



# Schwarz Problems for Poly-Hardy Space on the Unit Ball

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**Abstract.** In this paper we study the Schwarz boundary value problems (for short BVP) for the poly-Hardy space defined on the unit ball of higher dimensional Euclidean space  $\mathbb{R}^n$ . We first discuss the boundary behavior of functions belonging to the poly-Hardy class. Then we construct the Schwarz kernel function, and describe the boundary properties of the Schwarz-type integrable operator. Finally, we study the Schwarz BVPs for the Hardy class and the poly-Hardy class on the unit ball of higher dimensional Euclidean space  $\mathbb{R}^n$ , and obtain the expressions of solutions, explicitly. As an application, the monogenic signals considered for the Hardy spaces defined on the unit sphere are reconstructed when the scalar- and sub-algebra-valued data are given, which is the extension of the analytic signals for the Hardy spaces on the unit circle of the complex plane.

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## 1. Introduction

One of the fundamental boundary value problems in the classical complex analysis is the Schwarz boundary value problem, for short, the Schwarz problem. When a real valued continuous function on the unit circle of the complex plane is given, an analytic function is found on the unit disc, satisfying that

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the boundary value of the real part of it on the unit circle coincides with the prescribed function. This problem is as a particular and simplest case of the Riemann–Hilbert problem first proposed by Riemann in 1851, where an analytic function is sought on the unit disc of the complex plane obtaining a given linear combination of its real and imaginary parts on the boundary. In 1872, Schwarz solved this particular problem long before the general Riemann–Hilbert problem was successfully dealt with, see Refs. e.g. [1, 2]. He made a modification of the classical Cauchy kernel, and the real part of his kernel, nowadays known as the Schwarz kernel, turns out to coincide with the classical Poisson kernel for the harmonic functions [3–5]. Later on, Refs. [6, 7] studied the Schwarz problems for the Hardy spaces and the poly-Hardy spaces on the unit disc of the complex plane, and got the explicit integrable solutions. Hereby, what interests us lies in the observation that the solutions to the Schwarz BVPs in the case of the Hardy spaces on the unit disc or on the half plane is equivalent to the reconstruction of the analytic signals (The analytic signals are regarded as the non-tangential boundary values of functions belonging to the Hardy spaces on the unit disc or on the real line [8, 9].) for the Hardy spaces on the unit circle or on the real line.

The Riemann–Hilbert BVPs including the Schwarz problem as a special case, for the monogenic functions and the poly-monogenic functions defined in the sub-domains of higher dimensional Euclidean space were discussed in Refs. e.g. [10–12], making full use of Clifford analysis, which is an elegant generalization of the theory of complex analysis into higher dimensional Euclidean space (seen in Refs. e.g. [13–15]). However, no study of the Riemann–Hilbert BVPs has been done to link to the monogenic signals defined on the sphere of higher dimensional Euclidean space  $\mathbb{R}^n$  (Here, the monogenic signals could be regarded as the non-tangential limits of monogenic functions on the boundary [8, 9].). On the other hand, although the monogenic signals in three dimensions were studied in Refs. [16–18], utilizing the Clifford algebra valued Hilbert transforms (see Refs. e.g. [8, 13, 15, 19]), to the authors’ knowledge, only special ones have been explicitly presented. Thus, the natural questions arise as what the expressions of all of them would look like and how they link to the Riemann–Hilbert BVPs for the poly-Hardy spaces in higher dimensional Euclidean space  $\mathbb{R}^n$ . These will be not only purely theoretical questions, since such problems are closely linked to physical applications like signal processing [8, 16] or problems in fluid mechanics [14, 15]. Furthermore, to solve these questions gives impetus to consider respective discrete problems [20, 21]. Moreover, in Refs. [8, 22], the unique vector-valued monogenic signal in four dimensions was constructed under the additional Cauchy-type harmonic conjugate condition. Motivated by these considerations, we will extend the connection between analytic signals and the Schwarz BVPs for Hardy spaces to the case of higher dimensional Euclidean space  $\mathbb{R}^n$ , and use them to reconstruct all of the monogenic signals considered on the unit sphere of higher dimensional Euclidean space  $\mathbb{R}^n$  in explicit form. Referring to the case of Hilbert-type BVPs for the

poly-Hardy spaces on the unit circle [7], we will mainly study the Schwarz BVPs for the poly-Hardy spaces defined on the unit ball of higher dimensional Euclidean space  $\mathbb{R}^n$ . We first introduce the poly-Hardy class on the unit ball of higher dimensional Euclidean space  $\mathbb{R}^n$ , characterize the boundary behavior of functions in the poly-Hardy class, construct the Schwarz kernel function in the higher dimensions, and describe the boundary properties of the Schwarz integrable operator. Then making full use of them, we solve the Schwarz BVPs for the Hardy class on the unit ball and for the poly-Hardy class on the unit ball, of higher dimensional Euclidean space  $\mathbb{R}^n$ , respectively. Finally, we get the explicit expressions of solutions. As an application, all of the monogenic signals considered on the unit sphere of higher dimensional Euclidean space  $\mathbb{R}^n$  are reconstructed when the scalar- and sub-algebra-valued boundary data are given.

The paper is organized as follows. In Sect. 2 we simply recall some basic facts about Clifford analysis which will be needed in the sequel. In Sect. 3, we will introduce the poly-Hardy space on the unit ball of higher dimensional Euclidean space  $\mathbb{R}^n$ , derive the decomposition theorem and characterize the boundary behavior of the functions in the poly-Hardy class. In Sect. 4, we introduce the theory of the Schwarz boundary value problem for the Hardy class on the unit ball of higher dimensional Euclidean space  $\mathbb{R}^n$  with  $\mathbb{L}_p(1 < p < +\infty)$ -integrable boundary data, and construct the Schwarz kernel function to solve it. In the last section we will discuss the Schwarz boundary value problems for the poly-Hardy class on the unit ball of higher dimensional Euclidean space  $\mathbb{R}^n$  with boundary data belonging to  $\mathbb{L}_p(1 < p < +\infty)$  space. For this class of Schwarz boundary value problem, based on the decomposition theorem, we get the explicit expressions of their solutions.

## 2. Preliminaries

In this section we simply review some basic facts about Clifford algebras which will be needed in the sequel. More details could be seen in Refs. e.g. [13–15] or monographs elsewhere.

Let the Euclidean space  $\mathbb{R}^n$  ( $n \in \mathbb{N}, n \geq 2$ ) possess an orthogonal basis  $\{e_1, e_2, \dots, e_n\}$  satisfying  $e_i^2 = -1$  for  $i = 1, 2, \dots, n$ ,  $e_i e_j + e_j e_i = 0$  for  $1 \leq i < j \leq n$ , and  $e_{h_1} e_{h_2} \dots e_{h_r} = e_{h_1 h_2 \dots h_r}$  for  $1 \leq h_1 < h_2 < \dots < h_r \leq n$ . Thus leads to the  $2^n$ -dimensional real Clifford algebra  $\mathbb{R}_n$ , having the basis  $\{e_{\mathcal{A}} : \mathcal{A} = \{h_1, \dots, h_r\} \in \mathcal{PN}\}$ , where  $\mathcal{N}$  stands for the set  $\{1, 2, \dots, n\}$  and  $\mathcal{PN}$  denotes the family of all order-preserving subsets of  $\mathcal{N}$ . In particular, we denote  $e_{\emptyset}$  as  $e_0$ , which is the identity element now written as 1. All of the scalars, 1-vectors and 2-vectors in  $\mathbb{R}_n$  are denoted by  $\mathbb{R}_0, \mathbb{R}_1(\cong \mathbb{R}^n)$  and  $\mathbb{R}_2$ , respectively. An arbitrary element of  $\mathbb{R}_1$  is denoted by  $x = \sum_{j=1}^n x_j e_j \triangleq \underline{x} + x_n e_n, x_j \in \mathbb{R}_0 (j = 1, 2, \dots, n)$ . Let  $\mathbb{R}_{n-1}$  be a sub-algebra of  $\mathbb{R}_n$  constructed by  $\{e_1, e_2, \dots, e_{n-1}\}$ , then  $\mathbb{R}_n = \mathbb{R}_{n-1} \oplus e_n \mathbb{R}_{n-1}$ . Given a  $\lambda = \lambda_1 + e_n \lambda_2 \in \mathbb{R}_n$  with  $\lambda_1, \lambda_2 \in \mathbb{R}_{n-1}$ , we define  $X^{(n)}(\lambda) = \lambda_1$  and  $Y^{(n)}(\lambda) = \lambda_2$ , and  $\text{Sc}(\lambda) = \lambda_0, \lambda_0 \in \mathbb{R}_0$

being the scalar part of  $\lambda$ . The conjugation is defined by  $\bar{a} = \sum_{\mathcal{A}} a_{\mathcal{A}} \bar{e}_{\mathcal{A}}$ ,  $\bar{e}_{\mathcal{A}} = (-1)^{\frac{k(k+1)}{2}} e_{\mathcal{A}}$ ,  $N(\mathcal{A}) = k$ ,  $a_{\mathcal{A}} \in \mathbb{R}_0$ . The inner product  $\langle \cdot, \cdot \rangle$  in  $\mathbb{R}_n$  is defined by putting  $\langle a, \bar{b} \rangle = [ab]_0$  for arbitrary  $a, b \in \mathbb{R}_n$ . The Clifford product of  $x, y \in \mathbb{R}_1$  is defined by  $xy = -\langle x, y \rangle + x \wedge y$ ,  $x \wedge y = \sum_{1 \leq i < j \leq n} e_i e_j (x_i y_j - x_j y_i)$ . It is easy to derive the norm on  $\mathbb{R}_n$  by  $|a| = \left( \sum_{\mathcal{A}} |a_{\mathcal{A}}|^2 \right)^{\frac{1}{2}} = \sqrt{\langle a, \bar{a} \rangle}$ . Especially, we have  $|x|^2 = \left( \sum_{j=1}^n x_j^2 \right)^{\frac{1}{2}} = \langle x, \bar{x} \rangle$ ,  $x = \sum_{j=1}^n e_j x_j \in \mathbb{R}_1$ . Moreover,  $x^{-1} \triangleq \bar{x}|x|^{-2}$  is the inverse of  $x \in \mathbb{R}_1 \setminus \{0\}$ , i.e.,  $xx^{-1} = x^{-1}x = 1$ . Up to the conjugation it corresponds to the Kelvin inverse in real analysis.

