Adv. Appl. Clifford Algebras 27 (2017), 1073–1101

0188-7009/021073-29
published online November 9, 2016
DOI 10.1007/s00006-016-0736-0

Advances in Applied Clifford Algebras

A Frame Theory of Hardy Spaces with the Quaternionic and the Clifford Algebra Settings

Qiuhui Chen, Pei Dang* and Tao Qian

Communicated by Wolfgang Sprössig.

Abstract. The purpose of this article is twofold. The first is to construct frames of the $L^2(\mathbb{R}^n)$ by using dilation and modulation starting from a single function of a certain type. The second is to construct frames of the $H^2(\mathbb{R}^n_{1,+})$ by using a Cauchy type integral. The work is motivated by the recent development of sparse representation of Hardy space functions, and, especially, by adaptive Fourier decomposition in relation to rational orthogonal systems. We work in two contexts. One is the quaternionic space and the other is the Euclidean space in the Clifford algebra setting. We also investigate what type of functions can give rise to frames of $L^2(\mathbb{R}^n)$ by dilation and modulation.

Mathematics Subject Classification. 42A38, 44A15, 62P30, 94A20.

Keywords. Frame, Quaternion-Hardy space, Fourier transform, Group representation.

1. Introduction

Frames were firstly introduced in 1952 by Duffin and Schaeffer in [12] where frames are used as a tool in the study of nonharmonic Fourier series. A sequence $\{f_n : n \in I\}$ is called a frame of a Hilbert space H if, for any $f \in H$, the inequalities $A\|f\|_H^2 \leq \sum_{n\in I} |\langle f, f_n\rangle_H|^2 \leq B\|f\|_H^2$ hold for some positive constants A and B, where, I is some index set, $\langle\cdot,\cdot\rangle_H$ and $\|\cdot\|_H$ are the inner product and norm of H. The next work to develop the frame theory was nearly 30 years later. In 1980, Young presented frames in an abstract setting in his book [30] that also concerns the topic of nonharmonic Fourier series. In 1985, as the wavelet era began, Daubechies et al. [7] observed that frames can be used to find series expansions of functions in $L^2(\mathbb{R})$ which are very similar to the expansions using orthonormal bases. Due to Daubechies' important works [8,9] and the successive paper [21] by Heil and Walnut, mathematicians

*Corresponding author.

Birkhäuser

found the potential of this topic, see [1,2,4,10,13–20,22,26,28]. There are a lot of results about the wavelet theory in terms of Lie group [29]. The subject of frame has been developing in two different aspects. One is to study frame theory as a branch of functional analysis. The other aspect is to construct specific frames using harmonic analysis, time-frequency analysis and signal processing, including Gabor and wavelet frames.

In construction of frames, it concerns three basic operators: modulation, dilation and translation. They play crucial roles. They are in the above order defined, respectively, as

$$\mathcal{M}_{\underline{\omega}} f(\cdot) = e^{i\langle \underline{\omega}, \cdot \rangle} f(\cdot),\ \mathcal{D}_A f(\cdot) = |A|^{-\frac{1}{2}} f(A^{-1}\cdot),\ \mathcal{T}_{\underline{\mathrm{b}}} f(\cdot) = f(\cdot - \underline{\mathrm{b}})$$

for parameters $\underline{\omega}, \underline{\mathrm{b}} \in \mathbb{R}^n$ and a dilation matrix $A \in \mathbb{R}^{n\times n}$. Here $\mathbb{R}^{n\times n}$ denotes the set of $n \times n$ matrices with positive determinants. The windowed Fourier transform [17,20] is based on translation and modulation, but wavelet transform [8,9] is based on dilation and translation. Precisely, for a fixed windowed function ϕ and a wavelet function ψ in $L^2(\mathbb{R}^n)$, the windowed Fourier transform and the wavelet transform are defined by

$$\mathcal{T}_{\phi}^{\mathrm{win}} f(\underline{\mathrm{b}}, \underline{\omega}) = \int_{\mathbb{R}^n} f(\underline{\mathrm{x}}) \phi(\underline{\mathrm{x}} - \underline{\mathrm{b}}) e^{-i\langle \underline{\omega}, \underline{\mathrm{x}} \rangle} d\underline{\mathrm{x}}$$

and

$$\mathcal{T}_{\psi}^{\mathrm{wav}} f(A, \underline{\mathrm{b}}) = |A|^{-\frac{1}{2}} \int_{\mathbb{R}^n} f(\underline{\mathrm{x}}) \psi\left(A^{-1}\underline{\mathrm{x}} - \underline{\mathrm{b}}\right) d\underline{\mathrm{x}},$$

respectively. Each of them involves two of the three basic operators: modulation $\mathcal{M}$, translation $\mathcal{T}$ and dilation $\mathcal{D}$.

We remark that, in the windowed Fourier transform, the variables in translation–modulation stand for time and frequency variables, respectively. In the wavelet transform, the variables in translation–dilation are called time and dilation variables, respectively. Usually, the dilation variable is regarded as "frequency". The windowed Fourier transform is treated as the coefficients of the Schrödinger representation of the Heisenberg group up to a phase factor with modulus 1. The wavelet transform can be interpreted as the representation coefficients of the group of affine transform. We remark that in the Heisenberg group the multiplication law complies with the composition principle of translation and modulation. Similarly, in the affine group, the multiplication law corresponds to the composition principle of translation and dilation (see [17]).

In this paper, we investigate frame theory in the Clifford algebra setting. Due to the fact that $\mathbb{R}_1^n$ is not closed under Clifford multiplication, the functions are restricted to those defined in $\mathbb{R}_1^n$ and with values in $\mathbb{R}_1^n$ as well. We establish a frame-type inequality for such functions. In the quaternionic case we establish a frame theory for $\mathbb{H}^2(\mathbb{Q}_+)$.

Our basic idea is as follows. Find radial functions $\varphi(|\cdot|)$ in $L^2(\mathbb{R}^n)$ such that the family of functions $\{\varphi_{j,\underline{\mathrm{k}}} : (j,\underline{\mathrm{k}}) \in \mathbb{Z}\times\mathbb{Z}^n\}$ in $L^2(\mathbb{R}^n)$ defined by

$$\varphi_{j,\underline{\mathrm{k}}}(|\underline{\mathrm{x}}|) = a_0^{-\frac{jn}{2}}\varphi(\frac{|\underline{\mathrm{x}}|}{a_0^j})e^{i\langle a_0^{-j}\omega_0\underline{\mathrm{k}},\underline{\mathrm{x}}\rangle},\ \underline{\mathrm{x}}\in\mathbb{R}^n. \tag{1.1}$$

forms a frame of $L^2(\mathbb{R}^n)$. Note that in such formulation the dilation-modulation lattice set is

$$\Lambda = \left\{(a_0^{-j}\omega_0\underline{\mathrm{k}}, a_0^{-j}) : (j,\underline{\mathrm{k}})\in\mathbb{Z}\times\mathbb{Z}^n\right\}. \tag{1.2}$$

Next, by using a Cauchy-type formula (6.7), generate a frame for the Hardy space defined in $\mathbb{R}^n_{1,+}$ consisting of certain vector-valued monogenic functions corresponding to the conjugate harmonic systems of Stein–Weiss [27]. Consequently, such type monogenic frames are available for the quaternionic Hardy space $\mathbb{H}^2(\mathbb{Q}_+)$. We specially note that what are used are only dilation and modulation, and, that is, to the authors knowledge, the first such formulation among the literature on frames on $\mathbb{R}^n$ with the Clifford algebra setting.

The present study on frames of the type (1.1) with monogenic extensions is, in fact, motivated by their one dimensional counterparts, and their analytic extensions, and especially the TM systems in one complex variable (see [25] and thereafter references). Firstly, we observe that the Takenaka–Malmquist(TM) system is essentially generated from dilation–modulation of the exponential function. Here, the TM system is defined by

$$e_n(z) = \frac{\sqrt{\frac{1}{\pi}Im\{\lambda_n\}}}{z-\bar{\lambda}_n}B_n(z),\ z\in\mathbb{C}^+,\ n\in\mathbb{Z}_+,$$

where $\{\lambda_n\}\subset\mathbb{C}^+$ and B_n is an order $n-1$ Blaschke product on the upper-half complex plane defined by

$$B_n(z) = \prod_{j=0}^{n-1}\frac{z-\lambda_j}{z-\bar{\lambda}_j}.$$

A TM system is an orthonormal system of the conventional Hardy space $H^2(\mathbb{C}^+)$ on the upper half plane that is defined by

$$\|F\|^2_{H^2(\mathbb{C}^+)} := \sup_{y>0}\int_{-\infty}^{\infty}|F(x+iy)|^2dx < \infty.$$

For each function $F\in H^2(\mathbb{C}^+)$, its non-tangential boundary limit $f := \lim_{y\to 0}F(\cdot+iy)$ exists. The set of non-tangential boundary limits of $H^2(\mathbb{C}^+)$ also forms a Hilbert space, a closed subspace of $L^2(\mathbb{R})$, which is denote by $H^2(dt)$ and called the boundary Hardy space of $H^2(\mathbb{C}^+)$. Moreover, the inner products in $H^2(\mathbb{C}^+)$ and $H^2(dt)$ are related by

$$\langle F, G\rangle_{\mathbb{C}^+} = \langle f, g\rangle,$$

where $\langle f, g\rangle$ means the usual inner product in $L^2(\mathbb{R})$. Correspondingly, we can define the boundary value TM system (an orthonormal basis of $H^2(dt)$), still denoted by $\{e_n : n\in\mathbb{Z}_+\}$. In Sect. 3, we prove that each basic function e_n

essentially comes from dilation–modulation of the exponential function $e^{-|\cdot|}$. Secondly, in Sect. 4, we observe that radial functions as kernel functions are sufficient to ensure the existence of the inverse formula of the continuous time-frequency transform defined by (4.12). Correspondingly, as discretization of (4.12), the generating atom in our frame system (1.1) is a radial function.

About the construction of frames $L^2(\mathbb{R}^n)$ of the type (1.1), we investigate a continuous time-frequency transformation with modulated–dilated kernel in the space $L^2(\mathbb{R})$. We point out that the basic kernel function has to be radial in order to ensure the synthesis formula. This continuous transform is also the group representation coefficient of a kind of affine group from dilation and modulation. The reproducing formula implies that the dilation–modulation kernel is dense in $L^2(\mathbb{R}^n)$ when the dilation parameter and modulation parameter run over $\mathbb{R}_+$ and $\mathbb{R}^n$, respectively. As a separable space, it is intuitively to look for suitable discretization of the dilation parameter and modulation parameter in the transformation that we study. By choosing the lattices Λ, it leads to the system $\{\varphi_{j,\underline{k}} : (j, \underline{k}) \in \mathbb{Z} \times \mathbb{Z}^n\}$ defined in (1.1).

The writing plan is as follows. Section 2 gives some basic knowledge in Clifford algebra. Section 3 explains that the TM system is essentially generated by dilation–modulation of an exponential function. Section 4 is devoted to a generalized Heisenberg group, its representation and an important integral transform. Section 5 deals with necessary conditions and sufficient conditions for a radial function to be able to induce a frame as defined in (1.1). The formulation of this part is standard in accordance with the idea and techniques in [9] and [5]. Section 6 is devoted to establishing a frame-type inequality. Section 7 is construction of frames in the Quaternion Hardy space.

2. Preliminary

Most of the basic knowledge and notation recalled in this section are referred to [3,11,23,24]. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be basis elements satisfying $\mathbf{e}_j\mathbf{e}_k + \mathbf{e}_k\mathbf{e}_j = -2\delta_{j,k}$, where $\delta_{j,k} = 1$ if $j = k$ and $\delta_{j,k} = 0$ otherwise, $j, k = 1, 2, \ldots, n$. Let

$$\mathbb{R}^n = \{\underline{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n : x_j \in \mathbb{R}, \quad j = 1, 2, \ldots, n\}$$

be identical with the usual Euclidean space $\mathbb{R}^n$. The space $\mathbb{R}^n_1$ is defined as

$$\mathbb{R}^n_1 = \{x = x_0 + \underline{x} : x_0 \in \mathbb{R}, \underline{x} \in \mathbb{R}^n\}.$$

The upper half space $\mathbb{R}^n_{1,+}$ is defined as $\mathbb{R}^n_1$ with the constraint $x_0 > 0$. For convenience, we sometimes use $\mathbf{e}_0 := 1$.

An element in $\mathbb{R}^n_1$ is called a vector. The real (complex) Clifford algebra generated by $\mathbf{e}_1, \ldots, \mathbf{e}_n$, denoted by $\mathbb{R}^{(n)}$ ($\mathbb{C}^{(n)}$), is the associative algebra generated by $\mathbf{e}_1, \ldots, \mathbf{e}_n$, over the real (complex) field $\mathbb{R}$ ($\mathbb{C}$). A general element in $\mathbb{R}^{(n)}$, therefore, is of the form $x = \sum_{\mathbf{s}} x_{\mathbf{s}}\mathbf{e}_{\mathbf{s}}$, where $\mathbf{e}_{\mathbf{s}} = \mathbf{e}_{j_1}\mathbf{e}_{j_2}\cdots\mathbf{e}_{j_k}$, and $\mathbf{s}$ runs over all the ordered subsets of $\{1, 2, \ldots, n\}$, namely,

$$\mathbf{s} = \{1 \leq j_1 < j_2 < \cdots < j_k \leq n\},\ 1 \leq k \leq n.$$

The natural inner product between x and y in $\mathbb{C}^{(n)}$, denoted by $\langle x, y\rangle$, is the complex number $\sum_{\mathbf{s}} x_{\mathbf{s}}\overline{y_{\mathbf{s}}}$, where $x = \sum_{\mathbf{s}} x_{\mathbf{s}}\mathbf{e}_s$ and $y = \sum_{\mathbf{s}} y_{\mathbf{s}}\mathbf{e}_s$. The norm associated with this inner product is

$$|x| = \langle x, x\rangle^{\frac{1}{2}} = \left(\sum_{\mathbf{s}} |x_{\mathbf{s}}|^2\right)^{\frac{1}{2}}.$$

If $x, y, \ldots, u$ are vectors, then

$$|xy\cdots u| = |x|\ |y|\cdots|u|.$$

The conjugates of a vector $x = x_0 + \underline{x}$ is defined as $\bar{x} = x_0 - \underline{x}$. It is easy to verify that $0 \neq x \in \mathbb{R}^n_1$ implies

$$x^{-1} = \frac{\bar{x}}{|x|^2}.$$

Let $\mathbf{f}(x)$ be a function defined in $\mathbb{R}^n$ or $\mathbb{R}^n_1$ taking values in $\mathbb{C}^{(n)}$ with the form $\mathbf{f}(x) = \sum_{\mathbf{s}} f_{\mathbf{s}}(x)\mathbf{e}_{\mathbf{s}}$, where $f_{\mathbf{s}}$ are complex-valued functions. We will use the Dirac operator

$$\partial = \partial_0 + \underline{\partial}$$

where $\partial_0 = \partial/\partial x_0 = (\partial/\partial x_0)\mathbf{e}_0$ and $\underline{\partial} = (\partial/\partial x_1)\mathbf{e}_1 + \cdots + (\partial/\partial x_1)\mathbf{e}_n$.

