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Backward shift invariant subspaces with applications to band preserving and phase retrieval problems

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The band preserving and phase retrieval problems have long been interested and studied. In this paper, we, for the first time, give solutions to these problems in terms of backward shift invariant subspaces. The backward shift method among other methods seems to be direct and natural. We show that a function $g \in L^p(\mathbb{R})$, $1 \le p \le \infty$, with $fg \in L^2(\mathbb{R})$, that makes the band of fg to be within that of f if and only if g divided by an inner function related to f, belongs to some backward shift invariant subspace in relation to f. By the construction of backward shift invariant space, the solution g is further explicitly represented through the span of the rational function system whose zeros are those of the Laplace transform of f. As an application, we also use the backward shift method to give a characterization for the solutions of the phase retrieval problem. Copyright © 2015 John Wiley & Sons, Ltd.

Keywords: backward shift invariant subspace; band preserving; Laplace transform; phase retrieval

1. Introduction

Instantaneous amplitude and phase are basic concepts in functional analysis, signal processing and physics. A classical way of defining these concepts without ambiguity is through analytic signal [1]. Assume that f is a signal in $L^p(\mathbb{R})$, $1 \le p \le \infty$. The analytic signal associated with f is denoted by f_+, defined as

$$f_+(x) := f(x) + iHf(x),$$

where Hf, the Hilbert transform of f, is defined by

$$H(f)(x) := \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|t-x|>\varepsilon} \frac{f(t)}{x-t}\, dt. \tag{1.1}$$

It can be easily shown that Hf is well-defined by (1.1) if f is in $L^p(\mathbb{R})$, $1 < p < \infty$. Restricted to such function classes, H is a bounded operator and thus extendable to the whole $L^p(\mathbb{R})$. For $p = 1$, the definition of Hilbert transformation is based on its weak-boundedness. In the $p = \infty$ case, Hilbert transformation maps bounded functions to BMO functions [2]. With an analytic signal, there is a unique pair (ρ, θ) such that

$$f_+(x) = f(x) + iHf(x) = \rho(x)e^{i\theta(x)},$$

where $\rho = \sqrt{f^2 + (Hf)^2}$ and $\theta = \arctan[(Hf)/f]$ are respectively called the (analytic) *amplitude* and *phase* of f. Let $H^p(\mathbb{R}) := \{f \in L^p(\mathbb{R}) \mid f = g + iHg\}$ be the class of analytic signals in $L^p(\mathbb{R})$, $1 \le p \le \infty$. It is known that $f \in H^2(\mathbb{R})$ if and only if $f \in L^2(\mathbb{R})$ with $\operatorname{supp} \widehat{f} \subseteq [0, \infty)$, where $\widehat{f}$ is the Fourier transform of f defined by

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx.$$

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In applications, one often deals with signals of finite energy whose Fourier transform has compact supports, viz, the so-called *bandlimited signals*. If $f \in L^2(\mathbb{R})$ is bandlimited with supp $\hat{f} \subset [A, B]$, where $A = \inf \hat{f}$ and $B = \sup \hat{f}$, then, we say that f has band $[A, B]$. The band of f is denoted as Band$\{f\}$. We call $B - A$ the *bandwidth* of f. For the purpose of this paper, we use $\mathcal{FH}^2[A, B] = \{f \in L^2(\mathbb{R}) |$ Band$\{f\} \subset [A, B]\}$ for the set of the bandlimited signals whose bands are contained in $[A, B]$. Two classical problems of long interest in a number of practical areas, including optics, antenna theory and physics, are formulated as follows: The first is to find all functions g such that Band$\{fg\} \subset$ Band$\{f\}$. The second is referred as *phase retrieval problem,* that is, to find all-pass filters $e^{i\theta(x)}$ such that Band$\{fe^{i\theta(\cdot)}\} \subset$ Band$\{f\}$. According to the descriptions of these two problems, the solution of the second problem is closely related to that of the first problem. About the first problem, we learn that, if f and $g \in L^2(\mathbb{R})$ with, respectively, bands $[A, B]$ and $[C, D]$, then fg has band $[A + C, B + D]$ by the well-known Titchmarsh's convolution theorem on compact supported distributions. This shows that if A, B are finite numbers, then g cannot be of finite band. In order to get concrete and structural information of g, an efficient and classical way is to make use of knowledge in complex analysis. The Paley–Winer theorem asserts that if f in $L^2(\mathbb{R})$, then $f \in \mathcal{FH}^2[0, A]$ if and only if f is the restriction to the real line of an entire functions $F(z)$ of the exponential type that belongs to the Hardy space $\mathcal{H}^2(\mathbb{C}^+)$. This allows to use the Hadamard factorization theorem of entire functions. The existing results on band preserving and phase retrieval problems are, therefore, in the form of quotient of two entire functions [3–6]. Our results are in terms of backward shift invariant spaces with explicit representations.