We introduce the Dirac operator  $\mathcal{D} = \sum_{j=1}^n e_j \partial_{x_j}$ , where  $\partial_{x_j}$ ,  $j = 1, 2, \dots, n$ , denotes the partial differential operator  $\frac{\partial}{\partial x_j}$ . It results in the decomposition of the negative Laplacian in  $\mathbb{R}_1$ , i.e.,  $\mathcal{D}^2 = -\sum_{j=1}^n \partial_{x_j}^2$ .

Let  $\mathbb{B} = \{x \in \mathbb{R}_1 : |x| < 1\}$  be the unit ball centered at the origin with its boundary  $\partial\mathbb{B} = \{x \in \mathbb{R}_1 : |x| = 1\}$ . The  $\mathbb{L}_p(1 < p < +\infty)$ -integrability, continuity, differentiability, and so on of a function  $\phi(x) = \sum_{\mathcal{A}} \phi_{\mathcal{A}}(x) e_{\mathcal{A}} : \mathbb{B} \rightarrow \mathbb{R}_n$ , are ascribed to each component  $\phi_{\mathcal{A}}(x) : \mathbb{B} \rightarrow \mathbb{R}_0$ , which is continuous, differentiable, and so on, respectively. Let  $\mathcal{C}^k(\mathbb{B}, \mathbb{R}_n)$  ( $k \geq 1, k \in \mathbb{N}$ ) denote the space of all  $k$ -times continuously differentiable functions defined in  $\mathbb{B}$ . The null solutions to the higher order Dirac operator  $\mathcal{D}^k$ , i.e.,  $\mathcal{D}^k \phi(x) = 0$ , are the so-called poly-monogenic functions, particularly monogenic when  $k = 1$ . The set of all these functions is denoted by  $\mathbb{H}_k(\mathbb{B}) = \{\phi : \mathbb{B} \rightarrow \mathbb{R}_n \mid \mathcal{D}^k \phi = 0\}$ .

### 3. Poly-Hardy Spaces

In this section we will introduce the poly-Hardy space on the unit ball of higher dimensional Euclidean space  $\mathbb{R}^n$ , get a decomposition theorem for it, and study its boundary behavior.

Let  $\phi$  be a function defined on the unit ball  $\mathbb{B}$  of higher dimensional Euclidean space  $\mathbb{R}^n$ , the monogenic Hardy space on the unit ball is defined as

$$\mathbb{H}^p(\mathbb{B}) = \left\{ \phi \in \mathbb{H}_1(\mathbb{B}) : |\phi|_{1,p} < +\infty \right\} \quad (1)$$

where  $|\phi|_{1,p} = \sup_{0 \leq r < 1} |\phi_r|_p$ , with

$$|\phi_r|_p = \left( \int_{\partial\mathbb{B}} |\phi_r(\eta)|^p dS_{\eta} \right)^{\frac{1}{p}}, \quad \phi_r(\eta) = \phi(r\eta), \quad 0 \leq r < 1, \eta \in \partial\mathbb{B}, 1 < p < +\infty. \quad (2)$$

Then the space  $\mathbb{H}^p(\mathbb{B})$  is a Banach space under the norm of (2). In the special case of  $p = 2$ ,  $\mathbb{H}^2(\mathbb{B})$  is a Hilbert space under the inner product  $(f, g) = \int_{\partial\mathbb{B}} \bar{f}(\eta) g(\eta) dS_{\eta}$ ,  $f, g \in \mathbb{H}^2(\mathbb{B})$ .

Let  $\mathbb{H}_k(\mathbb{B})$  ( $k \geq 1, k \in \mathbb{N}$ ) be as in Sect. 2. For arbitrary function  $\phi \in \mathbb{H}_k(\mathbb{B})$ , we define functions as follows

$$\phi^j = \mathcal{D}^j \phi, \quad x \in \mathbb{B}, \quad j = 0, 1, 2, \dots, k-1, \quad (3)$$

and, hereby,  $\phi^0 = \phi, x \in \mathbb{B}$ .

**Definition 3.1.** For arbitrary  $1 < p < +\infty$  and  $k \geq 1, k \in \mathbb{N}$ , the set of all functions defined on the unit ball  $\mathbb{B}$  of higher dimensional Euclidean space  $\mathbb{R}^n$ , satisfying

$$\left\{ \phi \in \mathbb{H}_k(\mathbb{B}) : |\phi^j|_{1,p} < +\infty, \quad j = 0, 1, 2, \dots, k-1 \right\}, \quad (4)$$

is the so-called poly-Hardy space of order  $k$  defined on the unit ball, where the norm  $|\cdot|_{1,p}$  is defined as (2) and  $\phi^j$  ( $j = 0, 1, 2, \dots, k-1$ ) given in (3). Therefore, such a poly-Hardy space defined on the unit ball of higher dimensional Euclidean space  $\mathbb{R}^n$  will be denoted as  $\mathbb{H}_k^p(\mathbb{B})$ .

It is obvious that the space  $\mathbb{H}_k^p(\mathbb{B})$  ( $1 < p < +\infty$ ) is linear. Define

$$|\phi|_{k,p} = \sum_{j=0}^{k-1} |\phi^j|_{1,p}, \quad \phi \in \mathbb{H}_k^p(\mathbb{B}), \quad (5)$$

where the norm  $|\cdot|_{1,p}$  is given in (2). In the following context we assume  $\mathbb{H}_1^p(\mathbb{B}) = \mathbb{H}^p(\mathbb{B})$  without confusion and for brevity. In particular, when  $k = 1$ , the norm  $|\cdot|_{k,p}$  in (5) reduces to the case of (2). When  $k = 2$ , the space  $\mathbb{H}_2^p(\mathbb{B})$  ( $1 < p < +\infty$ ) is actually the harmonic Hardy space defined on the unit ball  $\mathbb{B}$  of higher dimensional Euclidean space  $\mathbb{R}^n$ . Generally, when  $k = 2m, m \in \mathbb{N}$ , the space  $\mathbb{H}_k^p(\mathbb{B})$  ( $1 < p < +\infty$ ) is the poly-harmonic Hardy space defined on the unit ball  $\mathbb{B}$  of higher dimensional Euclidean space  $\mathbb{R}^n$ . From Definition 4.1, we get  $\mathbb{H}_k^p(\mathbb{B}) = \{\phi \in \mathbb{H}_k(\mathbb{B}) : |\phi|_{k,p} < +\infty\}$ . In order to avoid the trivial case, we assume  $k > 1, k \in \mathbb{N}$  without explanation.

Before characterizing the boundary behavior of the poly-Hardy space  $\mathbb{H}_k^p(\mathbb{B})$ , we need several lemmas as follows.

**Lemma 3.2** (see Refs. [23–27]). *The shifted Euler operator defined on the space  $\mathcal{C}^1(\mathbb{B}, \mathbb{R}_n)$  is given by*

$$E_s = sI + \sum_{j=1}^n x_j \partial_{x_j} \quad (s > 0), \quad (6)$$

with  $I$  being the identity operator defined on the space  $\mathcal{C}^1(\mathbb{B}, \mathbb{R}_n)$ . The operator  $I_s : \mathcal{C}(\mathbb{B}, \mathbb{R}_n) \rightarrow \mathcal{C}(\mathbb{B}, \mathbb{R}_n)$  is defined by

$$I_s \phi = \int_0^1 \phi(tx) t^{s-1} dt (s > 0), \quad x \in \mathbb{B}. \quad (7)$$

Then on the space  $\mathcal{C}^1(\mathbb{B}, \mathbb{R}_n)$  we have

- (i)  $E_s I_s = I_s E_s = I$ ,
- (ii)  $\mathcal{D} E_s \phi = E_{s+1} \mathcal{D} \phi$  and  $E_s x \phi = x E_{s+1} \phi$ .

Furthermore, if  $\phi \in \mathcal{C}^k(\mathbb{B}, \mathbb{R}_n)$ ,  $k \in \mathbb{N}$ , is a solution to  $\mathcal{D}^k \phi = 0$ , then  $E_s \phi$  and  $I_s \phi$  are both solutions to  $\mathcal{D}^k \phi = 0$ , where  $\mathcal{D}^k \phi \triangleq \mathcal{D}^{k-1}(\mathcal{D} \phi)$ .

**Lemma 3.3** (see Refs. [23, 26, 27]). Let  $j \in \mathbb{N}$  be arbitrary. If  $\phi \in \mathcal{C}^1(\mathbb{B}, \mathbb{R}_n)$  is monogenic, then

$$\mathcal{D}(x^j \phi) = \begin{cases} -2mx^{2m-1}\phi, & \text{if } j = 2m, \\ -2x^{2(m-1)}E_{\frac{n+1}{2} + [\frac{j}{2} - 1]}\phi, & \text{if } j = 2m - 1, \end{cases} \quad x \in \mathbb{B},$$

where  $m \in \mathbb{N}$ . Moreover, for  $l, p \in \mathbb{N}$  and  $2 \leq l \leq j$ , one gets

$$\mathcal{D}^l x^j \phi = C_{l,j} x^{j-l} E_{\frac{n+1}{2} + [\frac{j-l}{2}]}\dots E_{\frac{n+1}{2} + [\frac{j}{2} - 1]}\phi, \quad (8)$$

with

$$C_{l,j} = \begin{cases} 2^l m(m-1) \dots (m-p+1), & \text{if } j = 2m, l = 2p, \\ -2^l m(m-1) \dots (m-p), & \text{if } j = 2m, l = 2p+1, \\ 2^l (m-1) \dots (m-p), & \text{if } j = 2m-1, l = 2p, \\ -2^l (m-1) \dots (m-p+1), & \text{if } j = 2m-1, l = 2p-1. \end{cases}$$

Particularly, for  $l = j$ , we derive

$$\mathcal{D}^j x^j \phi = C_{j,j} E_{\frac{n+1}{2}} \dots E_{\frac{n+1}{2} + m - 1} \phi \quad \text{with} \quad C_{j,j} = \begin{cases} 2^j m!, & \text{if } j = 2m, \\ -2^j (m-1)!, & \text{if } j = 2m-1. \end{cases}$$

**Lemma 3.4.** (i) If  $\phi \in \mathbb{H}^p(\mathbb{B})$ , and  $E_s$  ( $s > 0$ ) defined by (6), then  $E_s \phi \in \mathbb{H}^p(\mathbb{B})$ .  
(ii) If  $\phi \in \mathbb{H}^p(\mathbb{B})$ , and  $I_s$  ( $s > 1$ ) defined by (7), then  $I_s \phi \in \mathbb{H}^p(\mathbb{B})$ .

