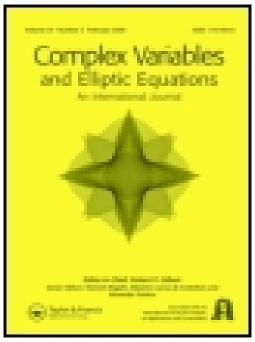

Taylor & Francis
Taylor & Francis Group

Complex Variables and Elliptic Equations
An International Journal

ISSN: 1747-6933 (Print) 1747-6941 (Online) Journal homepage: http://www.tandfonline.com/loi/gcov20

Hardy space decomposition of on the unit circle:

Hai-Chou Li, Guan-Tie Deng & Tao Qian

COMPLEX VARIABLES AND ELLIPTIC EQUATIONS, 2016
VOL. 61, NO. 4, 510–524
http://dx.doi.org/10.1080/17476933.2015.1102901

Hardy space decomposition of L^p on the unit circle: $0 < p \leq 1$

Hai-Chou Li[a], Guan-Tie Deng[b] and Tao Qian[a]

[a]Department of Mathematics, University of Macau, Macao; [b]School of Mathematical Sciences, Beijing Normal University, Beijing

ABSTRACT

In this paper, we consider Hardy space decomposition of $L^p(\partial\mathbb{D})$, $0 < p \leq 1$, where $\mathbb{D}$ stands for the open unit disc, and $\partial\mathbb{D}$ is its boundary. Hardy spaces decompositions for $L^p(\partial\mathbb{D})$ and $L^p(\mathbb{R})$ for $1 \leq p \leq \infty$ are, as classical results, available in the literature. For $1 \leq p \leq \infty$, the basic tools are the Plemelj formula and the boundedness of the Hilbert transformation. For $0 < p \leq 1$, neither on the real line, nor on the unit circle, a Plemelj formula, or Hilbert transformation are available. In a recent paper, Deng and Qian obtain Hardy spaces decomposition for $L^p(\mathbb{R})$, $0 < p < 1$, on the real line by means of rational approximation. In the present paper using rational functions, we achieve the same goal for $L^p(\partial\mathbb{D})$ for the range $0 < p \leq 1$. The work on the unit circle exposes the particular features of the kind of decomposition in the compact situation adaptable to higher dimensions.

ARTICLE HISTORY
Received 22 August 2015
Accepted 24 September 2015

KEYWORDS
Hardy space; unit disc; decomposition theorem; rational approximation; analytic signal

COMMUNICATED BY
Y. Xu

1. Introduction

The Paley–Wiener Theorem for $L^2(\mathbb{R})$ functions states that f is the non-tangential limit of a function in $H^2(\mathbb{C}^+)$ if and only if supp$\hat{f} \subset [0,\infty)$. The latter condition is equivalent with

$$\hat{f} = \chi_{[0,\infty)}\hat{f}, \tag{1}$$

where for any set A, the notation χ_A denotes the characteristic function of A, i.e. $\chi_A(x) = 1$, if $x \in A$; and $\chi_A(x) = 0$, if $x \notin A$.

Similarly, for an $L^2(\mathbb{R})$ function f, it is the non-tangential limit of a function in $H^2(\mathbb{C}^-)$ if and only if supp$\hat{f} \subset (-\infty, 0]$, the latter being equivalent with

$$\hat{f} = \chi_{(-\infty,0]}\hat{f}. \tag{2}$$

Let now $f \in L^2(\mathbb{R})$. We have the decomposition $f = f^+ + f^-$, where

$$f^+ = (\chi_{[0,\infty)}\hat{f})^\vee, \quad f^- = (\chi_{(-\infty,0]}\hat{f})^\vee. \tag{3}$$

From (1) and (2) we see that f^+ and f^- belong to, respectively, the Hardy spaces $H^2(\mathbb{C}^+)$ and $H^2(\mathbb{C}^-)$. Such decomposition can also be obtained through Hilbert transformation,

CONTACT Guan-Tie Deng denggt@bnu.edu.cn

denoted by H:

$$\begin{aligned} f^{\pm} &= \left(\frac{1}{2}(1 \pm i(-i)\mathrm{sgn}(\cdot))\hat{f}\right)^{\vee} \\ &= \frac{1}{2}\left(f \pm iHf\right), \end{aligned}$$

where Hf is the Hilbert transform of f, given by

$$\begin{aligned} Hf(x) &\triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}(-i\mathrm{sgn}(\xi))\hat{f}(\xi)\mathrm{d}\xi \\ &= \frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|x-t|>\epsilon}\frac{f(t)}{x-t}\mathrm{d}t \\ &\triangleq \frac{1}{\pi}\mathrm{p.v.}\int_{-\infty}^{\infty}\frac{f(t)}{x-t}\mathrm{d}t, \end{aligned}$$

where the above first integral is taken in the L^2-sense. For the relevant knowledge we refer the reader to, say, [1].

