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On the inversion of Fueter's theorem

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ABSTRACT

The well known Fueter theorem allows to construct quaternionic regular functions or monogenic functions with values in a Clifford algebra defined on open sets of Euclidean space $\mathbb{R}^{n+1}$, starting from a holomorphic function in one complex variable or, more in general, from a slice hyperholomorphic function. Recently, the inversion of this theorem has been obtained for odd values of the dimension n. The present work extends the result to all dimensions n by using the Fourier multiplier method. More precisely, we show that for any axially monogenic function f defined in a suitable open set in $\mathbb{R}^{n+1}$, where n is a positive integer, we can find a slice hyperholomorphic function $\vec{f}$ such that $f = \Delta^{(n-1)/2}\vec{f}$. Both the even and the odd dimensions are treated with the same, viz., the Fourier multiplier, method. For the odd dimensional cases the result obtained by the Fourier multiplier method coincides with the existing result obtained through the pointwise differential method.

1. Introduction

The Fueter theorem is a useful tool to generate Cauchy–Fueter regular functions from holomorphic functions in the upper half complex plane $\mathbb{C}^+$, see [1]. Furthermore, let O be an open subset of $\mathbb{C}^+$, $f(z) = u(s,t)+iv(s,t)$ be a holomorphic function defined on O and $\mathbb{H}$ be the set of all quaternions. Ω_q is an open subset of $\mathbb{H}$ and is induced by O, i.e., $\Omega_q = \{q = q_0 + \underline{q} \in \mathbb{H} \mid (q_0, |\underline{q}|) \in O\}$, where $q_0, \underline{q} := q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ denote the real and the imaginary part of the quaternion q, respectively. In Ω_q, the function

$$F(q_0, \underline{q}) := \Delta_q\left(u(q_0, |\underline{q}|) + \frac{\underline{q}}{|\underline{q}|}v(q_0, |\underline{q}|)\right)$$

is both left and right regular (or monogenic) with respect to the quaternionic Cauchy–Riemann operator

$$D_q = \partial_{q_0} + \mathbf{i}\partial_{q_1} + \mathbf{j}\partial_{q_2} + \mathbf{k}\partial_{q_3},$$

i.e., F satisfies $D_qF = FD_q = 0$, where $\Delta_q = \partial_{q_0}^2 + \partial_{q_1}^2 + \partial_{q_2}^2 + \partial_{q_3}^2$ denotes the Laplacian operator in the four real variables $q_0, \ldots, q_3$.

Qian by means of Fueter's theorem developed a singular integral theory on the quaternionic unit sphere and its Lipschitz perturbations that corresponds to the operator algebra of the spherical Dirac operator, which is identical with the

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Cauchy–Dunford H^∞ functional calculus of the spherical Dirac operator. He also developed a theory of bounded holomorphic Fourier multipliers in the Coifman–Meyer formulation of Fourier transformation on the Lipschitz surfaces [2].

Under the above assumptions on f, in 1957, Fueter's theorem was extended to $\mathbb{R}^{n+1}$ by Sce [3] for odd values of the dimension n. Specifically, taken a function f as above, the Clifford algebra valued function

$$G(x_0, \underline{x}) := \Delta^{\frac{n-1}{2}} \left(u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|) \right), \quad x = x_0 + \underline{x} \in \mathbb{R}^{n+1}, \tag{1.1}$$

is left and right monogenic with respect to the generalized Cauchy–Riemann operator in $\mathbb{R}^{n+1}$

$$D := \partial_{x_0} + \sum_{i=1}^{n} \mathbf{e}_i \partial_{x_i}, \tag{1.2}$$

i.e., $DG = GD = 0$, where

$$\Delta := \sum_{i=0}^{n} \partial_{x_i}^2 \tag{1.3}$$

is the Laplacian in the $n+1$ real variables. Moreover, the function G as in (1.1) turns out to be axially monogenic, namely it is monogenic and it has the form $A(x_0, |\underline{x}|) + (\underline{x}/|\underline{x}|)B(x_0, |\underline{x}|)$ where A and B are real-valued (more in general, for an axially monogenic function A and B may have values in a Clifford algebra).

Qian in 1997 extended Sce's result to $\mathbb{R}^{n+1}$ for all positive integers n. In fact the Fourier multiplier method used by Qian for n even is also valid for n odd, see [4,5] and also [6]. To the author's knowledge, the approach of using Fueter's theorem is, so far, unique, in establishing the singular integral operator algebra theory on the sphere and its Lipschitz perturbations. In contrast, the analogous theories for various contexts of unbounded Lipschitz graphs of one and higher dimensions were established with a considerable variety of methods [7–11].

Fueter theorem can also be understood in terms of representation theory as an intertwining map between some sl(2)-modules, see [12]. In particular, it is related with some properties of Gegenbauer polynomials. These polynomials are important in the representation theory for the spin group Spin(n), within the setting of branching rules and axially monogenic polynomials on $\mathbb{R}^n$. Fueter theorem is also connected with some Appell sequences, see [13].

For further generalizations of Fueter's Theorem beyond Sce and Qian we refer the reader to [14–19].

It is natural to ask whether there exists any converse result: given a monogenic function, is it possible to find its Fueter's primitive? The main goal of this paper is to show that the Fueter mapping is surjective on the set of axially monogenic functions, and it is possible to solve the following inverse problem for all dimensions n:

Problem 1.1. Given an axially monogenic function

$$f(x) = A(x_0, r) + \underline{\omega} B(x_0, r),$$

where $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$, $r := |\underline{x}|$, $\underline{\omega} := \underline{x}/r$, and $A(x_0, r)$, $B(x_0, r)$ are Clifford algebra valued functions, determine a slice hyperholomorphic function $\vec{f}$ (the so-called Fueter's primitive) such that

$$f(x) = \Delta^{\frac{n-1}{2}} \vec{f}(x).$$

We note that for n odd integer this result was previously proved by Colombo et al. in [20,21] in which $\Delta^{(n-1)/2}$ is a pointwise differential operator. In this paper, we give a uniform treatment for all integers n in which $\Delta^{(n-1)/2}$ is defined by the corresponding Fourier multiplier and show that when n is odd our result coincides with the above mentioned result in [21].

The proof is a combination of the method used by Colombo, Sabadini, Sommen in [21], as main strategy, and Qian's Fourier multiplier method as given in [4,5], as technical approach. Although the Fourier multiplier method is applicable only to the Clifford Cauchy kernel type meromorphic functions, through the Clifford Cauchy integral formula for general domains one can obtain Fueter's inversion for a general class of axially monogenic functions.

It is worthwhile mentioning that the method used in [21] can be further generalized to obtain Fueter's inversion for axially monogenic functions of degree k, see [22], and then obtain results in the case of monogenic functions, when the dimension n is odd. Using a completely different approach, it is also possible to relate the class of slice hyperholomorphic functions to the class of monogenic functions by using the Radon transform and its dual, see [23]. This approach does not depend on the dimension n considered.

The plan of the paper is organized as follows. Section 2 contains preliminary results on Clifford algebras, monogenic functions, Fueter's theorem and its inversion. In Section 3 we recall some basic facts on slice hyperholomorphic functions. Section 4 is the core of the paper and is devoted to obtain the inversion of Fueter's theorem.

2. Preliminary results

We start the section by revising some basic facts on (universal) Clifford algebras. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be an orthonormal basis of Euclidean space $\mathbb{R}^n$, satisfying the relations $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij},\ i, j \in \{1, 2, \ldots, n\}$, where δ_{ij} is the Kronecker delta. That is $\mathbf{e}_i^2 = -1$ for $i = 1, 2, \ldots, n$ and $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 0$ for $1 \leq i \neq j \leq n$. The real Clifford algebra $\mathbb{R}_{0,n}$ is the real algebra constructed over these elements. A basis for the $\mathbb{R}_{0,n}$ as a real vector space is given by the elements $\mathbf{e}_S := \mathbf{e}_{j_1}\mathbf{e}_{j_2}\cdots\mathbf{e}_{j_k}$, where $S := \{j_1, j_2, \ldots, j_k\} \subseteq \{1, 2, \ldots, n\}$ with $1 \leq j_1 < j_2 < \cdots < j_k \leq n$; or $S = \emptyset$, and we denote $\mathbf{e}_\emptyset := 1$. Hence the real linear space $\mathbb{R}_{0,n}$ has dimension 2^n. An element $a \in \mathbb{R}_{0,n}$ can be written as

$$a = \sum_S a_S \mathbf{e}_S, \quad a_S \in \mathbb{R}.$$

For each $k \in \{1, 2, \ldots, n\}$, denote the subset of k-vectors of $\mathbb{R}_{0,n}$ by

$$\mathbb{R}_{0,n}^{(k)} := \left\{ a \in \mathbb{R}_{0,n} :\ a = \sum_{|S|=k} a_S \mathbf{e}_S \right\}$$

which means that the set $\mathbb{R}_{0,n}^{(k)}$ is spanned by the products of k different basis vectors of Euclidean space $\mathbb{R}^n$. In particular, when $k = 1$, $\mathbb{R}_{0,n}^{(1)}$ is identified with $\mathbb{R}^n$. And when $k = 0$, $\mathbb{R}_{0,n}^{(0)}$ is actually $\mathbb{R}$ and for this reason a 0-vector is usually called scalar. Thus we have

$$\mathbb{R}_{0,n} = \bigoplus_{k=0}^{n} \mathbb{R}_{0,n}^{(k)}$$

and for an element $a \in \mathbb{R}_{0,n}$ we can also rewrite it by

$$a = \sum_{k=0}^{n} [a]_k$$

where $[a]_k$ is the projection of a on $\mathbb{R}_{0,n}^{(k)}$.