Define the left and right roles of the operator ∂ by

$$\partial\mathbf{f} = \sum_{j=0}^{n}\sum_{\mathbf{s}} \frac{\partial f_{\mathbf{s}}}{\partial x_j}\mathbf{e}_j\mathbf{e}_{\mathbf{s}}$$

and

$$\mathbf{f}\partial = \sum_{j=0}^{n}\sum_{\mathbf{s}} \frac{\partial f_{\mathbf{s}}}{\partial x_j}\mathbf{e}_{\mathbf{s}}\mathbf{e}_j.$$

If $\partial\mathbf{f} = 0$ in a domain (open and connected) Λ, then we say that $\mathbf{f}$ is left-monogenic in Λ; and if $\mathbf{f}\partial = 0$ in Λ, we say that $\mathbf{f}$ is right-monogenic in Λ. If $\mathbf{f}$ is both left- and right-monogenic, then we say that $\mathbf{f}$ is monogenic.

The Cauchy Theorem holds in the present case. Let Λ be a domain with Lipschitz boundary $\partial\Lambda$ and $\mathbf{g}$ be right- and $\mathbf{f}$ be left-monogenic in a neighborhood of $\Lambda \cup \partial\Lambda$. Then

$$\int_{\partial\Lambda} \mathbf{g}(y)n(y)\mathbf{f}(y)d\sigma(y) = 0,$$

where $n(y)$ is the outward unit normal to the surface $\partial\Lambda$ at y and $d\sigma(y)$ is the area measure. We also have the Cauchy formula. Under the above assumptions,

$$\mathbf{g}(x) = \frac{1}{\Omega_{n+1}}\int_{\partial\Lambda} \mathbf{g}(y)n(y)E(y-x)d\sigma(y),\ x \in \Lambda,$$

and

$$\mathbf{f}(x) = \frac{1}{\Omega_{n+1}}\int_{\partial\Lambda} E(y-x)(y)n(y)\mathbf{f}d\sigma(y),\ x \in \Lambda,$$

where

$$E(x) = \frac{\bar{x}}{|x|^{n+1}}$$

is the Cauchy kernel, and $\Omega_{n+1} = 2\pi^{\frac{n+1}{2}}/\Gamma(\frac{n+1}{2})$ is the area of the n-dimensional unit sphere in $\mathbb{R}_1^n$. The Fourier transformation in $\mathbb{R}^n$ is defined by

$$\mathcal{F}(f)(\underline{\xi}) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-i\langle \underline{x}, \underline{\xi}\rangle} f(\underline{x}) d\underline{x},$$

and the inverse of Fourier transform is defined by

$$\mathcal{F}^{-1}(g)(\underline{x}) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{i\langle \underline{x}, \underline{\xi}\rangle} g(\underline{x}) d\underline{x}.$$

For a function $\mathbf{f} : R^n \to \mathbb{R}_1^n$ with $\mathbf{f}(\underline{x}) = \sum_{j=0}^{n} f_j(\underline{x})\mathbf{e}_j$, its Fourier transform is defined by $\mathcal{F}(\mathbf{f})(\underline{x}) = \sum_{j=0}^{n} \mathcal{F}(f_j)(\underline{x})\mathbf{e}_j$.

3. An Observation

For a given parameter sequence $\{\lambda_n\}_{n=0}^{\infty}$ in the upper half plane $\mathbb{C}^+$, we consider the iterative system

$$\frac{d}{dt} g_n(t) + i\bar{\lambda}_n g_n(t) = \sqrt{\frac{\operatorname{Im}(\lambda_n)}{\operatorname{Im}(\lambda_{n-1})}} \left(\frac{d}{dt} g_{n-1}(t) + i\lambda_{n-1} g_{n-1}(t) \right), \ t \in \mathbb{R},\ n \geq 1 \tag{3.1}$$

with initial function $g_0 = -\sqrt{2}ie^{-|\cdot|}\chi_{[0,\infty)}$. Here χ_E denotes the characteristic function of the set $E \subset \mathbb{R}$. By the method of variation of parameters for inhomogeneous linear ordinary differential equations, we get an alternative representation of (3.1)

$$\begin{aligned} g_n(t) &= ce^{-i\bar{\lambda}_n t}\chi_{[0,\infty)}(t) \\ &\quad + \sqrt{\frac{\operatorname{Im}(\lambda_n)}{\operatorname{Im}(\lambda_{n-1})}} e^{-i\bar{\lambda}_n t} \int e^{i\bar{\lambda}_n t} \left(g'_{n-1}(t) + i\lambda_{n-1} g_{n-1}(t) \right) dt \\ &= ce^{-i\bar{\lambda}_n t}\chi_{[0,\infty)}(t) + \sqrt{\frac{\operatorname{Im}(\lambda_n)}{\operatorname{Im}(\lambda_{n-1})}} g_{n-1}(t) \\ &\quad + \sqrt{\frac{\operatorname{Im}(\lambda_n)}{\operatorname{Im}(\lambda_{n-1})}} \int i(\lambda_{n-1} - \bar{\lambda}_n) g_{n-1}(t) e^{i\bar{\lambda}_n t} dt. \end{aligned}$$

By induction, we can prove that, for integer $n \geq 1$,

$$g_n(t) = \sum_{j=0}^{n} c_j^{(n)} e^{-i\bar{\lambda}_j t}\chi_{[0,\infty)}(t),\ t \in \mathbb{R}. \tag{3.2}$$

Specifically, we can choose

$$c_j^{(n)} = \sqrt{\mathrm{Im}(\lambda_n)} \prod_{k=0}^{n-1} (\bar{\lambda}_j - \lambda_k) / \prod_{\substack{k=0 \\ k \neq j}}^{n} (\bar{\lambda}_j - \bar{\lambda}_k),\ j = 0, 1, \ldots, n,$$

which corresponds to the initial condition $g_n(0) = \sum_{j=0}^{n} c_j^{(n)}$.

Denote the Fourier–Laplace transform $\mathcal{L} : L^2(\mathbb{R}^+) \to H^2(\mathbb{C}^+)$ by

$$\mathcal{L}f(z) = \int_{\mathbb{R}^+} f(t) e^{itz} dt,\ z \in \mathbb{C}^+.$$

By calculation, the image of g_n under the transform $\mathcal{L}$ is

$$\mathcal{L}g_n(z) = e_n(z),\ n \in \mathbb{Z}_+. \tag{3.3}$$

The system $\{\mathcal{L}g_n : n \in \mathbb{N}\}$ is called Takenaka–Malmquist system, which is an orthonormal basis in the Hardy space $H^2(\mathbb{C}^+)$ under the condition $\sum_{k \in \mathbb{Z}_+} \frac{\sqrt{\mathrm{Im}(\lambda_k)}}{1 + |\lambda_k|^2} = \infty$.

Remark that $\mathcal{L}$ is an isometry between $L^2(\mathbb{R}^+)$ and $H^2(\mathbb{C}^+)$. We therefore conclude that the system $\{g_n : n \in \mathbb{Z}_+\}$ is also an orthogonal basis of $L^2(\mathbb{R}^+)$. It is interesting to investigate the time-frequency structure of g_n. By (3.2), we know that each function g_n is a finite superposition of some basic atoms from the set $\{e^{-i\bar{\lambda}_k \cdot} \chi_{[0,\infty)} : k \in \mathbb{Z}_+\}$. The basic atom $e^{-i\bar{\lambda}_k \cdot} \chi_{[0,\infty)}$ can be written as modulation–dilation form of one fixed function, that is,

$$e^{-i\bar{\lambda}_k t} \chi_{[0,\infty)}(t) = e^{-i\mathrm{Re}(\lambda_k)t} e^{-\mathrm{Im}(\lambda_k)t} \chi_{[0,\infty)}(t).$$

This example essentially suggests a passage to construct bases or frames in $H^2(\mathbb{C}^+)$ by using the Cauchy formula. A general theory in higher dimensions will be established in the following sections.

4. Integral Transform and Group Representation

We need the following canonical commutation relations among translation, modulation and dilation. Translation and modulation satisfy

$$\begin{aligned} \mathcal{T}_{\underline{b}} \mathcal{M}_{\underline{\omega}} &= e^{-i\langle \underline{b}, \underline{\omega} \rangle} \mathcal{M}_{\underline{\omega}} \mathcal{T}_{\underline{b}} = \mathcal{R}_{-\langle \underline{b}, \underline{\omega} \rangle} \mathcal{M}_{\underline{\omega}} \mathcal{T}_{\underline{b}},\ \mathcal{M}_{\underline{\omega}} \mathcal{T}_{\underline{b}} = e^{i\langle \underline{b}, \underline{\omega} \rangle} \mathcal{T}_{\underline{b}} \mathcal{M}_{\underline{\omega}} \\ &= \mathcal{R}_{\langle \underline{b}, \underline{\omega} \rangle} \mathcal{T}_{\underline{b}} \mathcal{M}_{\underline{\omega}}. \end{aligned} \tag{4.1}$$

Modulation and dilation ensure

$$\mathcal{D}_A \mathcal{M}_{\underline{\omega}} = \mathcal{M}_{(A^T)^{-1}\underline{\omega}} \mathcal{D}_A,\ \mathcal{M}_{\underline{\omega}} \mathcal{D}_A = \mathcal{D}_A \mathcal{M}_{A^T \underline{\omega}}. \tag{4.2}$$

For dilation and translation we have

$$\mathcal{D}_A \mathcal{T}_{\underline{b}} = \mathcal{T}_{A\underline{b}} \mathcal{D}_A,\ \mathcal{T}_{\underline{b}} \mathcal{D}_A = \mathcal{D}_A \mathcal{T}_{A^{-1}\underline{b}}. \tag{4.3}$$

There holds the symmetric relation

$$\mathcal{M}_{\frac{1}{2}\underline{\omega}} \mathcal{T}_{\underline{b}} \mathcal{M}_{\frac{1}{2}\underline{\omega}} = \mathcal{R}_{-\frac{1}{2}\langle \underline{b}, \underline{\omega} \rangle} \mathcal{M}_{\underline{\omega}} \mathcal{T}_{\underline{b}} = \mathcal{R}_{\frac{1}{2}\langle \underline{b}, \underline{\omega} \rangle} \mathcal{T}_{\underline{b}} \mathcal{M}_{\underline{\omega}}, \tag{4.4}$$

where the rotation transform is defined by

$$\mathcal{R}_d f(\cdot) = e^{id} f(\cdot).$$

The composition rule for the time-frequency operation $\mathcal{R}_d\mathcal{M}_{\frac{1}{2}\underline{\omega}}\mathcal{T}_{\underline{\mathrm{b}}}\mathcal{M}_{\frac{1}{2}\underline{\omega}}$, namely,

$$\begin{aligned}&\left(\mathcal{R}_d\mathcal{M}_{\frac{1}{2}\underline{\omega}}\mathcal{T}_{\underline{\mathrm{b}}}\mathcal{M}_{\frac{1}{2}\underline{\omega}}\right)\left(\mathcal{R}_{\tilde{d}}\mathcal{M}_{\frac{1}{2}\underline{\tilde{\omega}}}\mathcal{T}_{\underline{\tilde{\mathrm{b}}}}\mathcal{M}_{\frac{1}{2}\underline{\tilde{\omega}}}\right)\\ &\quad=\mathcal{R}_{d+\tilde{d}+\frac{1}{2}(\langle\underline{\tilde{\mathrm{b}}},\underline{\omega}\rangle-\langle\underline{\mathrm{b}},\underline{\tilde{\omega}}\rangle)}\mathcal{M}_{\frac{1}{2}(\underline{\omega}+\underline{\tilde{\omega}})}T_{\underline{\mathrm{b}}+\underline{\tilde{\mathrm{b}}}}\mathcal{M}_{\frac{1}{2}(\underline{\omega}+\underline{\tilde{\omega}})}\end{aligned}$$

suggests a multiplication on $(\mathbb{R}^n)^2\times\mathbb{R}$

$$(\underline{\mathrm{b}},\underline{\omega},d)(\underline{\tilde{\mathrm{b}}},\underline{\tilde{\omega}},\tilde{d})=\left(\underline{\mathrm{b}}+\underline{\tilde{\mathrm{b}}},\underline{\omega}+\underline{\tilde{\omega}},d+\tilde{d}+\frac{1}{2}(\langle\underline{\tilde{\mathrm{b}}},\underline{\omega}\rangle-\langle\underline{\mathrm{b}},\underline{\tilde{\omega}}\rangle)\right).$$

This group is called the full Heisenberg group. The unitary operator $\mathcal{R}_d\mathcal{M}_{\frac{1}{2}\underline{\omega}}\mathcal{T}_{\underline{\mathrm{b}}}\mathcal{M}_{\frac{1}{2}\underline{\omega}}$ is called the Schrödinger representation of the full Heisenberg group. The representation coefficient is

$$\langle f,\mathcal{R}_d\mathcal{M}_{\frac{1}{2}\underline{\omega}}\mathcal{T}_{\underline{\mathrm{b}}}\mathcal{M}_{\frac{1}{2}\underline{\omega}}\phi\rangle=R_{-d+\frac{1}{2}\langle\underline{\mathrm{b}},\underline{\omega}\rangle}\langle f,\mathcal{M}_{\underline{\omega}}\mathcal{T}_{\underline{\mathrm{b}}}\phi\rangle=R_{-d+\frac{1}{2}\langle\underline{\mathrm{b}},\underline{\omega}\rangle}\mathcal{T}_{\phi}^{\mathrm{win}}f(\underline{\mathrm{b}},\underline{\omega}).$$

Up to the phase factor $e^{i(-d+\frac{1}{2}\langle\underline{\mathrm{b}},\underline{\omega}\rangle)}$, the coefficients of the Schrödinger representation coincide with the windowed Fourier transform in $L^2(\mathbb{R}^n)$.