The above decomposition can also be reached through the Plemelj formula: For $f \in L^2(\mathbb{R})$ one can well define a Hardy $H^2(\mathbb{C}^+)$ function through the Cauchy transformation

$$F^+(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-z}\mathrm{d}t, \quad z = x + iy,\ y > 0. \tag{4}$$

Then the Plemelj formula asserts

$$\lim_{y\to 0+} F(z) = \frac{1}{2}\left(f + iHf\right) = f^+(x), \quad \text{a.e.} \tag{5}$$

Similarly, for $z \in \mathbb{C}^-, z = x + iy,\ y < 0$, we have

$$F^-(z) = \frac{1}{2\pi i}\int_{\infty}^{-\infty}\frac{f(t)}{t-z}\mathrm{d}t, \quad z = x + iy,\ y < 0, \tag{6}$$

and

$$\lim_{y\to 0-} F(z) = \frac{1}{2}\left(f - iHf\right) = f^-(x), \quad \text{a.e.} \tag{7}$$

The significance of the $L^2(\mathbb{R})$ space decomposition into the Hardy spaces $H^2(\mathbb{C}^{\pm})$ lays on the fact that the Hardy space functions have very good properties vs. the L^2-functions: The functions $f^{\pm}$ are non-tangential boundary limits of analytic functions, the latter being analytically defined in their respective domains. We note that a function in L^p is a.e. determined, that is, if $f = g$, a.e., then f and g are considered as the same function in L^p. In this regard one cannot assume a general L^p-function to have desired smoothness. On the other hand, a.e. identical L^p functions correspond to the same Hardy space decomposition, the latter having any kinds of smoothness in their domains of definition. Furthermore, Cauchy's theory and techniques are applicable to Hardy space functions. Finally, there is a one-to-one correspondence between the Hardy space functions and their non-tangential boundary limits. Based on what just-mentioned delicate analysis on the L^2-functions can be carried out via their Hardy space components. Such treatment of L^2-functions, for instance, has ample applications in both the theoretical and applicable mathematics. In particular, some demonstrative results in signal analysis have recently been obtained.[2–6]

Both the above-mentioned Fourier spectrum characterization and the Plemelj formula can be extended to Hardy $H^p(\mathbb{R})$ spaces with some cases of $p \neq 2$. Systematic studies on the spectrum properties as well as the L^p decomposition are carried out in [7,8] for $1 \leq p \leq \infty$. In particular, (5) and (7) hold for $f \in L^p(\mathbb{R}), 1 < p < \infty$; while $f \in H^p(\mathbb{R})$ if and only if $f \in L^p(\mathbb{R})$ and in the distributional sense supp$\hat{f} \subset [0, \infty)$ for all $1 \leq p \leq \infty$. Comparatively, the Plemelj formula approach is more effective and realizable as Fourier transformation on the $L^p(\mathbb{R})$ spaces for $p > 2$ gives rise to, in general, only distributions. The Fourier spectrum characterization, as well as the Hardy spaces decompositions are generalized to Hardy spaces on tubes in higher dimensions.[9]

There exists an analogous theory on the unit circle for $1 \leq p \leq \infty$. They have particular features as they are expressed by Fourier series.[1,10] Now we turn to the index range $0 < p \leq 1$.

Aleksandrov [11] and Deng and Qian [12] have studied the case $L^p(\mathbb{R}), 0 < p < 1$. There is no Fourier transformation theory for functions in such $L^p(\mathbb{R})$ spaces as they are not even distributions. One, however, can have H^p spaces decompositions. The paper [11] uses the real method of harmonic analysis by making use of a dense subclass of the $L^p(\mathbb{R})$-functions with vanishing moment conditions and Hilbert transforms. Comparatively, [12] uses the complex analysis methods, and, in particular, rational function approximation to achieve the function decomposition goal. The methods in [12] are direct and constructive. For the index range $0 < p \leq 1$, the present paper obtains the $L^p(\partial\mathbb{D})$ Hardy spaces decomposition using rational approximation.

The writing plan of the paper is as follows. The second section contains preliminary knowledge on the analytic Hardy spaces. We state our main results in the third section, that is the Hardy spaces decomposition theorem of the $L^p(\partial\mathbb{D})$, $0 < p \leq 1$. The fourth section contains main technical lemmas. The last two sections, as key part, provide the proofs of the main theorems and lemmas.

2. Preliminaries on the analytic function spaces

Throughout this paper, we denote the unit disc and the upper-half plane by, respectively, $\mathbb{D}$ and $\mathbb{C}^+$. That is,

$$\mathbb{D} = \left\{z = re^{i\theta} : 0 < r < 1, -\pi \leq \theta < \pi\right\}$$

and

$$\mathbb{C}^+ = \{z = x + iy : x \in \mathbb{R}, y > 0\}.$$

The boundary of the unit disc, that is the unit circle, is denoted as

$$\partial\mathbb{D} = \left\{z = e^{i\theta} : -\pi \leq \theta < \pi\right\}.$$

Moreover, $\mathbb{D}_O$ denotes the outside of the closure of the unite disc:

$$\mathbb{D}_O = \left\{z = re^{i\theta} : r > 1, -\pi < \theta \leq \pi\right\}.$$

$H^p(\mathbb{D})$ and $H^p(\mathbb{C}^+)$ denote the Hardy spaces on the unit disc and the upper-half plane, respectively. Now, we introduce their classical definitions.[1,10,13].

Definition 2.1: When $0 < p < \infty$, we say $f \in H^p(\mathbb{D})$ if $f(z)$ is an analytic function on $\mathbb{D}$, and satisfies

$$\|f\|_{H_I^p} = \sup_{0<r<1} M_p(f,r) < \infty,$$

where

$$M_p(f,r) = \left(\int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}}.$$

When $p = \infty$, we say $f \in H^\infty(\mathbb{D})$ if $f(z)$ is a bounded analytic function on $\mathbb{D}$ and write

$$\|f\|_{H_I^\infty} = \sup_{z\in\mathbb{D}} |f(z)|.$$

For $p \geq 1$, $\|\cdot\|_p$ is the norm of $H^p(\mathbb{D})$, and $H^p(\mathbb{D})$ is a Banach space.

For $0 < p < 1$, the inequality

$$|z_1 + z_2|^p \leq |z_1|^p + |z_2|^p$$

holds that implies that $H^p(\mathbb{D})$ is a metric space with the metric

$$d(f,g) = \|f - g\|_p^p.$$

It is further shown that $H^p(\mathbb{D})$ is a complete metric space.