There are three involutions defined on $\mathbb{R}_{0,n}$: the main involution, the reversion and the conjugation. For an element $a \in \mathbb{R}_{0,n}$ the main involution $\sim$: $a \to \tilde{a}$ is given by

$$\tilde{a} = \sum_S a_S \tilde{\mathbf{e}}_S$$

where $\tilde{\mathbf{e}}_S := (-1)^k \mathbf{e}_S$ where the length $|S|$ of S is such that $|S| = k$. For an element $a \in \mathbb{R}_{0,n}$ the reversion $* :\ a \to a^*$ is given by

$$a^* = \sum_S a_S \mathbf{e}_S^*$$

where $\mathbf{e}_S^* := (-1)^{k(k-1)/2}\mathbf{e}_S$ with $\mathbf{e}_S = \mathbf{e}_{j_1}\mathbf{e}_{j_2}\cdots\mathbf{e}_{j_k}$. Finally, the conjugation of a is a combination by the main involution and the reversion of a. In details, for an element $a \in \mathbb{R}_{0,n}$, the conjugation $- :\ a \to \overline{a}$ is given by

$$\overline{a} = (\tilde{a})^* = \sum_S a_S (\tilde{\mathbf{e}}_S)^*.$$

For elements $a, b \in \mathbb{R}_{0,n}$, suppose that $a = \sum_A a_A \mathbf{e}_A$ and $b = \sum_B b_B \mathbf{e}_B$, the product of a and b is given by

$$ab = \sum_A \sum_B a_A b_B \mathbf{e}_A \mathbf{e}_B.$$

Then, it is easy to show that

$$\begin{aligned} \widetilde{ab} &= \tilde{a}\tilde{b}, \\ (ab)^* &= b^* a^*, \\ \overline{ab} &= \overline{b}\,\overline{a}. \end{aligned}$$

Now we can give the definition of norm of $\mathbb{R}_{0,n}$. For elements $a, b \in \mathbb{R}_{0,n}$, a norm of a, denoted by $|a|$, is given by

$$|a| = ([a\overline{a}]_0)^{\frac{1}{2}} = \left(\sum_S |a_S|^2 \right)^{\frac{1}{2}}.$$

The complex Clifford algebra over $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is denoted by $\mathbb{C}_{0,n}$, and it is defined by

$$\mathbb{C}_{0,n} := \mathbb{C} \otimes \mathbb{R}_{0,n} = \mathbb{R}_{0,n} \oplus \mathbf{i}\mathbb{R}_{0,n}$$

where i is the imaginary unit of $\mathbb{C}$. An element $a \in \mathbb{C}_{0,n}$, can be written as

$$a = \sum_S a_S \mathbf{e}_S, \quad a_S \in \mathbb{C}.$$

All the concepts introduced in the case of $\mathbb{R}_{0,n}$ can be reformulated in the case of $\mathbb{C}_{0,n}$. In the case of the conjugation, for an element $a = \sum_S a_S \mathbf{e}_S \in \mathbb{C}_{0,n}$, the conjugate of a is defined as $\overline{a} = \sum_S \overline{a}_S \overline{\mathbf{e}}_S$ with $\overline{\mathrm{i}} = -\mathrm{i}$.

An important subset of the real Clifford algebra $\mathbb{R}_{0,n}$ is $\mathbb{R}_{0,n}^{(0)} \oplus \mathbb{R}_{0,n}^{(1)} = \mathbb{R} \oplus \mathbb{R}^n$, whose elements are called paravectors. Actually, a paravector is the sum of a scalar and a 1-vector, thus $\mathbb{R}\oplus\mathbb{R}^n$ is the real-linear span of $1, \mathbf{e}_1, \ldots, \mathbf{e}_n$ and $x \in \mathbb{R}\oplus\mathbb{R}^n$ can be written as $x := x_0 + \underline{x}$, where $x_0 \in \mathbb{R}$ and $\underline{x} := \sum_{j=1}^n x_j \mathbf{e}_j \in \mathbb{R}^n$. Thus $\mathbb{R} \oplus \mathbb{R}^n$ may be naturally identified with $\mathbb{R}^{n+1}$ by associating all paravector $x = x_0 + \underline{x}$ with the element $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$. In view of this identification, we will write $\mathbb{R}^{n+1}$ instead of $\mathbb{R} \oplus \mathbb{R}^n$. It is immediate from the definition of norm that

$$|x| := ([x\overline{x}]_0)^{1/2} = \sqrt{x_0^2 + x_1^2 + \cdots + x_n^2},$$

for all $x \in \mathbb{R}^{n+1}$, where $\overline{x} := x_0 - \underline{x}$. If $x \in \mathbb{R}^{n+1}\backslash\{0\}$, then the inverse x^{-1} exists and $x^{-1} := \overline{x} \cdot |x|^{-2}$.

We denote by $C^1(\Omega, Cl_{0,n})$ (resp. $C^1(\underline{\Omega}, Cl_{0,n})$) the continuously differentiable functions which are defined on an open set $\Omega \subset \mathbb{R}_1^n$ (resp. $\underline{\Omega} \subset \mathbb{R}^n$) and take values in the Clifford algebra $Cl_{0,n}$ which means either $\mathbb{R}_{0,n}$ or $\mathbb{C}_{0,n}$. For all $f \in C^1(\Omega, Cl_{0,n})$ it has the form

$$f = \sum_S f_S \mathbf{e}_S$$

where the functions f_S are $\mathbb{R}$-valued or $\mathbb{C}$-valued. Let k be a nonnegative integers, we denote ∂_k be the derivative for the kth variables, i.e, $\partial_k := \partial_{x_k}$ for x_k be the kth variables of $x \in \mathbb{R}^{n+1}$. The Dirac operator is denoted by

$$D_{\underline{x}} := \partial_{x_1}\mathbf{e}_1 + \partial_{x_2}\mathbf{e}_2 + \cdots + \partial_{x_n}\mathbf{e}_n, \quad \underline{x} \in \underline{\Omega}.$$

For all $x \in \Omega$, the generalized Cauchy–Riemann operator is denoted by

$$D := \partial_{x_0} + D_{\underline{x}}.$$

We also need to introduce the Laplacian in the $n + 1$ real variables $x_0, x_1, \ldots, x_n$

$$\Delta := \partial_{x_0}^2 + \partial_{x_1}^2 + \cdots + \partial_{x_n}^2.$$

Monogenic functions are a crucial object in Clifford analysis.

Definition 2.1 (*Monogenic Function*)**.** Let $f(x) \in C^1(\Omega, Cl_{0,n})$ (resp. $f(\underline{x}) \in C^1(\underline{\Omega}, Cl_{0,n})$). Then $f(x)$ (resp. $f(\underline{x})$) is called a (left) monogenic function if and only if

$$Df(x) = 0 \text{ (resp. } D_{\underline{x}}f(\underline{x}) = 0).$$

Axially monogenic functions are special case of monogenic functions.

Definition 2.2 (*Axially Monogenic Function*)**.** A function $f(x) \in C^1(\Omega, Cl_{0,n})$ is said to be axially monogenic if it is monogenic and has the form

$$f(x) = A(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} B(x_0, |\underline{x}|)$$

where A, B are $Cl_{0,n}$-valued functions. Let $\Omega \subset \mathbb{R}^{n+1}$, we denote by $\mathcal{AM}(\Omega)$ the set of all axially monogenic functions in Ω.

By the definition of monogenic functions the axially monogenic functions satisfy the Vekua's system.

Theorem 2.3 (*see [21, Theorem 2.10]*)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$. If $\tilde{f}(x) = A(x_0, r) + \underline{\omega}B(x_0, r) \in \mathcal{AM}(\Omega)$, then the pair of the functions $A(x_0, r)$ and $B(x_0, r)$ forms a solution of the Vekua's system, i.e.*

$$\begin{cases} \partial_{x_0}A(x_0, r) - \partial_r B(x_0, r) = \dfrac{n-1}{r} B(x_0, r), \\ \partial_{x_0}B(x_0, r) + \partial_r A(x_0, r) = 0 \end{cases}$$

and vice versa.