The wavelet transform can also be interpreted as a representation coefficient $\langle f,\mathcal{T}_{\underline{\mathrm{b}}}\mathcal{D}_A\psi\rangle$ of a unitary representation $\mathcal{T}_{\underline{\mathrm{b}}}\mathcal{D}_A$ of the group of the affine transformation, which is the set $\mathbb{R}^{n\times n}\times\mathbb{R}^n$ with the multiplication by

$$(A,\underline{\mathrm{b}})(\tilde{A},\underline{\tilde{\mathrm{b}}})=(A\tilde{A},\underline{\mathrm{b}}+A\underline{\tilde{\mathrm{b}}}).$$

The multiplication is based on the composition law for the operator $\mathcal{T}_{\underline{\mathrm{b}}}\mathcal{D}_A$

$$(\mathcal{T}_{\underline{\mathrm{b}}}\mathcal{D}_A)(\mathcal{T}_{\underline{\tilde{\mathrm{b}}}}\mathcal{D}_{\tilde{A}})=\mathcal{T}_{\underline{\mathrm{b}}+A\underline{\tilde{\mathrm{b}}}}\mathcal{D}_{A\tilde{A}},\ (A,\underline{\mathrm{b}}),(\tilde{A},\underline{\tilde{\mathrm{b}}})\in\mathbb{R}^{n\times n}\times\mathbb{R}^n.$$

The Heisenberg group and the affine transformation group are imbedding subgroups of the following more complicated group, called the polarized group $\mathbb{H}^{pol}$, i.e. the set $\mathbb{R}^{n\times n}\times(\mathbb{R}^n)^2\times\mathbb{R}$ with the multiplication

$$(A,\underline{\mathrm{b}},\underline{\omega},d)\cdot(\tilde{A},\underline{\tilde{\mathrm{b}}},\underline{\tilde{\omega}},\tilde{d})=(A\tilde{A},\underline{\mathrm{b}}+A\underline{\tilde{\mathrm{b}}},\underline{\omega}+(A^T)^{-1}\underline{\tilde{\omega}},d+\tilde{d}-\langle A^{-1}b,\underline{\tilde{\omega}}\rangle).\quad(4.5)$$

In the scalar case, this group, called affine Weyl–Heisenberg group, first appeared in [6]. The inverse of $(A,\underline{\mathrm{b}},\underline{\omega},d)$ is

$$(A,\underline{\mathrm{b}},\underline{\omega},d)^{-1}=\left(A^{-1},-A^{-1}\underline{\mathrm{b}},-A^T\underline{\omega},-d-\langle\underline{\mathrm{b}},\underline{\omega}\rangle\right).\quad(4.6)$$

The group representation is the unitary operator defined by

$$\rho_1(A,\underline{\mathrm{b}},\underline{\omega},d):=\mathcal{R}_{d+\frac{1}{2}\langle b,\underline{\omega}\rangle}\mathcal{M}_{\frac{1}{2}\underline{\omega}}\mathcal{T}_{\underline{\mathrm{b}}}\mathcal{M}_{\frac{1}{2}\underline{\omega}}D_A.\quad(4.7)$$

We remark that, by direct calculation with patience by using (4.1), (4.2), (4.3) and (4.4), the identities (4.5),(4.6) and (4.7) can be verified.

In fact, we can construct another group structure $\mathbb{H}^{full}$, which is the set $\mathbb{R}^{n\times n}\times(\mathbb{R}^n)^2\times\mathbb{R}$ under the multiplication

$$\begin{aligned}&(A,\underline{\mathrm{b}},\underline{\omega},d)\cdot(\tilde{A},\underline{\tilde{\mathrm{b}}},\underline{\tilde{\omega}},\tilde{d})\\ &\quad=\left(A\tilde{A},\underline{\mathrm{b}}+A\underline{\tilde{\mathrm{b}}},\underline{\omega}+(A^T)^{-1}\underline{\tilde{\omega}},d+\tilde{d}+\frac{1}{2}(\langle A\underline{\tilde{\mathrm{b}}},\underline{\omega}\rangle-\langle A^{-1}\underline{\mathrm{b}},\underline{\tilde{\omega}}\rangle)\right).\end{aligned}\quad(4.8)$$

We call it the full Group. The inverse is

$$(A, \underline{b}, \underline{\omega}, d)^{-1} = \left(A^{-1}, -A^{-1}\underline{b}, -A^T\underline{\omega}, -d\right). \tag{4.9}$$

The group representation is

$$\rho_2(A, \underline{b}, \underline{\omega}, d) := \mathcal{R}_d \mathcal{M}_{\frac{1}{2}\underline{\omega}} \mathcal{T}_{\underline{b}} \mathcal{M}_{\frac{1}{2}\underline{\omega}} \mathcal{D}_A. \tag{4.10}$$

By adopting the techniques in [17] for Heisenberg group, we note that $\mathbb{H}^{full}$ and $\mathbb{H}^{pol}$ are isomorphic via the group isomorphism $\imath : \mathbb{H}^{full} \to \mathbb{H}^{pol}$ with

$$\imath : (A, \underline{b}, \underline{\omega}, d) \to \left(A, \underline{b}, \underline{\omega}, d - \frac{1}{2}\langle \underline{b}, \underline{\omega}\rangle\right).$$

The reason is as below. Firstly, the mapping $\imath$ is clearly bijective on $\mathbb{R}^{n\times n} \times (\mathbb{R}^n)^2 \times \mathbb{R}$. Secondly, for $\mathbf{h}, \tilde{\mathbf{h}} \in \mathbb{H}^{full}$, the relations

$$\begin{aligned}
&\imath(\mathbf{h}\cdot\tilde{\mathbf{h}}) \\
&\quad= \imath\left(A\tilde{A}, \underline{b} + A\tilde{\underline{b}}, \underline{\omega} + (A^T)^{-1}\tilde{\underline{\omega}}, d + \tilde{d} + \frac{1}{2}\left(\langle A\tilde{\underline{b}}, \underline{\omega}\rangle - \langle A^{-1}\underline{b}, \tilde{\underline{\omega}}\rangle\right)\right) \\
&\quad= \left(A\tilde{A}, \underline{b} + A\tilde{\underline{b}}, \underline{\omega} + (A^T)^{-1}\tilde{\underline{\omega}}, d + \tilde{d} + \frac{1}{2}\left(\langle A\tilde{\underline{b}}, \underline{\omega}\rangle - \langle A^{-1}\underline{b}, \tilde{\underline{\omega}}\rangle\right)\right. \\
&\qquad\left.-\frac{1}{2}\langle \underline{b} + A\tilde{\underline{b}}, \underline{\omega} + (A^T)^{-1}\tilde{\underline{\omega}}\rangle\right) \\
&\quad= \left(A\tilde{A}, \underline{b} + A\tilde{\underline{b}}, \underline{\omega} + (A^T)^{-1}\tilde{\underline{\omega}}, d + \tilde{d} - \langle A^{-1}\underline{b}, \tilde{\underline{\omega}}\rangle - \frac{1}{2}\langle \underline{b}, \underline{\omega}\rangle - \frac{1}{2}\langle \tilde{\underline{b}}, \tilde{\underline{\omega}}\rangle\right)
\end{aligned}$$

and

$$\begin{aligned}
&\imath(\mathbf{h})\cdot\imath(\tilde{\mathbf{h}}) \\
&\quad= \left(A, \underline{b}, \underline{\omega}, d - \frac{1}{2}\langle \underline{b}, \underline{\omega}\rangle\right)\cdot\left(\tilde{A}, \tilde{\underline{b}}, \tilde{\underline{\omega}}, \tilde{d} - \frac{1}{2}\langle \tilde{\underline{b}}, \tilde{\underline{\omega}}\rangle\right) \\
&\quad= \left(A\tilde{A}, \underline{b} + A\tilde{\underline{b}}, \underline{\omega} + (A^T)^{-1}\tilde{\underline{\omega}}, [d - \frac{1}{2}\langle \underline{b}, \underline{\omega}\rangle] + [\tilde{d} - \frac{1}{2}\langle \tilde{\underline{b}}, \tilde{\underline{\omega}}\rangle] - \langle A^{-1}b, \tilde{\underline{\omega}}\rangle\right)
\end{aligned}$$

imply $\imath(\mathbf{h})\cdot\imath(\tilde{\mathbf{h}}) = \imath(\mathbf{h}\cdot\tilde{\mathbf{h}})$.

Our interest lies in a common imbedding subgroup of $\mathbb{H}^{full}$ and $\mathbb{H}^{pol}$ when setting $b = d = 0$, where the multiplication on $\mathbb{R}^{n\times n} \times \{0\} \times \mathbb{R}^n \times \{0\}$ is defined by

$$(A, 0, \underline{\omega}, 0)\cdot(\tilde{A}, 0, \tilde{\underline{\omega}}, 0) = (A\tilde{A}, 0, \underline{\omega} + (A^T)^{-1}\tilde{\underline{\omega}}, 0).$$

For simplicity, we denote it by $\mathbb{H}$, the set $\mathbb{R}^{n\times n} \times \mathbb{R}^n$ with the multiplication

$$(A, \underline{\omega})\cdot(\tilde{A}, \tilde{\underline{\omega}}) = \left(A\tilde{A}, \underline{\omega} + (A^T)^{-1}\tilde{\underline{\omega}}\right). \tag{4.11}$$

The corresponding group representation of $\mathbb{H}$ is the unitary operator $\mathcal{M}_{\underline{\omega}}\mathcal{D}_A$ which has the composition rule of *frequency-dilation*

$$(\mathcal{M}_{\underline{\omega}}\mathcal{D}_A)(\mathcal{M}_{\tilde{\underline{\omega}}}\mathcal{D}_{\tilde{A}}) = \mathcal{M}_{\underline{\omega}+(A^T)^{-1}\tilde{\underline{\omega}}}\mathcal{D}_{A\tilde{A}}.$$

In order to design a useful time-frequency transform, we only consider the uniform scaling (isotropic scaling), that is, the case $A = aI$ with I being the identity matrix of order n. For a fixed atom $\phi \in L^2(\mathbb{R}^n)$, according to the representation coefficients of the group $\mathbb{H}$, we are suggested to investigate the integral transform in $L^2(\mathbb{R}^n)$

$$\mathcal{U}_\phi f(\underline{\omega}, a) = \langle f, \mathcal{M}_{\underline{\omega}} \mathcal{D}_{aI}\phi\rangle = a^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} f(\underline{\mathrm{x}}) \overline{\phi\left(\frac{\underline{\mathrm{x}}}{a}\right)} e^{-i\langle\underline{\omega}, \underline{\mathrm{x}}\rangle} d\underline{\mathrm{x}}, \quad \underline{\omega} \in \mathbb{R}^n, \quad a \in \mathbb{R}_+. \tag{4.12}$$

The next theorem indicates that f can be reconstructed from $\mathcal{U}_\phi f$.

Theorem 4.1. *Suppose that $\phi \in L^2(\mathbb{R}^n)$ is a radial function with $\phi(\underline{x}) = \varphi(|\underline{x}|)$ and $C_\varphi := (2\pi)^n \int_{\mathbb{R}_+} \frac{|\varphi(t)|^2}{t} dt < \infty$. Then the following formula*

$$f = \frac{1}{C_\varphi} \int\limits_{\mathbb{R}_+} a^{n-1} da \int\limits_{\mathbb{R}^n} d\underline{\omega} \mathcal{U}_\phi f(\underline{\omega}, a) \mathcal{M}_{\underline{\omega}} \mathcal{D}_{aI}\phi \tag{4.13}$$

holds in the weak sense for any $f \in L^2(\mathbb{R}^n)$.

Proof. Let the image space $\mathcal{U}_\phi(L^2(\mathbb{R}^n))$ be equipped with the inner product

$$\langle \mathcal{U}_\phi f, \mathcal{U}_\phi g\rangle_{(\underline{\omega}, a)} = \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}^n} \mathcal{U}_\phi f(\underline{\omega}, a) \overline{\mathcal{U}_\phi g(\underline{\omega}, a)} a^{n-1} da d\underline{\omega}.$$

It suffices to prove $\langle \mathcal{U}_\phi f, \mathcal{U}_\phi g\rangle_{(\underline{\omega}, a)} = C_\phi \langle f, g\rangle$. Direct computation gives

$$\begin{aligned}\langle \mathcal{U}_\phi f, \mathcal{U}_\phi g\rangle_{(\underline{\omega}, a)} &= \langle\langle f, \mathcal{M}_{\underline{\omega}}\mathcal{D}_{aI}\phi\rangle, \langle g, \mathcal{M}_{\underline{\omega}}\mathcal{D}_{aI}\phi\rangle\rangle_{(\underline{\omega}, a)} \\ &= (2\pi)^n \left\langle \left(f\overline{\mathcal{D}_{aI}\phi}\right)^\wedge(\underline{\omega}), \left(g\overline{\mathcal{D}_{aI}\phi}\right)^\wedge(\underline{\omega})\right\rangle_{(\underline{\omega}, a)}.\end{aligned}$$

The Plancherel formula leads to

$$\begin{aligned}\langle \mathcal{U}_\phi f, \mathcal{U}_\phi g\rangle_{(\underline{\omega}, a)} &= (2\pi)^n \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}^n} \left(f\overline{\mathcal{D}_{aI}\phi}\right)^\wedge(\underline{\omega}) \overline{\left(g\overline{\mathcal{D}_{aI}\phi}\right)^\wedge(\underline{\omega})} a^{n-1} da d\underline{\omega} \\ &= (2\pi)^n \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}^n} \left(f\overline{\mathcal{D}_{aI}\phi}\right)(\underline{\omega}) \overline{\left(g\overline{\mathcal{D}_{aI}\phi}\right)(\underline{\omega})} a^{n-1} da d\underline{\omega} \\ &= (2\pi)^n \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}^n} f(\underline{\omega})\overline{g(\underline{\omega})} |(\mathcal{D}_{aI}\phi)(\underline{\omega})|^2 a^{n-1} da d\underline{\omega} \\ &= (2\pi)^n \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}^n} f(\underline{\omega})\overline{g(\underline{\omega})} \frac{|\varphi(|\underline{\omega}|/a)|^2}{a} da d\underline{\omega} \\ &= \langle f, g\rangle (2\pi)^n \int\limits_{\mathbb{R}_+} \frac{|\varphi(t)|^2}{t} dt.\end{aligned}$$

This completes the proof of this theorem. □

Remark. In the synthesis formula (4.13), $a^{n-1} da d\underline{\omega}$ is used to replace the Haar measure $\frac{1}{a^{n+1}} da d\underline{\omega}$ in wavelet transform.