Moreover, according to the theory of subharmonic functions in the unit disc $\mathbb{D}$, there is another definition of $H^p(\mathbb{D})$. That is, $f \in H^p(\mathbb{D})$ if and only if the subharmonic function $|f(z)|^p$ has a harmonic majorant, and, for $p < \infty$, $\|f\|_p$ is the value of the least harmonic majorant at $z = 0$. This definition of $H^p(\mathbb{D})$ in term of harmonic majorants is conformally invariant. It is used to define H^p functions on any plane domain or Riemann surface.[1]

Similarly, the Banach or complete metric spaces $H^p(\mathbb{D}_O)$, $0 < p \leq \infty$, for outside the closed unit disc are defined as follows.

Definition 2.2: When $0 < p < \infty$, we say $f \in H^p(\mathbb{D}_O)$ if $f(z)$ is an analytic function on $\mathbb{D}_O$, and satisfies

$$\|f\|_{H_O^p} = \sup_{r>1} M_p(f,r) < \infty,$$

where

$$M_p(f,r) = \left(\int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}}.$$

When $p = \infty$, we say $f \in H^\infty(\mathbb{D}_O)$ if $f(z)$ is a bounded analytic function on $\mathbb{D}_O$. The norm of the space is defined by

$$\|f\|_{H_O^\infty} = \sup_{z\in\mathbb{D}_O} |f(z)|.$$

The Hardy spaces of the upper- and lower-half complex planes are irrelevant to this paper, we suppress their definitions.

Since $|f|^p$ is subharmonic for $f \in H^p(\mathbb{D})$, the function $M_p(f,r)$ increases in $r \in (0,1)$, and

$$\|f\|_{H^p_I} = \sup\left\{M_p(f,r) : 0 < r < 1\right\} = \lim_{0<r<1, r\to 1} M_p(f,r).$$

The existence of the non-tangential boundary limit of $f(z) \in H^p(\mathbb{D})$ is an important property of Hardy spaces. Denoting by $f(e^{i\theta})$ the boundary limit, we have $f(e^{i\theta}) \in L^p([-\pi,\pi])$, and

$$\|f\|_p = \left(\int_{-\pi}^{\pi} |f(e^{i\theta})|^p \mathrm{d}\theta\right)^{\frac{1}{p}} = \|f\|_{H^p_I}.$$

The correspondence between the Hardy space functions and their boundary limits are one-to-one, and, in fact, an isometric isomorphism. In the sequence, with an abuse of notation, we mix up the notations for the Hardy space functions and their boundary limits, as well as their respective norms.

For the Hardy spaces outside the closed unit disc the situation is similar. In the sequel we use, correspondingly, the self explanatory notations.

We denote by $L^p_I(\partial\mathbb{D})$ and $L^p_O(\partial\mathbb{D})$ the closed subspaces of $L^p(\partial\mathbb{D})$, consisting of, respectively, the non-tangential boundary limits of the functions of $H^p(\mathbb{D})$ and $H^p(\mathbb{D}_O)$. The task of this paper is to show

$$L^p(\partial\mathbb{D}) = L^p_I(\partial\mathbb{D}) + L^p_O(\partial\mathbb{D}),$$

where the right-hand side is not a direct sum, nor the intersection of $L^p_I(\partial\mathbb{D})$ and $L^p_O(\partial\mathbb{D})$ is the trivial function set containing only the zero function. Note that the notation $f \in L^p(\partial\mathbb{D})$ if and only if $F \in L^p([-\pi,\pi])$, where $F(t) = f(e^{it})$.

3. Main results: decomposition of the $L^p(\partial\mathbb{D})$

In this paper, similarly to [12], we use the rational approximation method to obtain $L^p(\partial\mathbb{D})$ space decomposition on the unit circle for the range of $0 < p \le 1$. Rational approximation, being a particular one of complex approximation in general, has a long history.[14] On the real line case it has been proved that any function $f \in L^p(\mathbb{R})$, $0 < p < 1$, has a Hardy space decomposition $f = f_1 + f_2$, where f_1 and f_2 are the non-tangential boundary limits of two Hardy space functions, respectively, in the upper- and lower-half planes.[11,12] In [12] such functions f_1 and f_2 are obtained through sequences of rational functions with poles in, respectively, the lower- and upper-half planes.

In the present paper, we obtain the following Hardy spaces decomposition: For any $f \in L^p([-\pi,\pi])$, $0 < p \le 1$, there holds $f = f_I + f_O$, where f_I and f_O are non-tangential boundary limits of functions in the Hardy spaces $H^p(\mathbb{D})$ and $H^p(\mathbb{D}_O)$, respectively.

Theorem 3.1: *For $0 < p < 1$ there holds*

$$L^p(\partial\mathbb{D}) = L^p_I(\partial\mathbb{D}) + L^p_O(\partial\mathbb{D})$$

in the following sense: There exist a positive constant A_p and two sequences of rational functions $\{P_k(z)\} \subseteq H^p(\mathbb{D})$ and $\{Q_k(z)\} \subseteq H^p(\mathbb{D}_O)$ such that

(i)

$$\sum_{k=1}^{\infty}\left(\|P_k\|_{H_I^p}^p + \|Q_k\|_{H_O^p}^p\right) \leq A_p\|f\|_p^p, \tag{8}$$

(ii)

$$\lim_{n\to\infty} ||f - \sum_{k=1}^{n}(P_k + Q_k)||_p = 0, \tag{9}$$

(iii)

$$g(z) = \sum_{k=1}^{\infty} P_k(z) \in H^p(\mathbb{D}), \;\; h(z) = \sum_{k=1}^{\infty} Q_k(z) \in H^p(\mathbb{D}_O), \tag{10}$$

(iv) *$g(e^{i\theta})$ and $h(e^{i\theta})$ are the non-tangential boundary limits of functions $g(z) \in H^p(\mathbb{D})$ and $h(z) \in H^p(\mathbb{D}_O)$, respectively, $f(e^{i\theta}) = g(e^{i\theta}) + h(e^{i\theta})$ almost everywhere for $\theta \in [-\pi, \pi]$.*

(v)

$$\|f\|_p^p \leq \|g\|_p^p + \|h\|_p^p \leq A_p\|f\|_p^p.$$

For $p = 1$ there holds a similar decomposition result under an extra vanishing moment condition. However, the second inequality in the inequality chain given in (v) does not hold.