There are two contexts for the theory of shift and backward shift operators, viz, the disc case and half complex plane case [7–11]. For the purpose of this paper, we concentrate in the half-plane case. Denote by S the forward shift operator on $H^p(\mathbb{R})$, defined by

$$Sf(t) = e^{iat} f(t), \qquad \forall \ a > 0.$$

Its conjugate operator S^* is defined in $H^{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$, by

$$S^* f(t) = e^{-iat} f(t).$$

The celebrating Beurling–Lax [10] theorem asserts that a subspace $M \subset H^2$ is a forward shift invariant subspace (i.e. $SM \subset M$) if and only if $M = IH^2$. Owing to this result and the relationship $\langle f, Sg \rangle = \langle S^* f, g \rangle, f, g \in L^2(\mathbb{R})$, we can easily deduce that all backward shift invariant subspaces (i.e. $S^* M \subset M$) are of the form

$$H^2 \bigcap \overline{IH^2},$$

where $I(x) = \lim_{y \to 0^+} I(x + iy)$ is the boundary value of inner function $I(z)$, $I(z) = e^{i(az+b)} B(z) S(z)$ with

$$S(z) = \exp\left\{\frac{i}{\pi} \int_{-\infty}^{\infty} \frac{1 + zt}{t - z}\, d\mu(t)\right\}, \quad B(z) = \prod_{\{\alpha_k\}} \frac{|\alpha_k^2 + 1|}{\alpha_k^2 + 1} \cdot \frac{z - \alpha_k}{z - \overline{\alpha_k}}, \tag{1.2}$$

where b is a real number, a is a nonnegative real number, $B(z)$ is Blaschke product formed by the Blaschke sequence $\{\alpha_k\}$ that satisfy $\sum(\frac{Im(\alpha_k)}{1+|\alpha_k|^2}) < \infty$, $(\frac{|\alpha_k^2+1|}{\alpha_k^2+1} \equiv 1$ is understood whenever $\alpha_k = i)$ and $S(z)$ is a singular inner function determined by its singular measure $\mu(t)$.

In this paper, we will take advantage of backward shift invariant subspaces to study the band preserving and the phase retrieval problems. We show that a function $g \in L^p(\mathbb{R}), 1 \le p \le \infty$, with $fg \in L^2(\mathbb{R})$, that makes Band$\{fg\} \subset$ Band$\{f\}$ if and only if $g_+ = g + iHg$ and $g_- = g - iHg$, or g divided by the inner function of f_+, belong to the backward shift invariant subspace $H^p(\mathbb{R}) \bigcap \overline{IH^p(\mathbb{R})}$, where I is some inner function related to f. These results will be given in Section 2. In Section 3, we deal with construction of the related backward shift invariant spaces. Under the imposed constraint $f \in \mathcal{FH}^2[A, B]$ in the paper, it will be shown that a solution g to the problem is in the closure of the span of the rational system whose zeros are among those of the Laplace transform of f, in, respectively, the upper-half and lower-half complex plane. This allows us to give a complete characterization of the solutions to the band preserving and phase retrieval problems in Section 3.

2. Band preserving in relation to backward shift invariant spaces

Indeed, analytic signals are related to holomorphic functions in the upper-half plane $\mathbb{C}^+ := \{z = x + iy \in \mathbb{C} : y > 0\}$. Let $\mathcal{H}^p(\mathbb{C}^+)$ denotes the Hardy space of all holomorphic functions $F(z)$ on $\mathbb{C}^+$ that satisfy

$$\|F\|_{\mathcal{H}^p(\mathbb{C}^+)} := \begin{cases} \sup\limits_{y>0} \left\{\int_{-\infty}^{+\infty} |F(x + iy)|^p \, dx\right\}^{1/p} < \infty, & \text{for } 1 \le p < \infty; \\ \sup\limits_{z \in \mathbb{C}^+} |F(z)| < \infty, & \text{for } p = \infty. \end{cases}$$

The linear mapping that takes a function $F \in \mathcal{H}^p(\mathbb{C}^+)$ to its boundary function $f(x)$ in $L^p(\mathbb{R})$ is an isometric isomorphism from $\mathcal{H}^p(\mathbb{C}^+)$ onto $H^p(\mathbb{R})$. Hereafter, we denote the boundary value function of $F \in \mathcal{H}^p(\mathbb{C}^+)$ by f. For a function $f \in H^p(\mathbb{R})$, the corresponding function in $\mathcal{H}^p(\mathbb{C}^+)$ is denoted by F. Note that $\mathcal{FH}^2[0, A] \subseteq H^2(\mathbb{R})$. The following lemma shows that the singular inner function of f reduces to the constant function 1 in this special case [3].

Lemma 2.1
Let a non-zero function $f \in \mathcal{FH}^2[0, A]$ and

$$F(z) = \frac{1}{2\pi}\int_0^A \widehat{f}(\omega)e^{j\omega z}d\omega. \tag{2.3}$$

Then, $f(x)$ has a factorization of the form

$$f(x) = O_f(x)I_f^u(x), \tag{2.4}$$

where $O_f(x)$ is the boundary value of the outer function of $F(z)$

$$O_f(x) = \exp\left\{\ln|f(x)| + \frac{i}{\pi}\lim_{\epsilon\to 0+}\int_{|x-t|>\epsilon}\frac{1+tx}{(x-t)(1+t^2)}\ln|f(t)|dt\right\},$$

$I_f^u(x) = e^{i(a_u x + b_u)}B_f^u(x)$ is the boundary value of the inner function of $F(z)$, where a_u is some nonnegative real number in $[0, A]$, b_u is a real number and $B_f^u(x)$ is the boundary value of the Blaschke product formed by the zeros of $F(z)$ in the upper-half plane $\mathbb{C}^+$.

Remark
If $f \in \mathcal{FH}^2[0, A]$ and $0 \in \operatorname{supp}\widehat{f}$, then the boundary inner function of $f(x)$ is $I_f^u(x) = e^{ib_u}B_f^u(x)$.

Lemma 2.2
Assume that $f \neq 0$ and $f \in \mathcal{FH}^2[0, A]$. The following result holds:

(i) If $g \in H^p(\mathbb{R}), 1 \le p \le \infty$ and $fg \in L^2(\mathbb{R})$, then $\operatorname{supp}\widehat{fg} \subseteq [0, \infty)$;
(ii) If $\overline{g} \in H^p(\mathbb{R}), 1 \le p \le \infty$ and $fg \in L^2(\mathbb{R})$, then $\operatorname{supp}\widehat{fg} \subseteq (-\infty, A]$.