*Proof.* (i) Since  $\phi \in \mathbb{H}^p(\mathbb{B})$ , due to Lemma 3.2, then  $\mathcal{D} E_s \phi = 0, x \in \mathbb{B}$ . From (5),  $|\phi|_{1,p} < +\infty$ . Therefore, using the Minkowski's inequality, we get  $E_s \phi \in \mathbb{H}^p(\mathbb{B})$ .

(ii) Since  $\phi \in \mathbb{H}^p(\mathbb{B})$ , then  $|\phi|_{1,p} < +\infty$ , that is,  $\sup_{0 \leq r < 1} (\int_{\partial \mathbb{B}} |\phi(r\eta)|^p d\eta)^{\frac{1}{p}} < +\infty$ .

As  $I_s \phi(x) = \int_0^1 \phi(tx) t^{s-1} dt$  ( $s > 1$ ),  $x \in \mathbb{B}$ , then applying the Hölder inequality, we have

$$\begin{aligned} \int_{\partial \mathbb{B}} |I_s \phi(x)|^p dx &\leq \int_{\partial \mathbb{B}} \int_0^1 |\phi(tx)|^p t^{p(s-1)} dt dx = \int_0^1 \int_{t\partial \mathbb{B}} |\phi(u)|^p du t^{p(s-1)-1} dt \\ &\leq |\phi|_{1,p}^p \int_0^1 t^{p(s-1)-1} dt = \frac{1}{p(s-1)} |\phi|_{1,p}^p < +\infty. \end{aligned} \quad (9)$$

Therefore,  $|I_s \phi(x)|_{1,p} < +\infty$ , that is,  $I_s \phi \in \mathbb{H}^p(\mathbb{B})$ .  $\square$

Now, let us derive the following theorem for the poly-Hardy space  $\mathbb{H}_k^p(\mathbb{B})$ .

**Theorem 3.5.** Let  $\mathbb{H}_k^p(\mathbb{B})$ ,  $1 < p < +\infty$ , be the poly-Hardy space of order  $k$  ( $k > 1$ ) defined on the unit ball of higher dimensional Euclidean space  $\mathbb{R}^n$ , then

$$\mathbb{H}_k^p(\mathbb{B}) = \mathbb{H}_1^p(\mathbb{B}) \oplus x\mathbb{H}_1^p(\mathbb{B}) \oplus \cdots \oplus x^{k-1}\mathbb{H}_1^p(\mathbb{B}), \quad (10)$$

where  $x^j\mathbb{H}_1^p(\mathbb{B}) = \{x^j\phi(x) : \phi \in \mathbb{H}_1^p(\mathbb{B})\}$ ,  $j = 0, 1, 2, \dots, k-1$ .

*Proof.* First of all, there needs to prove that

$$\mathbb{H}_k^p(\mathbb{B}) \subset \mathbb{H}_1^p(\mathbb{B}) \oplus x\mathbb{H}_1^p(\mathbb{B}) \oplus \cdots \oplus x^{k-1}\mathbb{H}_1^p(\mathbb{B}). \quad (11)$$

In fact, if  $\phi \in \mathbb{H}_k^p(\mathbb{B})$ , applying Definition 4.1 and Lemma 3.3 in [23] or [27], one has the unique decomposition

$$\phi = \sum_{j=0}^{k-1} x^j \phi_j, \quad x \in \mathbb{B}, \quad (12)$$

where each  $\phi_j$  is monogenic in  $\mathbb{B}$  ( $j = 0, 1, 2, \dots, k-1$ ) and given by

$$\begin{cases} \phi_0 = (I - x\mathcal{Q}_1\mathcal{D})(I - x^2\mathcal{Q}_2\mathcal{D}^2) \cdots (I - x^{k-1}\mathcal{Q}_{k-1}\mathcal{D}^{k-1})\phi, \\ \phi_1 = \mathcal{Q}_1\mathcal{D}(I - x^2\mathcal{Q}_2\mathcal{D}^2) \cdots (I - x^{k-1}\mathcal{Q}_{k-1}\mathcal{D}^{k-1})\phi, \\ \vdots \\ \phi_{k-2} = \mathcal{Q}_{k-2}\mathcal{D}^{k-2}(I - x^{k-1}\mathcal{Q}_{k-1}\mathcal{D}^{k-1})\phi, \\ \phi_{k-1} = \mathcal{Q}_{k-1}\mathcal{D}^{k-1}\phi, \end{cases}$$

with  $\mathcal{Q}_j = \frac{1}{a_j} I_{\frac{n}{2}} I_{\frac{n}{2}+1} \cdots I_{\frac{n}{2}+[\frac{j-1}{2}]}$ ,  $a_j = (-2)^k [\frac{j}{2}]!$  for  $j = 1, 2, \dots, k-1$ , and

$$[s] = \begin{cases} q, & \text{if } q \in \mathbb{N}, \\ q+1, & \text{if } s = q+t, q \in \mathbb{N}, 0 < t < 1. \end{cases}$$

Moreover, by Lemma 3.3, we get

$$\mathcal{D}\phi = \sum_{j=1}^{k-1} \mathcal{D}(x^j \phi_j) = \begin{cases} -\sum_{j=1}^{k-1} 2mx^{j-1}\phi_j, & \text{if } j = 2m, \\ -\sum_{j=1}^{k-1} 2x^{j-1}E_{\frac{n+1}{2}+[\frac{j}{2}]-1}\phi_j, & \text{if } j = 2m-1, \end{cases} \quad (13)$$

and for  $l \in \mathbb{N}$  and  $2 \leq l \leq j$ ,

$$\mathcal{D}^l \phi = \sum_{j=1}^{k-1} \mathcal{D}^l(x^j \phi_j) = \sum_{j=1}^{k-1} C_{l,j} x^{j-l} E_{\frac{n+1}{2}+[\frac{j-l}{2}]} \cdots E_{\frac{n+1}{2}+[\frac{j}{2}]-1} \phi_j, \quad x \in \mathbb{B}. \quad (14)$$

This implies

$$\begin{aligned} \phi_{k-1} &= C_{k-1,k-1}^{-1} I_{\frac{n+1}{2}+m-1} \cdots I_{\frac{n+1}{2}} \phi^{k-1}, \quad x \in \mathbb{B} \text{ with} \\ C_{k-1,k-1} &= \begin{cases} 2^{k-1}m!, & \text{if } k-1 = 2m, \\ -2^{k-1}(m-1)!, & \text{if } k-1 = 2m-1. \end{cases} \end{aligned}$$

Since  $\phi \in \mathbb{H}_k^p(\mathbb{B})$ , then  $|\phi^{k-1}|_{1,p} < +\infty$ . Therefore, associating with Lemma 3.4, we have  $|\phi_{k-1}|_{1,p} < +\infty$ . Secondly, we also get

$$\begin{aligned} \phi_{k-2} &= C_{k-2,k-2}^{-1} I_{\frac{n+1}{2}+m-1} \dots I_{\frac{n+1}{2}} \\ &\quad \times \left( \phi^{k-2} - C_{k-2,k-1} x E_{\frac{n+1}{2}+[\frac{1}{2}]} \dots E_{\frac{n+1}{2}+[\frac{k-2}{2}]-1} \phi_{k-1} \right), \quad x \in \mathbb{B}. \end{aligned} \quad (15)$$

From (12), we know  $\mathcal{D}(\phi^{k-2} - C_{k-2,k-1} x E_{\frac{n+1}{2}+[\frac{1}{2}]} \dots E_{\frac{n+1}{2}+[\frac{k-2}{2}]-1} \phi_{k-1}) = 0, x \in \mathbb{B}$ . Making full use of Lemmas 3.2 and 3.4, we have  $|x E_{\frac{n+1}{2}+[\frac{1}{2}]} \dots E_{\frac{n+1}{2}+[\frac{k-2}{2}]-1} \phi_{k-1}|_{1,p} < +\infty$ . Associating with  $|\phi_{k-1}|_{1,p} < +\infty$  and  $|\phi^{k-2}|_{1,p} < +\infty$ , we get  $|\phi_{k-2}|_{1,p} < +\infty$ .

Applying (14) and Lemma 3.4, by iteration of the same procedure, one has  $|\phi_j|_{1,p} < +\infty, j = 0, 1, \dots, k-1$ .

This implies  $\phi_j \in \mathbb{H}_1^p(\mathbb{B}), j = 0, 1, 2, \dots, k-1$ , which gives the validity of (11). Thus, we get

$$\mathbb{H}_k^p(\mathbb{B}) = \mathbb{H}_1^p(\mathbb{B}) + x \mathbb{H}_1^p(\mathbb{B}) + \dots + x^{k-1} \mathbb{H}_1^p(\mathbb{B}). \quad (16)$$

Finally, let  $0 = \sum_{j=0}^{k-1} x^j \phi_j(x)$  with  $\phi_j \in \mathbb{H}_1^p(\mathbb{B}), j = 0, 1, 2, \dots, k-1$ . Starting with (3), one gets that  $\phi_j \equiv 0, j = 0, 1, 2, \dots, k-1, x \in \mathbb{B}$ . The proof of the result is completed.  $\square$

Due to Theorem 3.5, we might characterize the boundary behavior of functions belonging to the poly-Hardy space  $\mathbb{H}_k^p(\mathbb{B})$ .