Fueter's theorem is a useful tool in Clifford analysis and it can be seen as a bridge from complex analysis of one complex variable to quaternionic analysis [1]. Then, Fueter's theorem was then extended to $\mathbb{R}^{n+1}$ separately by Sce [3] for odd values of n and by Qian [4] for even values of n. In details, let $\mathbb{C}$ be the complex plane and $\mathbb{C}^+$ be the upper half complex plane, i.e.,

$$\mathbb{C}^+ := \{z \in \mathbb{C} \mid z = x_0 + \mathrm{i}y_0, y_0 > 0\}.$$

Let O be a non-empty open set in $\mathbb{C}^+$ and let

$$f_0(z) = u(x_0, y_0) + \mathrm{i}v(x_0, y_0)$$

be a holomorphic function in O. Then the set O induces an open set in $\mathbb{R}^{n+1}$ defined as

$$\Omega := \{x = x_0 + \underline{x} \in \mathbb{R}^{n+1} \mid (x_0, |\underline{x}|) \in O\}.$$

We will say that Ω is the set induced by O and for $x \in \Omega$, we can define the so-called *induced function*

$$\vec{f}_0(x) := u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|).$$

By the mentioned generalizations in [3] and [4] of Fueter's theorem, the function

$$\Delta^{\frac{n-1}{2}} \vec{f}_0, \quad x \in \Omega$$

is axially monogenic.

In the paper, $\mathbb{N}$ denotes the set of positive integers. Let $n \in \mathbb{N}$ and τ be the mapping defined by

$$\tau(f_0) = \Delta^{\frac{n-1}{2}} \vec{f}_0,$$

where $\vec{f}_0$ is the function induced by f_0 and $\Delta^{(n-1)/2}$ is in the distributional sense of which the precise meaning will be given below in the same section; or, alternatively, the reader may refer to page 117 in [24]. We note that $\tau(f_0)$ is an axially monogenic function, while the function $\vec{f}_0$ belongs to the class of slice hyperholomorphic functions which will be rigorously defined in the next section.

A natural question is whether the mapping τ is surjective on the set of axially monogenic functions. In other words we ask:

Question 1. Given an axially monogenic function f, establish whether there exists a complex holomorphic function f_0 such that $f = \tau(f_0)$.

Closely related, the main question we address in this paper is Problem 1.1, i.e., to establish the existence of a *Fueter's primitive* of an axially monogenic function. As we explained in the introduction section, Colombo, Sabadini and Sommen give, in [21], a positive answer for the odd dimensions n. The full answer to this question is given by the study in the present paper. We show that for a given axially monogenic function f, there exists a slice hyperholomorphic function $\vec{f}$ such that $f = \Delta^{(n-1)/2}\vec{f}$, where n can be all positive integers. The existence of such function $\vec{f}$ establishes *the inversion of Fueter's theorem* in general, and the function $\vec{f}$ will be called a Fueter's primitive of f.

When $n \in \mathbb{N}$ is an even integer, $\Delta^{(n-1)/2}$ is not a pointwise differential operator. In order to solve the inversion problem, we need some knowledge on Fourier multiplier operators. Fourier transform and inverse Fourier transform of functions f defined on $\mathbb{R}^{n+1}$ are formally defined by

$$\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^{n+1}} e^{2\pi \mathrm{i}\langle x,\xi\rangle} f(x)\mathrm{d}x$$

and

$$\mathcal{F}^{-1}(f)(x) := \int_{\mathbb{R}^{n+1}} e^{-2\pi \mathrm{i}\langle x,\xi\rangle} f(\xi)\mathrm{d}\xi,$$

respectively. As is well known, the Fourier inversion formula $\mathcal{F}^{-1}\mathcal{F}(f) = f$ holds for f in the Schwarz class. We will use the so-called Fourier multiplier operator induced by g:

$$M_g(f) := \mathcal{F}^{-1}[g\mathcal{F}(f)].$$

In particular, for $n \in \mathbb{N}$ and $x \in \mathbb{R}^{n+1}$, the fractional Laplace operator $(-\Delta)^{(n-1)/2}$ is defined via the Fourier multiplier operator M_g with $g(x) := (2\pi|x|)^{n-1}$.

By using the Fourier multiplier operator the following key relation is established in [4]: for all positive integers k,

$$\Delta^{\frac{n-1}{2}}(x^{-k}) = \frac{(-1)^{k-1}\lambda_n}{(k-1)!} \cdot \left((\partial_0)^{k-1}E\right)(x),$$

where λ_n is the constant given in Lemma 4.5, and E is the Cauchy kernel of monogenic functions (see [25]):

$$E(x) := \frac{\overline{x}}{\omega_n |x|^{n+1}}, \quad x \in \mathbb{R}^{n+1}\setminus\{0\}, \tag{2.1}$$

where $\omega_n := 2\pi^{(n+1)/2}/\Gamma[(n+1)/2]$ is the surface area of the n dimensional unit sphere in $\mathbb{R}^{n+1}$. The expression (2.1) of the Cauchy kernel will be crucial in the proof of Lemma 4.5. The Cauchy kernel bears this name because it is the kernel used in the Cauchy integral formula for monogenic functions, which we recall below (see [25] for more details). This formula will be used in the proof of Theorem 4.15.

Theorem 2.4 (*see [25, Corollary 9.6]*). *Let S be a region of $\mathbb{R}^{n+1}$, $S \subset \Omega$, and ∂S be compact differentiable and oriented. If f is left monogenic in Ω, then*

$$\int_{\partial S} E(y-x)\mathrm{d}\sigma(y) f(y) = \begin{cases} f(x), & x \in S^o \\ 0, & x \in \Omega\setminus S, \end{cases}$$

where S^o denotes the interior of S, and the differential form $\mathrm{d}\sigma(y)$ is given by $\mathrm{d}\sigma(y) := \eta(y)\mathrm{d}S(y)$, $\eta(y)$ is the outer unit normal to ∂S at the point y and $\mathrm{d}S(y)$ the surface measure of ∂S.

3. Slice hyperholomorphic functions

We now recall the definitions and some properties of slice hyperholomorphic (or slice monogenic) functions, see [26–29]. Denote by $\mathbb{S}^{n-1}$ the $n-1$ dimensional unit sphere in $\mathbb{R}^n$, that is

$$\mathbb{S}^{n-1} := \{\underline{x} \in \mathbb{R}^n : |\underline{x}|^2 = 1\}.$$

If $\underline{\omega} \in \mathbb{S}^{n-1}$, it is well-know that $\underline{\omega}^2 = -1$. Let

$$\mathbb{C}_{\underline{\omega}} := \mathbb{R} + \underline{\omega}\mathbb{R} := \{u + \underline{\omega}v : u, v \in \mathbb{R}, \underline{\omega} \in \mathbb{S}^{n-1}\}. \tag{3.1}$$

For some $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$, define

$$\underline{\omega}_x =: \begin{cases} \dfrac{\underline{x}}{|\underline{x}|}, & \text{if } \underline{x} \neq 0 \\ \text{any element of } \mathbb{S}^{n-1}, & \text{if } \underline{x} = 0. \end{cases}$$

Thus, by (3.1) we know that $x \in \mathbb{C}_{\underline{\omega}_x}$ and $x = x_0 + \underline{\omega}_x|\underline{x}|$. We also need the following notation. Given an element $x \in \mathbb{R}^{n+1}$, we denote

$$[x] := \{y \in \mathbb{R}^{n+1} \mid y = \mathrm{Re}(x) + \mathrm{I}|\underline{x}|, \forall\, \mathrm{I} \in \mathbb{S}^{n-1}\},$$

where $\mathrm{Re}(x)$ is the real part of x. The set $[x]$ is a $n-1$ dimensional sphere in $\mathbb{R}^{n+1}$. If $x \in \mathbb{R}$, then $[x] = \{x\}$, and the radius of the sphere is zero. The notation reflects the fact that $[x]$ is an equivalence class, precisely it contains the elements y equivalent to x under the equivalence $y \sim x$ if and only if $y = s^{-1}xs$, for $s \in \mathbb{R}^{n+1}$, $s \neq 0$.

In this section we denote by O an open set in $\mathbb{R} \times \mathbb{R}$, then

$$\Omega := \{x = x_0 + \underline{x} \in \mathbb{R}^{n+1} : (x_0, |\underline{x}|) \in O\}$$

is said to be the set induced by O. It is an open set satisfying the property described in the following definition:

Definition 3.1 (*Axially Symmetric Open Sets*). An open set $\Omega \subset \mathbb{R}^{n+1}$ is said to be axially symmetric if the $(n-1)$-sphere $[u + \underline{\omega}v]$ is contained in Ω whenever $u + \underline{\omega}v \in \Omega$ for some $u, v \in \mathbb{R}$.

We define the set of slice holomorphic functions only on axially symmetric open sets Ω.