Proof

(i) If $p = \infty$, then $fg \in H^2(\mathbb{R})$, and consequently, $\operatorname{supp}\widehat{fg} \subset [0, \infty)$. For $1 \le p < \infty$, there exists $0 < r < \infty$ such that $\frac{1}{2} + \frac{1}{p} = \frac{1}{r}$, or, equivalently, $\frac{1}{2/r} + \frac{1}{p/r} = 1$. It can be easily shown, by Hölder's inequality and definition of the complex Hardy $\mathcal{H}^r(\mathbb{C}^+)$ space, $fg \in H^r(\mathbb{R})$. Because $fg \in L^2(\mathbb{R})$, we have $fg \in H^2(\mathbb{R})$ (Corollary II. 4.3, [12]), and therefore $\operatorname{supp}\widehat{fg} \subset [0, \infty)$.
(ii) Because $f \in \mathcal{FH}^2[0, A]$, we have $e^{iAx}\overline{f(x)} \in \mathcal{FH}^2[0, A]$. Let $h(x) := e^{iAx}\overline{f(x)g(x)}$. The result of (i) shows that $\operatorname{supp}\widehat{h} \subset [0, \infty)$. Because $(\widehat{fg})(\omega) = \widehat{h}(A - \omega)$, we have $\operatorname{supp}\widehat{fg} \subset (-\infty, A]$. The proof is finished. □

Set $\overline{H^p(\mathbb{R})} := \{f | \overline{f} \in H^p(\mathbb{R})\}$. With Lemma 2.1 and Lemma 2.2, we give a characterization of functions g such that $\text{Band}\{fg\} \subset \text{Band}\{f\}$.

Theorem 2.3
Let non-zero functions $f \in \mathcal{FH}^2[0, A]$ and $\overline{g} \in H^p(\mathbb{R}), 1 \le p \le \infty$. Assume that $fg \in L^2(\mathbb{R})$. Then, $fg \in \mathcal{FH}^2[0, A]$ if and only if $\overline{g} \in H^p(\mathbb{R}) \bigcap I_f^u \overline{H^p(\mathbb{R})}$, where $I_f^u(x) = e^{i(a_u x + b_u)}B_f^u(x)$ the boundary inner function of $f(x)$ given in Lemma 2.1.

Proof
Denote $h := fg \in \mathcal{FH}^2[0, A]$. Then, we have $H(z) \in \mathcal{H}^2(\mathbb{C}^+)$, where $H(z)$ is the Laplace transform of h defined by (2.3). The outer function of $H(z)$ can be represented by

$$O_H(z) = \exp\left\{\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{1+zt}{(z-t)(1+t^2)}\ln|(fg)(t)|dt\right\}.$$

Because $f \in H^2(\mathbb{R}), \overline{g} \in H^p(\mathbb{R})$ and $|g| = |\overline{g}|$, by the factorization theorem, we have

$$\begin{aligned} O_{fg}(x) :&= \lim_{y\to 0+} O_H(z) = \lim_{y\to 0+}\exp\left\{\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{1+zt}{(z-t)(1+t^2)}\ln|f(t)|dt\right\}\lim_{y\to 0+}\exp\left\{\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{1+zt}{(z-t)(1+t^2)}\ln|\overline{g(t)}|dt\right\} \\ &= O_f(x)O_{\overline{g}}(x), \end{aligned}$$

for almost all x, where $z = x + iy$. Let I_f^u be the boundary inner function of f. Hence

$$fg = O_{(fg)}I_{(fg)} = O_{\overline{g}}O_f I_{(fg)} = \overline{I_f^u}O_{\overline{g}}O_f I_f^u I_{(fg)} = f\overline{I_f^u}O_{\overline{g}}I_{(fg)},$$

for almost all x. Because $f \not\equiv 0$, the aforementioned equation implies that $g = \overline{I_f^u}O_{\overline{g}}I_{(fg)} \in \overline{I_f^u}H^p(\mathbb{R})$. Thus, $\overline{g} \in H^p(\mathbb{R}) \bigcap I_f^u\overline{H^2(\mathbb{R})}$. By Lemma 2.1, we have $I_f^u(x) := e^{i(a_u x + b_u)}B_f^u(x)$.

Conversely, suppose that $\overline{g} \in H^p(\mathbb{R}) \bigcap I_f^u \overline{H^p(\mathbb{R})}$, then $g \in H^p(\mathbb{R})$, and there exists $h \in H^p(\mathbb{R})$ such that $g = I_f^u\overline{h}$. Because $f \in \mathcal{FH}^2[0, A]$, then $f \in L^\infty(\mathbb{R}) \bigcap H^2(\mathbb{R}) \subseteq H^\infty(\mathbb{R})$. Thus, $gf = h\overline{I_f^u}I_f^u O_f = hO_f \in H^2(\mathbb{R})$ for $fg \in L^2(\mathbb{R}) \cap H^p(\mathbb{R})$. Hence, $\operatorname{supp}\widehat{fg} \subseteq [0, \infty)$. By the assumption and Lemma 2.2, we also have $\operatorname{supp}\widehat{fg} \subseteq (-\infty, A]$. This follows that $fg \in \mathcal{FH}^2[0, A]$. The proof is finished. □

Set $f_l(x) := e^{iAx}\overline{f(x)}$. Then, $f_l \in \mathcal{FH}^2[0,A]$ if and only if $f \in \mathcal{FH}^2[0,A]$. Invoking Theorem 2.3, we have