Theorem 3.2: *Let $f \in L^1(\partial\mathbb{D})$ and f satisfy the vanishing moment condition*

$$\int_{-\pi}^{\pi} f(x)\mathrm{d}x = 0.$$

Then, there exist a positive constant C and two sequences of rational functions $\{P_k(z)\} \subseteq H^1(\mathbb{D})$ and $\{Q_k(z)\} \subseteq H^1(\mathbb{D}_O)$ such that

(i)

$$\sum_{k=1}^{\infty}\left(\|P_k\|_{H_I^1} + \|Q_k\|_{H_O^1}\right) \leq C\|f\|_1, \tag{11}$$

(ii)

$$\lim_{n\to\infty} ||f - \sum_{k=1}^{n}(P_k + Q_k)||_1 = 0, \tag{12}$$

(iii)

$$g(z) = \sum_{k=1}^{\infty} P_k(z) \in H^1(\mathbb{D}), \;\; h(z) = \sum_{k=1}^{\infty} Q_k(z) \in H^1(\mathbb{D}_O), \tag{13}$$

(iv) *$g(e^{i\theta})$ and $h(e^{i\theta})$ are the non-tangential boundary limits of functions for $g(z) \in H^1(\mathbb{D})$ and $h(z) \in H^1(\mathbb{D}_O)$, respectively, $f(e^{i\theta}) = g(e^{i\theta}) + h(e^{i\theta})$ almost everywhere for $\theta \in [-\pi, \pi]$. In summary, we have*

$$L^1(\partial\mathbb{D}) = L_I^1(\partial\mathbb{D}) + L_O^1(\partial\mathbb{D})$$

in the sense of L^1.

4. Main lemmas

In order to prove the theorems we need the following lemmas. Lemmas 4.1–4.3 are used to prove Theorem 3.1, and Lemmas 4.4 and 4.5 are used to prove Theorem 3.2.

Lemma 4.1:

(1) *If* $R(e^{i\theta}) \in L^p([-\pi,\pi])$, $0 < p < 1$, *and the rational function* $R(z)$ *is analytic in the unit disc* $\mathbb{D} = \{z : |z| < 1,\}$. *Then* $R(z) \in H^p(\mathbb{D})$.

(2) *If* $R(e^{i\theta}) \in L^p([-\pi,\pi])$, $0 < p < 1$, *and the rational function* $R(z)$ *is analytic in* $\mathbb{D}_O = \{z : |z| > 1\}$. *Then* $R(z) \in H^p(\mathbb{D}_O)$.

Lemma 4.2: *Let* $0 < p < 1$, $f(e^{i\theta}) \in L^p([-\pi,\pi])$ *and* $\varepsilon > 0$. *there exists a sequence of trigonometric polynomials* $R_k(e^{i\theta}) = \sum_{|j|\le N_k} c_{kj}e^{ij\theta}$ *such that*

$$\sum_{k=1}^{\infty} ||R_k||_p^p \le (1+\varepsilon)\|f\|_p^p \tag{14}$$

and

$$\lim_{n\to\infty} ||f - \sum_{k=1}^{n} R_k||_p = 0. \tag{15}$$

With the help of Lemma 4.1, we can obtain the following important Lemma 4.3.

Lemma 4.3: *Suppose that* $0 < p < 1$ *and* R *is a rational function with* $R(e^{i\theta}) \in L^p([-\pi,\pi])$, *then there exist two rational functions* $P(z) \in H^p(\mathbb{D})$ *and* $Q(z) \in H^p(\mathbb{D}_O)$ *such that*

$$R(z) = P(z) + Q(z),$$

for $|z| = 1$, *and*

$$\|P\|_{H_I^p}^p + \|Q\|_{H_O^p}^p \le C_p\|R\|_p^p.$$

where C_p *is a constant only dependent on* p.

However, for the case $p = 1$ there does not hold the analogue results to Lemmas 4.2 and 4.3 under the same conditions. In order to have such result, we need to add the vanishing moment condition, viz., $\int_{-\pi}^{\pi} f(x)\mathrm{d}x = 0$.

Lemma 4.4: *Let* $\mathcal{A}$ *be the set of trigonometric polynomials* $R_k(e^{i\theta}) = \sum_{|k|\le N} c_k e^{ik\theta}$ *with the property* $\int_{-\pi}^{\pi} R(e^{i\theta})\mathrm{d}\theta = 0$. *If* $f(e^{i\theta}) \in L^1([-\pi,\pi])$ *and* $\int_{-\pi}^{\pi} f(e^{i\theta})\mathrm{d}\theta = 0$, *then, for any* $\varepsilon > 0$, *there exists a sequence of trigonometric polynomials* $\{R_k(e^{i\theta})\}$ *in* $\mathcal{A}$ *such that*

$$\sum_{k=1}^{\infty} ||R_k||_1 \le (1+\varepsilon)\|f\|_1 \tag{16}$$

and

$$\lim_{n\to\infty} ||f - \sum_{k=1}^{n} R_k||_1 = 0. \tag{17}$$

5. The proofs of the main lemmas

In this section, we are to prove the important lemmas.