Definition 3.2 (*Slice Hyperholomorphic Functions*). Let $O \subset \mathbb{R} \times \mathbb{R}^+$ and let Ω be the axially symmetric open sets induced by O. We say that a function $f : \Omega \to \mathbb{R}_{0,n}$ is slice hyperholomorphic if it is of the form

$$f_{\underline{\omega}}(x) = f(x_0 + \underline{\omega}r) = \alpha(x_0, r) + \underline{\omega}\beta(x_0, r), \tag{3.2}$$

where $r = |\underline{x}|$, $\underline{\omega} \in \mathbb{S}^{n-1}$, α and β are $\mathbb{R}_{0,n}$-valued differentiable functions, $\alpha(x_0, r) = \alpha(x_0, -r)$, $\beta(x_0, r) = -\beta(x_0, -r)$ for all $(x_0, r) \in O$ and satisfy the Cauchy–Riemann system

$$\begin{cases} \partial_{x_0}\alpha - \partial_r\beta = 0, \\ \partial_r\alpha + \partial_{x_0}\beta = 0. \end{cases}$$

Note that the hypothesis $\alpha(x_0, r) = \alpha(x_0, -r), \beta(x_0, r) = -\beta(x_0, -r)$ is necessary in order to have the function $f(x_0 + \underline{\omega} r)$ well defined, in fact:

$$f(x_0 + \underline{\omega} r) = f(x_0 + (-\underline{\omega})(-r)) = \alpha(x_0, -r) - \underline{\omega}\beta(x_0, -r),$$

if and only if α and β are even and odd in the second variable, respectively.

Functions f slice hyperholomorphic according to Definition 3.2 are such that their restriction $f_{\underline{\omega}}$ to the complex plane $\mathbb{C}_{\underline{\omega}}$ are in the kernel of the Cauchy–Riemann operator $\partial_{x_0} + \underline{\omega}\partial_r$, see [27], for every $\underline{\omega} \in \mathbb{S}^{n-1}$. The converse holds on axially symmetric slice domains. We recall that an open set Ω is said to be a slice domain if $\Omega \cap \mathbb{R}$ is non empty and $\mathbb{C}_{\underline{\omega}} \cap \Omega$ is a domain in $\mathbb{C}_{\underline{\omega}}$ for all $\underline{\omega} \in \mathbb{S}^{n-1}$.

Now we give the definition of Fueter's primitive for axially monogenic functions.

Definition 3.3 (*Fueter's Primitive*)**.** Let $n \in \mathbb{N}$, Ω be an axially symmetric domain in $\mathbb{R}^{n+1}$ and f be an axially monogenic function defined on Ω. Then a slice hyperholomorphic function $\vec{f}$, defined on Ω, is a Fueter's primitive of f on Ω if it satisfies

$$f = \Delta^{\frac{n-1}{2}}\vec{f}.$$

4. The inversion of Fueter's theorem

We start the section by recalling a result which will play an important role in the sequel, namely in the proof of Lemma 4.5. We refer the reader to page 73 in [24] for more details.

Lemma 4.1. *Let $0 < \alpha < n+1$, $k \in \mathbb{N}$ and $P_k(x)$ be a homogeneous harmonic polynomial of degree k. Then*

$$\int_{\mathbb{R}^{n+1}} \frac{P_k(x)}{|x|^{k+n+1-\alpha}} \mathcal{F}(\varphi)(x)\mathrm{d}x = \gamma_{k,\alpha} \int_{\mathbb{R}^{n+1}} \frac{P_k(x)}{|x|^{k+\alpha}} \varphi(x)\mathrm{d}x \tag{4.1}$$

for every φ which is sufficiently rapidly decreasing at infinity and

$$\gamma_{k,\alpha} := \mathrm{i}^k \pi^{(n+1)/2-\alpha}\Gamma(k/2+\alpha/2)/\Gamma(k/2+(n+1)/2-\alpha/2),$$

where i *is the imaginary unit of the complex plane $\mathbb{C}$ and Γ is the Gamma function.*

Remark 4.2. The formula (4.1) implies, in the tempered distribution sense,

$$\mathcal{F}\left[\frac{P_k(x)}{|x|^{k+n+1-\alpha}}\right](\xi) = \gamma_{k,\alpha}\frac{P_k(\xi)}{|\xi|^{k+\alpha}}$$

or

$$\frac{P_k(x)}{|x|^{k+n+1-\alpha}} = \gamma_{k,\alpha}\mathcal{F}^{-1}\left[\frac{P_k(\xi)}{|\xi|^{k+\alpha}}\right](x).$$

Let $n \in \mathbb{N}$, the following lemma states that the partial derivative ∂_{x_0} can commute with the fractional Laplace operator $(-\Delta)^{(n-1)/2}$.

Lemma 4.3. *Let $n \in \mathbb{N}$ and f be in the Schwarz class, then*

$$\partial_{x_0}\left[(-\Delta)^{\frac{n-1}{2}}f(x)\right] = (-\Delta)^{\frac{n-1}{2}}\left[\partial_{x_0}f(x)\right].$$

Proof. In fact, by the definition of $(-\Delta)^{(n-1)/2}$, we have

$$\begin{aligned}
\partial_{x_0}\left[(-\Delta)^{\frac{n-1}{2}}f(x)\right] &= \mathcal{F}^{-1}\left\{\mathcal{F}\left\{\partial_{x_0}\left[(-\Delta)^{\frac{n-1}{2}}f(x)\right]\right\}(\xi)\right\}(x)\\
&= \mathcal{F}^{-1}\left\{(-2\pi \mathrm{i}\xi_0)\,\mathcal{F}\left[(-\Delta)^{\frac{n-1}{2}}f(x)\right](\xi)\right\}(x)\\
&= \mathcal{F}^{-1}\left\{(2\pi|\xi|)^{n-1}(-2\pi \mathrm{i}\xi_0)\,\mathcal{F}\left[f(x)\right](\xi)\right\}(x)\\
&= \mathcal{F}^{-1}\left\{(2\pi|\xi|)^{n-1}\mathcal{F}\left[\partial_{x_0}f(x)\right](\xi)\right\}(x)\\
&= (-\Delta)^{\frac{n-1}{2}}\left[\partial_{x_0}f(x)\right]
\end{aligned}$$

where f belongs to the Schwarz class. □

Remark 4.4. Likewise, the lemma is also true for the generalized Cauchy–Riemann operator $D = \partial_{x_0} + \partial_{\underline{x}}$, i.e.

$$D\left[(-\Delta)^{\frac{n-1}{2}} f(x)\right] = (-\Delta)^{\frac{n-1}{2}} \left[Df(x)\right].$$

Now we prove the following lemma.

Lemma 4.5. *Let $n, k \in \mathbb{N}$. Then x^{-k}, defined on $\mathbb{R}^{n+1}\backslash\{0\}$, is a Fueter's primitive of*

$$\frac{(-1)^{k-1}\lambda_n}{(k-1)!} \cdot \left((\partial_0)^{k-1} E\right)(x), \quad \textit{where } \lambda_n = \frac{(2\pi)^{n-1}\omega_n \gamma_{1,n}}{\mathrm{i}^{(n-1)}\gamma_{1,1}}.$$

Proof. The essence of the proof is contained in [4] or [5], we repeat the computations for completeness and to justify the related constants as well. By the relation

$$x^{-k} = \left(\frac{\bar{x}}{|x|^2}\right)^k = \frac{(-1)^{k-1}}{(k-1)!}(\partial_{x_0})^{k-1}\left(\frac{\bar{x}}{|x|^2}\right)$$

and Lemma 4.3, we have

$$\begin{aligned}(-\Delta)^{\frac{n-1}{2}}(x^{-k}) &= \frac{(-1)^{k-1}}{(k-1)!}(\partial_{x_0})^{k-1}\left[(-\Delta)^{\frac{n-1}{2}}\left(\frac{\bar{x}}{|x|^2}\right)\right] \\ &= \frac{(-1)^{k-1}}{(k-1)!}(\partial_{x_0})^{k-1}\left\{\mathcal{F}^{-1}\left\{\mathcal{F}\left[(-\Delta)^{\frac{n-1}{2}}\left(\frac{\bar{x}}{|x|^2}\right)\right](\xi)\right\}(x)\right\} \\ &= \frac{(-1)^{k-1}}{(k-1)!}(\partial_{x_0})^{k-1}\left\{\mathcal{F}^{-1}\left[(2\pi|\xi|)^{n-1}\mathcal{F}\left(\frac{\bar{x}}{|x|^2}\right)(\xi)\right](x)\right\}.\end{aligned}$$

Applying Lemma 4.1 for $k = 1, \alpha = n$, we get

$$(-\Delta)^{\frac{n-1}{2}}(x^{-k}) = \frac{(-1)^{k-1}}{(k-1)!}(\partial_{x_0})^{k-1}\left[\mathcal{F}^{-1}\left(\gamma_{1,n}(2\pi|\xi|)^{n-1}\frac{\bar{\xi}}{|\xi|^{n+1}}\right)(x)\right].$$

By using Lemma 4.1 for $k = 1, \alpha = 1$, we obtain

$$(-\Delta)^{\frac{n-1}{2}}(x^{-k}) = \frac{(-1)^{k-1}}{(k-1)!}\frac{\gamma_{1,n}}{\gamma_{1,1}}(2\pi)^{n-1}(\partial_{x_0})^{k-1}\left(\frac{\bar{x}}{|x|^{n+1}}\right).$$

Thus we have

$$\Delta^{\frac{n-1}{2}}(x^{-k}) = \frac{(-1)^{k-1}\lambda_n}{(k-1)!} \cdot \left((\partial_0)^{k-1}E\right)(x),$$

where $\lambda_n := (2\pi)^{n-1}\omega_n\gamma_{1,n}/(\mathrm{i}^{(n-1)}\gamma_{1,1})$. In addition, it is immediate that x^{-k} is slice hyperholomorphic. Thus, the statement follows. □

Remark 4.6. It should be noted that the $\mathbb{R}_{0,n}$-valued function x^{-k} is a Fueter's primitive of the $\mathbb{C}_{0,n}$-valued function $[(-1)^{k-1}\lambda_n/(k-1)!] \cdot \left((\partial_0)^{k-1}E\right)(x)$ for n even. While for n odd this does not happen, and, instead, both the axial monogenic function and its Fueter's primitive take function values in the real Clifford algebra.