Corollary 2.4
Let f, g be non-zero functions, $f \in \mathcal{FH}^2[0,A]$, $g \in H^p(\mathbb{R})$, $1 \le p \le \infty$. Assume that $fg \in L^2(\mathbb{R})$. Then, $fg \in \mathcal{FH}^2[0,A]$ if and only if $g \in H^p(\mathbb{R}) \bigcap I_f^l \overline{H^p(\mathbb{R})}$, where $I_f^l := e^{i(a_l x + b_l)} B_f^l(x)$ is the boundary inner function of $f_l(x) := e^{iAx}\overline{f(x)}$, a_l is some nonnegative real number in $[0,A]$, b_l is a real number and $B_f^l(x)$ is the boundary value of the Blaschke product formed by the zeros of $F(z)$ in the lower-half plane $\mathbb{C}^- = \{z | z = x - iy,\ y > 0\}$.

Note that

$$F_l(z) := (\partial^{-1} f_l)(z) = \frac{1}{2\pi}\int_0^A \widehat{f_l}(\omega) e^{i\omega z} d\omega = \frac{1}{2\pi}\int_0^A \overline{\widehat{f}(A-\omega)} e^{i\omega z} = e^{iAz}\overline{F(\bar{z})}.$$

Thus, the zeroes of $F_l(z)$ in the upper-half complex plane are the conjugates of those of $F(z)$ in the lower-half complex plane. We denote by $\{\alpha_k\}$ and $\{\beta_k\}$ the sets of the zeros of $F(z)$ in the upper-half complex plane $\mathbb{C}^+$ and in the lower-half complex plane $\mathbb{C}^-$ (they repeat according to their respective multiples), respectively, where $F(z)$ is given by (2.3). Then, B_f^u in Theorem 2.3 and B_f^l in Corollary 2.4 are respectively given by

$$B_f^u(x) = \prod_{\alpha_k} \frac{|\alpha_k^2 + 1|}{\alpha_k^2 + 1} \cdot \frac{x - \alpha_k}{x - \overline{\alpha_k}}, \qquad B_f^l(x) = \prod_{\beta_k} \frac{|\overline{\beta_k}^2 + 1|}{\overline{\beta_k}^2 + 1} \cdot \frac{x - \overline{\beta_k}}{x - \beta_k}. \tag{2.5}$$

Let $f \in \mathcal{FH}^2[0,A]$ be a non-zero function. By Theorem 2.3 and Corollary 2.4, we learn that if $g \in H^p(\mathbb{R})$ or $\overline{g} \in H^p(\mathbb{R}), 1 \le p \le \infty$, a function g making $fg \in \mathcal{FH}^2[0,A]$ can be completely characterized by a backward shift invariant subspace $H^p(\mathbb{R}) \bigcap I\overline{H^p(\mathbb{R})}$, where I is an inner function related to f. Next, we extend the just obtained results to general functions $g \in L^p(\mathbb{R}), 1 \le p \le \infty$. Because the operator H is bounded on $L^p(\mathbb{R}), 1 < p < \infty$, for any $g \in L^p(\mathbb{R})$, we have the projectional Hardy spaces decomposition

$$g(x) = \frac{1}{2}(g_+(x) + g_-(x)), \tag{2.6}$$

where $g_+ := g + iHg$ and $g_- := g - iHg$ with $g_+, \overline{g_-} \in H^p(\mathbb{R})$. They are, respectively, called *the analytic signal* and *the dual analytic signal* of g. We first have

Lemma 2.5
Let f be non-zero, $f \in \mathcal{FH}^2[0,A]$. There exists a function $g \in L^p(\mathbb{R}), 1 < p < \infty$, such that $fg \in \mathcal{FH}^2[0,A]$ if and only if both the relations $fg_+ \in \mathcal{FH}^2[0,A]$ and $fg_- \in \mathcal{FH}^2[0,A]$ hold.

Proof
Suppose that $fg \in \mathcal{FH}^2[0,A]$, then supp$\widehat{fg} \subseteq [0,A]$. Because $f \in \mathcal{FH}^2[0,A]$, $g_+ \in H^p(\mathbb{R})$ and $g_- \in \overline{H^p(\mathbb{R})}$, by Lemma 2.2, we have

$$\text{supp}(\widehat{fg_+}) \subseteq [0,\infty), \quad \text{supp}(\widehat{fg_-}) \subseteq (-\infty, A].$$

Thus, $(\widehat{fg})(\omega) = (\widehat{fg_-})(\omega) = 0$ for $\omega < 0$ and $(\widehat{fg})(\omega) = (\widehat{fg_+})(\omega)$ for $\omega > A$. These yield that $fg_- \in \mathcal{FH}^2[0,A]$ and $fg_+ \in \mathcal{FH}^2[0,A]$.
Conversely, if $fg_+ \in \mathcal{FH}^2[0,A]$ and $fg_- \in \mathcal{FH}^2[0,A]$, then $fg = fg_+ + fg_- \in \mathcal{FH}^2[0,A]$. The proof is complete. □

In virtue of Theorem 2.3, Corollary 2.4 and Lemma 2.5, we obtain

Theorem 2.6
Let f, g be non-zero, $f \in \mathcal{FH}^2[0,A]$, $g \in L^p(\mathbb{R}), 1 < p < \infty$ and $fg \in L^2(\mathbb{R})$. Then, $fg \in \mathcal{FH}^2[0,A]$ if and only if

$$\overline{g_-} \in H^p(\mathbb{R}) \bigcap e^{ia_u x} B_f^u(x) \overline{H^p(\mathbb{R})},$$

and

$$g_+ \in H^p(\mathbb{R}) \bigcap e^{ia_l x} B_f^l(x) \overline{H^p(\mathbb{R})},$$

where a_u and a_l are two nonnegative real constants in $[0,A]$ and $B_f^u(x)$ and $B_f^l(x)$ are respectively given in (2.5).