5.1. Proof of Lemma 4.1

Proof: By the assumption, we can let $R(z)$ be of the following form:

$$R(z) = \frac{P(z)}{Q(z)},$$

where $P(z),\ Q(z)$ are polynomials of z satisfying

$$Q(z) = Q_1(z)(z - e^{i\theta_1})^{l_1} \cdots (z - e^{i\theta_k})^{l_k},$$

$Q_1(e^{i\theta_j}) \neq 0,\ l_j < \infty,\ j = 1, 2, \ldots, k$; and, there exists a positive constant $M_1 > 0$ such that

$$M_1 \geq \left|\frac{P(z)}{Q_1(z)}\right|, \quad z \in \overline{\mathbb{D}}.$$

It is clear that there exists $\varepsilon_0 \in (0, 1)$ such that

$$-\pi < \theta_1 - 2\varepsilon_0 < \theta_1 + 2\varepsilon_0 < \theta_2 - 2\varepsilon_0 < \theta_2 + 2\varepsilon_0 < \ldots < \theta_k - 2\varepsilon_0 < \theta_k + 2\varepsilon_0 < \pi.$$

Moreover, there exists a positive constant M_0 such that

$$\left|\frac{P(e^{i\theta})}{Q_1(e^{i\theta})}\right| \geq M_0 \quad \text{for} \quad \theta \in \bigcup_{j=1}^{k} [\theta_j - 2\varepsilon_0, \theta_j + 2\varepsilon_0].$$

Let $J_k = \bigcup_{j=1}^{k} I_j$, where $I_j = \{\theta \in (-\pi, \pi) : |\theta - \theta_j| < \varepsilon_0\}$, then,

$$\begin{aligned}
\int_{-\pi}^{\pi} |R(e^{i\theta})|^p \, d\theta &= \int_{-\pi}^{\pi} \left|\frac{P(e^{i\theta})}{Q_1(e^{i\theta})}\right|^p \frac{d\theta}{\prod_{j=1}^{k} |e^{i\theta} - e^{i\theta_j}|^{pl_j}} \\
&\geq \sum_{j=1}^{k} \int_{I_j} \left|\frac{P(e^{i\theta})}{Q_1(e^{i\theta})}\right|^p \frac{d\theta}{\prod_{j=1}^{k} |e^{i\theta} - e^{i\theta_j}|^{pl_j}} \\
&\geq M_0^p \sum_{j=1}^{k} \int_{I_j} \frac{d\theta}{\prod_{j=1}^{k} |e^{i\theta} - e^{i\theta_j}|^{pl_j}} \\
&\geq M_0^p \int_{J_k} \frac{d\theta}{\prod_{j=1}^{k} |e^{i\theta} - e^{i\theta_j}|^{pl_j}}.
\end{aligned} \tag{18}$$

Observing that, for $j \in \{1, 2, \ldots, k\}$, there is

$$\frac{1}{|e^{i\theta} - e^{i\theta_j}|} = \frac{1}{\left|e^{i\frac{\theta+\theta_j}{2}}(e^{i\frac{\theta-\theta_j}{2}} - e^{-i\frac{\theta-\theta_j}{2}})\right|} = \frac{1}{\left|2\sin\frac{\theta-\theta_j}{2}\right|} \geq \frac{1}{|\theta - \theta_j|}. \tag{19}$$

Thus, fixing $j \in \{1, 2, \ldots, k\}$,

$$\int_{-\pi}^{\pi} |R(e^{i\theta})|^p \, d\theta \geq \frac{M_0^p}{\varepsilon_0^{pL_k}} \int_{I_j} \frac{d\theta}{|\theta - \theta_j|^{pl_j}}, \tag{20}$$

where $L_k = \sum_{s=1}^{k} l_s$, The fact $R(e^{i\theta}) \in L^p([-\pi,\pi])$ implies $pl_j < 1$ for all $j \in \{1,2,\ldots,k\}$. Next, we are to show

$$\sup_{0<r<1} \int_{-\pi}^{\pi} |R(re^{i\theta})|^p \, d\theta < \infty. \tag{21}$$

If $0 < r < \frac{1}{2}$, then $|re^{i\theta} - e^{i\theta_j}| > \frac{1}{2}$, so

$$\begin{aligned}\int_{-\pi}^{\pi} |R(re^{i\theta})|^p \, d\theta &= \int_{-\pi}^{\pi} \left|\frac{P(re^{i\theta})}{Q_1(re^{i\theta})}\right|^p \frac{d\theta}{\prod_{j=1}^{k} |re^{i\theta} - e^{i\theta_j}|^{pl_j}} \\ &\le M_1^p \int_{-\pi}^{\pi} \frac{d\theta}{\prod_{j=1}^{k} |re^{i\theta} - e^{i\theta_j}|^{pl_j}} \le 2\pi M_1^p 2^{L_k};\end{aligned} \tag{22}$$

if $\frac{1}{2} \le r < 1$, then

$$\int_{-\pi}^{\pi} |R(re^{i\theta})|^p \, d\theta \le M_1^p \int_{-\pi}^{\pi} \frac{d\theta}{\prod_{j=1}^{k} |re^{i(\theta-\theta_j)} - 1|^{pl_j}}. \tag{23}$$

Observe that, for $|\varphi| < \pi$, $\frac{1}{2} \le r < 1$,

$$\begin{aligned}|re^{i\varphi} - 1|^2 &= (1-r)^2 + 2r(1-\cos\varphi) \\ &> 2r(1-\cos\varphi) = 4r\sin^2\left(\frac{\varphi}{2}\right) \ge \frac{\varphi^2}{\pi^2}.\end{aligned}$$