**Theorem 3.6.** *If  $\phi \in \mathbb{H}_k^p(\mathbb{B}) (1 < p < +\infty)$ , then  $\phi$  has the non-tangential limit  $\phi^+$  almost everywhere on the sphere  $\partial\mathbb{B}$  under the  $|\cdot|_p$ -norm given by (2), and*

$$\lim_{r \rightarrow 1} |\phi^+ - \phi_r|_p = 0, \quad (17)$$

where  $\phi_r$  is same with (2).

*Proof.* Since  $\phi \in \mathbb{H}_k^p(\mathbb{B}) (1 < p < +\infty)$ , making full use of Theorem 3.5, we have

$$\phi(x) = \sum_{j=0}^{k-1} x^j \phi_j(x), \phi_j \in \mathbb{H}_1^p(\mathbb{B}), \quad j = 0, 1, 2, \dots, k-1. \quad (18)$$

Applying the boundary behavior of the monogenic Hardy space  $\mathbb{H}_1^p(\mathbb{B})$  (see Theorem 7.9 of Chapter 2 in Ref. [15]),  $\phi_j (j = 0, 1, 2, \dots, k-1)$  has non-tangential limit  $\phi_j^+(t)$  almost everywhere on the sphere  $\partial\mathbb{B}$ , and

$$\lim_{r \rightarrow 1} |\phi_j^+ - (\phi_j)_r|_p = 0, \quad (19)$$

where  $(\phi_j)_r (j = 0, 1, 2, \dots, k-1)$  is defined similarly to (2). Hence, associating with the decomposition (18), we obtain

$$\phi^+(t) = \sum_{j=0}^{k-1} t^j \phi_j^+(t), \quad \text{a.e. } t \in \partial\mathbb{B}. \quad (20)$$

This implies that  $\phi$  has non-tangential limit  $\phi^+$  almost everywhere on the sphere  $\partial\mathbb{B}$ . Applying (19) and (20), one has

$$|\phi^+ - \phi_r|_p \leq \sum_{j=0}^{k-1} \left[ \left| \phi_j^+ - (\phi_j)_r \right|_p + (1 - r^j) \left| (\phi_j)_r \right|_p \right], \quad (21)$$

which leads to the validity of (17) by (19). It follows the result.  $\square$

**Corollary 3.7.** *If  $\phi \in \mathbb{H}_k^p(\mathbb{B}) (1 < p < +\infty)$ , then  $\mathcal{D}^j \phi$  have the non-tangential limit  $(\mathcal{D}^j \phi)^+$ ,  $j = 1, 2, \dots, k-1$ , almost everywhere on the sphere  $\partial\mathbb{B}$  under the  $|\cdot|_p$ -norm given by (2). This implies that the non-tangential limits of the right-hand sides of (13) and (14) all exit under the  $|\cdot|_p$ -norm given by (2).*

*Proof.* Since  $\phi \in \mathbb{H}_k^p(\mathbb{B}) (1 < p < +\infty)$ , associating with (5), then  $\mathcal{D}^j \phi \in \mathbb{H}_{k-j}^p(\mathbb{B}) (1 < p < +\infty)$ ,  $j = 1, 2, \dots, k-1$ . According to Theorem 3.6, it follows the result.  $\square$

*Remark 3.8.* Theorem 3.6 gives the characterization of boundary behavior of the poly-Hardy space  $\mathbb{H}_k^p(\mathbb{B}) (1 < p < +\infty)$ . The boundary value  $\phi^+$  seen in (20) is the so-called non-tangential boundary value of  $\phi$ . In what follows, the symbol  $\phi^+$  will be understood as the non-tangential boundary value of  $\phi$  if no explanation. Moreover, if  $k = 1$ , when the dimension of the space considered is  $n = 4$ , it reduces to that in [8]. When the dimension of the space considered is  $n = 2$ , it reduces to that in [7].

## 4. Schwarz BVPs for Hardy Space

In this section, we study the non-homogeneous Schwarz BVPs for the Hardy class on the unit ball of higher dimensional Euclidean space  $\mathbb{R}^n$  with  $\mathbb{L}_p (1 < p < +\infty)$ -integrable boundary data. For this kind of Schwarz BVPs, we obtain the explicit expressions of the solutions.

First, let us give a theorem about the harmonic conjugate of a monogenic function defined on the unit ball of higher dimensional Euclidean space  $\mathbb{R}^n$ .

**Definition 4.1.** Let  $\Omega$  be a sub-domain of  $\mathbb{R}^n (\mathbb{R}^n \cong \mathbb{R}_1)$ , and  $\phi, \psi \in \mathcal{C}^2(\Omega, \mathbb{R}_{n-1})$  are harmonic. If the function  $\phi + e_n \psi$  is monogenic in  $\Omega$ , then  $\phi$  is called harmonic conjugate to  $\psi$ .

Obviously, then also  $\psi$  is harmonic conjugate to  $\phi$ . According to Definition 4.1, for the function

$$P(x, y) = \frac{1 - |x|^2}{\omega_n |y - x|^n} \quad (n \geq 2, n \in \mathbb{N}), \quad x \in \mathbb{B}, y \in \partial\mathbb{B}, \quad (22)$$

where  $\omega_n$  denotes the area of unit sphere  $\partial\mathbb{B}$ , a harmonic conjugate function  $Q(x, y)$  of  $P(x, y)$  could be given by

$$\begin{aligned} Q(x, y) = \frac{1}{\omega_n} & \left[ \frac{(2-n)(\underline{x}-\underline{y})}{|\underline{y}-\underline{x}|^{n-1}} + \frac{(n-1)(\underline{x}-\underline{y})\langle \underline{x}-\underline{y}, 2\underline{x} \rangle}{|\underline{y}-\underline{x}|^{n+1}} \right. \\ & - \frac{2\underline{x}}{|\underline{y}-\underline{x}|^{n-1}} F\left(\frac{n}{2}, \frac{x_n - y_n}{|\underline{y}-\underline{x}|^n}\right) \\ & \left. + \frac{2(\underline{x}-\underline{y})}{|y-x|^n} \left( \frac{(x_n - y_n)\langle \underline{x}-\underline{y}, \underline{x} \rangle}{|\underline{y}-\underline{x}|^2} - x_n \right) \right], \end{aligned} \quad (23)$$

where

$$F(\alpha, t) = \begin{cases} \frac{1}{2\alpha-1} \frac{t}{(1+t^2)^{\alpha-1}} + \frac{2\alpha-3}{2\alpha-2} F(\alpha-1, t), & 2\alpha \in \mathbb{N}+2, \alpha \neq 1, \\ \frac{1}{2\alpha-2} \frac{t}{(1+t^2)^{\alpha-1}} F\left(\frac{3}{2}-\alpha, 1; 2-\alpha; 1+t^2\right), & \alpha \in \mathbb{N}+\frac{3}{2}, \\ \frac{1}{2\alpha-2} \frac{t}{(1+t^2)^{\alpha-1}} \left( \sum_{k=0}^{\alpha-2} \frac{(\frac{3}{2}-\alpha)_k}{(2-\alpha)_k} (1+t^2)^k \right) \\ \quad + \frac{1}{2\alpha-2} \frac{(\frac{3}{2}-\alpha)_{\alpha-2}}{(2-\alpha)_{\alpha-2}} \arctan t, & \alpha \in \mathbb{N}+1, \end{cases} \quad (24)$$

with

$$(a)_k \triangleq \begin{cases} 1, & k=0, \\ a(a+1)\dots(a+k-1), & k \in \mathbb{N}, \end{cases} \quad (25)$$

and  $F(a, b; c; t)$  stands for the hyper-geometric function, see Refs. e.g. [12, 28]. Similar results could be also seen in Refs. e.g. [28–33].

**Theorem 4.2.** *Let  $P(x, y), Q(x, y)$  be as (22) and (23), respectively. Introduce the function*

$$K(x, y) = P(x, y) + e_n Q(x, y), \quad x \in \mathbb{B}, y \in \partial\mathbb{B}. \quad (26)$$

*If  $f \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_{n-1})$ , then*

$$p.v. \int_{\partial\mathbb{B}} Q(x, y) f(y) dS_y, \quad x \in \partial\mathbb{B} \quad (27)$$

*is well defined, where p.v. is short for the Cauchy principle value. Moreover,*

$$Qf(x) \triangleq p.v. \int_{\partial\mathbb{B}} Q(x, y) f(y) dS_y \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_{n-1}). \quad (28)$$

*Proof.* Observing that  $Q(x, y)$  of the singular integrable operator (27) behaves like the Cauchy kernel  $G_1(y - x)$ , which is given by

$$G_1(y - x) = \frac{1}{\omega_n} \frac{\overline{y - x}}{|y - x|^n}, \quad y \neq x, \quad (29)$$

at the singular point when we consider the Cauchy principle value of the singular integrable operator (27), we get (28).  $\square$

Furthermore, Theorem 4.2 leads to

$$\text{p.v.} \int_{\partial \mathbb{B}} K(x, y) f(y) dS_y \in \mathbb{L}_p(\partial \mathbb{B}, \mathbb{R}_n), \quad (30)$$

and

$$\begin{aligned} & \lim_{\mathbb{B} \ni x \rightarrow y \in \partial \mathbb{B}} \int_{\partial \mathbb{B}} K(x, y) f(y) dS_y \\ &= f(y) + e_n \text{ p.v.} \int_{\partial \mathbb{B}} Q(x, y) f(y) dS_y \in \mathbb{L}_p(\partial \mathbb{B}, \mathbb{R}_n). \end{aligned} \quad (31)$$