Remark
Let $f \in \mathcal{FH}^2[0,A]$. If $0 \in$ supp$\widehat{f}$, then $a_u = 0$. If $A \in$ supp$\widehat{f}$, then $0 \in$ supp$\widehat{f_l}$ and $a_l = 0$.

The aforementioned theorem gives a characterization for the solutions $g \in L^p(\mathbb{R}), 1 < p < \infty$, to the band preserving problem. It, however, does not cover the cases $p = 1$ and $p = \infty$ due to the failure of the projectional Hardy spaces decomposition. The case $p = \infty$ is directly related to the phase retrieval problem. In the succeeding section, we will treat the two exceptional cases as follows.

Theorem 2.7
Let $f \in \mathcal{FH}^2[0,A]$, $g \in L^p(\mathbb{R}), 1 \le p \le \infty$ be non-zero functions and $fg \in L^2(\mathbb{R})$. Then, $fg \in \mathcal{FH}^2[0,A]$ if and only if

$$g \in \overline{I_f^u} H^p(\mathbb{R}) \bigcap I_f^l \overline{H^p(\mathbb{R})} = \overline{I_f^u}\left[H^p(\mathbb{R}) \bigcap I_f^u I_f^l \overline{H^p(\mathbb{R})}\right], \tag{2.7}$$

where $I_f^u := e^{i(a_u x + b_u)} B_f^u(x)$ is the inner function of f and $I_f^l(x) := e^{i(a_l x + b_l)} B_f^l(x)$ is the inner function of $f_1(x) := e^{iAx}\overline{f(x)}$.

Proof
Let $h := fg$. Because $f,\ h \in \mathcal{FH}^2[0,A]$, by Lemma 2.1, we have $f = O_f I_f^u$ and $h = O_h I_h$, where I_f^u is the inner function of f with the form $e^{i(a_u x+b_u)}B_f^u(x)$ and I_h is the inner function of h. From the facts that $g \in L^p(\mathbb{R})$, $\ln|h| = \ln|fg|, \ln|f| \in L^2\left(\frac{dt}{1+t^2}\right)$, we have $O_g := \frac{O_h}{O_f} \in H^p(\mathbb{R})$. Thus

$$g = \frac{h}{f} = \frac{O_g I_h}{I_f^u} \in \overline{I_f^u} H^p(\mathbb{R}).$$

On the other hand, for $h_l(x) := e^{iAx}\overline{h(x)}, f_l(x) := e^{iAx}\overline{f(x)}$, there hold $h_l, f_l \in \mathcal{FH}^2[0,A]$. Because $\ln|h_l| = \ln|h|, \ln|f_l| = \ln|f|$. Then, $f_l = O_f I_f^l$ and $h = O_h I_h^l$, where I_f^l is the inner function of f_l with the form $e^{i(a_l x+b_l)}B_f^l(x)$ and I_h^l is the inner function of h_l. Hence

$$\overline{g} = \frac{\overline{h}}{\overline{f}} = \frac{h_l}{f_l} = \frac{O_h I_h^l}{O_f I_f^l} = \frac{O_g I_h^l}{I_f^l} \in \overline{I_f^l} H^p(\mathbb{R}).$$

By combining with $g \in \overline{I_f^u} H^p(\mathbb{R})$, we have $g \in \overline{I_f^u} H^p(\mathbb{R}) \bigcap I_f^l \overline{H^p(\mathbb{R})} = \overline{I_f^u}\left[H^p(\mathbb{R}) \bigcap I_f^u I_f^l \overline{H^p(\mathbb{R})}\right]$.

Conversely, if $g \in \overline{I_f^u} H^p(\mathbb{R}) \bigcap I_f^l \overline{H^p(\mathbb{R})}$, then there exist $g_1, g_2 \in H^p(\mathbb{R})$ such that $g = \overline{I_f^u} g_1$ and $\overline{g} = \overline{I_f^l} g_2$. Let $f_l(x) := e^{iAx}\overline{f(x)}$. Because $f, f_l \in \mathcal{FH}^2[0,A]$ and $fg \in L^2(\mathbb{R})$, as assumed, we have $fg = O_f I_f^u \overline{I_f^u} g_1 = g_1 O_f \in H^2(\mathbb{R})$ and $e^{iAx}\overline{f(x)g(x)} = O_{f_l} I_f^l \overline{I_f^l} g_2 = O_{f_1} g_2 \in H^2(\mathbb{R})$. Hence, $\operatorname{supp}\widehat{fg} \subseteq [0,A]$ and $fg \in \mathcal{FH}^2[0,A]$. The proof is complete. □

3. Characterization of backward shift invariant subspace and its application to band preserving and phase retrieval problems

From the previous analysis, we learn that the solutions g of the band preserving problem are related to backward shift invariant spaces $H^p(\mathbb{R}) \bigcap I\overline{H^p(\mathbb{R})}$, where I is some inner function. Under the condition $f \in FH^2([0,A])$, the related inner function I is with the simplified form $I(x) = e^{i(ax+b)}B(x), a \geq 0, b \in \mathbb{R}, B$ is a Blaschke product. To know more about the solutions g is to know more about the construction of the backward shift invariant spaces. Many relevant references are in Russian and are for the disc case [8, 9, 13]. Specifically, for the half-plane case, the literature on construction of $H^p(\mathbb{R}) \bigcap e^{i(ax+b)}B(x)\overline{H^p(\mathbb{R})}$ in terms of the system consisting of shifted Cauchy kernels does not seem to be available. In this paper, we provide the proof for such construction on the upper-half plane.