The resolution of the identity (4.13) can be rewritten in another way

$$C_{\varphi}^{-1}\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}^n} a^{n-1}dad\underline{\omega}\langle\cdot,\mathcal{M}_{\underline{\omega}}\mathcal{D}_{aI}\phi\rangle\mathcal{M}_{\underline{\omega}}\mathcal{D}_{aI}\phi = I_d, \tag{4.14}$$

where $\langle\cdot,g\rangle g$ stands for the operator on $L^2(\mathbb{R}^n)$ that sends f to $\langle f,g\rangle g$.

The synthesis formula (4.13) indicates that the system $\{\mathcal{M}_{\underline{\omega}}\mathcal{D}_{aI}\varphi(|\cdot|):\underline{\omega}\in\mathbb{R}^n, a\in\mathbb{R}_+\}$ is dense in $L^2(\mathbb{R})$. Due to the separability of the space $L^2(\mathbb{R}^n)$, it is natural to expect a countable system from discretization of $\mathcal{M}_{\underline{\omega}}\mathcal{D}_{aI}\varphi(|\cdot|)$. To this end, we consider the lattices Λ defined in (1.2), which leads to the atoms

$$\varphi_{j,\underline{\mathrm{k}}}(\underline{\mathrm{x}}) = \mathcal{M}_{a_0^{-j}\omega_0\underline{\mathrm{k}}}\mathcal{D}_{a_0^jI}\varphi(|\underline{\mathrm{x}}|) = a_0^{-\frac{jn}{2}}\varphi\left(\frac{|\underline{\mathrm{x}}|}{a_0^j}\right)e^{i\langle a_0^{-j}\omega_0\underline{\mathrm{k}},\underline{\mathrm{x}}\rangle},\ \underline{\mathrm{x}}\in\mathbb{R}^n. \tag{4.15}$$

Here $a_0>1$ and $\omega_0>0$.

5. Conditions on Generator and Lattice Parameters

For numerical stability, we require that the atoms $\{\varphi_{j,\underline{\mathrm{k}}}:(j,\underline{\mathrm{k}})\in\mathbb{Z}\times\mathbb{Z}^n\}$ defined in (4.15) form a frame of $L^2(\mathbb{R}^n)$, that is,

$$A\|f\|^2\le\sum_{j\in\mathbb{Z}}\sum_{\underline{\mathrm{k}}\in\mathbb{Z}^n}|\langle f,\varphi_{j,\underline{\mathrm{k}}}\rangle|^2\le B\|f\|^2 \tag{5.1}$$

holding for any $f\in L^2(\mathbb{R}^n)$. It is well known [17,20] that, if imposing the linearly independent condition on $\{\phi_{j,\underline{\mathrm{k}}}\}$, the system $\{\phi_{j,\underline{\mathrm{k}}}:(j,\underline{\mathrm{k}})\in\mathbb{Z}\times\mathbb{Z}^n\}$ becomes a Riesz basis of $L^2(\mathbb{R}^n)$, i.e.

$$A\|\{c_{j,\underline{\mathrm{k}}}\}\|_{l^2}^2\le\|\sum_{j\in\mathbb{Z}}\sum_{\underline{\mathrm{k}}\in\mathbb{Z}^n}c_{j,\underline{\mathrm{k}}}\varphi_{j,\underline{\mathrm{k}}}\|^2\le B\|\{c_{j,\underline{\mathrm{k}}}\}\|_{l^2}^2.$$

We will look for conditions on the generator φ and the sampling lattice parameters $\omega_0\in(0,\infty)$, $a_0\in(1,\infty)$, that make (5.1) hold.

5.1. Necessary Conditions on Generator and Lattice Parameters

Before establishing our main theorem we first prove some useful lemmas. The first lemma is related to trace-class operators. We first recall some notation and terminology. A operator $\mathcal{C}$ mapping $L^2(\mathbb{R}^n)$ to itself is said to be self-adjoint if $\mathcal{C}^*=\mathcal{C}$. If a self-adjoint operator $\mathcal{C}$ satisfies $\langle\mathcal{C}f,f\rangle\ge 0$ for all $f\in L^2(\mathbb{R}^n)$, then it is called a positive operator. Trace-class operators are special operators for which $\sum_l|\langle\mathcal{C}u_l,u_l\rangle|$ is finite for all orthonormal bases in $L^2(\mathbb{R}^n)$.

For a trace-class operator the quantity $\sum_l\langle\mathcal{C}u_l,u_l\rangle$ is independent of any particularly chosen orthonormal basis, which is named as the trace of $\mathcal{C}$ and denoted by

$$Tr\mathcal{C}=\sum_l\langle\mathcal{C}u_l,u_l\rangle.$$

For a positive operator $\mathcal{C}$, if $\sum_l \langle \mathcal{C}u_l, u_l\rangle$ is finite for a particular orthonormal basis, then $\mathcal{C}$ is a trace-class operator [9]. Any function $f \in L^2(\mathbb{R}^n)$ has the following decomposition

$$\mathcal{C}f = \sum_l d_l \langle f, u_l\rangle u_l \tag{5.2}$$

in terms of the eigenvectors of the positive trace-class $\mathcal{C}$. Here $\mathcal{C}u_l = d_l u_l$ and $d_l \geq 0$.

Lemma 5.1. *Suppose that* $\{\varphi_{j,\underline{k}} : (j, \underline{k}) \in \mathbb{Z} \times \mathbb{Z}^n\}$ *is a frame of* $L^2(\mathbb{R}^n)$ *with the bounds* A *and* B *as given in* (5.1)*. Then for any positive trace-class operator* $\mathcal{C}$*, there holds*

$$A\, Tr\mathcal{C} \leq \sum_{j\in\mathbb{Z}} \sum_{\underline{k}\in\mathbb{Z}^n} \langle \mathcal{C}\varphi_{j,\underline{k}}, \varphi_{j,\underline{k}}\rangle \leq B\, Tr\mathcal{C}. \tag{5.3}$$

Proof. Let $\{u_l : l \in \mathbb{Z}\}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$. Choosing coefficients $d_l \geq 0$ satisfying $\sum_{l\in\mathbb{Z}} d_l < \infty$, setting $f = u_l$ in (5.1) and then taking the weighted sum (with the weights d_l) on the obtained inequalities, we get

$$A \sum_{l\in\mathbb{Z}} d_l \leq \sum_{l\in\mathbb{Z}} d_l \sum_{j\in\mathbb{Z}} \sum_{\underline{k}\in\mathbb{Z}^n} |\langle u_l, \varphi_{j,\underline{k}}\rangle|^2 \leq B \sum_{l\in\mathbb{Z}} d_l.$$

By invoking (5.2) and $Tr\mathcal{C} = \sum_{l\in\mathbb{Z}} d_l$, we obtain the relation (5.3). □

Next we introduce a special positive trace-class operator related to the synthesis formula (4.14). Define

$$\mathcal{C} = \int_{\mathbb{R}_+} \int_{\mathbb{R}^n} a^{n-1} \langle \cdot, h^{a,\underline{\omega}}\rangle h^{a,\underline{\omega}} c(a, \underline{\omega}) da d\underline{\omega} \tag{5.4}$$

where $h^{a,\underline{\omega}} = M_{\underline{\omega}} D_{aI} h$, h is any function in $L^2(\mathbb{R}^n)$ and c is an arbitrary positive function in $L^1(\mathbb{R} \times \mathbb{R}^n)$ such that the integral $\int_{\mathbb{R}_+} \int_{\mathbb{R}^n} a^{n-1} c(a, \underline{\omega}) da d\underline{\omega}$ converges.

Lemma 5.2. *The operator defined in* (5.4) *is a positive trace-class operator with*

$$Tr\mathcal{C} = \|h\|_2^2 \int_{\mathbb{R}_+} \int_{\mathbb{R}^n} a^{n-1}\, c(a, \underline{\omega})\, da d\underline{\omega}. \tag{5.5}$$

Proof. The positivity of $\mathcal{C}$ can be verified by

$$\langle \mathcal{C}f, f\rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}^n} a^{n-1} |\langle f, h^{a,\underline{\omega}}\rangle|^2 c(a, \underline{\omega}) da d\underline{\omega} \geq 0.$$

The following calculation indicates that $\mathcal{C}$ is a trace-class operator:

$$Tr\mathcal{C} = \sum_{l\in\mathbb{Z}} \langle \mathcal{C}u_l, u_l\rangle = \int_{\mathbb{R}_+}\int_{\mathbb{R}^n} a^{n-1}c(a,\underline{\omega}) \sum_{l\in\mathbb{Z}} |\langle h^{a,\underline{\omega}}, u_l\rangle|^2 da d\underline{\omega}$$
$$= \int_{\mathbb{R}_+}\int_{\mathbb{R}^n} a^{n-1}c(a,\underline{\omega}) \|h^{a,\underline{\omega}}(\cdot)\|_2^2 da d\underline{\omega}$$
$$= \|h\|_2^2 \int_{\mathbb{R}_+}\int_{\mathbb{R}^n} a^{n-1}c(a,\underline{\omega}) da d\underline{\omega},$$

where we utilized the Lebesgue dominated convergence theorem and the Parseval identity for orthonormal basis. □

Specifically, set

$$c(a,\underline{\omega}) = \begin{cases} \eta(a|\underline{\omega}|), & 1 \le a \le a_0, \quad \underline{\omega} \in \mathbb{R}^n \\ 0, & \text{otherwise}, \end{cases} \tag{5.6}$$

where $\eta \in L^1(\mathbb{R})$ is a positive function with $\eta(r) = O((1+r^2)^{-\frac{n+1}{2}})$. We have the following corollary.

Corollary 5.3. *Let $\mathcal{C}$ be the operator defined in (5.4) with $c(a,\underline{\omega})$ defined in (5.6). Then*

$$\mathcal{C} = \int_1^{a_0}\int_{\mathbb{R}^n} a^{n-1}\langle \cdot, h^{a,\underline{\omega}}\rangle h^{a,\underline{\omega}} \eta(a|\underline{\omega}|) da d\underline{\omega} \tag{5.7}$$

and

$$Tr\mathcal{C} = \Omega_n \|h\|_2^2 \ln a_0 \int_{\mathbb{R}_+} r^{n-1}\eta(r)dr, \tag{5.8}$$

where $\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$, the area of the unit sphere in $\mathbb{R}^n$.

Next lemma establishes an identity about the inner product of $\varphi_{j,\underline{k}}$ and $h^{a,\underline{\omega}}$.

Lemma 5.4. *The following identity holds*

$$\langle \varphi_{j,\underline{k}}, h^{a,\underline{\omega}}\rangle = \langle \varphi(|\cdot|), h^{a_0^{-j}a,\ a_0^j\underline{\omega}-\omega_0\underline{k}}\rangle. \tag{5.9}$$

Proof. The definitions of $\varphi_{j,\underline{k}}$ and $h^{a,\underline{\omega}}$ imply that

$$\langle \varphi_{j,\underline{k}}, h^{a,\underline{\omega}}\rangle = \int_{\mathbb{R}^n} a_0^{-\frac{jn}{2}} \varphi(\frac{|\underline{x}|}{a_0^j}) e^{i a_0^{-j}\omega_0\langle \underline{k},\underline{x}\rangle} a^{-\frac{n}{2}} \overline{h(\frac{\underline{x}}{a})} e^{-i\langle \underline{\omega},\underline{x}\rangle} d\underline{x}$$
$$= a_0^{-\frac{jn}{2}} a^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(\frac{|\underline{x}|}{a_0^j}) \overline{h(\frac{\underline{x}}{a})} e^{-i\langle \underline{\omega} - a_0^{-j}\omega_0\underline{k},\ \underline{x}\rangle} d\underline{x}.$$

Changing $\underline{x}$ to $a_0^j\underline{x}$ leads to

$$\langle \varphi_{j,\underline{k}}, h^{a,\underline{\omega}}\rangle = a_0^{\frac{jn}{2}} a^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(|\underline{x}|) \overline{h(\frac{a_0^j\underline{x}}{a})} e^{-i\langle a_0^j\underline{\omega}-\omega_0\underline{k},\ \underline{x}\rangle} d\underline{x}.$$

We therefore conclude (5.9). □

The next lemma offers an integral representation for the quantity $\sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n}\langle\mathcal{C}\varphi_{j,\underline{k}},\varphi_{j,\underline{k}}\rangle$ in terms of the functions η and h.