*Remark 4.3.* Also, we could give the definition of the harmonic conjugate, starting with a scalar-valued function. Let  $\Omega$  be a sub-domain of  $\mathbb{R}^n$  ( $\mathbb{R}^n \cong \mathbb{R}_1$ ) with Lipschitz boundary  $\partial \Omega$ . If  $\phi \in \mathcal{C}^2(\Omega, \mathbb{R}_n)$  is a scalar-valued harmonic, and  $\psi \in \mathcal{C}^2(\Omega, \mathbb{R}_n)$  is also harmonic whose scalar part is zero, satisfying

$$\phi + \psi = \frac{1}{\omega_n} \int_{\partial \Omega} G_1(y - x) d\sigma_y f(y) \text{ for a scalar-valued } f, \quad x \in \mathbb{R}^n \setminus \partial \Omega, \quad (32)$$

where  $\omega_n$  is the area of unit sphere of  $\mathbb{R}^n$  ( $\mathbb{R}^n \cong \mathbb{R}_1$ ) and  $G_1(y - x)$  is given by (29), then  $\phi$  is so-called to be the Cauchy-type harmonic conjugate of  $\psi$ . In particular, for the function

$$P(x, y) = \frac{1}{\omega_n} \frac{1 - |x|^2}{|y - x|^n}, \quad (33)$$

its unique Cauchy-type harmonic conjugate is

$$\tilde{Q}(x, y) = \frac{1}{\omega_n} \left( \frac{2}{|y - x|^n} - \frac{m - 2}{r^{m-1}} \int_0^r \frac{\rho^{m-1}}{|\rho \zeta - y|^n} d\rho \right) x \wedge y, \quad (34)$$

where  $x = r\zeta, y, \zeta \in \partial \mathbb{B}, 0 \leq r < 1$ . Results similar to Theorem 4.2 could be seen in Ref. [22]. Both of Theorem 4.2 and Remark 4.3 allow us to define a Hilbert transform  $\mathcal{H}(f) = Qf$  on the sphere  $\partial \mathbb{B}$ .

Next, let us consider our Schwarz boundary value problem for Hardy space.

**Problem I.** Given the boundary data  $f \in \mathbb{L}_p(\partial \mathbb{B}, \mathbb{R}_{n-1})$ , find a function  $\phi \in \mathbb{H}^p(\mathbb{B})$ , satisfying the Schwarz boundary value condition

$$X^{(n)}\{\lambda\phi\} = f, \quad \text{a.e. } x \in \partial \mathbb{B}, \quad (35)$$

where  $\lambda \in \mathbb{R}_{n-1}$  is an arbitrary invertible constant with its inverse  $\lambda^{-1}$ .

**Theorem 4.4.** *For the given function  $f \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_{n-1})$ , Problem (35) has the general solution and its explicit form is given by*

$$\phi(x) = \int_{\partial\mathbb{B}} K(x, y) \lambda^{-1} f(y) dS_y + \lambda^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}, \quad x \in \mathbb{B}, \quad (36)$$

where  $K(x, y)$  is given by (26), and

$$V_{l_1, \dots, l_s}(\underline{x}) = \frac{1}{s!} \sum_{\pi(l_1, \dots, l_s)} z_{l_1} z_{l_2} \dots z_{l_s}$$

with  $z_s = x_s e_0 - x_1 e_1 e_s$  ( $s = 1, 2, \dots, n-1$ ),  $a_{l_1, \dots, l_s} \in \mathbb{R}_{n-1}$  being constants, respectively.

*Proof.* From Theorem 4.2, we know that

$$\psi_1(x) = \int_{\partial\mathbb{B}} K(x, y) \lambda^{-1} f(y) dS_y \in \mathbb{H}^p(\mathbb{B}) \quad (37)$$

satisfies the condition of Problem (35). Hence, we only need to consider the homogeneous Schwarz problem

$$X^{(n)}\{\Phi\} = 0, \quad \text{a.e. } x \in \partial\mathbb{B}. \quad (38)$$

In what follows, we will first prove that if there exists a solution to the homogeneous case of Problem (38), then it could be given explicitly.

In fact, Let  $\Phi$  is a solution to the homogeneous case of Problem (35), then we have  $\mathcal{D}\Phi = 0, x \in \mathbb{B}$  and

$$\begin{cases} \Delta(X^{(n)}\Phi) = 0, & x \in \mathbb{B}, \\ (X^{(n)}\Phi) = 0, & \text{a.e. } x \in \partial\mathbb{B}. \end{cases} \quad (39)$$

Therefore,  $X^{(n)}\Phi \equiv 0, x \in \mathbb{B}$ , that is,  $\Phi = e_n Y^{(n)}\Phi, x \in \mathbb{B}$ .

Secondly, beginning with Definition 4.1, we get that the solution to Problem (38) is presented by

$$Y^{(n)}\Phi(x) = \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}, \quad a_{l_1, \dots, l_s} \in \mathbb{R}_{n-1}, \quad (40)$$

where  $V_{l_1, \dots, l_s}(\underline{x}) = \frac{1}{s!} \sum_{\pi(l_1, \dots, l_s)} z_{l_1} z_{l_2} \dots z_{l_s}$ ,  $z_s = x_s e_0 - x_1 e_1 e_s$  ( $s = 1, 2, \dots, n-1$ ),  $a_{l_1, \dots, l_s} \in \mathbb{R}_{n-1}$ .

Thus the result follows. □

*Remark 4.5.* When  $p = 2, \lambda = 1$ , Problem (35) changes into the case

$$\begin{cases} \phi \in \mathbb{H}^2(\mathbb{B}), \\ X^{(n)}\{\phi\} = f, & \text{a.e. } x \in \partial\mathbb{B}, \end{cases} \quad (41)$$

where  $f \in \mathbb{L}_2(\partial\mathbb{B}, \mathbb{R}_{n-1})$ . Its solving actually corresponds to reconstruct the real algebra  $\mathbb{R}_n$ -valued monogenic signals of the Hardy space defined on  $\mathbb{B}$

when a real sub-algebra valued  $\mathbb{R}_{n-1}$  boundary datum is given on the sphere  $\partial\mathbb{B}$ .

Similarly, when a scalar-valued initial datum is given on the sphere  $\partial\mathbb{B}$ , to find a paravector valued monogenic signal is equivalent to solve a Schwarz BVP as follows.

**Problem II.** Find a function  $\phi : \mathbb{B} \rightarrow \mathbb{R}_0 \oplus \mathbb{R}_2$ , satisfying

$$\begin{cases} \phi \in \mathbb{H}^2(\mathbb{B}), \\ \text{Sc}\{\phi\} = f, \quad \text{a.e. } x \in \partial\mathbb{B}, \end{cases} \quad (42)$$

where  $f \in \mathbb{L}_2(\partial\mathbb{B}, \mathbb{R}_0)$ . Its unique solution is presented by

$$\phi(x) = \int_{\partial\mathbb{B}} \tilde{K}(x, y) f(y) dS_y, \quad x \in \mathbb{B}, \quad (43)$$

where

$$\tilde{K}(x, y) = P(x, y) + \tilde{Q}(x, y), \quad x \in \mathbb{B}, y \in \partial\mathbb{B}, \quad (44)$$

with  $\tilde{Q}(x, y)$  seen in (34).

*Remark 4.6.* When the dimension of the considered space is equal to 2, Problems (35) and (42) are both changed into the classical Schwarz BVP [7]: to find a function  $\phi : \mathbb{D} \rightarrow \mathbb{C}$ , where  $\mathbb{C} \cong \{x_0 + e_1 x_1 : x_0, x_1 \in \mathbb{R}_0\}$  or  $\mathbb{R}_2$ ,  $\mathbb{D} = \{|x| < 1 : x = x_0 + e_1 x_1, x_0, x_1 \in \mathbb{R}_0\}$  or  $\mathbb{D} = \{|x| < 1 : x = x_1 e_1 + e_2 x_2, x_1, x_2 \in \mathbb{R}_0\}$  with its boundary  $\partial\mathbb{D}$ , satisfying

$$\begin{cases} \phi \in \mathbb{H}^p(\mathbb{D}, \mathbb{C}), \\ \text{Re}\{\phi\} = f, \quad \text{a.e. } x \in \partial\mathbb{D}, \end{cases} \quad (45)$$

where  $\text{Re}(x) = x_0$  or  $x_1, x \in \mathbb{C}$  with  $\mathbb{C}$  as above, which corresponds to the real part of a complex number in the planar complex analysis. After direct observation,  $K(x, y)$  and  $\tilde{K}(x, y)$  defined by (26) and (44) play the role analogous to the classical Schwarz kernel function in Problems (35) and (42), respectively. Therefore, hereby, they are the so-called Schwarz kernel functions in the higher dimensions, for short, still the Schwarz kernel functions. Thus, Theorem 4.2 gives the characterization of the boundary properties of the Schwarz integrable operator, which corresponds to the Plemelj–Sokhotsky formula in complex analysis.

## 5. Schwarz BVPs for Poly-Hardy Space

In this section, we extend the results obtained in the previous section. We study the Schwarz BVP for the poly-Hardy class on the unit ball with  $\mathbb{L}_p$  ( $1 < p < +\infty$ )-integrable boundary data, and derive the explicit expressions of the solutions.

**Problem III.** Given the boundary data  $f_j \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_{n-1}), j = 0, 1, 2, \dots, k-1$ , find a function  $\phi : \mathbb{B} \rightarrow \mathbb{R}_n$  satisfying the Schwarz boundary value conditions

$$\begin{cases} \phi \in \mathbb{H}_k^p(\mathbb{B}), & 1 < p < +\infty, \\ X^{(n)}\{\lambda\phi\} = f_0, & \text{a.e. } x \in \partial\mathbb{B}, \\ \vdots & \vdots \\ X^{(n)}\{\lambda\mathcal{D}^{k-1}\phi\} = f_{k-1}, & \text{a.e. } x \in \partial\mathbb{B}, \end{cases} \quad (46)$$

where  $\lambda \in \mathbb{R}_{n-1}$  is a constant with its inverse  $\lambda^{-1}$ .