Thus, (23) becomes

$$\begin{aligned}\int_{-\pi}^{\pi} |R(re^{i\theta})|^p \, d\theta &\le M_1^p \int_{-\pi}^{\pi} \frac{d\theta}{\prod_{j=1}^{k} \left(\frac{\theta-\theta_j}{\pi}\right)^{pl_j}} \\ &= M_2 \int_{(-\pi,\pi)\setminus J_k} \frac{d\theta}{\prod_{j=1}^{k} |\theta-\theta_j|^{pl_j}} + M_2 \int_{J_k} \frac{d\theta}{\prod_{j=1}^{k} |\theta-\theta_j|^{pl_j}} \\ &\le 2\pi M_2 \varepsilon_0^{-pL_k} + M_2 \varepsilon_0^{-pL_k} \sum_{j=1} k \int_{I_j} \frac{d\theta}{|\theta-\theta_j|^{pl_j}} < \infty,\end{aligned} \tag{24}$$

where $M_2 = M_1^p \pi^{L_k}$.

Combining (22) and (24), we obtain (21). Together with the analyticity of $R(z)$ in $\mathbb{D}$, we obtain $R(z) \in H^p(\mathbb{D})$. The proof of Lemma 4.1 is complete. □

5.2. *Proof of Lemma 4.2*

Proof: Let $f(e^{i\theta}) \in L^p([-\pi,\pi])$, $0 < p < 1$, $\|f\|_p > 0$ and $\varepsilon > 0$, since the set of trigonometric polynomials is dense in $L^p([-\pi,\pi])$, there exists a sequence of trigonometric polynomials $r_k(e^{i\theta}) = \sum_{|j|\le N_k} c_{kj} e^{ij\theta}$ such that

$$\|f - r_k\|_p^p < \frac{\varepsilon \|f\|_p^p}{2^{k+2}}.$$

Then

$$\|r_k\|_p^p = \|r_k - f + f\|_p^p \leq \|f - r_k\|_p^p + \|f\|_p^p \leq (1 + \frac{\varepsilon}{2^{k+2}}\|f\|_p^p,$$

and

$$\|r_{k+1} - r_k\|_p^p = \|r_{k+1} - f + f - r_k\|_p^p \leq \|r_{k+1} - f\|_p^p + \|f - r_k\|_p^p \leq \frac{\varepsilon}{2^{k+1}}\|f\|_p^p,$$

for $k = 2, 3, \ldots$. So if we take

$$R_1(z) = r_1(z), \quad R_k(z) = r_k(z) - r_{k-1}(z), \quad (k = 2, 3, \ldots)$$

then

$$\sum_{k=1}^{n} R_k(z) = r_1(z) + (r_2(z) - r_1(z)) + \cdots + (r_n(z) - r_{n-1}(z)) = r_n(z).$$

Thus, we can obtain that

$$\sum_{k=1}^{\infty} ||R_k||_p^p \leq (1 + \varepsilon)\|f\|_p^p,$$

and

$$\lim_{n\to\infty} ||f - \sum_{k=1}^{n} R_k||_p = \lim_{n\to\infty} ||f - r_n||_p = 0.$$

That is, (14) and (15) can be obtained, and then, the proof of Lemma 4.2 is complete. □

5.3. Proof of Lemma 4.3

Proof: If the trigonometric polynomial $R(e^{i\theta}) \in L^p([-\pi, \pi])$, $0 < p < 1$, and can be represented as $R(e^{i\theta}) = \sum_{|k|\leq N} c_k e^{ik\theta}$, then for $z \in \mathbb{C}$, we let, using the same symbol,

$$R(z) = \sum_{|k|\leq N} c_k z^k.$$

In order to decompose this rational function $R(z)$ into a sum of two rational functions $P(z)$ and $Q(z)$ which are in the Hardy spaces $H^p(\mathbb{D})$ and $H^p(\mathbb{D}_O)$, respectively, we first let

$$P(z,\varphi) = \frac{z^{N_1}}{z^{N_1} - e^{i\varphi}} R(z), \quad Q(z,\varphi) = \frac{z^{-N_1}}{z^{-N_1} - e^{-i\varphi}} R(z),$$

for $z \in \mathbb{C}$ and $\varphi \in \mathbb{R}$, where $N_1 > N$ is a positive integer. Then, the equality

$$P(z,\varphi) + Q(z,\varphi) = R(z) \tag{25}$$

holds for $z \in \mathbb{C}$ and $\varphi \in \mathbb{R}$ with $z^{N_1} \neq e^{i\varphi}$.

Next, we want to find a special real number $\varphi_0 \in \mathbb{R}$ such that $P(z,\varphi_0) \in H^p(\mathbb{D})$, $Q(z,\varphi_0) = R(z) - P(z,\varphi_0) \in H^p(\mathbb{D}_O)$ for this fixed $\varphi_0 \in \mathbb{R}$ and the estimation in Lemma 4.3 holds. The proof of Lemma 4.3 will be completed by letting $P(z) = P(z,\varphi_0)$ and $Q(z) = Q(z,\varphi_0)$.

We observe that the functions $P(z,\varphi)$ and $Q(z,\varphi)$ are both rational functions. Moreover, the poles of $P(z,\varphi)$ are contained in the boundary of the unit disc, $\partial\mathbb{D}$. Similarly, the poles of $Q(z,\varphi)$ are contained in $\partial\mathbb{D}\bigcup\{0\}$. Therefore, $P(z,\varphi)$ and $Q(z,\varphi)$ are analytic in the unit disc $\mathbb{D}$ and $\mathbb{D}_O$, respectively.