Lemma 5.5. *Suppose that $\mathcal{C}$ is defined by (5.7). Then*

$$\sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n}\langle\mathcal{C}\varphi_{j,\underline{k}},\varphi_{j,\underline{k}}\rangle=\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}a^{n-1}\sum_{\underline{k}\in\mathbb{Z}^n}\eta\left(a|\underline{\omega}+\omega_0\underline{k}|\right)\left|\langle\varphi(|\cdot|),h^{a,\underline{\omega}}\rangle\right|^2 dad\underline{\omega}. \tag{5.10}$$

Proof. Using (5.7) and (5.9), we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n}\langle\mathcal{C}\varphi_{j,\underline{k}},\varphi_{j,\underline{k}}\rangle&=\sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n}\int_1^{a_0}\int_{\mathbb{R}^n}a^{n-1}|\langle\varphi_{j,\underline{k}},h^{a,\underline{\omega}}\rangle|^2\eta(a|\underline{\omega}|)dad\underline{\omega}\\&=\sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n}\int_1^{a_0}\int_{\mathbb{R}^n}a^{n-1}|\langle\varphi(|\cdot|),h^{a_0^{-j}a,a_0^j\underline{\omega}-\omega_0\underline{k}}\rangle|^2\eta(a|\underline{\omega}|)dad\underline{\omega}.\end{aligned}$$

Changing the variable pair $(a,\underline{\omega})$ to $(a_0^ja,a_0^{-j}(\underline{\omega}+\omega_0\underline{k}))$ and noting that the Jacobi is $a_0^{-j(n-1)}$, we have

$$\sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n}\langle\mathcal{C}\varphi_{j,\underline{k}},\varphi_{j,\underline{k}}\rangle=\sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n}\int_{a_0^{-j}}^{a_0^{-j+1}}\int_{\mathbb{R}^n}a^{n-1}\left|\langle\varphi(|\cdot|),h^{a,\underline{\omega}}\rangle\right|^2\eta\left(a|\underline{\omega}+\omega_0\underline{k}|\right)dad\underline{\omega}.$$

Note that the function in the right hand side of the above integral is independent of j. By applying the Lebesgue dominated convergence theorem, we obtain (5.10). □

We need the following property for the Gaussian function.

Lemma 5.6. *Suppose that $\eta(x)=\lambda^ne^{-\pi\lambda^2x^2}$ with the parameter $\lambda>0$. Then*

$$\sum_{\underline{k}\in\mathbb{Z}^n}\eta\left(a|\underline{\omega}+\omega_0\underline{k}|\right)=\frac{1}{a^n\omega_0^n}+\rho(a,\underline{\omega}) \tag{5.11}$$

with $|\rho(a,\underline{\omega})|<g_\lambda$ and $g_\lambda=(\frac{1}{a\omega_0}+\lambda)^n-\frac{1}{a^n\omega_0^n}$.

Proof. Using the property of the Gaussian function, it can be proved that (see, for instance, Daubechies, see [8, Lemma 2.2])

$$\sum_{j\in\mathbb{Z}}\lambda e^{-\pi\lambda^2a^2(t+\omega_0j)^2}=\frac{1}{a\omega_0}+\rho_1(a,t),\ t\in\mathbb{R}$$

with $|\rho_1(a,t)|<\lambda$. Therefore,

$$\begin{aligned}\sum_{\underline{k}\in\mathbb{Z}^n}\eta\left(a|\underline{\omega}+\omega_0\underline{k}|\right)&=\prod_{j=1}^n\sum_{k_j\in\mathbb{Z}}\lambda e^{-\pi\lambda^2a^2(\omega_j+\omega_0k_j)^2}\\&=\prod_{j=1}^n\left(\frac{1}{a\omega_0}+\rho_1(a,\omega_j)\right).\end{aligned}$$

Here ω_j is the jth element of the $\underline{\omega}$ for $1 \le j \le n$. Write $\sum_{\underline{k}\in\mathbb{Z}^n} \eta\left(a|\underline{\omega} + \omega_0\underline{k}|\right)$ as the sum of $\frac{1}{a^n\omega_0^n}$ and $\rho(a,\underline{\omega})$ with

$$\rho(a,\underline{\omega}) = \prod_{j=1}^{n}\left(\frac{1}{a\omega_0} + \rho_1(a,\omega_j)\right) - \frac{1}{a^n\omega_0^n}.$$

The estimation for $\rho(a,\underline{\omega})$ can be obtained from the calculation

$$\begin{aligned}|\rho(a,\underline{\omega})| &\le \prod_{j=1}^{n}\left(\frac{1}{a\omega_0} + |\rho_1(a,\omega_j)|\right) - \frac{1}{a^n\omega_0^n}\\ &\le \left(\frac{1}{a\omega_0} + \lambda\right)^n - \frac{1}{a^n\omega_0^n} = g_\lambda.\end{aligned}$$

□

The final lemma of this section as given below investigates the operator $\mathcal{C}$ when η is specified as the Gaussian function.

Lemma 5.7. *Suppose that* $\mathcal{C}$ *is defined by* (5.7) *with* $\eta(x) = \lambda^n e^{-\pi\lambda^2x^2}$ *for a parameter* $\lambda > 0$. *Let* $h \in L^2(\mathbb{R}^n)$ *be a radial function* $h = \hbar(|\cdot|)$ *with* $C_\hbar := \int_{\mathbb{R}_+}\frac{|\hbar(x)|^2}{x} < \infty$. *Then*

$$\sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n}\langle\mathcal{C}\varphi_{j,\underline{k}},\varphi_{j,\underline{k}}\rangle = \frac{(2\pi)^n}{\omega_0^n}\|h\|_2^2\int_{\mathbb{R}_+}\frac{\varphi(t)}{t}dt + R \tag{5.12}$$

with $0 < |R| \le C_\hbar g_\lambda\|\varphi(|\cdot|)\|_2^2$, *and the trace of the operator* $\mathcal{C}$ *is*

$$Tr\mathcal{C} = \|h\|_2^2\ln a_0. \tag{5.13}$$

Proof. Combing (5.10) with (5.11), we get

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n}\langle\mathcal{C}\varphi_{j,\underline{k}},\varphi_{j,\underline{k}}\rangle\\ &= \int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\frac{1}{a\omega_0^n}\left|\langle\varphi(|\cdot|),h^{a,\underline{\omega}}\rangle\right|^2 dad\underline{\omega}\\ &\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}^n}a^{n-1}\rho(a,\underline{\omega})\left|\langle\varphi(|\cdot|),h^{a,\underline{\omega}}\rangle\right|^2 dad\underline{\omega}\\ &:= J + R.\end{aligned}$$

We first deal with R. Using $|\rho(a,\underline{\omega})| < g_\lambda$ and the synthesis formula (4.13) (note that the condition $C_\hbar := \int_{\mathbb{R}_+}\frac{|\hbar(x)|^2}{x} < \infty$ holds), it follows

$$\begin{aligned}|R| &\le g_\lambda\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}a^{n-1}\left|\langle\varphi(|\cdot|),h^{a,\underline{\omega}}\rangle\right|^2 dad\underline{\omega}\\ &= C_\hbar g_\lambda\|\varphi(|\cdot|)\|_2^2.\end{aligned}$$

Next, by Plancherel formula, we rewrite J as

$$\begin{aligned} J &= \int_{\mathbb{R}_+}\int_{\mathbb{R}^n} \frac{1}{a\omega_0^n} \left|\langle \varphi(|\cdot|), h^{a,\underline{\omega}}\rangle\right|^2 da d\underline{\omega} \\ &= \frac{1}{\omega_0^n}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} \frac{1}{a^{n+1}} \left|\int_{\mathbb{R}^n} \varphi(|\underline{x}|)\overline{\hbar\left(\frac{\underline{x}}{a}\right)} e^{-i\langle\underline{\omega},\underline{x}\rangle} d\underline{x}\right|^2 da d\underline{\omega} \\ &= \frac{1}{\omega_0^n}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} (2\pi)^n \frac{1}{a^{n+1}} \left|\left(\varphi(|\cdot|)\hbar\left(\frac{|\cdot|}{a}\right)\right)^{\wedge}(\underline{\omega})\right|^2 d\underline{\omega} da \\ &= \frac{(2\pi)^n}{\omega_0^n}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} \frac{1}{a^{n+1}} \left|\varphi(|\underline{x}|)\hbar\left(\frac{|\underline{x}|}{a}\right)\right|^2 d\underline{x} da. \end{aligned}$$

Setting $\underline{x} \to a\underline{y}$ in the above integral leads to

$$\begin{aligned} J &= \frac{(2\pi)^n}{\omega_0^n}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} \frac{1}{a}|\varphi(a|\underline{y}|)|^2|\hbar(|\underline{y}|)|^2 d\underline{y} da \\ &= \frac{(2\pi)^n}{\omega_0^n}\int_{\mathbb{R}_+} \frac{1}{a|\underline{y}|}|\varphi(a|\underline{y}|)|^2 d(a|\underline{y}|) \int_{\mathbb{R}^n} |\hbar(|\underline{y}|)|^2 d\underline{y} \\ &= \frac{(2\pi)^n}{\omega_0^n}\|h\|_2^2 \int_{\mathbb{R}_+} \frac{\varphi(t)}{t} dt. \end{aligned}$$

To calculate the trace, we recall (5.8) and the definition of the Gamma function to obtain

$$\begin{aligned} Tr\mathcal{C} &= \Omega_n\|h\|_2^2 \ln a_0 \int_{\mathbb{R}_+} r^{n-1}\eta(r) dr \\ &= \Omega_n\|h\|_2^2 \ln a_0 \int_{\mathbb{R}_+} r^{n-1}\lambda^n e^{-\pi\lambda^2 r^2} dr \\ &= \Omega_n\|h\|_2^2 \frac{1}{2}\pi^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)\ln a_0 = \|h\|_2^2 \ln a_0. \end{aligned}$$

The proof is complete. □

Through all these preparations, we now establish a theorem concerning necessary conditions.

Theorem 5.8. *If the system* $\{\varphi_{j,\underline{k}} : (j,\underline{k}) \in \mathbb{Z}\times\mathbb{Z}^n\}$ *defined in* (4.15) *constitute a frame for* $L^2(\mathbb{R}^n)$ *with frame bounds* A, B*, then*

$$\frac{\omega_0^n \ln a_0}{(2\pi)^n} A \le \int_{\mathbb{R}_+} \frac{|\varphi(t)|^2}{t} dt \le \frac{\omega_0^n \ln a_0}{(2\pi)^n} B. \tag{5.14}$$

Proof. By (5.3), (5.13) and (5.12), we get the inequality

$$A\|h\|_2^2 \ln a_0 \le \frac{(2\pi)^n}{\omega_0^n}\|h\|_2^2 \int\limits_{\mathbb{R}_+} \frac{|\varphi(t)|^2}{t} dt + R \le B\|h\|_2^2 \ln a_0$$

with $0 < |R| \le C_\hbar g_\lambda \|\varphi(|\cdot|)\|_2^2$. Note that $\lim_{\lambda\to 0} R = 0$ (see the definition of g_λ in (5.11)). Letting λ tend to 0 and dividing by $\|h\|_2^2$ in the above inequalities, we find

$$A \ln a_0 \le \frac{(2\pi)^n}{\omega_0^n} \int\limits_{\mathbb{R}_+} \frac{|\varphi(t)|^2}{t} dt \le B \ln a_0.$$

Therefore, (5.14) follows. □

5.2. Sufficient Conditions

We now discuss sufficient conditions that ensure $\{\varphi_{j,\underline{k}} : (j,\underline{k}) \in \mathbb{Z}\times\mathbb{Z}^n\}$ to be a frame of $L^2(\mathbb{R}^n)$. The key point is to estimate the quantity $\sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n} \left|\langle f, \varphi_{j,\underline{k}}\rangle\right|^2$.

Let $\diamondsuit$ be the unit hypercube $[0,1]^n$ in $\mathbb{R}^n$ and $\diamondsuit_{\underline{s}}$ be $\underline{s}$-shift of the box $\diamondsuit$. Denote by $2\pi\omega_0^{-1}a_0^j\diamondsuit_{\underline{s}}$ the $2\pi\omega_0^{-1}a_0^j$ dilation of $\diamondsuit_{\underline{s}}$. By utilizing the partition of $\mathbb{R}^n = \cup_{\underline{s}\in Z^n}^{\infty} 2\pi\omega_0^{-1}a_0^j\diamondsuit_{\underline{s}}$, a direct computation gives rise to

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n} \left|\langle f, \varphi_{j,\underline{k}}\rangle\right|^2 \\
&= \sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n} \left| \int\limits_{\mathbb{R}^n} f(\underline{x}) a_0^{-\frac{jn}{2}} \overline{\varphi\left(\frac{|\underline{x}|}{a_0^j}\right)} e^{-i\langle a_0^{-j}\omega_0\underline{k},\underline{x}\rangle} d\underline{x}\right|^2 \\
&= \sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n} \left| \sum_{\underline{s}\in\mathbb{Z}^n} \int\limits_{2\pi\omega_0^{-1}a_0^j\diamondsuit_{\underline{s}}} f(\underline{x}) a_0^{-\frac{jn}{2}} \overline{\varphi\left(\frac{|\underline{x}|}{a_0^j}\right)} e^{-i\langle a_0^{-j}\omega_0\underline{k},\underline{x}\rangle} d\underline{x}\right|^2 \\
&= \sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n} \left| \sum_{\underline{s}\in\mathbb{Z}^n} \int\limits_{2\pi\omega_0^{-1}a_0^j\diamondsuit} f(\underline{x}+2\pi\omega_0^{-1}a_0^j\underline{s}) a_0^{-\frac{nj}{2}} \overline{\varphi\left(\frac{|\underline{x}+2\pi\omega_0^{-1}a_0^j\underline{s}|}{a_0^j}\right)} e^{-i\langle a_0^{-j}\omega_0\underline{k},\underline{x}+2\pi\omega_0^{-1}a_0^j\underline{s}\rangle} d\underline{x}\right|^2 \\
&= \sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n} a_0^{-jn} \left| \int\limits_{2\pi\omega_0^{-1}a_0^j\diamondsuit} \sum_{\underline{s}\in\mathbb{Z}^n} f(\underline{x}+2\pi\omega_0^{-1}a_0^j\underline{s}) \overline{\varphi(|a_0^{-j}\underline{x}+2\pi\omega_0^{-1}\underline{s}|)} e^{-i\langle a_0^{-j}\omega_0\underline{k},\underline{x}\rangle} d\underline{x}\right|^2.
\end{aligned}$$

