N}-1}g_{s_1,N_1} + Q_{s-t^1_{s,N}-2}g_{s_1,N_1}\Big)\Big] \in L^1(\mathbb{R}^n).$$

Therefore we have

$$g^3_{s,N} - \Big(Q_{s-t^1_{s,N}-1}g_{s_1,N_1} + Q_{s-t^1_{s,N}-2}g_{s_1,N_1}\Big) \in H^1(\mathbb{R}^n).$$

Hence $g^3_{s,N} \in H^1(\mathbb{R}^n)$.

For $i = 1, \dots, n$, we have

$$T^1_{s,t^1_{s,N},N} R_i g_{s_1,N_1}(x) = T^1_{s,t^1_{s,N},N} R_i\Big[g^1_{s,N}(x) + g^2_{s,N}(x) + Q_{s-t^1_{s,N}-1}g_{s_1,N_1}(x)\Big].$$

So the conditions

$$T^1_{s,t^1_{s,N},N} R_i\Big[g^1_{s,N} + g^2_{s,N} + Q_{s-t^1_{s,N}-1}g_{s_1,N_1}\Big] \in P^\alpha(\mathbb{R}^n), \quad i = 1, \dots, n$$

and $g^1_{s,N} \in P^\alpha(\mathbb{R}^n)$ imply

$$T^1_{s,t^1_{s,N},N} R_i\Big[g^2_{s,N} + Q_{s-t^1_{s,N}-1}g_{s_1,N_1}\Big] \in P^\alpha(\mathbb{R}^n).$$

For $i = 1, \dots, n$, we have

$$\begin{aligned} T^1_{s,t^1_{s,N},N} R_i\Big[g^2_{s,N}(x) + Q_{s-t^1_{s,N}-1}g_{s_1,N_1}(x)\Big] &= R_i\Big[g^2_{s,N}(x) - Q_{s-t^1_{s,N}-2}g_{s_1,N_1}(x)\Big] \\ &\quad + T^1_{s,t^1_{s,N},N} R_i\Big[Q_{s-t^1_{s,N}-2}g_{s_1,N_1}(x) + Q_{s-t^1_{s,N}-1}g_{s_1,N_1}(x)\Big]. \end{aligned}$$

Applying (6.13), we obtain

$$R_i\Big[g^2_{s,N} - Q_{s-t^1_{s,N}-2}g_{s_1,N_1}\Big] \in P^\alpha(\mathbb{R}^n).$$

Hence $g^2_{s,N} - Q_{s-t^1_{s,N}-2}g_{s_1,N_1} \in P^\alpha(\mathbb{R}^n)$, that is, g_{s_1,N_1} satisfies (6.12) and (6.13). By (6.5) and (6.6), $g^1_{s,N} \in P^\alpha(\mathbb{R}^n)$ and $g^3_{s,N} \in H^1(\mathbb{R}^n)$. Putting together, we complete the proof of (6.3).

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