When $a = 0$ and $B(x)$ of the boundary value of the Blaschke product given in (1.2). Let $B_0(x) = 1$.

$$B_n(x) = \prod_{j=1}^{n} \frac{|\alpha_j^2+1|}{\alpha_j^2+1} \cdot \frac{x-\alpha_j}{x-\overline{\alpha_j}}, \quad e_n(x) = \frac{\sqrt{2Im(\alpha_n)}}{x-\overline{\alpha_n}} B_{n-1}(x), \quad n \geq 1.$$

$\{e_1, \ldots, e_n, \ldots\}$ is obtained through the Gram–Schmidt orthogonalization process on $\{B_n\}$, called a Takenaka–Malmquist system. We will be working with the induced conjugate paring $\langle \cdot, \cdot \rangle$ on $H^p(\mathbb{R})$ and $H^{p'}(\mathbb{R})$, namely

$$\langle f(x), g(x) \rangle = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx,$$

where $f \in H^p(\mathbb{R}), g \in H^{p'}(\mathbb{R}), 1/p + 1/p' = 1$. Furthermore, each e_n in the system is in H^p, $1 < p < \infty$, and $\{e_1, \ldots, e_n, \ldots\}$ is orthogonal with respect to the pairing between H^p and $H^{p'}$.

Theorem 3.1
Let $\alpha_1, \ldots, \alpha_n, \ldots$ be a Blaschke sequence of points in the upper-half complex plane that defines a Blaschke product $B(z)$ given in (1.2). Then, for $1 < p < \infty$

$$H^p(\mathbb{R}) \bigcap B(x)\overline{H^p(\mathbb{R})} = (BH^{p'})^{\perp} = \overline{\operatorname{span}}^p\{e_n\}_{n=1}^{\infty}, \tag{3.8}$$

where the closure is in the L^p topology and $(BH^{p'})^{\perp} = \{f \in H^p | \langle f, Bg \rangle = 0, \ \forall \ g \in H^{p'}\}$.

Proof
To prove the first identical relation of (3.8), we first show

$$H^p(\mathbb{R}) \bigcap B\overline{H^p(\mathbb{R})} \subset (BH^{p'})^{\perp}.$$

This is all clear. In fact, for $f = B\overline{h}_p \in H^p(\mathbb{R}), h_p \in H^p(\mathbb{R})$, we have, for any $g = Bh_{p'}, h_{p'} \in H^{p'}(\mathbb{R})$

$$\langle f, g \rangle = \langle B\overline{h}_p, Bh_{p'} \rangle = \langle \overline{h}_p, h_{p'} \rangle = 0.$$

Next, we show

$$H^p(\mathbb{R}) \bigcap B\overline{H^p(\mathbb{R})} \supset (BH^{p'})^{\perp}.$$

Let $f \in (BH^{p'}(\mathbb{R}))^{\perp}$. Then, for any function of the form $Bh_{p'}, h_{p'} \in H^{p'}(\mathbb{R}), 1 < p' < \infty$

$$0 = \langle f, Bh_{p'} \rangle = \langle \overline{B}f, h_{p'} \rangle.$$

From Lemma 4.1 (p.241, [12]), we know $\overline{f}B = h \in H^p(\mathbb{R})$ or $f = B\overline{h}$. This completes the first identical relation of (3.8).

Now, we prove the second identical relation of (3.8). Because $1 < p < \infty$, each e_n is in $(BH^{p'})^{\perp}$. In fact, for any $f = Bh_{p'}$

$$\langle f, e_n \rangle = \langle Bh_{p'}, e_n \rangle = c\left(\frac{B}{B_{n-1}} h_{p'}\right)(\alpha_n) = 0,$$

where c is a constant and B/B_{n-1} is a Blaschke product with α_n as its zero. Because $(BH^{p'})^{\perp}$ is closed in H^p, we have

$$H^p(\mathbb{R}) \bigcap (BH^{p'})^{\perp} \supset \overline{\text{span}}^p\{e_n\}_{n=1}^{\infty}.$$

Next, we prove the opposite set inclusion. Let $f \in H^p(\mathbb{R}) \bigcap \overline{BH^p}$. Thus, $f = B\overline{h}_p \in H^p$. We are to show that f is in the L^p-closure of $\{e_n\}_{n=1}^{\infty}$. By Theorem 4.2 (Chapter VI, p.242, [12]), it suffices to show that if $g \in H^{p'}, 1 < p' < \infty$, such that $\langle g, e_n \rangle = 0$, then $\langle g, f \rangle = 0$. The assumption $\langle g, e_n \rangle = 0$ implies that g has all zeros of B together with the multiples. Then

$$\langle g, f \rangle = \langle \overline{B}g, \overline{h}_p \rangle = 0.$$

The proof is complete. □

When $p = 2$, we have

Corollary 3.2

$$H^2(\mathbb{R}) = \left(H^2(\mathbb{R}) \bigcap \overline{BH^2(\mathbb{R})}\right) \oplus BH^2(\mathbb{R}) = \overline{\text{span}}^2\{e_n\}_{n=1}^{\infty} \oplus BH^2(\mathbb{R}). \tag{3.9}$$

We further have

Corollary 3.3

For $1 < p < \infty$, $\overline{\text{span}}^p\{e_n\}_{n=1}^{\infty} = H^p(\mathbb{R})$ if and only if the sequence $\{\alpha_1, \ldots, \alpha_n, \ldots\}$ cannot be zeros of a Blaschke product.