**Theorem 5.1.** For the given function  $f_j \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_{n-1}), j = 0, 1, 2, \dots, k-1$ , Problem (46) has the general solution and its explicit form is given by

$$\phi(x) = \sum_{j=0}^{k-1} x^j \phi_j(x), x \in \mathbb{B}, \quad (47)$$

with

$$\phi_j(x) = \begin{cases} \int_{\partial\mathbb{B}} K(x, y) \lambda^{-1} \tilde{f}_0(y) dS_y + \lambda^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}^{(0)}, & \text{if } j = 0, \\ C_{1,1}^{-1} I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} K(x, y) \lambda^{-1} \tilde{f}_1(y) dS_y + \lambda^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}^{(1)} \right), & \text{if } j = 1, \\ C_{j,j}^{-1} I_{\frac{n+1}{2} + [\frac{j}{2}] - 1} \dots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} K(x, y) \lambda^{-1} \tilde{f}_j(y) dS_y + \lambda^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}^{(j)} \right), & \text{if } 2 \leq j \leq k-1, \end{cases} \quad (48)$$

where  $K(x, y)$  as (26),  $a_{l_1, \dots, l_s}^{(j)} \in \mathbb{R}_{n-1}, j = 0, 1, 2, \dots, k-1$  are all constants, and for arbitrary  $x \in \partial\mathbb{B}$ ,

$$\tilde{f}_j(x) = \begin{cases} f_0(x) - \sum_{l=1}^{k-1} C_{1,l}^{-1} X^{(n)} \left\{ x^l \left( \int_{\partial\mathbb{B}} K(x, y) \lambda^{-1} \tilde{f}_l(y) dS_y + \lambda^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}^{(l)} \right) \right\}, & \text{if } j = 0, \\ f_1(x) - \sum_{l=2}^{k-1} C_{1,l} C_{l,l}^{-1} X^{(n)} \left\{ x^l I_{\frac{n+1}{2} + [\frac{l}{2}] - 1} \dots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} K(x, y) \lambda^{-1} \tilde{f}_l(y) dS_y \right. \right. \\ \left. \left. + \lambda^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}^{(l)} \right) \right\}, & \text{if } j = 1, k \text{ odd}, \\ f_1(x) - \sum_{l=2}^{k-1} C_{1,l} C_{l,l}^{-1} X^{(n)} \left\{ x^l I_{\frac{n+1}{2} + [\frac{l}{2}]} \dots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} K(x, y) \lambda^{-1} \tilde{f}_l(y) dS_y \right. \right. \\ \left. \left. + \lambda^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}^{(l)} \right) \right\}, & \text{if } j = 1, k \text{ even}, \\ f_j(x) - \sum_{l=j+1}^{k-1} C_{j,l} C_{l,l}^{-1} X^{(n)} \left\{ x^l I_{\frac{n+1}{2} + [\frac{l-j}{2}]} \dots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} K(x, y) \lambda^{-1} \tilde{f}_l(y) dS_y \right. \right. \\ \left. \left. + \lambda^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}^{(l)} \right) \right\}, & \text{if } 2 \leq j \leq k-1. \end{cases} \quad (49)$$

*Proof.* Since  $\phi \in \mathbb{H}_k^p(\mathbb{B}), 1 < p < +\infty$ , in virtue of Theorem 3.5, there exists unique functions  $\phi_j$  satisfying  $\phi_j \in \mathbb{H}^p(\mathbb{B}), j = 0, 1, 2, \dots, k-1$  and  $\phi = \sum_{j=0}^{k-1} x^j \phi_j$ .

Using Lemma 3.2 for arbitrary  $l \in \mathbb{N}, l \leq j$ , we have

$$\mathcal{D}^l \phi = \sum_{j=0}^{k-1} \mathcal{D}^l (x^j \phi_j) = \sum_{j=l}^{k-1} C_{l,j} x^{j-l} \left( E_{\frac{n+1}{2} + [\frac{j-l}{2}]} \dots E_{\frac{n+1}{2} + [\frac{j}{2}] - 1} \phi_j \right). \quad (50)$$

Then, making use of Theorem 3.6 and Corollary 3.7, Problem (46) is equivalent to the case

$$\left\{ \begin{array}{ll} \phi_j \in \mathbb{H}^p(\mathbb{B}), & j = 0, 1, 2, \dots, k-1, \\ X^{(n)} \left\{ \lambda x^j \phi_j \right\} = f_0, & x \in \partial\mathbb{B}, \\ X^{(n)} \left\{ \lambda \sum_{j=1}^{k-1} C_{1,j} x^{j-1} E_{\frac{n+1}{2} + [\frac{j-1}{2}]} \dots E_{\frac{n+1}{2} + [\frac{j}{2}] - 1} \phi_j \right\} = f_1, & x \in \partial\mathbb{B}, \\ \vdots & \vdots \\ X^{(n)} \left\{ \lambda \sum_{j=l}^{k-1} C_{l,j} x^{j-l} E_{\frac{n+1}{2} + [\frac{j-l}{2}]} \dots E_{\frac{n+1}{2} + [\frac{j}{2}] - 1} \phi_j \right\} = f_l, & x \in \partial\mathbb{B}, \\ \vdots & \vdots \\ X^{(n)} \left\{ \lambda C_{k-1,k-1} E_{\frac{n+1}{2}} \dots E_{\frac{n+1}{2} + [\frac{k-1}{2}] - 1} \phi_{k-1} \right\} = f_{k-1}, & x \in \partial\mathbb{B}. \end{array} \right. \quad (51)$$

We now proceed by induction. First, we consider

$$\left\{ \begin{array}{l} \phi_{k-1} \in \mathbb{H}^p(\mathbb{B}), \\ X^{(n)} \left\{ \lambda C_{k-1,k-1} E_{\frac{n+1}{2}} \dots E_{\frac{n+1}{2} + [\frac{k-1}{2}] - 1} \phi_{k-1} \right\} = f_{k-1}, \end{array} \right. \quad x \in \partial\mathbb{B}. \quad (52)$$

By Theorem 4.4, Problem (52) has the explicit solution

$$\begin{aligned} \phi_{k-1}(x) &= C_{k-1,k-1}^{-1} I_{\frac{n+1}{2} + [\frac{k-1}{2}] - 1} \dots I_{\frac{n+1}{2}} \\ &\times \left( \int_{\partial\mathbb{B}} K(x, y) \lambda^{-1} f_{k-1}(y) dS_y + \lambda^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}^{(k-1)} \right), \quad x \in \mathbb{B}. \end{aligned} \quad (53)$$

Applying Lemma 3.4, and (37), we get that

$$\begin{aligned} &I_{\frac{n+1}{2} + [\frac{k-1}{2}] - 1} \dots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} K(t, y) \lambda^{-1} f_{k-1}(y) dS_y + \lambda^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{t}) a_{l_1, \dots, l_s}^{(k-1)} \right) \\ &= \lim_{\mathbb{B} \ni x \rightarrow t \in \partial\mathbb{B}} \left\{ I_{\frac{n+1}{2} + [\frac{k-1}{2}] - 1} \dots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} K(x, y) \lambda^{-1} f_{k-1}(y) dS_y \right. \right. \\ &\quad \left. \left. + \lambda^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}^{(k-1)} \right) \right\}, \end{aligned}$$

belongs to  $\mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_n)$ . Therefore,  $\phi_{k-1} \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_n)$ , thus leading to  $t\phi_{k-1} \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_n)$ .

Then, we solve the second boundary value problem

$$\left\{ \begin{array}{l} \phi_{k-2} \in \mathbb{H}^p(\mathbb{B}), \\ X^{(n)} \left\{ \lambda C_{k-2,k-2} \left( E_{\frac{n+1}{2}} \dots E_{\frac{n+1}{2} + [\frac{k-2}{2}] - 1} \phi_{k-2} \right) \right\} = \tilde{f}_{k-2}, \end{array} \right. \quad x \in \partial\mathbb{B}. \quad (54)$$

where

$$\begin{aligned} \tilde{f}_{k-2}(x) = f_{k-2}(x) - C_{k-1,k-2}C_{k-1,k-1}^{-1}X^{(n)} \left\{ xI_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} K(x,y)\lambda^{-1}f_{k-1}(y)dS_y \right. \right. \\ \left. \left. + \lambda^{-1}e_n \sum_{s=0}^{+\infty} V_{l_1,\dots,l_s}(\underline{x})a_{l_1,\dots,l_s}^{(k-1)} \right) \right\} \end{aligned} \quad (55)$$

on  $\partial\mathbb{B}$ . Hence, Problem (54) has the explicit solution

$$\begin{aligned} \phi_{k-2}(x) = C_{k-2,k-2}^{-1}I_{\frac{n+1}{2}+[\frac{k-2}{2}]-1} \dots I_{\frac{n+1}{2}} \\ \times \left( \int_{\partial\mathbb{B}} K(x,y)\lambda^{-1}\tilde{f}_{k-2}(y)dS_y + \lambda^{-1}e_n \sum_{s=0}^{+\infty} V_{l_1,\dots,l_s}(\underline{x})a_{l_1,\dots,l_s}^{(k-2)} \right), \quad x \in \mathbb{B}. \end{aligned} \quad (56)$$