Let n and k be positive integers, then Lemma 4.5 yields

$$\Delta^{\frac{n-1}{2}}(x^{-k}) = \frac{(-1)^{k-1}\lambda_n}{(k-1)!} \cdot \left((\partial_0)^{k-1}E\right)(x). \tag{4.2}$$

Thus the formula (4.2) can be seen as the definition of $\Delta^{(n-1)/2}(x^{-k})$ for all $k, n \in \mathbb{N}$. However, for $k \in \mathbb{N}$, x^k is not in the Schwarz class, so the method in Lemma 4.5 cannot be used to give the definition of $\Delta^{(n-1)/2}\left(x^k\right)$. We note that when n is odd, and for positive powers k the function $\Delta^{(n-1)/2}(x^k)$ may be defined through pointwise differentiation. Thus we need to define $\Delta^{(n-1)/2}(x^k)$ when n is an even positive integer. Inspired by the method in [4], we will give the definition of $\Delta^{(n-1)/2}(x^k)$ for all $k, n \in \mathbb{N}$. And when n is odd the defined functions coincide with those obtained through pointwise differentiation.

For a given function f, its Kelvin inversion is denoted by $I(f)(\cdot) := E(\cdot)f((\cdot)^{-1})$. Let $k \in \mathbb{N}$, and let

$$P^{(-k)}(\cdot) := \Delta^{\frac{n-1}{2}}((\cdot)^{-k}), \qquad P^{(k-1)} := I(P^{(-k)}).$$

Definition 4.7. Let $n, k \in \mathbb{N}$. For all $x \in \mathbb{R}^{n+1}$ and $k \geq n-1$, we define

$$\Delta^{\frac{n-1}{2}}(x^k) := P^{(k+1-n)}(x).$$

Remark 4.8. The above definition is natural. For n odd, the right hand side is equal to the left hand side under the pointwise differentiation $\Delta^{(n-1)/2}$ (see [4]). For n even, it may be shown from Lemma 4.1 that $\Delta^{(n-1)/2}(x^k) = P^{(k+1-n)}(x)$ indeed in the distributional sense. We note that the rule $P^{(k+1-n)} = I(P^{(-(k+2-n))})$ together with $-(k+2-n) \leq -1$ gives the restriction $k \geq n-1$. In the sequel we will use only values of k in such a range. The restriction of the range $k \geq n-1$ in the definition of the mapping from x^k to $P^{(k+1-n)}(x)$ reflects the fact that in $\mathbb{R}^{n+1}$ there are no monogenic functions with homogeneity degrees $-1, -2, \ldots, -(n-1)$ (see [30]). We will use this definition of $\Delta^{(n-1)/2}(x^k)$, $k \geq n-1$, as a crucial tool in the proof of Lemma 4.12.

The functions $\mathcal{K}_n^{\pm}$ defined below will play an important role in the proof of Theorem 4.15.

Definition 4.9. Let $n \in \mathbb{N}$ and $E(x)$ the Cauchy kernel defined in (2.1). For all $x \in \mathbb{R}^{n+1} \backslash \mathbb{S}^{n-1}$, we define the kernels

$$\mathcal{K}_n^{+}(x) := \int_{\mathbb{S}^{n-1}} E(x-\underline{\omega})\, \mathrm{d}S(\underline{\omega})$$

and

$$\mathcal{K}_n^{-}(x) := \int_{\mathbb{S}^{n-1}} E(x-\underline{\omega})\, \underline{\omega}\, \mathrm{d}S(\underline{\omega})$$

where $\mathrm{d}S(\underline{\omega})$ is the surface measure on $\mathbb{S}^{n-1}$.

Remark 4.10. From Theorem 3.6 in [21], we know that $\mathcal{K}_n^{+}(x)$ and $\mathcal{K}_n^{-}(x)$ are axially monogenic functions. Furthermore, for all $x_0 \in \mathbb{R}$ we have

$$\lim_{|\underline{x}|\to 0} \mathcal{K}_n^{+}(x) = C_n \frac{x_0}{(x_0^2+1)^{(n+1)/2}},$$
$$\lim_{|\underline{x}|\to 0} \mathcal{K}_n^{-}(x) = -C_n \frac{1}{(x_0^2+1)^{(n+1)/2}},$$

where

$$C_n := \frac{\Gamma[(n+1)/2]}{\sqrt{\pi}\,\Gamma(n/2)}.$$

Next lemma is crucial to prove Lemma 4.12. Before introducing it, we need some notations. Let $k \in \mathbb{N} \cup \{0\}$. A monogenic homogeneous polynomial P_k of degree k in $\mathbb{R}^n$ (we recall that $\mathbb{R}^n$ is identified with the set of 1-vectors $\underline{x}$ via the map $(x_1, \ldots, x_n) \mapsto x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$) is called a solid inner spherical monogenic of degree k. We denote by M_k the set of solid inner spherical monogenics of degree k.

Lemma 4.11 *(See [31, Theorem 2.1]).* *Let $P_k(\underline{x}) \in M_k$ be fixed and $W_0(x_0)$ an analytic function in $\widetilde{\Omega} \subset \mathbb{R}$. Then there exists a unique sequence $\{W_s(x_0)\}_{s>0}$ of analytic functions such that the series*

$$f(x_0, \underline{x}) = \sum_{s=0}^{\infty} \underline{x}^s W_s(x_0) P_k(\underline{x})$$

is convergent in an open set U in $\mathbb{R}^{n+1}$ containing the set $\widetilde{\Omega}$, and its sum f is monogenic in U. The function $W_0(x_0)$ is determined by the relation

$$P_k(\underline{\omega}) W_0(x_0) = \lim_{|\underline{x}|\to 0} \frac{1}{|\underline{x}|^k} f(x_0, \underline{x}), \quad \underline{\omega} = \frac{\underline{x}}{|\underline{x}|} \in \mathbb{S}^{n-1}.$$

The series $f(x_0, \underline{x})$ is the generalized axial CK-extension of the function $W_0(x_0)$.

The following Lemma is devoted to find Fueter's primitives of $\mathcal{K}_n^{+}(x)$ and $\mathcal{K}_n^{-}(x)$.

Lemma 4.12. *For all $x_0 \in \mathbb{R}$, denote*

$$\mathcal{P}_n^{+}(x_0) := \frac{C_n}{\lambda_n'} \cdot D_{x_0}^{-(n-1)} \left\{ \frac{x_0}{(1+x_0^2)^{(n+1)/2}} \right\},$$
$$\mathcal{P}_n^{-}(x_0) := \frac{C_n}{\lambda_n'} \cdot D_{x_0}^{-(n-1)} \left\{ \frac{1}{(1+x_0^2)^{(n+1)/2}} \right\},$$

where $\lambda'_n := \lambda_n/(n-1)!$ *when* $x_0 < 1$, $\lambda'_n := (-1)^{n-1}\lambda_n/(n-1)!$ *when* $x_0 > 1$, *and* $D_{x_0}^{-(n-1)}$ *stands for the* $(n-1)$*-fold antiderivative operation with respect to variable* x_0. *Let* $\mathcal{P}_n^+(x)$ *and* $\mathcal{P}_n^-(x)$ *be the functions obtained by replacing* x_0 *by* x *in* $\mathcal{P}_n^+(x_0)$ *and* $\mathcal{P}_n^-(x_0)$, *respectively. Then, for all* $x \neq 0$, $|x| \neq 1$, $\mathcal{P}_n^+(x)$ *is a Fueter's primitive of* $\mathcal{K}_n^+(x)$, *and* $\mathcal{P}_n^-(x)$ *is a Fueter's primitive of* $\mathcal{K}_n^-(x)$.

Before giving a proof of Lemma 4.12 we note that $D_{x_0}^{-(n-1)}$ is the $(n-1)$-fold antiderivative operation with respect to variable x_0. Hence, the representations of $\mathcal{P}_n^{\pm}(x)$ may differ by a term in the kernel of $(\partial_{x_0})^{n-1}$. However, the notation $D_{x_0}^{-(n-1)}$ does not give rise to any ambiguity. This due to the fact that Fueter's primitives are not unique and, in fact, any two Fueter's primitives differ by a function in the kernel of $\Delta^{(n-1)/2}$.