Proof

If the sequence $\{\alpha_n\}_{n=1}^{\infty}$ consists of the zeros, together with their multiples, of a Blaschke product, say B, then

$$H^p(\mathbb{R}) \bigcap \overline{BH^p(\mathbb{R})} = \overline{\text{span}}^p\{e_n\}_{n=1}^{\infty}.$$

Now the left-hand-side cannot be identical with $H^p(\mathbb{R})$ for not all functions in $H^p(\mathbb{R})$ are of the form $B\overline{h}_p$. This shows that the closure of the span is not $H^p(\mathbb{R})$.

On the other hand, suppose that the sequence $\{\alpha_n\}_{n=1}^{\infty}$ cannot define a Blaschke product. In the case, if $f \in H^p(\mathbb{R}), 1 < p < \infty$, is orthogonal with all $e_n, n = 1, 2, \ldots$, then f has to be zero function. Otherwise, f would have zeros of the same multiples at α_n. This shows that the sequence forms the zeros of a Blaschke product of f, contrary to the assumption. □

Let $\{\alpha_k\}$ and $\{\beta_k\}$ respectively denote the zero sequence of $F(z) = (1/2\pi)\int_0^{2\pi} \widehat{f}(\omega)e^{i\omega z}dz$ in the upper-half complex plane $\mathbb{C}^+$ and in the lower-half complex plane $\mathbb{C}^-$(they repeat according to its multiplicities). With Theorems 2.7 and 3.1, we have the following result.

Theorem 3.4

Let $f \in \mathcal{FH}^2[0, A]$ and $g \in L^p(\mathbb{R}), 1 < p < \infty$, be non-zero functions. If the endpoints $0, A \in \text{supp}\widehat{f}$. Then, $fg \in \mathcal{FH}^2[0, A]$ if and only if

$$gB_f^u \in \overline{span}^p\left\{e_n(x) = \frac{\sqrt{2\Im z_n}}{x - \overline{z_n}} \prod_{k=1}^{n} \frac{x - \overline{z_k}}{x - z_k} | n \in \mathbb{N}\right\}, \tag{3.10}$$

where $\{z_k\} = \{\overline{\alpha_k}\} \bigcup \{\beta_k\}$ and $B_f^u(x)$ is given in (2.5).

Proof

By Theorem 2.7, we obtain that $fg \in \mathcal{FH}^2[0, A]$ if and only if $g \in \overline{I_f^u}\left[H^p(\mathbb{R}) \bigcap I_f^u I_f^l \overline{H^p(\mathbb{R})}\right]$, where $I_f^u = e^{i(a_u x + b_u)}B_f^u(x)$ and $I_f^l = e^{i(a_l x + b_l)}B_f^l$. Because $0, A \in \text{supp}\widehat{f}$, thus, $a_l = a_u = 0$. By Theorem 3.1, the assertion is proved. □

Specially, if f and g are real functions, we have $\overline{\widehat{f}(\omega)} = \widehat{f}(-\omega)$, $\overline{F(\overline{z})} = F(z)$ and $g = g_+ + \overline{g_+}$. By Theorems 2.6 and 3.1, we have

Corollary 3.5

Let $f \in \mathcal{FH}^2[-A, A]$ and $g \in L^p(\mathbb{R}), 1 < p < \infty$, be non-zero real functions. If the endpoints $-A, A \in \text{supp}\widehat{f}$. Then, $fg \in \mathcal{FH}^2[-A, A]$ if and only if

$$g_+ \in \overline{span}^p\left\{e_n(x) = \frac{\sqrt{2\Im z_n}}{x - \overline{\beta_n}} \prod_{k=1}^{n} \frac{x - \overline{\beta_k}}{x - \beta_k} | n \in \mathbb{N}\right\},$$

where $\{\beta_k\}$ are the zero sequence of $F(z) = (1/2\pi)\int_{-A}^{A} \hat{f}(\omega)e^{i\omega z}d\omega$ in the lower-half plane.

Notice that the space $H^p(\mathbb{R})\bigcap B\overline{H^p(\mathbb{R})}$ depends upon the point sets

$$E=\left\{\alpha_k:\alpha_k\in\mathbb{C}^+,k\in\mathbb{N}\right\}.$$

Each α_k may repeat a number of times, where the time is identical with its multiple in the Blaschke product. So, we could rearrange them and make the repetition explicit by setting

$$E=\left\{\underbrace{\alpha_1,\cdots,\alpha_1}_{n_1},\underbrace{\alpha_2,\cdots,\alpha_2}_{n_2},\cdots,\cdots\right\}.$$

Thus, we can accordingly form another possible basis to characterize $H^p(\mathbb{R})\bigcap B\overline{H^p(\mathbb{R})}$ for $1<p<\infty$.

Corollary 3.6
Let α_k be different zeros of $B(z)$ given by (1.2) of which each has a multiple n_k. Then

$$H^p(\mathbb{R})\bigcap B\overline{H^p(\mathbb{R})}=\overline{span}^p\left\{\frac{1}{(x-\overline{\alpha_k})^j}:j=1,\cdots,n_k;k\in\mathbb{N}\right\}.$$

Indeed, the Takenaka–Malmquist system is the Gram–Schmidt orthogonalization of the system given in Corollary 3.6. Hence, if $g\in L^2(\mathbb{R})$, we can also give an equivalent characterization of Theorem 3.4 in the frequency domain.