Inductively on  $2 \leq l \leq k-2$ , the following boundary value problem

$$\begin{cases} \phi_l \in \mathbb{H}^p(\mathbb{B}), \\ X^{(n)} \left\{ \lambda C_{l,l} E_{\frac{n+1}{2}} \dots E_{\frac{n+1}{2}+[\frac{l}{2}]-1} \phi \right\} = \tilde{f}_l, \quad x \in \partial\mathbb{B}, \end{cases} \quad (57)$$

where

$$\begin{aligned} \tilde{f}_l(x) = f_l(x) - \sum_{j=l+1}^{k-1} C_{k-1,j}C_{j,j}^{-1}X^{(n)} \left\{ x^j I_{\frac{n+1}{2}+[\frac{j-l}{2}]} \dots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} K(x,y)\lambda^{-1}\tilde{f}_j(y)dS_y \right. \right. \\ \left. \left. + \lambda^{-1}e_n \sum_{s=0}^{+\infty} V_{l_1,\dots,l_s}(\underline{x})a_{l_1,\dots,l_s}^{(j)} \right) \right\} \end{aligned} \quad (58)$$

on  $\partial\mathbb{B}$ , has the explicit solution

$$\begin{aligned} \phi_l(x) = C_{l,l}^{-1}I_{\frac{n+1}{2}+[\frac{l}{2}]-1} \dots I_{\frac{n+1}{2}} \\ \times \left( \int_{\partial\mathbb{B}} K(x,y)\lambda^{-1}\tilde{f}_l(y)dS_y + \lambda^{-1}e_n \sum_{s=0}^{+\infty} V_{l_1,\dots,l_s}(\underline{x})a_{l_1,\dots,l_s}^{(l)} \right), \quad x \in \mathbb{B}. \end{aligned} \quad (59)$$

Now, the remaining two cases when  $l = 0, 1$  are treated as follows. For  $l = 1$ , the boundary value problem

$$\begin{cases} \phi_1 \in \mathbb{H}^p(\mathbb{B}), \\ X^{(n)} \left\{ \lambda C_{1,1} E_{\frac{n+1}{2}} \phi_1 \right\} = \tilde{f}_1, \quad x \in \partial\mathbb{B}, \end{cases}$$

where

$$\tilde{f}_1(x) = \begin{cases} f_1(x) - \sum_{j=2}^{k-1} C_{1,j}C_{j,j}^{-1}X^{(n)} \left\{ x^j I_{\frac{n+1}{2}+[\frac{j}{2}]-1} \dots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} K(x,y)\lambda^{-1}\tilde{f}_j(y)dS_y \right. \right. \\ \left. \left. + \lambda^{-1}e_n \sum_{s=0}^{+\infty} V_{l_1,\dots,l_s}(\underline{x})a_{l_1,\dots,l_s}^{(j)} \right) \right\}, & \text{if } k \text{ odd,} \\ f_1(x) - \sum_{j=2}^{k-1} C_{1,j}C_{j,j}^{-1}X^{(n)} \left\{ x^j I_{\frac{n+1}{2}+[\frac{j}{2}]} \dots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} K(x,y)\lambda^{-1}\tilde{f}_j(y)dS_y \right. \right. \\ \left. \left. + \lambda^{-1}e_n \sum_{s=0}^{+\infty} V_{l_1,\dots,l_s}(\underline{x})a_{l_1,\dots,l_s}^{(j)} \right) \right\}, & \text{if } k \text{ even,} \end{cases} \quad x \in \partial\mathbb{B}, \quad (60)$$

has the explicit solution

$$\phi_1(x) = C_{1,1}^{-1} I_{\frac{n+1}{2}} \left( \int_{\partial \mathbb{B}} K(x, y) \lambda^{-1} \tilde{f}_1(y) dS_y + \lambda^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}^{(1)} \right),$$

$$x \in \mathbb{B}. \quad (61)$$

For  $l = 0$ , the boundary value problem

$$\begin{cases} \phi_0 \in \mathbb{H}^p(\mathbb{B}), \\ X^{(n)} \{\lambda \phi_0\} = \tilde{f}_0, \quad x \in \partial \mathbb{B}, \end{cases} \quad (62)$$

where

$$\begin{aligned} \tilde{f}_0(x) &= f_0(x) - \sum_{j=1}^{k-1} C_{j,j}^{-1} X^{(n)} \\ &\times \left\{ x^j \left( \int_{\partial \mathbb{B}} K(x, y) \lambda^{-1} \tilde{f}_j(y) dS_y + \lambda^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_k}(\underline{x}) a_{l_1, \dots, l_s}^{(j)} \right) \right\}, \quad x \in \partial \mathbb{B}, \end{aligned} \quad (63)$$

has the explicit solution

$$\phi_0(x) = \int_{\partial \mathbb{B}} K(x, y) \lambda^{-1} \tilde{f}_0(y) dS_y + \lambda^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}^{(0)}, \quad x \in \mathbb{B}. \quad (64)$$

Associating with (53)–(64), we obtain that Problem (46) has the explicit solution  $\phi(x) = \sum_{j=0}^{k-1} x^j \phi_j$ . It follows the result.  $\square$

**Corollary 5.2.** *When  $p = 2, \lambda_j = 1, j = 0, 1, \dots, k-1$ , Problem (46) reduces into the case*

$$\begin{cases} \phi \in \mathbb{H}_k^2(\mathbb{B}), \quad 1 < p < +\infty, \\ X^{(n)} \{\phi\} = f_0, \quad a.e. \ x \in \partial \mathbb{B}, \\ X^{(n)} \{\mathcal{D}\phi\} = f_1, \quad a.e. \ x \in \partial \mathbb{B}, \\ \vdots \\ X^{(n)} \{\mathcal{D}^{k-1}\phi\} = f_{k-1}, \quad a.e. \ x \in \partial \mathbb{B}. \end{cases} \quad (65)$$

*Its solutions actually correspond to reconstruct the real algebra  $\mathbb{R}_n$ -valued monogenic signals of the Hardy space defined on  $\mathbb{B}$  when a group of real sub-algebra valued  $\mathbb{R}_{n-1}$  initial datum is given on the sphere  $\partial \mathbb{B}$  of higher dimensional Euclidean space  $\mathbb{R}^n$ . Similar results also refer to [28].*

Moreover, we can consider the following problem, analogous to Theorem 5.1.

**Problem IV.** Given the boundary data  $f_j \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_{n-1}), j = 0, 1, 2, \dots, k-1$ , find a function  $\phi : \mathbb{B} \rightarrow \mathbb{R}_n$  satisfying the Schwarz boundary value conditions

$$\begin{cases} \phi \in \mathbb{H}_k^p(\mathbb{B}), & 1 < p < +\infty, \\ X^{(n)}\{\lambda_0\phi\} = f_0, & \text{a.e. } x \in \partial\mathbb{B}, \\ \vdots & \vdots \\ X^{(n)}\{\lambda_{k-1}\mathcal{D}^{k-1}\phi\} = f_{k-1}, & \text{a.e. } x \in \partial\mathbb{B}, \end{cases} \quad (66)$$

where  $\lambda_j \in \mathbb{R}_{n-1}$  is a constant with its inverse  $\lambda_j^{-1}, j = 0, 1, 2, \dots, k-1$ .

**Theorem 5.3.** For the given function  $f_j \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_{n-1}), j = 0, 1, 2, \dots, k-1$ , Problem (46) has the general solution and its explicit form is given by

$$\phi(x) = \sum_{j=0}^{k-1} x^j \phi_j(x), \quad x \in \mathbb{B}, \quad (67)$$

with

$$\phi_j(x) = \begin{cases} \int_{\partial\mathbb{B}} K(x, y) \lambda_0^{-1} \tilde{f}_0(y) dS_y + \lambda_0^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}^{(0)}, & \text{if } j = 0, \\ C_{1,1}^{-1} I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} K(x, y) \lambda_1^{-1} \tilde{f}_1(y) dS_y + \lambda_1^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}^{(1)} \right), & \text{if } j = 1, \\ C_{j,j}^{-1} I_{\frac{n+1}{2} + [\frac{j}{2}] - 1} \dots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} K(x, y) \lambda_j^{-1} \tilde{f}_j(y) dS_y + \lambda_j^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}^{(j)} \right), & \text{if } 2 \leq j \leq k-1, \end{cases} \quad (68)$$

where  $K(x, y)$  as (26),  $a_{l_1, \dots, l_s}^{(j)} \in \mathbb{R}_{n-1}, j = 0, 1, 2, \dots, k-1$  are all constants, and for arbitrary  $x \in \partial\mathbb{B}$ ,

$$\tilde{f}_j(x) = \begin{cases} f_0(x) - \sum_{l=1}^{k-1} C_{l,l}^{-1} X^{(n)} \left\{ x^l \left( \int_{\partial\mathbb{B}} K(x, y) \lambda_l^{-1} \tilde{f}_l(y) dS_y + \lambda_l^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}^{(l)} \right) \right\}, & \text{if } j = 0, \\ f_1(x) - \sum_{l=2}^{k-1} C_{1,l} C_{l,l}^{-1} X^{(n)} \left\{ x^l I_{\frac{n+1}{2} + [\frac{l}{2}] - 1} \dots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} K(x, y) \lambda_l^{-1} \tilde{f}_l(y) dS_y \right. \right. \\ \left. \left. + \lambda_l^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}^{(l)} \right) \right\}, & \text{if } j = 1, k \text{ odd}, \\ f_1(x) - \sum_{l=2}^{k-1} C_{1,l} C_{l,l}^{-1} X^{(n)} \left\{ x^l I_{\frac{n+1}{2} + [\frac{l}{2}]} \dots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} K(x, y) \lambda_l^{-1} \tilde{f}_l(y) dS_y \right. \right. \\ \left. \left. + \lambda_l^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}^{(l)} \right) \right\}, & \text{if } j = 1, k \text{ even}, \\ f_j(x) - \sum_{l=j+1}^{k-1} C_{k-1,l} C_{l,l}^{-1} X^{(n)} \left\{ x^l I_{\frac{n+1}{2} + [\frac{l-j}{2}]} \dots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} K(x, y) \lambda_l^{-1} \tilde{f}_l(y) dS_y \right. \right. \\ \left. \left. + \lambda_l^{-1} e_n \sum_{s=0}^{+\infty} V_{l_1, \dots, l_s}(\underline{x}) a_{l_1, \dots, l_s}^{(l)} \right) \right\}, & \text{if } 2 \leq j \leq k-1. \end{cases} \quad (69)$$

Similarly, we solve the Schwarz BVP as follows.