Besides, our proof of Lemma 4.12 splits into two cases, namely $|x_0| < 1$ and $|x_0| > 1$, each of which leads to a value of the constant λ'_n.

Proof. We will only prove that $\mathcal{P}_n^+(x)$ is a Fueter's primitive of $\mathcal{K}_n^+(x)$ for $x \neq 0, |x| \neq 1$, that is, $\mathcal{P}_n^+(x)$ is slice hyperholomorphic, and $\Delta^{(n-1)/2}\mathcal{P}_n^+(x) = \mathcal{K}_n^+(x)$. Following a similar procedure, one can show that $\mathcal{P}_n^-(x)$ is a Fueter's primitive of $\mathcal{K}_n^-(x)$ for $x \neq 0$, $|x| \neq 1$.

We will see that the main part of proof it to prove $\lim_{|\underline{x}|\to 0} \Delta^{(n-1)/2}\mathcal{P}_n^+(x) = \lim_{|\underline{x}|\to 0} \mathcal{K}_n^+(x)$ for $x \neq 0$, $|x| \neq 1$. We separate it into two parts. When $|x_0| < 1$, we have

$$\begin{aligned}
\mathcal{P}_n^+(x_0) &:= \frac{C_n}{\lambda'_n} \cdot D_{x_0}^{-(n-1)}\left\{\frac{x_0}{(1+x_0^2)^{(n+1)/2}}\right\}\\
&= \frac{C_n}{\lambda'_n} \cdot D_{x_0}^{-(n-1)}\left\{x_0 \sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k} x_0^{2k}\right\}\\
&= \frac{C_n}{\lambda'_n} \cdot D_{x_0}^{-(n-1)}\left\{\sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k} x_0^{2k+1}\right\}\\
&= \frac{C_n}{\lambda'_n} \cdot \sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k} \frac{x_0^{2k+n}}{(2k+2)(2k+3)\cdots(2k+n)},
\end{aligned}$$

where we used the fact that the power series $\sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k} x_0^{2k}$ converges absolutely in $|x_0| < 1$. Replacing x_0 by x, we have

$$\mathcal{P}_n^+(x) = \frac{C_n}{\lambda'_n} \cdot \sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k} \frac{x^{2k+n}}{(2k+2)(2k+3)\cdots(2k+n)}. \tag{4.3}$$

For any given $k, n \in \mathbb{N}$, x^{2k+n} is slice hyperholomorphic. Thus, $\mathcal{P}_n^+(x)$ is a slice hyperholomorphic function for $|x| < 1$. Indeed, the power series (4.3) converges absolutely, and uniformly on all compact sets in $|x| < 1$. We now note that we can apply the operator $\Delta^{(n-1)/2}$ to both sides of (4.3). In fact, $\Delta^{(n-1)/2}\left(x^{2k+n}\right)$ is well defined:

$$\Delta^{\frac{n-1}{2}}\mathcal{P}_n^+(x) = \frac{C_n}{\lambda'_n} \cdot \sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k} \frac{\Delta^{\frac{n-1}{2}}\left(x^{2k+n}\right)}{(2k+2)(2k+3)\cdots(2k+n)}. \tag{4.4}$$

Now we calculate $\Delta^{(n-1)/2}\left(x^{2k+n}\right)$. By Definition 4.7 we have

$$\begin{aligned}
\Delta^{\frac{n-1}{2}}\left(x^{2k+n}\right) &= P^{(2k+1)}(x)\\
&= I\left(P^{(-(2k+2))}\right)(x)\\
&= -\frac{\lambda_n}{(2k+1)!}E(x)\cdot\left((\partial_0)^{2k+1}E\right)(x^{-1}).
\end{aligned}$$

Then we have

$$\begin{aligned}
\lim_{|\underline{x}|\to 0}\left[\Delta^{\frac{n-1}{2}}\left(x^{2k+n}\right)\right] &= -\frac{\lambda_n}{(2k+1)!}\lim_{|\underline{x}|\to 0}\left\{E(x)\cdot\left((\partial_0)^{2k+1}E\right)(x^{-1})\right\}\\
&= -\frac{\lambda_n}{(2k+1)!}E(x_0)\cdot\left((\partial_0)^{2k+1}E\right)\left(\frac{x_0}{|x_0|^2}\right)\\
&= -\frac{\lambda_n}{(2k+1)!}E(x_0)\cdot\left(-\frac{(n+2k)!}{(n-1)!}|x_0|^{n+2k+1}\right)\\
&= \frac{\lambda_n(n+2k)!}{(2k+1)!(n-1)!}x_0^{2k+1}
\end{aligned}$$

where we obtain the second equality by the continuously differentiable function E and the third equality by the definition of partial derivative.

Taking the limit $|\underline{x}| \to 0$ on both sides of (4.4), and setting $\lambda_n' := \lambda_n/(n-1)!$, we have

$$\begin{aligned}\lim_{|\underline{x}|\to 0}\left[\Delta^{\frac{n-1}{2}}\mathcal{P}_n^+(x)\right] &= \frac{C_n}{\lambda_n'}\cdot\sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k}\frac{\lim_{|\underline{x}|\to 0}\left[\Delta^{\frac{n-1}{2}}\left(x^{2k+n}\right)\right]}{(2k+2)(2k+3)\cdots(2k+n)}\\ &= \frac{C_n}{\lambda_n'}\cdot\sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k}\frac{\lambda_n(n+2k)!x_0^{2k+1}}{(2k+2)(2k+3)\cdots(2k+n)(2k+1)!(n-1)!}\\ &= C_n\cdot x_0\sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k}x_0^{2k}\\ &= C_n\cdot\frac{x_0}{(1+x_0^2)^{(n+1)/2}}\\ &= \lim_{|\underline{x}|\to 0}\mathcal{K}_n^+(x).\end{aligned}$$

When $|x_0| > 1$, we reason in an analogous way and we have

$$\begin{aligned}\mathcal{P}_n^+(x_0) &:= \frac{C_n}{\lambda_n'}\cdot D_{x_0}^{-(n-1)}\left\{\frac{x_0}{(1+x_0^2)^{(n+1)/2}}\right\}\\ &= \frac{C_n}{\lambda_n'}\cdot D_{x_0}^{-(n-1)}\left\{x_0|x_0|^{-(n+1)}\sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k}x_0^{-2k}\right\}.\end{aligned}$$

If $x_0 > 1$, we have

$$\begin{aligned}\mathcal{P}_n^+(x_0) &= \frac{C_n}{\lambda_n'}\cdot D_{x_0}^{-(n-1)}\left\{x_0|x_0|^{-(n+1)}\sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k}x_0^{-2k}\right\}\\ &= \frac{C_n}{\lambda_n'}\cdot D_{x_0}^{-(n-1)}\left\{\sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k}x_0^{-(2k+n)}\right\}\\ &= \frac{C_n}{\lambda_n'}\cdot\sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k}\frac{(-1)^{n-1}x_0^{-(2k+1)}}{(2k+n-1)(2k+n-2)\cdots(2k+1)}.\end{aligned}$$

Replacing x_0 by x, we have

$$\mathcal{P}_n^+(x) = \frac{C_n}{\lambda_n'}\cdot\sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k}\frac{(-1)^{n-1}x^{-(2k+1)}}{(2k+n-1)(2k+n-2)\cdots(2k+1)}.$$

For any given $k \in \mathbb{N}$, $x^{-(2k+1)}$ is slice hyperholomorphic for $|x| > 1$. And the power series expressing $\mathcal{P}_n^+(x)$ converges absolutely, and uniformly on all compact sets in $|x| > 1$. Thus $\mathcal{P}_n^+(x)$ is a slice hyperholomorphic function for $|x| > 1$.