Let

$$I=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|P(e^{i\theta},\varphi)|^p\,\mathrm{d}\theta\mathrm{d}\varphi,$$

then

$$\begin{aligned}I&=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|R(e^{i\theta})|^p}{|e^{iN_1\theta}-e^{i\varphi}|^p}\,\mathrm{d}\theta\mathrm{d}\varphi\\&=\int_{-\pi}^{\pi}|R(e^{i\theta})|^p\left(\int_{-\pi}^{\pi}\frac{1}{|e^{iN_1\theta}-e^{i\varphi}|^p}\,\mathrm{d}\varphi\right)\mathrm{d}\theta.\end{aligned}$$

Observing

$$\begin{aligned}\int_{-\pi}^{\pi}\frac{1}{|e^{iN_1\theta}-e^{i\varphi}|^p}\,\mathrm{d}\varphi&=\frac{1}{2^p}\int_{-\pi}^{\pi}\frac{2^p}{|1-e^{i\varphi}|^p}\,\mathrm{d}\varphi\\&=\frac{1}{2^p}\int_{-\pi}^{\pi}\frac{1}{|\sin\frac{\varphi}{2}|^p}\,\mathrm{d}\varphi\\&=\frac{4}{2^p}\int_{0}^{\frac{\pi}{2}}\frac{1}{\sin^p\varphi}\,\mathrm{d}\varphi\\&\leq\frac{4}{2^p}\int_{0}^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\left(\frac{2\varphi}{\pi}\right)^p}=\frac{2^{1-p}\pi}{1-p},\end{aligned}$$

we obtain

$$I\leq\frac{2^{1-p}\pi}{1-p}\int_{-\pi}^{\pi}|R(e^{i\theta})|^p\,\mathrm{d}\theta=\frac{2^{1-p}\pi}{1-p}\|R\|_p^p.$$

Therefore, there exists a real number φ_0 such that

$$\|P\|_p^p\leq\frac{2^{1-p}\pi}{1-p}\|R\|_p^p.$$

where $P(z)=P(z,\varphi_0)$. Finally, let $Q(z)=Q(z,\varphi_0)$, since

$$Q(z)=R(z)-P(z)\ \text{ and }\ |R(z)-P(z)|^p\leq|R(z)|^p+|P(z)|^p,$$

we have

$$\|Q\|_p^p=\|R-P\|_p^p\leq\|R\|_p^p+\|P\|_p^p\leq\left(1+\frac{2^{1-p}\pi}{1-p}\right)\|R\|_p^p.$$

so

$$\|P\|_p+\|Q\|_p\leq C_p\|R\|_p,$$

where the constant

$$C_p=\left(1+\frac{2^{1-p}\pi}{1-p}\right)^{\frac{1}{p}}+\left(\frac{2^{1-p}\pi}{1-p}\right)^{\frac{1}{p}}.$$

Lemma 4.1 assures that

$$P(z) \in H^p(\mathbb{D}), \ \ Q(z) \in H^p(\mathbb{D}_O).$$

Therefore, the proof of Lemma 4.3 is completed by noticing that $\|P\|_{H_I^p} = \|P\|_p$ and $\|Q|_{H_O^p} = \|Q\|_p$. □

5.4. Proof of Lemma 4.4

Proof: We can assume that $f \not\equiv 0$ is a real function, which satisfies $f(e^{i\theta}) \in L^1([-\pi,\pi])$ and $\int_{-\pi}^{\pi} f(e^{i\theta})\mathrm{d}\theta = 0$. Let $f(e^{i\theta}) = f^+(e^{i\theta}) - f^-(e^{i\theta})$, where $f^+(e^{i\theta}) = \max\{f(e^{i\theta}), 0\}$, $f^-(e^{i\theta}) = \max\{-f(e^{i\theta}), 0\}$.

Since the set of trigonometric polynomials is dense in $L^1([-\pi,\pi])$, it follows that, for $\varepsilon > 0$, there exist two sequences of trigonometric polynomials $\{r_{1k}\}, \{r_{2k}\}$ such that

$$\|r_{1k} - f^+\|_1 < \frac{\varepsilon_k}{2}, \ \ \|r_{2k} - f^-\|_1 < \frac{\varepsilon_k}{2},$$

where $\varepsilon_k = \frac{\varepsilon\|f\|_1}{2^{k+3}}$. Then we have

$$\|r_{1k} - r_{2k} - f\|_1 \le \|r_{1k} - f^+\|_1 + \|r_{2k} - f^-\|_1 < \varepsilon_k.$$

If we let

$$\alpha_k = \int_{-\pi}^{\pi} \left(r_{1k}(e^{i\theta}) - r_{2k}(e^{i\theta})\right) \mathrm{d}\theta,$$

then

$$\begin{aligned} |\alpha_k| &= \left| \int_{-\pi}^{\pi} \left(r_{1k}(e^{i\theta}) - f^+(e^{i\theta})\right) \mathrm{d}\theta + \int_{-\pi}^{\pi} \left(f^-(e^{i\theta}) - r_{2k}(e^{i\theta})\right) \mathrm{d}\theta \right| \\ &\le \|r_{1k} - f^+\|_1 + \|r_{2k} - f^-\|_1 < \varepsilon_k. \end{aligned}$$