Corollary 3.7
Let $f\in\mathcal{FH}^2[0,A]$, $g\in L^2(\mathbb{R})$ be non-zero functions and the endpoints $0,A\in Supp\widehat{f}$. Suppose $\{z_k'|k\in\mathbb{N}\}$ be different zeros of $F(z)=(1/2\pi)\int_0^A\widehat{f}(\omega)e^{i\omega z}d\omega$ in $\mathbb{C}\setminus\mathbb{R}$ and each of which has a multiple n_k. Then, $fg\in\mathcal{FH}^2[0,A]$ if and only if

$$\widehat{g}\in\overline{span}^p\{u[(-1)^d\omega]\omega^j e^{-i\omega z_k'}:j=0,\cdots,n_k-1;k\in\mathbb{N}\},\tag{3.11}$$

where $d=-1$ if $\Im z_k'>0$ and $d=0$ if $\Im z_k'<0$.

As application of Theorem 2.3, Corollary 2.4 and Theorem 2.4, next, we solve the phase retrieval problem. Namely, under what conditions Band{f} and Band$\{g\}$ both are contained in $[0,A]$ for some positive A and $|f|=|g|$. Trivial solutions are $g(t)=cf(t+a)$ and $g(t)=c\overline{f(-t+a)}$ with $|c|=1$ and $a\in\mathbb{R}$. It has been showed that more complicated solutions could be obtained from any one of them by flipping non-real zeros of its Laplace transform [3, 4, 14]. All these existing results rely on the Paley–Wiener theorem and the Hadamard factorization theorem. Comparatively, the backward shift invariant space method is more direct and explicit.

Corollary 3.8
Let f,g be non-zero functions, $f\in\mathcal{FH}^2[0,A]$, $g=e^{i\theta(x)}\in\overline{H^\infty(\mathbb{R})}$. Then, $fg\in\mathcal{FH}^2[0,A]$ if and only if

$$g(x)=c_1e^{ia_1x}\prod_{\alpha_k'}\frac{\overline{\alpha_k'}^2+1}{|\overline{\alpha_k'}^2+1|}\cdot\frac{x-\overline{\alpha_k'}}{x-\alpha_k'},$$

where $|c_1|=1$, a_1 is a nonpositive real number in $[-a_u,0]$ and $\{\alpha_k'\}$ is any subsequence of zero sequence $\{\alpha_k\}$ of $F(z)=1/2\pi\int_0^A\widehat{f}(\omega)e^{i\omega z}$ in the upper-half plane.

Proof
The 'if' part is easy. Now, we prove the 'only if' part. Theorem 2.3 implies that $fg\in\mathcal{FH}^2[0,A]$ if and only if $\overline{g}\in H^\infty(\mathbb{R})\bigcap e^{ia_ux}B_f^u(x)\overline{H^\infty(\mathbb{R})}$. Because $\overline{g}\in H^\infty(\mathbb{R})$ and $|\overline{g}|=1$, $\overline{g}$ is an inner function. We, at the same time, have

$$e^{ia_ux}B_f^u(x)g(x)\in H^\infty(\mathbb{R}).$$

It follows that $\overline{g(x)}$ is a divisor of $e^{ia_ux}B_f^u(x)$. This completes the proof. □

By choosing $\overline{g}=I_f^u=e^{i(a_ux+b_u)}B_f^u$ in Corollary 3.8, we obtain

Corollary 3.9
Let a non-zero function $f\in\mathcal{FH}^2[0,A]$. Then, its boundary outer function $O_f\in\mathcal{FH}^2[0,A]$.

By the same method, we have

Corollary 3.10
Let f,g be non-zero functions, $f\in\mathcal{FH}^2[0,A]$, $g=e^{i\theta(x)}\in H^\infty(\mathbb{R})$. Then, $fg\in\mathcal{FH}^2[0,A]$ if and only if

$$g(x)=c_2e^{ia_2x}\prod_{\beta_k'}\frac{|\overline{\beta_k'}^2+1|}{\overline{\beta_k'}^2+1}\cdot\frac{x-\overline{\beta_k'}}{x-\beta_k'},$$

where $|c_2| = 1$, a_2 is some nonnegative real constant in $[0, a_l]$ and $\{\beta'_k\}$ is any subsequence of zero sequence $\{\beta_k\}$ of $F(z) = 1/2\pi \int_0^A \widehat{f}(\omega)e^{i\omega z}$ in the lower-half plane.

Corollary 3.11

Let f, g be non-zero functions, $f \in \mathcal{FH}^2[0, A]$, $g = e^{i\theta(x)} \in L^\infty(\mathbb{R})$. Then, $fg \in \mathcal{FH}^2[0, A]$ if and only if

$$g(x) = ce^{iax} \prod_{\alpha'_k} \frac{\overline{\alpha'_k}^2 + 1}{|\overline{\alpha'_k}^2 + 1|} \cdot \frac{x - \overline{\alpha'_k}}{x - \alpha'_k} \prod_{\beta'_k} \frac{|\overline{\beta'_k}^2 + 1|}{\overline{\beta'_k}^2 + 1} \cdot \frac{x - \overline{\beta'_k}}{x - \beta'_k},$$

where $|c| = 1$, a is some real constant in $[-a_u, a_l]$, $\{\alpha'_k\}$ is any subsequence of $\{\alpha_k\}$ and $\{\beta'_k\}$ is any subsequence of $\{\beta_k\}$.

Remark

Because $f \in \mathcal{FH}^2[A, B]$ and $fg \in \mathcal{FH}^2[A, B]$ if and only if $h \in \mathcal{FH}^2[0, B - A]$ and $hg \in \mathcal{FH}^2[0, B - A]$, where $h(x) := e^{-iAx}f(x)$. It is easy to generalize the earlier discussions to $f \in \mathcal{FH}^2[A, B]$ and $fg \in \mathcal{FH}^2[A, B]$.

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