By the Plancherel theorem for periodic functions, we obtain

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n} \left|\langle f, \varphi_{j,\underline{k}}\rangle\right|^2 \\
&= \left(\frac{2\pi}{\omega_0}\right)^n \sum_{j\in\mathbb{Z}} \int\limits_{2\pi\underline{\omega}_0^{-1}a_0^j\diamondsuit} \left| \sum_{\underline{s}\in\mathbb{Z}^n} f(\underline{x}+2\pi\omega_0^{-1}a_0^j\underline{s}) \overline{\varphi(|a_0^{-j}x+2\pi\omega_0^{-1}\underline{s}|)}\right|^2 d\underline{x} \\
&= \left(\frac{2\pi}{\omega_0}\right)^n \sum_{j\in\mathbb{Z}} \int\limits_{2\pi\omega_0^{-1}a_0^j\diamondsuit} \sum_{\underline{s}\in\mathbb{Z}^n} f(\underline{x}+2\pi\omega_0^{-1}a_0^j\underline{s}) \overline{\varphi(|a_0^{-j}x+2\pi\omega_0^{-1}\underline{s}|)}
\end{aligned}$$

$$\sum_{\underline{k}\in\mathbb{Z}^n} \overline{f(\underline{x}+2\pi\omega_0^{-1}a_0^j\underline{k})}\varphi(|a_0^{-j}\underline{x}+2\pi\omega_0^{-1}\underline{k}|)d\underline{x}$$
$$= \left(\frac{2\pi}{\omega_0}\right)^n \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} f(\underline{x})\overline{\varphi(|a_0^{-j}\underline{x}|)} \sum_{\underline{k}\in\mathbb{Z}^n} \overline{f(\underline{x}+2\pi\omega_0^{-1}a_0^j\underline{k})}\varphi(|a_0^{-j}\underline{x}+2\pi\omega_0^{-1}\underline{k}|)d\underline{x}.$$

We rewrite it as

$$\sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n}\left|\langle f,\varphi_{j,\underline{k}}\rangle\right|^2 := M_1 + M_2, \tag{5.15}$$

where

$$M_1 = \left(\frac{2\pi}{\omega_0}\right)^n \int_{\mathbb{R}^n} |f(\underline{x})|^2 \sum_{j\in\mathbb{Z}} |\varphi(|a_0^{-j}\underline{x}|)|^2 d\underline{x}$$

and

$$M_2 = \left(\frac{2\pi}{\omega_0}\right)^n \sum_{j\in\mathbb{Z}} \sum_{\underline{k}\in\mathbb{Z}^n\setminus\{0\}} \int_{\mathbb{R}^n} f(\underline{x})\overline{f(\underline{x}+2\pi\omega_0^{-1}a_0^j\underline{k})}\ \overline{\varphi(|a_0^{-j}\underline{x}|)}\varphi(|a_0^{-j}\underline{x}+2\pi\omega_0^{-1}\underline{k}|)d\underline{x}.$$

Obviously M_1 has the estimation

$$\left(\frac{2\pi}{\omega_0}\right)^n \|f\|_2^2 \ \mathrm{ess}\inf_{\underline{x}\in\mathbb{R}^n} \sum_{j\in\mathbb{Z}} |\varphi(|a_0^{-j}\underline{x}|)|^2$$
$$\leq M_1 \leq \left(\frac{2\pi}{\omega_0}\right)^n \|f\|_2^2 \ \mathrm{ess}\sup_{\underline{x}\in\mathbb{R}^n} \sum_{j\in\mathbb{Z}} |\phi(|a_0^{-j}\underline{x}|)|^2. \tag{5.16}$$

By the Cauchy–Schwartz inequality, we get

$$|M_2| \leq \left(\frac{2\pi}{\omega_0}\right)^n \sum_{j\in\mathbb{Z}} \sum_{\underline{k}\in\mathbb{Z}^n\setminus\{0\}} \int_{\mathbb{R}^n} \left| f(\underline{x})\overline{f(\underline{x}+2\pi\omega_0^{-1}a_0^j\underline{k})\varphi(|a_0^{-j}\underline{x}|)}\varphi(|a_0^{-j}\underline{x}+2\pi\omega_0^{-1}\underline{k}|)\right| d\underline{x}$$
$$\leq \left(\frac{2\pi}{\omega_0}\right)^n \sum_{j\in\mathbb{Z}} \sum_{\underline{k}\in\mathbb{Z}^n\setminus\{0\}} \left(\int_{\mathbb{R}^n} |f(\underline{x})|^2 \left|\varphi(|a_0^{-j}\underline{x}|)\varphi(a_0^{-j}\underline{x}+2\pi\omega_0^{-1}\underline{k}|)\right| d\underline{x}\right)^{\frac{1}{2}}$$
$$\left(\int_{\mathbb{R}^n} |f(\underline{x}+2\pi\omega_0^{-1}a_0^j\underline{k})|^2 \left|\varphi(|a_0^{-j}\underline{x}|)\varphi(|a_0^{-j}\underline{x}+2\pi\omega_0^{-1}\underline{k}|)\right| d\underline{x}\right)^{\frac{1}{2}}$$
$$\leq \left(\frac{2\pi}{\omega_0}\right)^n \sum_{\underline{k}\in\mathbb{Z}^n\setminus\{0\}} \sum_{j\in\mathbb{Z}} \left(\int_{\mathbb{R}^n} |f(\underline{x})|^2 \left|\varphi(|a_0^{-j}\underline{x}|)\varphi(|a_0^{-j}\underline{x}+2\pi\omega_0^{-1}\underline{k}|)\right| d\underline{x}\right)^{\frac{1}{2}}$$
$$\left(\int_{\mathbb{R}^n} |f(\underline{x})|^2 \left|\varphi(|a_0^{-j}\underline{x}-2\pi\omega_0^{-1}\underline{k}|)\varphi(|a_0^{-j}\underline{x}|)\right| d\underline{x}\right)^{\frac{1}{2}}.$$

Using the Cauchy–Schwartz inequality to the summation over j, we have

$$|M_2| \leq \left(\frac{2\pi}{\omega_0}\right)^n \sum_{\underline{k}\in\mathbb{Z}^n\setminus\{0\}} \left(\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n} |f(\underline{x})|^2 \left|\varphi(|a_0^{-j}\underline{x}|)\varphi(|a_0^{-j}\underline{x}+2\pi\omega_0^{-1}\underline{k}|)\right| d\underline{x}\right)^{\frac{1}{2}}$$

$$\left(\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n} |f(\underline{x})|^2 \left|\varphi(|a_0^{-j}\underline{x}-2\pi\omega_0^{-1}\underline{k}|)\varphi(|a_0^{-j}\underline{x}|)\right| d\underline{x}\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{2\pi}{\omega_0}\right)^n \|f\|_2^2 \sum_{\underline{k}\in\mathbb{Z}^n\setminus\{0\}} \left(\operatorname{ess}\sup_{\underline{x}\in\mathbb{R}^n}\sum_{j\in\mathbb{Z}} \left|\varphi(|a_0^{-j}\underline{x}|)\right| \left|\varphi(|a_0^{-j}\underline{x}+2\pi\omega_0^{-1}\underline{k}|)\right|\right)^{\frac{1}{2}}$$

$$\left(\operatorname{ess}\sup_{\underline{x}\in\mathbb{R}^n}\sum_{j\in\mathbb{Z}} \left|\varphi(|a_0^{-j}\underline{x}|)\right| \left|\varphi(|a_0^{-j}\underline{x}-2\pi\omega_0^{-1}\underline{k}|)\right|\right)^{\frac{1}{2}}.$$

By setting

$$\eta(\underline{t}) = \operatorname{ess}\sup_{\underline{x}\in\mathbb{R}^n}\sum_{j\in\mathbb{Z}} \left|\varphi(|a_0^{-j}\underline{x}|)\right| \left|\varphi(|a_0^{-j}\underline{x}+\underline{t}|)\right|, \tag{5.17}$$

we get

$$|M_2| \leq \left(\frac{2\pi}{\omega_0}\right)^n \|f\|_2^2 \sum_{\underline{k}\in\mathbb{Z}^n\setminus\{0\}} \left(\eta(2\pi\omega_0^{-1}\underline{k})\eta(-2\pi\omega_0^{-1}\underline{k})\right)^{\frac{1}{2}}. \tag{5.18}$$

Combing (5.15) with (5.16) and (5.18), we obtain

$$\left(\frac{2\pi}{\omega_0}\right)^n \|f\|_2^2 C_1 \leq \sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n} \left|\langle f, \varphi_{j,\underline{k}}\rangle\right|^2 \leq \left(\frac{2\pi}{\omega_0}\right)^n \|f\|_2^2 C_2 \tag{5.19}$$

with

$$C_1 = \operatorname{ess}\inf_{\underline{x}\in\mathbb{R}^n}\sum_{j\in\mathbb{Z}} |\varphi(|a_0^{-j}\underline{x}|)|^2 - \sum_{\underline{k}\in\mathbb{Z}^n\setminus\{0\}} \left[\eta(2\pi\omega_0^{-1}\underline{k})\eta(-2\pi\omega_0^{-1}\underline{k})\right]^{\frac{1}{2}} \tag{5.20}$$

and

$$C_2 = \operatorname{ess}\sup_{\underline{x}\in\mathbb{R}^n}\sum_{j\in\mathbb{Z}} |\varphi(|a_0^{-j}\underline{x}|)|^2 + \sum_{\underline{k}\in\mathbb{Z}^n\setminus\{0\}} \left[\eta(2\pi\omega_0^{-1}\underline{k})\eta(-2\pi\omega_0^{-1}\underline{k})\right]^{\frac{1}{2}}. \tag{5.21}$$

We therefore establish the following theorem on sufficiency.

Theorem 5.9. *If C_1 and C_2 defined in (5.20) and (5.21) are strictly positive and bounded, then $\{\varphi_{j,\underline{k}} : (j,\underline{k}) \in \mathbb{Z}\times\mathbb{Z}^n\}$ constitutes a frame of $L^2(\mathbb{R}^n)$.*

We remark that the conditions in Theorem 5.9 are complicated and inconvenient. We need to impose some conditions on the generator φ to ensure the strict positivity and boundedness of C_1 and C_2 in (5.20) and (5.21). To this end, it is natural to have the following two sets of requirements, that is,

(i) $\operatorname{ess}\inf_{\underline{x}\in\mathbb{R}^n}\sum_{j\in\mathbb{Z}} |\varphi(|a_0^{-j}\underline{x}|)|^2 > 0$ and $\operatorname{ess}\sup_{\underline{x}\in\mathbb{R}^n}\sum_{j\in\mathbb{Z}} |\varphi(|a_0^{-j}\underline{x}|)|^2 < \infty$;

(ii) η should decay sufficiently fast so to ensure the convergence of $\sum_{\underline{k}\in\mathbb{Z}^n\setminus\{0\}}\left[\eta(2\pi\omega_0^{-1}\underline{k})\eta(-2\pi\omega_0^{-1}\underline{k})\right]^{\frac{1}{2}}$ with the bound $\operatorname{ess\,inf}_{\underline{x}\in\mathbb{R}^n}\sum_{j\in\mathbb{Z}}|\varphi(|a_0^{-j}\underline{x}|)|^2$.

The first requirement (i) can be reduced to

$$\operatorname{ess\,inf}_{\underline{x}\in\mathbb{B}_{a_0}\setminus\mathbb{B}_1}\sum_{j\in\mathbb{Z}}|\varphi(a_0^{-j}|x|)|^2>0 \text{ and } \operatorname{ess\,sup}_{\underline{x}\in\mathbb{B}_{a_0}\setminus\mathbb{B}_1}\sum_{j\in\mathbb{Z}}|\varphi(a_0^{-j}|x|)|^2<\infty, \tag{5.22}$$

where the ball shell $\mathbb{B}_{a_0}\setminus\mathbb{B}_1\subset\mathbb{R}^n$ denotes the set of which the elements are in the ball centered at the origin with radius a_0 but not in the unit ball centered at the origin. The reason is that any non-zero $\underline{x}\in\mathbb{R}$ can be reduced to the range $\mathbb{B}_{a_0}\setminus\mathbb{B}_1$ by multiplying it with a suitable a_0^j.

Complying with the requirement (ii), we impose on φ the condition

$$\eta(\underline{t})=\operatorname*{ess\,sup}_{\underline{x}\in\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\left|\varphi(a_0^{-j}|\underline{x}|)\right|\left|\varphi(|a_0^{-j}\underline{x}+\underline{t}|)\right|=O\left(\prod_{j=1}^n(1+|t_j|)^{-(1+\epsilon)}\right) \tag{5.23}$$

for some $\epsilon>0$.

Corollary 5.10. *Suppose that* (5.22) *and* (5.23) *hold. Then there exists* $\tilde{\omega}_0>0$ *such that, for* $\omega_0\le\tilde{\omega}_0$*, the countable set* $\{\varphi_{j,\underline{k}}:(j,\underline{k})\in\mathbb{Z}\times\mathbb{Z}^n\}$ *defined in* (4.15) *is a frame of* $L^2(\mathbb{R}^n)$.