**Problem V.** Given the boundary data  $f_j \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_0), j = 0, 1, 2, \dots, k-1$ , find a function  $\phi : \mathbb{B} \rightarrow \mathbb{R}_0 \oplus \mathbb{R}_2$ , satisfying the Schwarz boundary value conditions

$$\begin{cases} \phi \in \mathbb{H}_k^p(\mathbb{B}), & 1 < p < +\infty, \\ \text{Sc}\{\lambda\phi\} = f_0, & \text{a.e. } x \in \partial\mathbb{B}, \\ \vdots & \vdots \\ \text{Sc}\{\lambda\mathcal{D}^{k-1}\phi\} = f_{k-1}, & \text{a.e. } x \in \partial\mathbb{B}, \end{cases} \quad (70)$$

where  $\lambda \in \mathbb{R}_0 \setminus \{0\}$  is a constant with its inverse  $\lambda^{-1}$ .

**Theorem 5.4.** For the given function  $f_j \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_0)$ ,  $j = 0, 1, 2, \dots, k-1$ , Problem (70) has the general solution and its unique form is given by

$$\phi(x) = \sum_{j=0}^{k-1} x^j \phi_j(x), \quad x \in \mathbb{B}, \quad (71)$$

with

$$\phi_j(x) = \begin{cases} \int_{\partial\mathbb{B}} \tilde{K}(x, y) \lambda^{-1} \tilde{f}_0(y) dS_y, & \text{if } j = 0, \\ C_{1,1}^{-1} I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} \tilde{K}(x, y) \lambda^{-1} \tilde{f}_1(y) dS_y \right), & \text{if } j = 1, \\ C_{j,j}^{-1} I_{\frac{n+1}{2} + [\frac{j}{2}] - 1} \dots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} \tilde{K}(x, y) \lambda^{-1} \tilde{f}_j(y) dS_y \right), & \text{if } 2 \leq j \leq k-1, \end{cases} \quad (72)$$

where  $\tilde{K}(x, y)$  as (44), and for arbitrary  $x \in \partial\mathbb{B}$ ,

$$\tilde{f}_j(x) = \begin{cases} f_0(x) - \sum_{l=1}^{k-1} C_{l,l}^{-1} S_c \left\{ x^l \left( \int_{\partial\mathbb{B}} \tilde{K}(x, y) \lambda^{-1} \tilde{f}_l(y) dS_y \right) \right\}, & \text{if } j = 0, \\ f_1(x) - \sum_{l=2}^{k-1} C_{1,l} C_{l,l}^{-1} S_c \left\{ x^l I_{\frac{n+1}{2} + [\frac{l}{2}] - 1} \dots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} \tilde{K}(x, y) \lambda^{-1} \tilde{f}_l(y) dS_y \right) \right\}, & \text{if } j = 1, k \text{ odd}, \\ f_1(x) - \sum_{l=2}^{k-1} C_{1,l} C_{l,l}^{-1} S_c \left\{ x^l I_{\frac{n+1}{2} + [\frac{l}{2}] - 1} \dots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} \tilde{K}(x, y) \lambda^{-1} \tilde{f}_l(y) dS_y \right) \right\}, & \text{if } j = 1, k \text{ even}, \\ f_j(x) - \sum_{l=j+1}^{k-1} C_{k-1,l} C_{l,l}^{-1} S_c \left\{ x^l I_{\frac{n+1}{2} + [\frac{l-j}{2}] - 1} \dots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} \tilde{K}(x, y) \lambda^{-1} \tilde{f}_l(y) dS_y \right) \right\}, & \text{if } 2 \leq j \leq k-1. \end{cases} \quad (73)$$

Furthermore, we solve a boundary value problem as follows.

**Problem VI.** Given the boundary data  $f_j \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_0)$ ,  $j = 0, 1, 2, \dots, k-1$ , find a function  $\phi : \mathbb{B} \rightarrow \mathbb{R}_0 \oplus \mathbb{R}_2$ , satisfying the Schwarz boundary value conditions

$$\begin{cases} \phi \in \mathbb{H}_k^p(\mathbb{B}), \quad 1 < p < +\infty, \\ \text{Sc}\{\lambda_0 \phi\} = f_0, \quad \text{a.e. } x \in \partial\mathbb{B}, \\ \text{Sc}\{\lambda_1 \mathcal{D}\phi\} = f_1, \quad \text{a.e. } x \in \partial\mathbb{B}, \\ \vdots \\ \text{Sc}\{\lambda_{k-1} \mathcal{D}^{k-1}\phi\} = f_{k-1}, \quad \text{a.e. } x \in \partial\mathbb{B}, \end{cases} \quad (74)$$

where  $\lambda_j \in \mathbb{R}_0 \setminus \{0\}$  is a constant with its inverse  $\lambda_j^{-1}$ ,  $j = 0, 1, 2, \dots, k-1$ .

**Theorem 5.5.** For the given function  $f_j \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_0)$ ,  $j = 0, 1, 2, \dots, k-1$ , Problem (70) has the general solution and its unique form is given by

$$\phi(x) = \sum_{j=0}^{k-1} x^j \phi_j(x), \quad x \in \mathbb{B}, \quad (75)$$

with

$$\phi_j(x) = \begin{cases} \int_{\partial\mathbb{B}} \tilde{K}(x, y) \lambda_0^{-1} \tilde{f}_0(y) dS_y, & \text{if } j = 0, \\ C_{1,1}^{-1} I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} \tilde{K}(x, y) \lambda_1^{-1} \tilde{f}_1(y) dS_y \right), & \text{if } j = 1, \\ C_{j,j}^{-1} I_{\frac{n+1}{2} + [\frac{j}{2}] - 1} \cdots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} \tilde{K}(x, y) \lambda_j^{-1} \tilde{f}_j(y) dS_y \right), & \text{if } 2 \leq j \leq k-1, \end{cases} \quad (76)$$

where  $\tilde{K}(x, y)$  as (44), and for arbitrary  $x \in \partial\mathbb{B}$ ,

$$\tilde{f}_j(x) = \begin{cases} f_0(x) - \sum_{l=1}^{k-1} C_{l,l}^{-1} \text{Sc} \left\{ x^l \left( \int_{\partial\mathbb{B}} \tilde{K}(x, y) \lambda_0^{-1} \tilde{f}_l(y) dS_y \right) \right\}, & \text{if } j = 0, \\ f_1(x) - \sum_{l=2}^{k-1} C_{1,l} C_{l,l}^{-1} \text{Sc} \left\{ x^l I_{\frac{n+1}{2} + [\frac{l}{2}] - 1} \cdots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} \tilde{K}(x, y) \lambda_l^{-1} \tilde{f}_l(y) dS_y \right) \right\}, & \text{if } j = 1, k \text{ odd}, \\ f_1(x) - \sum_{l=2}^{k-1} C_{1,l} C_{l,l}^{-1} \text{Sc} \left\{ x^l I_{\frac{n+1}{2} + [\frac{l}{2}]} \cdots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} \tilde{K}(x, y) \lambda_l^{-1} \tilde{f}_l(y) dS_y \right) \right\}, & \text{if } j = 1, k \text{ even}, \\ f_j(x) - \sum_{l=j+1}^{k-1} C_{k-1,l} C_{l,l}^{-1} \text{Sc} \left\{ x^l I_{\frac{n+1}{2} + [\frac{l-j}{2}]} \cdots I_{\frac{n+1}{2}} \left( \int_{\partial\mathbb{B}} \tilde{K}(x, y) \lambda_l^{-1} \tilde{f}_l(y) dS_y \right) \right\}, & \text{if } 2 \leq j \leq k-1. \end{cases} \quad (77)$$

*Proof.* Applying Theorem 5.4, the proof of the result is complete.  $\square$

*Remark 5.6.* When  $p = 2$ ,  $\lambda_j = 1$ ,  $j = 0, 1, \dots, k-1$ , Problem (74) turns into the case

$$\begin{cases} \phi \in \mathbb{H}_k^2(\mathbb{B}), & 1 < p < +\infty, \\ \text{Sc}\{\phi\} = f_0, & \text{a.e. } x \in \partial\mathbb{B}, \\ \text{Sc}\{\mathcal{D}\phi\} = f_1, & \text{a.e. } x \in \partial\mathbb{B}, \\ \vdots & \vdots \\ \text{Sc}\{\mathcal{D}^{k-1}\phi\} = f_{k-1}, & \text{a.e. } x \in \partial\mathbb{B}. \end{cases} \quad (78)$$

This could be understood as the reconstruction of the monogenic signals considered on the unit sphere when a group of scalar valued initial data is given, which is appearing in engineering applications.

*Remark 5.7.* When  $k = 2m$ ,  $m \in \mathbb{N}$ , Problems (46), (66), (70) and (74) change into the Schwarz BVPs for the poly-harmonic Hardy spaces defined on the unit ball  $\mathbb{B}$  of higher dimensional Euclidean space  $\mathbb{R}^n$ , respectively. This means that the solutions to Problems (46), (66), (70) and (74) derive the solutions to the corresponding Schwarz BVPs for the poly-harmonic Hardy spaces defined on the unit ball  $\mathbb{B}$  of higher dimensional Euclidean space  $\mathbb{R}^n$ . Moreover, when the dimension of the space considered is  $n = 2$ , Problems (46), (70) and (74) reduce to those discussed in Refs. e.g. [5, 7], respectively.

*Remark 5.8.* In this context, when the given boundary data take Clifford sub-algebra values and scalar values, respectively, two different kinds of Schwarz

BVPs for the poly-Hardy spaces on the unit ball of higher dimensional Euclidean space  $\mathbb{R}^n$  are considered. Compared to the classical Schwarz problems in complex analysis (see, Refs. e.g. [5, 7]), the explicit solutions to these two kinds of Schwarz BVPs are gotten while the uniqueness of them are different. One contains the Schwarz BVPs (70) and (74), and the general solutions to it are unique. Another contains the Schwarz BVPs (35), (46) and (66). Although the general solutions to it could be also presented explicitly, they are not unique. This is because they contain infinitely many arbitrary constants overdetermined. If one wants to determine the arbitrary constants involved in the solutions to the Schwarz BVPs considered, there needs to impose more constraints to the Schwarz BVPs. This is not within the scope of the present article, and will be further discussed in a forthcoming paper.

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