The rational function

$$r_k(e^{i\theta}) = r_{1k}(e^{i\theta}) - r_{2k}(e^{i\theta}) - \frac{\alpha_k}{2\pi}$$

satisfies

$$\int_{-\pi}^{\pi} r_k(e^{i\theta})\mathrm{d}\theta = \int_{-\pi}^{\pi} \left(r_{1k}(e^{i\theta}) - r_{2k}(e^{i\theta})\right) \mathrm{d}\theta - \alpha_k = 0,$$

that is, $r_k(e^{i\theta}) \in \mathcal{A}$. Furthermore, we have

$$\begin{aligned} \|r_k - f\|_1 &= \int_{-\pi}^{\pi} \left| r_{1k}(e^{i\theta}) - r_{2k}(e^{i\theta}) - \frac{\alpha_k}{2\pi} - f(e^{i\theta}) \right| \mathrm{d}\theta \\ &\le \int_{-\pi}^{\pi} \left| r_{1k}(e^{i\theta}) - r_{2k}(e^{i\theta}) - f(e^{i\theta}) \right| \mathrm{d}\theta + \int_{-\pi}^{\pi} \left| \frac{\alpha_k}{2\pi} \right| \mathrm{d}\theta < 2\varepsilon_k, \\ \|r_k\|_1 &= \|r_k - f + f\|_1 \le \|r_k - f\|_1 + \|f\|_1 < \left(\frac{\varepsilon}{2^{k+2}} + 1\right) \|f\|_1, \end{aligned}$$

and

$$\|r_k - r_{k-1}\|_1 = \|r_k - f + f - r_{k-1}\|_1 \le \|r_k - f\|_1 + \|f - r_{k-1}\|_1 < \frac{\varepsilon}{2^k} \|f\|_1.$$

Letting

$$R_1(z) = r_1(z),\ \ R_k(z) = r_k(z) - r_{k-1}(z),\ \ k = 2, 3, \ldots,$$

the sequence $\{R_k(z)\}$ is one consisting of rational functions satisfying (16) and (17). This completes the proof. □

6. Proofs of the main theorems

Based on the above technical lemmas, we now prove Theorems 3.1 and 3.2.

Proof of Theorem 3.1: By Lemmas 4.2 and 4.3, we obtain that, there exist two sequences of rational functions $P_k(z) \in H^p(\mathbb{D})$ and $Q_k(z) \in H^p(\mathbb{D}_O)$ such that

$$\sum_{k=1}^{\infty} (\|P_k\|_p^p + \|Q_k\|_p^p) \le A_\varepsilon \|f\|_p^p,$$

and

$$\lim_{n\to\infty} \|f - \sum_{k=1}^{n} (P_k + Q_k)\|_p = 0.$$

Since $P_k(z) \in H^p(\mathbb{D})$ and $Q_k(z) \in H^p(\mathbb{D}_O)$, we have

$$\|P_k\|_p^p = \|P_k\|_{H^p}^p,\ \ \|Q_k\|_p^p = \|Q_k\|_{H^p}^p.$$

Thus (8) and (9) hold, consequently, (10) as well. Therefore, the non-tangential boundary limits $g(e^{i\theta})$ and $h(e^{i\theta})$ of functions $g(z)$ and $h(z)$ exist almost everywhere, respectively. Moreover, (9) implies that $f(e^{i\theta}) = g(e^{i\theta}) + h(e^{i\theta})$ almost everywhere. This completes the proof. □

Proof of Theorem 3.2: To prove the case $p = 1$ we follow the same idea as we prove Theorem 3.1 but invoke Lemmas 4.3 and 4.4. □

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was partially supported by NSFC [grant number 11271045] and by SRFDP [grant number 20100003110004]; Research Grant of University of Macau, MYRG115(Y1-L4)-FST13-QT, Macao Government FDCT 098/2012/A3.

References

[1] Garnett JB. Bounded analytic functions. Vol. 96, Pure and applied mathematics [Harcourt Brace Jovanovich, Publishers]. New York (NY): Academic Press, Inc.; 1981.

[2] Qian T, Wang Y-B, Dang P. Adaptive decomposition into mono-components. Adv. Adapt. Data Anal. 2009;1:703–709.

[3] Dang P, Qian T, You Z. Hardy–Sobolev spaces decomposition in signal analysis. J. Fourier Anal. Appl. 2011;17:36–64.

[4] Dang P, Qian T. Analytic phase derivatives, all-pass filters and signals of minimum phase. IEEE Trans. Signal Process. 2011;59:4708–4718.
[5] Papoulis A. Signal analysis. Vol. 191, McGraw-Hill; 1977.
[6] Poletti MA. The homomorphic analytic signal. IEEE Trans. Signal Process. 1997;45:1943–1953.
[7] Qian T. Characterization of boundary values of functions in Hardy spaces with applications in signal analysis. J. Integral Equ. Appl. 2005;17:159–198.
[8] Qian T, Xu YS, Yan DY, et al. Fourier spectrum characterization of Hardy spaces and applications. Proc. Am. Math. Soc. 2009;137:971–980.
[9] Deng GT, Li HC, Qian T. Fourier spectrum characterization of H^p spaces on tubes over cones. Forthcoming 2016.
[10] Cima JA, Ross WT. The backward shift on the Hardy space. Vol. 79, Mathematical surveys and monographs. Providence (RI): American Mathematical Society; 2000.
[11] Aleksandrov AB. Invariant subspaces of the backward shift operator in the space H^p $p \in (0, 1)$ (Russian), Investigations on linear operators and the theory of functions, IX, Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov (LOMI). 1979;92:7–29.
[12] Deng GT, Qian T. Rational approximation of functions in hardy spaces. Complex Anal. Oper. Theorey. Forthcoming 2015.
[13] Deng GT. Complex analysis. 2010. Chinese.
[14] Walsh JL. Interpolation and approximation by rational functions in the complex domain. 4th ed. Vol. XX, American Mathematical Society Colloquium Publications. Providence (RI): American Mathematical Society; 1965.
[15] Aleksandrov AB. Approximation by rational functions and an analogue of M. Riesz's theorem on conjugate functions for the space L^p with $p \in (0, 1)$. Mat. Sb. (N.S.). 1978;107:3–19.
[16] Aleksandrov AB. Essays on non locally convex Hardy spaces. Vol. 864, Lecture notes in mathematics. Berlin: Springer-Verlag; 1981.