Proof. The condition (5.23) implies that there exists some positive constant C such that

$$\eta(\underline{t})\le C\prod_{j=1}^n(1+|t_j|)^{-(1+\epsilon)}.$$

Thus we have

$$\begin{aligned}\sum_{\underline{k}\in\mathbb{Z}^n\setminus\{0\}}\left[\eta(2\pi\omega_0^{-1}\underline{k})\eta(-2\pi\omega_0^{-1}\underline{k})\right]^{\frac{1}{2}}&\le\sum_{\underline{k}\in\mathbb{Z}^n\setminus\{0\}}C\prod_{j=1}^n(1+|2\pi\omega_0^{-1}|k_j||)^{-(1+\epsilon)}\\&\le C\prod_{j=1}^n\left(\sum_{k_j\in\mathbb{Z}\setminus\{0\}}\frac{1}{(1+2\pi\omega_0^{-1}|k_j|)^{1+\epsilon}}\right).\end{aligned}$$

Since the series $\sum_{k=1}^\infty\frac{1}{(t+k)^{1+\epsilon}}$ is uniformly convergent in R_+, we can define the function $u(t)=\sum_{k=1}^\infty\frac{1}{(1+2\pi kt^{-1})^{1+\epsilon}}, t\in\mathbb{R}_+$. Note that u is continuous and increasing in R_+ and satisfies that $\lim_{t\to0}u(t)=0$ and $\lim_{t\to+\infty}u(t)=+\infty$. Then the range of $u(R_+)=[0,+\infty)$. Considering $\prod_{j=1}^n(\sum_{k_j\in\mathbb{Z}\setminus\{0\}}\frac{1}{(1+2\pi\omega_0^{-1}|k_j|)^{1+\epsilon}})$ as a function in terms of ω_0, we know that its range is also $[0, +\infty)$. Therefore, there exists a constant $\tilde{\omega}_0$ satisfying

$$\sum_{\underline{k}\in\mathbb{Z}^n\setminus\{0\}} \left[\eta(2\pi\tilde{\omega}_0^{-1}\underline{k})\eta(-2\pi\tilde{\omega}_0^{-1}\underline{k})\right]^{\frac{1}{2}} < \operatorname{ess\,inf}_{\underline{x}\in\mathbb{B}_{a_0}\setminus\mathbb{B}_1} \sum_{j\in\mathbb{Z}} |\varphi(a_0^{-j}|\underline{x}|)|^2. \tag{5.24}$$

Therefore, C_1 and C_2 in (5.20) and (5.21) are strictly positive and bounded. Hence $\{\varphi_{j,\underline{k}} : (j, \underline{k}) \in \mathbb{Z} \times \mathbb{Z}^n\}$ defined in (4.15) with $\omega_0 = \tilde{\omega}_0$ constitutes a frame of $L^2(\mathbb{R}^n)$. For the case $\omega_0 < \tilde{\omega}_0$, since the function u is increasing, it indicates that (5.24) is still valid. We therefore conclude that for all $\omega_0 < \tilde{\omega}_0$ the set $\{\varphi_{j,\underline{k}} : (j, \underline{k}) \in \mathbb{Z} \times \mathbb{Z}^n\}$ is a frame of $L^2(\mathbb{R}^n)$. □

6. Frame-Type Inequalities in the Clifford Algebra Setting

For function pairs $\mathbf{f}(\underline{x}) = \sum_{\mathbf{s}} f_{\mathbf{s}}(\underline{x})\mathbf{e}_{\mathbf{s}}$ and $\mathbf{g}(\underline{x}) = \sum_{\mathbf{s}} g_{\mathbf{s}}(\underline{x})\mathbf{e}_{\mathbf{s}}$ with $f_{\mathbf{s}}, g_{\mathbf{s}} \in L^2(\mathbb{R}^n)$, mapping $\mathbb{R}^n$ to $\mathbb{R}^{(n)}$, define their inner product by

$$[\mathbf{f}, \mathbf{g}] := \int_{\mathbb{R}^n} \mathbf{f}(\underline{x})\overline{\mathbf{g}(\underline{x})}d\underline{x}. \tag{6.1}$$

When $\mathbf{f}(\underline{x}) = \sum_{j=1}^n f_j(\underline{x})\mathbf{e}_j$ and $\mathbf{g}(\underline{x}) = \sum_{k=1}^n g_k(\underline{x})\mathbf{e}_k$ with $f_j, g_k \in L^2(\mathbb{R}^n)$, we have

$$[\mathbf{f}, \mathbf{g}] = \int_{\mathbb{R}^n} \mathbf{f}(\underline{x})\overline{\mathbf{g}(\underline{x})}d\underline{x} = \sum_{j=1}^n \sum_{k=1}^n \langle f_j, g_k\rangle(-\mathbf{e}_j\mathbf{e}_k), \tag{6.2}$$

and in particular,

$$[\mathbf{f}, \mathbf{f}] = \sum_{j=1}^n \|f_j\|_2^2.$$

For a Clifford number of the form $x_0 + \sum_{h=1}^n x_h\mathbf{e}_h + \sum_{j=1}^n \sum_{k=1}^n x_{j,k}\mathbf{e}_j\mathbf{e}_k$, we introduce a quantity, that is needed in the following, denoted by $\ddagger \cdot \ddagger$, that is

$$\ddagger x_0 + \sum_{h=1}^n x_h\mathbf{e}_h + \sum_{j=1}^n \sum_{k=1}^n x_{j,k}\mathbf{e}_j\mathbf{e}_k \ddagger^2 \triangleq |x_0|^2 + \sum_{h=1}^n |x_h|^2 + \sum_{j=1}^n \sum_{k=1}^n |x_{j,k}|^2.$$

For $\mathbf{f}(\underline{x}) = \sum_{j=1}^n f_j(\underline{x})\mathbf{e}_j$ and $\mathbf{g}(\underline{x}) = \sum_{k=1}^n g_k(\underline{x})\mathbf{e}_k$ with $f_j, g_k \in L^2(\mathbb{R}^n)$, by (6.2), we have

$$\ddagger\, [\mathbf{f}, \mathbf{g}]\ddagger^2 := \sum_{j=1}^n \sum_{k=1}^n |\langle f_j, g_k\rangle|^2. \tag{6.3}$$

The Hardy space $\mathbb{H}^2(\mathbb{R}^n_{1,+})$ is defined as the set of left monogenic functions with the constraint

$$\|\mathbf{F}\|_{\mathbb{H}^2}^2 = \sup_{x_0>0} \int_{\mathbb{R}^n} |\mathbf{F}(x_0 + \underline{x})|^2 d\underline{x} < +\infty.$$

For two Hardy space functions from $\mathbb{R}^n_{1,+}$ to $\mathbb{R}^n_1$, their inner product is defined as that of the corresponding boundary-valued functions. That is, if

$\mathbf{F}(x) = \sum_{j=1}^{n} F_j(x)\mathbf{e}_j \in H^2(\mathbb{R}^n_{1,+})$ and $\mathbf{G}(x) = \sum_{j=1}^{n} G_j(x)\mathbf{e}_j \in H^2(\mathbb{R}^n_{1,+})$, and $\mathbf{f}(\underline{x}) = \lim_{x_0 \to 0^+} F(x)$ and $\mathbf{g}(\underline{x}) = \lim_{x_0 \to 0^+} \mathbf{G}(x)$, the inner product of F and G is

$$[\mathbf{F}, \mathbf{G}] := [\mathbf{f}, \mathbf{g}]. \tag{6.4}$$

For $j = 1, \ldots, n$, the jth Riesz transform of a complex-valued function f on $\mathbb{R}^n$ is defined as

$$\mathcal{R}_j f(\underline{x}) = c_n \lim_{\epsilon \to 0^+} \int\limits_{\mathbb{R}^n \setminus \mathbb{B}(0,\epsilon)} \frac{x_j - t_j}{|\underline{x} - \underline{t}|^{n+1}} f(\underline{t}) d\underline{t}, \ \underline{x} \in \mathbb{R}^n,$$

where $c_n = \frac{2}{\Omega_{n+1}} = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}$ and $\mathbb{B}(0, \epsilon)$ is the ball centered at the origin with radius ϵ.

The Hilbert transform $\mathcal{H}$ in Clifford sense is the transformation

$$\mathcal{H} f(\underline{x}) := -\sum_{j=1}^{n} \mathcal{R}_j f(\underline{x}) \mathbf{e}_j, \ \underline{x} \in \mathbb{R}^n$$

for $f \in L^2(\mathbb{R}^n)$, which maps $\mathbb{R}^n$ to $\mathbb{R}^n$. In the Fourier domain, $\mathcal{H}f$ is identified by

$$\mathcal{F}(\mathcal{H}f)(\underline{\xi}) = i \frac{\underline{\xi}}{|\underline{\xi}|} \mathcal{F}(f)(\underline{\xi}), \ \underline{\xi} \in \mathbb{R}^n.$$

For each $\mathbb{R}^n$-valued function $\mathbf{f}(\underline{x}) = \sum_{j=1}^{n} f_j(\underline{x})\mathbf{e}_j$ with $f_j(x) \in L^2(\mathbb{R}^n)$, define

$$\tilde{\mathcal{H}} \mathbf{f}(\underline{x}) := \sum_{j=1}^{n} \mathcal{H} f_j(\underline{x}) \mathbf{e}_j = -\sum_{k=1}^{n} \sum_{j=1}^{n} \mathcal{R}_k f_j(\underline{x}) \mathbf{e}_k \mathbf{e}_j. \tag{6.5}$$

Therefore, for any $f \in L^2(\mathbb{R}^n)$

$$\tilde{\mathcal{H}} \mathcal{H} f(\underline{x}) = \sum_{k=1}^{n} \sum_{j=1}^{n} \mathcal{R}_k \mathcal{R}_j f(\underline{x}) \mathbf{e}_j \mathbf{e}_k = -\sum_{k=1}^{n} \mathcal{R}_k^2 f(\underline{x}) = f(\underline{x}).$$

Combining (6.1) and (6.5), we have

$$[\mathbf{f}, \mathcal{H} g] = -[\tilde{\mathcal{H}} \mathbf{f}, g], \tag{6.6}$$

where $\mathbf{f} = \sum_{j=1}^{n} f_j \mathbf{e}_j : \mathbb{R}^n \to \mathbb{R}^n$ is a Clifford-valued function with $f_j \in L^2(\mathbb{R}^n)$ and g is a scalar-valued function in $L^2(\mathbb{R}^n)$. Now we prove (6.6):

$$\begin{aligned}
[\mathbf{f}, \mathcal{H}g] &= \int_{R^n} \mathbf{f}\overline{\mathcal{H}g} d\underline{x} \\
&= \int_{R^n} \sum_{j=1}^n f_j(\underline{x})\mathbf{e}_j \overline{[-\sum_{k=1}^n \mathcal{R}_k g(\underline{x})\mathbf{e}_k]} d\underline{x} \\
&= \int_{R^n} \sum_{j=1}^n \sum_{k=1}^n f_j(\underline{x})\overline{\mathcal{R}_k g(\underline{x})}\mathbf{e}_j\mathbf{e}_k d\underline{x} \\
&= \int_{R^n} \sum_{j=1}^n \sum_{k=1}^n \hat{f}_j(\underline{\xi})\overline{[-i\frac{\xi_k}{|\underline{\xi}|} g(\underline{\xi})]}\mathbf{e}_j\mathbf{e}_k d\underline{\xi} \\
&= -\int_{R^n} \sum_{j=1}^n \sum_{k=1}^n (-i\frac{\xi_k}{|\underline{\xi}|})\hat{f}_j(\underline{\xi})\overline{\hat{g}(\underline{\xi})}\mathbf{e}_j\mathbf{e}_k d\underline{\xi} \\
&= -\int_{R^n} \sum_{j=1}^n \sum_{k=1}^n \mathcal{R}_k f_j(\underline{x})\overline{g(\underline{x})}\mathbf{e}_j\mathbf{e}_k d\underline{x} \\
&= \int_{R^n} \sum_{k=1}^n \sum_{j=1}^n \mathcal{R}_k f_j(\underline{x})\overline{g(\underline{x})}\mathbf{e}_k\mathbf{e}_j d\underline{x} \\
&= -[\tilde{\mathcal{H}}\mathbf{f}, g].
\end{aligned}$$

We try to extend the inequality (5.1) to the Clifford setting. Suppose that $\{\varphi_{j,\underline{k}} : (j,\underline{k}) \in \mathbb{Z} \times \mathbb{Z}^n\}$ is a frame of $L^2(\mathbb{R}^n)$. For $x = x_0 + \underline{x} \in \mathbb{R}^n_{1,+}$, define

$$\Phi_{j,\underline{k}}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle \underline{x}, \underline{\xi}\rangle} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi}) \mathcal{F}(\varphi_{j,\underline{k}})(\underline{\xi}) d\underline{\xi} \tag{6.7}$$

where $\chi_+(\underline{\xi}) = \frac{1}{2}(1 + i\frac{\underline{\xi}}{|\underline{\xi}|})$. In particular, set $\Phi(x) = \Phi_{0,0}(x), x \in \mathbb{R}^n_{1,+}$. The functions $\Phi_{j,\underline{k}}$ are well defined since $\varphi_{j,\underline{k}} \in L^2(\mathbb{R}^n)$. It is easy to show that each $\Phi_{j,\underline{k}}$ is left monogenic for $x_0 > 0$. By using the identity

$$\mathcal{F}\left(e^{i\langle \underline{x}, \cdot\rangle} e^{-x_0|\cdot|} \chi_+(\cdot)\right)(\underline{\xi}) = \tilde{c}_n \frac{\overline{x - \underline{\xi}}}{|\underline{\xi} - x|^{n+1}} = \tilde{c}_n E(\underline{\xi} - x)$$

with $\tilde{c}_n = 2^{n-1}\pi^{(n-1)/2}\Gamma((n+1)/2)$, a Cauchy-type formula for $\Phi_{j,\underline{k}}$ arises

$$\Phi_{j,\underline{k}}(x) = \frac{1}{\Omega_{n+1}} \int_{\mathbb{R}^n} E(x - \underline{\xi})\varphi_{j,\underline{k}}(\underline{\xi}) d\underline{\xi}.$$

The Sokhotskyi–Plemelj formula then gives

$$\lim_{x_0 \to 0^+} \Phi_{j,\underline{k}}(x) = \frac{1}{2}(\varphi_{j,\underline{k}}(\underline{x}) + i\mathcal{H}(\varphi_{j,\underline{k}})(\underline{x})). \tag{6.8}$$

For $\mathbf{F} = \sum_{j=1}^n F_j\mathbf{e}_j \in \mathbb{H}^2(\mathbb{R}^n_{1,+})$, we have the following theorem on the frame-type inequalities.