Again reasoning as in the case $|x_0| < 1$, we obtain

$$\Delta^{\frac{n-1}{2}}\mathcal{P}_n^+(x) = \frac{C_n}{\lambda_n'}\cdot\sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k}\frac{(-1)^{n-1}\Delta^{\frac{n-1}{2}}\left[x^{-(2k+1)}\right]}{(2k+n-1)(2k+n-2)\cdots(2k+1)}.$$

Now we compute $\Delta^{(n-1)/2}\left[x^{-(2k+1)}\right]$. By Lemma 4.5 we have

$$\Delta^{\frac{n-1}{2}}\left[x^{-(2k+1)}\right] = P^{(-(2k+1))}(x) = \frac{\lambda_n}{(2k)!}\cdot\left((\partial_0)^{2k}E\right)(x). \tag{4.5}$$

Taking the limit $|\underline{x}| \to 0$ to the both side of (4.5) we have

$$\lim_{|\underline{x}|\to 0} \Delta^{\frac{n-1}{2}} \left[x^{-(2k+1)}\right] = \frac{\lambda_n}{(2k)!} \cdot \lim_{|\underline{x}|\to 0} \left((\partial_0)^{2k} E\right)(x) = \frac{\lambda_n}{(2k)!} \cdot \left((\partial_0)^{2k} E\right)(x_0) = \frac{\lambda_n}{(2k)!} \cdot \frac{(n+2k-1)!}{(n-1)!} x_0^{-(2k+n)} = \frac{\lambda_n (n+2k-1)!}{(2k)!(n-1)!} x_0^{-(2k+n)}.$$

By setting $\lambda_n' = (-1)^{n-1}\lambda_n/(n-1)!$, we obtain

$$\begin{aligned}\lim_{|\underline{x}|\to 0} \Delta^{\frac{n-1}{2}} \mathcal{P}_n^+(x) &= \frac{C_n}{\lambda_n'} \cdot \sum_{k=0}^{\infty} \binom{-\frac{n+1}{2}}{k} \frac{(-1)^{n-1} \lim_{|\underline{x}|\to 0} \left[\Delta^{\frac{n-1}{2}} x^{-(2k+1)}\right]}{(2k+n-1)(2k+n-2)\cdots(2k+1)} \\ &= \frac{C_n}{\lambda_n'} \cdot \sum_{k=0}^{\infty} \binom{-\frac{n+1}{2}}{k} \frac{(-1)^{n-1}\lambda_n (n+2k-1)! x_0^{-(2k+n)}}{(2k+n-1)(2k+n-2)\cdots(2k+1)(2k)!(n-1)!} \\ &= C_n \cdot x_0^{-n} \sum_{k=0}^{\infty} \binom{-\frac{n+1}{2}}{k} x_0^{-2k} \\ &= C_n \cdot \frac{x_0}{(1+x_0^2)^{(n+1)/2}} \\ &= \lim_{|\underline{x}|\to 0} \mathcal{K}_n^+(x).\end{aligned}$$

If $x_0 < -1$, we have

$$\begin{aligned}\mathcal{P}_n^+(x_0) &= \frac{C_n}{\lambda_n'} \cdot D_{x_0}^{-(n-1)} \left\{ x_0 |x_0|^{-(n+1)} \sum_{k=0}^{\infty} \binom{-\frac{n+1}{2}}{k} x_0^{-2k} \right\} \\ &= \frac{C_n}{\lambda_n'} \cdot D_{x_0}^{-(n-1)} \left\{ (-1)^{n+1} \sum_{k=0}^{\infty} \binom{-\frac{n+1}{2}}{k} x_0^{-(2k+n)} \right\} \\ &= \frac{C_n}{\lambda_n'} \cdot \sum_{k=0}^{\infty} \binom{-\frac{n+1}{2}}{k} \frac{x_0^{-(2k+1)}}{(2k+n-1)(2k+n-2)\cdots(2k+1)}.\end{aligned}$$

By the same way in the case $x_0 > 1$, we obtain

$$\Delta^{\frac{n-1}{2}} \mathcal{P}_n^+(x) = \frac{C_n}{\lambda_n'} \cdot \sum_{k=0}^{\infty} \binom{-\frac{n+1}{2}}{k} \frac{\Delta^{\frac{n-1}{2}} \left[x^{-(2k+1)}\right]}{(2k+n-1)(2k+n-2)\cdots(2k+1)}$$

where $\Delta^{(n-1)/2}\left[x^{-(2k+1)}\right]$ is already obtained by (4.5).

Taking the limit $|\underline{x}| \to 0$ to the both side of (4.5) we have

$$\begin{aligned}\lim_{|\underline{x}|\to 0} \Delta^{\frac{n-1}{2}} \left[x^{-(2k+1)}\right] &= \frac{\lambda_n}{(2k)!} \cdot \lim_{|\underline{x}|\to 0} \left((\partial_0)^{2k} E\right)(x) \\ &= \frac{\lambda_n}{(2k)!} \cdot \left((\partial_0)^{2k} E\right)(x_0) \\ &= (-1)^{n+1} \frac{\lambda_n}{(2k)!} \cdot \frac{(n+2k-1)!}{(n-1)!} x_0^{-(2k+n)} \\ &= (-1)^{n+1} \frac{\lambda_n (n+2k-1)!}{(2k)!(n-1)!} x_0^{-(2k+n)}.\end{aligned}$$

By setting $\lambda_n' = \lambda_n/(n-1)!$, we obtain

$$\begin{aligned}
\lim_{|\underline{x}|\to 0} \Delta^{\frac{n-1}{2}} \mathcal{P}_n^+(x) &= \frac{C_n}{\lambda_n'} \cdot \sum_{k=0}^{\infty} \binom{-\frac{n+1}{2}}{k} \frac{\lim_{|\underline{x}|\to 0}\left[\Delta^{\frac{n-1}{2}} x^{-(2k+1)}\right]}{(2k+n-1)(2k+n-2)\cdots(2k+1)} \\
&= \frac{C_n}{\lambda_n'} \cdot \sum_{k=0}^{\infty} \binom{-\frac{n+1}{2}}{k} \frac{(-1)^{n+1}\lambda_n (n+2k-1)! x_0^{-(2k+n)}}{(2k+n-1)(2k+n-2)\cdots(2k+1)(2k)!(n-1)!} \\
&= C_n \cdot x_0 (-1)^{n+1} x_0^{-(n+1)} \sum_{k=0}^{\infty} \binom{-\frac{n+1}{2}}{k} x_0^{-2k} \\
&= C_n \cdot x_0 |x_0|^{-(n+1)} \sum_{k=0}^{\infty} \binom{-\frac{n+1}{2}}{k} x_0^{-2k} \\
&= C_n \cdot \frac{x_0}{(1+x_0^2)^{(n+1)/2}} \\
&= \lim_{|\underline{x}|\to 0} \mathcal{K}_n^+(x).
\end{aligned}$$

Theorem 1 in [4] yields that $P^{(-(2k+1))}$ and $P^{(2k+2)}$ are both monogenic. It is easy to see that they are both axially monogenic. Therefore, $\Delta^{(n-1)/2}\mathcal{P}_n^+(x)$ is axially monogenic. Invoking Lemma 4.11 for the homogeneity 0 case, we have $\Delta^{(n-1)/2}\mathcal{P}_n^+(x) = \mathcal{K}_n^+(x)$. Since $\mathcal{P}_n^+(x)$ is also a slice hyperholomorphic function for $x \neq 0$, $|x| \neq 1$, we conclude that it is a Fueter's primitive of $\mathcal{K}_n^+(x)$ and this concludes the proof. □

Remark 4.13. When the dimension n is odd and $k \in \mathbb{N}$, in [4], Qian proved that $P^{(k-1)}(x) = \Delta^{(n-1)/2}(x^{n+k-2})$. Computations similar to those in the proof of Lemma 4.12 then give $\mathcal{P}_n^+(x) = \frac{K_n}{\lambda_n'}\mathcal{W}_n^+(x)$ and $\mathcal{P}_n^-(x) = \frac{K_n}{\lambda_n'}\mathcal{W}_n^-(x)$, where $\mathcal{W}_n^+(x)$ and $\mathcal{W}_n^-(x)$ are the functions obtained in [21] as Fueter's primitive of $\mathcal{K}_n^+(x)$ and $\mathcal{K}_n^-(x)$, where K_n is a constant depending on n. For more details, one can see [21]. Note that when n is odd, λ_n' assumes the same value for $|x| < 1$ and for $|x| > 1$. This coincidence shows that, for n odd, the method based on Fourier multiplier used in Lemma 4.12 gives rise to the same result proved in [21], the latter being based on the pointwise differential operator.

Remark 4.14. We note that the two slice hyperholomorphic extensions $\mathcal{P}_n^{\pm}(x)$ obtained as sums of the corresponding power series may be defined in a set larger than $\mathbb{R}^{n+1} \setminus \{x = 0, |x| = 1\}$. See for example the case $n = 3$, where we have $\mathcal{P}_3^+(x) = \frac{1}{2\pi}\arctan x$, $\mathcal{P}_3^-(x) = \frac{1}{2\pi} x \arctan x$, see [20,21].

We can now state and prove the main theorem of this paper. The proof will follow the lines of the proof of Theorem 4.2 of [21].

Theorem 4.15. *Let $n \in \mathbb{N}$, Ω be an axially symmetric open set in $\mathbb{R}^{n+1}$ and let $f(y) = f(y_0 + \underline{\omega} r) = A(y_0, r) + \underline{\omega} B(y_0, r)$ be a $Cl_{0,n}$-valued axially monogenic function on Ω. Let Γ be the boundary of an open bounded set $\mathcal{V}_{\underline{\omega}} \subset \mathbb{R} + \underline{\omega}\mathbb{R}^+$ and $V := \cup_{\underline{\omega}\in\mathbb{S}^{n-1}} \mathcal{V}_{\underline{\omega}} \subset \Omega$. Moreover, suppose that Γ is a regular curve whose parametric equations in the upper complex plane $\mathbb{C}_{\underline{\omega}}^+ = \{y_0 + \underline{\omega} r,\ y_0, r \in \mathbb{R}\}$ are given by $y_0 = y_0(s)$, $r = r(s)$ and are expressed in terms of the arc-length $s \in [0, L]$, $L > 0$. Then, for all $x \in V$,*

$$\vec{f}(x) := \int_\Gamma \mathcal{P}_n^-\left(\frac{x-y_0}{r}\right) r^{n-2}[\mathrm{d}y_0 A(y_0, r) - \mathrm{d}r B(y_0, r)] - \int_\Gamma \mathcal{P}_n^+\left(\frac{x-y_0}{r}\right) r^{n-2}[\mathrm{d}y_0 B(y_0, r) + \mathrm{d}r A(y_0, r)]$$

is a Fueter's primitive of $f(x)$, where $\mathcal{P}_n^{\pm}(x)$ are defined in Lemma 4.12.

Proof. We need the following notations.

(1) Σ is the manifold defined by

$$\Sigma := \{y_0 + \underline{\omega} r \mid (y_0, r) \in \Gamma, \underline{\omega} \in \mathbb{S}^{n-1}\}.$$

(2) $\mathrm{d}s$ is the infinitesimal arc-length, $\mathrm{d}S(\underline{\omega})$ is the infinitesimal element of surface area on $\mathbb{S}^{n-1}$.

(3) $\mathbf{t} = \frac{\mathrm{d}}{\mathrm{d}s}(y_0 + \underline{\omega} r)$ is the unit tangent vector at a point of $\Gamma \subset \mathbb{C}_{\underline{\omega}}$, while the normal unit vector $\mathbf{n}$ is given by

$$\mathbf{n} = -\underline{\omega}\mathbf{t} = \frac{\mathrm{d}}{\mathrm{d}s}[r(s) - \underline{\omega} y_0(s)].$$

(4) The scalar infinitesimal element of the manifold Σ, expressed in terms of ds and dS, is given by

$$\mathrm{d}\Sigma = r^{n-1}\mathrm{d}s\mathrm{d}S(\underline{\omega}).$$

(5) The oriented infinitesimal element of manifold $\mathrm{d}\sigma(s,\underline{\omega})$ is given by

$$\mathrm{d}\sigma(s,\underline{\omega}) = \mathbf{n}\mathrm{d}\Sigma = \frac{\mathrm{d}}{\mathrm{d}s}[r(s) - \underline{\omega}y_0(s)]r^{n-1}\mathrm{d}s\mathrm{d}S(\underline{\omega})$$

or

$$\mathrm{d}\sigma(s,\underline{\omega}) = [\mathrm{d}r(s) - \underline{\omega}\mathrm{d}y_0(s)]r^{n-1}\mathrm{d}S(\underline{\omega}).$$

Since f is monogenic, for all $x = x_0 + \underline{\nu}\tau \in V$, we can write it by using the Cauchy integral formula in Theorem 2.4:

$$f(x_0 + \underline{\nu}\rho) = \int_\Gamma \int_{\mathbb{S}^{n-1}} E(y_0 + \underline{\omega}r - x_0 - \underline{\nu}\rho)\mathrm{d}\sigma(s,\underline{\omega})f(y_0 + \underline{\omega}r). \tag{4.6}$$

We can rewrite (4.6) splitting into two parts:

$$\begin{aligned} f(x_0 + \underline{\nu}\rho) = & -\int_\Gamma \left[\int_{\mathbb{S}^{n-1}} E(y_0 + \underline{\omega}r - x_0 - \underline{\nu}\rho)\underline{\omega}\mathrm{d}S(\underline{\omega})\right] r^{n-1}[\mathrm{d}y_0 A(y_0,r) - \mathrm{d}rB(y_0,r)] \\ & + \int_\Gamma \left[\int_{\mathbb{S}^{n-1}} E(y_0 + \underline{\omega}r - x_0 - \underline{\nu}\rho)\mathrm{d}S(\underline{\omega})\right] r^{n-1}[\mathrm{d}y_0 B(y_0,r) + \mathrm{d}rA(y_0,r)]. \end{aligned}$$

Since the Cauchy kernel has homogeneity degree $-n$, i.e. $E(tx) = t^{-n}E(x)$ for $t > 0$, we have

$$\begin{aligned} f(x_0 + \underline{\nu}\rho) = & \int_\Gamma \left[\int_{\mathbb{S}^{n-1}} r^{-n}E\left(\frac{x-y_0}{r} - \underline{\omega}\right)\underline{\omega}\mathrm{d}S(\underline{\omega})\right] r^{n-1}[\mathrm{d}y_0 A(y_0,r) - \mathrm{d}rB(y_0,r)] \\ & - \int_\Gamma \left[\int_{\mathbb{S}^{n-1}} r^{-n}E\left(\frac{x-y_0}{r} - \underline{\omega}\right)\mathrm{d}S(\underline{\omega})\right] r^{n-1}[\mathrm{d}y_0 B(y_0,r) + \mathrm{d}rA(y_0,r)]. \end{aligned}$$

Recalling the definitions of $\mathcal{K}_n^{\pm}$, we may rewrite the above relation as

$$f(x) = \int_\Gamma \mathcal{K}_n^-\left(\frac{x-y_0}{r}\right) r^{-1}[\mathrm{d}y_0 A(y_0,r) - \mathrm{d}rB(y_0,r)] - \int_\Gamma \mathcal{K}_n^+\left(\frac{x-y_0}{r}\right) r^{-1}[\mathrm{d}y_0 B(y_0,r) + \mathrm{d}rA(y_0,r)].$$

Set $x' := \frac{x-y_0}{r}$ and let $\Delta_{x'}$ be the Laplacian in the variable x'. Due to Lemma 4.12, $\mathcal{P}_n^{\pm}$ are Fueter's primitives of, respectively, $\mathcal{K}_n^{\pm}$. Note that Lemma 4.12 asserts that $\mathcal{P}_n^{\pm}$ may only be defined outside a set consisting of the origin and the sphere $|x| = 1$. This restriction affects the integral below through the fixed x but upon the related integral variable s on the curve Γ. The restriction, in fact, just excludes a set of Lebesgue measure zero on Γ and thus does not actually affect the integral value. We obtain

$$f(x) = \int_\Gamma \Delta_{x'}^{\frac{n-1}{2}}\left[\mathcal{P}_n^-(x')\right] r^{-1}[\mathrm{d}y_0 A(y_0,r) - \mathrm{d}rB(y_0,r)] - \int_\Gamma \Delta_{x'}^{\frac{n-1}{2}}\left[\mathcal{P}_n^+(x')\right] r^{-1}[\mathrm{d}y_0 B(y_0,r) + \mathrm{d}rA(y_0,r)].$$

We now note that the power of the Laplacian and r can be interchanged, i.e.,

$$\Delta_{x'}^{(n-1)/2} = r^{n-1}\Delta^{(n-1)/2}.$$

In fact, from the definition

$$\Sigma := \{y_0 + \underline{\omega}r \mid (y_0,r) \in \Gamma, \underline{\omega} \in \mathbb{S}^{n-1}\}$$

we can see that r is identical with $|\underline{y}|$. It is, therefore, a constant for the Laplacian in x or x'. Hence we have $\Delta_{x'}^{(n-1)/2} = r^{n-1}\Delta_x^{(n-1)/2}$. Besides, since none of the involved integrands have singularities, we can exchange the order of the integration and the differential operation $\Delta^{(n-1)/2}$. As a consequence, the following equation holds:

$$f(x) = \Delta^{\frac{n-1}{2}}\left\{\int_\Gamma \mathcal{P}_n^-\left(\frac{x-y_0}{r}\right) r^{n-2}[\mathrm{d}y_0 A(y_0,r) - \mathrm{d}rB(y_0,r)] - \int_\Gamma \mathcal{P}_n^+\left(\frac{x-y_0}{r}\right) r^{n-2}[\mathrm{d}y_0 B(y_0,r) + \mathrm{d}rA(y_0,r)]\right\}.$$

By setting

$$\vec{f}(x) := \int_\Gamma \mathcal{P}_n^-\left(\frac{x-y_0}{r}\right) r^{n-2}[\mathrm{d}y_0 A(y_0,r) - \mathrm{d}rB(y_0,r)] - \int_\Gamma \mathcal{P}_n^+\left(\frac{x-y_0}{r}\right) r^{n-2}[\mathrm{d}y_0 B(y_0,r) + \mathrm{d}rA(y_0,r)],$$

we have $f = \Delta^{(n-1)/2}\vec{f}$. From Lemma 4.12, we know that $\vec{f}(x)$ is a slice hyperholomorphic function on V. Therefore, $\vec{f}$ is a Fueter's primitive of f. The statement follows. □

Remark 4.16. If we can find a Fueter's primitive $\vec{f}$ of an axially monogenic function f, where $\vec{f}$ is such that the involved functions α and β are real-valued, then it is immediate that $\vec{f}$ is the induced function of the holomorphic function $f_0(x+iy) = \alpha(x, y) + i\beta(x, y)$. This shows the existence of f_0 such that $\tau(f_0) = f$. Hence Question 1 in Section 2 is positively answered.

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