Theorem 6.1. *Suppose that* $\{\varphi_{j,\underline{k}} : (j,\underline{k}) \in \mathbb{Z}\times\mathbb{Z}^n\}$ *is a frame of* $L^2(\mathbb{R}^n)$ *with bounds* A *and* B *and* $\Phi_{j,\underline{k}}$ *is defined in* (6.7). *Then for any* $\mathbf{F} = \sum_{j=1}^{n} F_j\mathbf{e}_j \in \mathbb{H}^2(\mathbb{R}^n_{1,+})$, *the following frame-type inequalities hold*

$$\frac{A}{2}[\mathbf{F},\mathbf{F}] \le \sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n} \ddagger[\mathbf{F},\Phi_{j,\underline{k}}]\ddagger^2 \le \frac{B}{2}[\mathbf{F},\mathbf{F}]. \tag{6.9}$$

Proof. Denote by $\mathbf{f} = \sum_{j=1}^{n} f_j\mathbf{e}_j$ the boundary limit of $\mathbf{F}$ when $x_0 \to 0^+$. By (6.4), Sokhotskyi–Plemelj formula and (6.6), we have

$$[\mathbf{F},\Phi_{j,\underline{k}}] = \left[\mathbf{f}, \frac{1}{2}(\varphi_{j,\underline{k}} + i\mathcal{H}\varphi_{j,\underline{k}})\right] = \frac{1}{2}[\mathbf{f} - i\tilde{\mathcal{H}}\mathbf{f}, \varphi_{j,\underline{k}}].$$

Recalling that

$$\mathbf{f}(\underline{x}) - i\tilde{\mathcal{H}}\mathbf{f}(\underline{x}) = \sum_{s=1}^{n} f_s(\underline{x})\mathbf{e}_s + i\sum_{l=1}^{n}\sum_{s=1}^{n}\mathcal{R}_l f_s(\underline{x})\mathbf{e}_l\mathbf{e}_s,$$

it gives that

$$\ddagger[\mathbf{F},\Phi_{j,\underline{k}}]\ddagger^2 = \frac{1}{4}\sum_{s=1}^{n}|\langle f_s,\varphi_{j,\underline{k}}\rangle|^2 + \frac{1}{4}\sum_{l=1}^{n}\sum_{s=1}^{n}|\langle\mathcal{R}_l f_s,\varphi_{j,\underline{k}}\rangle|^2.$$

We therefore obtain

$$\sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n}\ddagger[\mathbf{F},\Phi_{j,\underline{k}}]\ddagger^2 = \frac{1}{4}\sum_{s=1}^{n}\sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n}|\langle f_s,\varphi_{j,\underline{k}}\rangle|^2$$
$$+\frac{1}{4}\sum_{l=1}^{n}\sum_{s=1}^{n}\sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n}|\langle\mathcal{R}_l f_s,\varphi_{j,\underline{k}}\rangle|^2.$$

Applying the frame inequalities (5.1) to the functions $f_s, \mathcal{R}_l f_s$ in $L^2(\mathbb{R}^n)$, it follows

$$\frac{A}{4}\sum_{s=1}^{n}\|f_s\|_2^2 + \frac{A}{4}\sum_{l=1}^{n}\sum_{s=1}^{n}\|\mathcal{R}_l f_s\|_2^2 \le \sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n}\ddagger[\mathbf{F},\Phi_{j,\underline{k}}]\ddagger^2$$
$$\le \frac{B}{4}\sum_{s=1}^{n}\|f_s\|_2^2 + \frac{B}{4}\sum_{l=1}^{n}\sum_{s=1}^{n}\|\mathcal{R}_l f_s\|_2^2.$$

Using the identity $\sum_{j=1}^{n}\|\mathcal{R}_j f\|_2^2 = \|f\|_2^2$ for any $f \in L^2(\mathbb{R}^n)$, we get

$$\frac{A}{2}\sum_{s=1}^{n}\|f_s\|_2^2 \le \sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n}\ddagger[\mathbf{F},\Phi_{j,\underline{k}}]\ddagger^2 \le \frac{B}{2}\sum_{s=1}^{n}\|f_s\|_2^2.$$

Recall that $[\mathbf{F},\mathbf{F}] = [\mathbf{f},\mathbf{f}] = \sum_{s=1}^{n}\|f_s\|_2^2$, where $\mathbf{f}$ is the boundary limit of $\mathbf{F}$. We conclude the frame-type inequalities (6.9). □

For any Clifford-valued function $\mathbf{F} = \sum_{j=0}^{n} F_j\mathbf{e}_j \in \mathbb{H}^2(\mathbb{R}^n_{1,+})$, we have the following result.

Theorem 6.2. *Suppose that* $\{\varphi_{j,\underline{k}} : (j,\underline{k}) \in \mathbb{Z}\times\mathbb{Z}^n\}$ *is a frame of* $L^2(\mathbb{R}^n)$ *with bounds* A *and* B *and* $\Phi_{j,\underline{k}}$ *is defined in* (6.7)*. Then for any* $\mathbf{F} = \sum_{j=0}^{n} F_j\mathbf{e}_j \in \mathbb{H}^2(\mathbb{R}^n_{1,+})$*, the following frame-type inequalities hold*

$$\frac{A}{2}[\mathbf{F},\mathbf{F}] \le \sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n} \ddagger[\mathbf{F},\Phi_{j,\underline{k}}]\ddagger^2 \le \frac{B}{2}[\mathbf{F},\mathbf{F}]. \tag{6.10}$$

Proof. Rewrite $\mathbf{F}$ as $\mathrm{sca}(\mathbf{F}) + \mathrm{vec}(\mathbf{F})$ with $\mathrm{sca}(\mathbf{F}) = F_0\mathbf{e}_0$ and $\mathrm{vec}(\mathbf{F}) = \sum_{j=1}^{n} F_j\mathbf{e}_j$. Then the boundary limit of $\mathbf{F}$ has the similar decomposition $\mathbf{f} = \mathrm{sca}(\mathbf{f})+\mathrm{vec}(\mathbf{f})$ with $\mathrm{sca}(\mathbf{f}) = f_0\mathbf{e}_0$ and $\mathrm{vec}(\mathbf{f}) = \sum_{j=1}^{n} f_j\mathbf{e}_j$. We establish two inequalities. The first one is

$$\frac{A}{2}[\mathrm{vec}(\mathbf{F}),\mathrm{vec}(\mathbf{F})] \le \sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n} \ddagger[\mathrm{vec}(\mathbf{F}),\Phi_{j,\underline{k}}]\ddagger^2 \le \frac{B}{2}[\mathrm{vec}(\mathbf{F}),\mathrm{vec}(\mathbf{F})], \tag{6.11}$$

which is directly from (6.9). The second one is

$$\frac{A}{2}\|\mathrm{sca}(\mathbf{F})\|^2 \le \sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^n} \ddagger[\mathrm{sca}(\mathbf{F}),\Phi_{j,\underline{k}}]\ddagger^2 \le \frac{B}{2}\|\mathrm{sca}(\mathbf{F})\|^2. \tag{6.12}$$

To prove (6.12), using the definition of $\Phi_{j,\underline{k}}$, Sokhotskyi–Plemelj formula and $\mathcal{R}_l^* = \mathcal{R}_l$, we get

$$\ddagger[\mathrm{sca}(\mathbf{F}),\Phi_{j,\underline{k}}]\ddagger^2 = \frac{1}{4}|\langle\mathrm{sca}(\mathbf{f}),\varphi_{j,\underline{k}}\rangle|^2 + \frac{1}{4}\sum_{l=1}^{n}|\langle\mathcal{R}_l\left(\mathrm{sca}(\mathbf{f})\right),\varphi_{j,\underline{k}}\rangle|^2.$$

Summing both sides of above identity over $j,\underline{k}$, recalling the frame properties of the system $\{\varphi_{j,\underline{k}}\}$ and utilizing the identity $\sum_{j=1}^{n}\|\mathcal{R}_j f\|_2^2 = \|f\|_2^2$ again, it gives the inequalities (6.12).

Finally, recalling two identities $\ddagger[\mathbf{F},\Phi_{j,\underline{k}}]\ddagger^2 = \ddagger[\mathrm{sca}(\mathbf{F}),\Phi_{j,\underline{k}}]\ddagger^2 + \ddagger[\mathrm{vec}(\mathbf{F}),\Phi_{j,\underline{k}}]\ddagger^2$ and $[\mathbf{F},\mathbf{F}] = \|\mathrm{sca}(\mathbf{F})\|^{\mathbf{2}} + [\mathrm{vec}(\mathbf{F}),\mathrm{vec}(\mathbf{F})]$, combining (6.11) with (6.12) together, we conclude (6.10). □

7. Frames in Quaternion-Hardy Space

We will build up a theory of frames in the quaternion-valued Hardy space. We work on the quaternions $\mathbb{Q}$ over the real numbers $\mathbb{R}$, the only associative normed division algebra that extends the complex numbers. To distinguish it from the complexified quaternions, we call it real-quaternions. A real-quaternion $x \in \mathbb{Q}$ is of the form $x = \sum_{j=0}^{3} x_j\mathbf{e}_j$, where $x_j \in \mathbb{R}$ and the basis elements $\mathbf{e}_j (0 \le j \le 3)$ satisfy

$$\mathbf{e}_0 = 1, \quad \mathbf{e}_j\mathbf{e}_k + \mathbf{e}_k\mathbf{e}_j = -2\delta_{j,k}, \quad j,k = 1,2,3, \tag{7.1}$$

$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3\mathbf{e}_1 = \mathbf{e}_2. \tag{7.2}$$

Note that (7.1) means that the basis elements $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ satisfy the usual assumptions on the basis elements of the Clifford algebra. The additional condition (7.2) amounts to saying that they are, in fact, the basis elements of the quaternionic algebra, viz., and respectively, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the usual notation. Based on these relations one defines the quaternionic multiplication and addition by linearity and distributive law, which is the same as the general Clifford algebra case. Together with the multiplication low, real-quaternions $\mathbb{Q}$ becomes a four-dimensional, normed division, associative but non-commutative algebra. It is easy to see that the embedding relations $\mathbb{R} \subset \mathbb{C} \subset \mathbb{Q}$ hold. As a vector space, or a topological space, $\mathbb{Q}$ is linearly identified with the four dimensional Euclidean space $\mathbb{R}^4$.

We are benefited from the *closeness* of multiplication of quaternion numbers, that is, any pair $\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{Q}$ implies $\mathbf{q}_1\mathbf{q}_2 \in \mathbb{Q}$.

Denote $\mathbb{Q}_+$ the half space

$$\mathbb{Q}_+ = \{x = x_0 + \underline{x} \in \mathbb{Q} : x_0 > 0\}.$$

A function $\mathbf{F} : \mathbb{Q}_+ \to \mathbb{Q}$ is called a left monogenic function if it satisfies

$$\partial \mathbf{F} = \sum_{j=0}^{3} \mathbf{e}_j \frac{\partial \mathbf{F}}{\partial x_j} = 0.$$

The quaternionic Hardy space $\mathbb{H}^2(\mathbb{Q}_+)$ is defined as the set of left monogenic functions with the constraint

$$\|\mathbf{F}\|^{\mathbf{2}}_{\mathbb{H}^{\mathbf{2}}} = \sup_{\mathbf{x_0}>\mathbf{0}} \int_{\mathbb{R}^{\mathbf{3}}} |\mathbf{F}(\mathbf{x_0} + \underline{\mathrm{x}})|^{\mathbf{2}} \mathbf{d}\underline{\mathrm{x}} < +\infty.$$

Adopting the notation in (6.1) and (6.4), it is easy to see

$$\|\mathbf{F}\|^{\mathbf{2}}_{\mathbb{H}^{\mathbf{2}}} = [\mathbf{F}, \mathbf{F}] = [\mathbf{f}, \mathbf{f}],$$

where $\mathbf{f}(\underline{\mathrm{x}}) = \lim_{x_0 \to 0} \mathbf{F}(x_0 + \underline{\mathrm{x}})$, the non-tangential boundary limit of $\mathbf{F}$. There exists a natural isometry between $\mathbb{H}^2(\mathbb{Q}_+)$ and its boundary-value space $\mathbb{H}^2(\partial\mathbb{Q}_+)$.

For $x = x_0 + \underline{\mathrm{x}} \in \mathbb{Q}_+$, define

$$\Phi_{j,\underline{\mathrm{k}}}(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\langle \underline{\mathrm{x}}, \underline{\xi}\rangle} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi}) \mathcal{F}(\varphi_{j,\underline{\mathrm{k}}})(\underline{\xi}) d\underline{\xi} \tag{7.3}$$

with $\chi_+(\underline{\xi}) = \frac{1}{2}(1 + i\frac{\underline{\xi}}{|\underline{\xi}|}),\ \underline{\xi} = \xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2 + \xi_3\mathbf{e}_3$.

We write (7.3) into its Cauchy integral form

$$\Phi_{j,\underline{\mathrm{k}}}(x) = \frac{1}{\Omega_4} \int_{\mathbb{R}^3} E(x - \underline{\xi}) \varphi_{j,\underline{\mathrm{k}}}(\underline{\xi}) d\underline{\xi},\ x = x_0 + \underline{\mathrm{x}} \in \mathbb{Q}_+$$

with $E(x) = \frac{\bar{x}}{|x|^4},\ x \in \mathbb{Q}_+$. Applying the Sokhotskyi–Plemelj formula in quaternionic case and adopting the same arguments for the proofs of (6.9) and (6.10), we have

Theorem 7.1. *Suppose that* $\{\varphi_{j,\underline{k}} : (j,\underline{k}) \in \mathbb{Z}\times\mathbb{Z}^3\}$ *is a frame of* $L^2(\mathbb{R}^3)$ *with bounds* A *and* B *and* $\Phi_{j,\underline{k}}$ *is defined in* (7.3). *Then for any* $\mathbf{F} = \sum_{j=0}^{3} F_j\mathbf{e}_j \in \mathbb{H}^2(\mathbb{Q}_+)$, *the following frame-type inequalities hold*

$$\frac{A}{2}[\mathbf{F},\mathbf{F}] \le \sum_{j\in\mathbb{Z}}\sum_{\underline{k}\in\mathbb{Z}^3} \ddagger[\mathbf{F},\Phi_{j,\underline{k}}]\ddagger^2 \le \frac{B}{2}[\mathbf{F},\mathbf{F}]. \tag{7.4}$$

Acknowledgements

This work was supported in part by NSFC Grant 11571083, Macao Science and Technology Development Fund (Grant No. 045/2015/A2), Multi-Year Research Grant (MYRG) MYRG116(Y1-L3)-FST13-QT, and Macao Government FDCT 098/2012/A3.

References

[1] Akcay, H., Ninness, B.: Orthonormal basis functions for modelling continuous-time systems. Signal Process. **77**, 261–274 (1999)

[2] Balan, R., Casazza, P.G., Heil, C., Landau, Z.: Density, overcompleteness, and localization of frames, II. Gabor systems. J. Fourier Anal. Appl. **12**, 307–344 (2006)

[3] Brackx, F., Delanghe, R., Sommen, F.: Clifford Analysis: Research Notes in Mathematics, vol. 76. Pitman Advanced Publishing Program, Boston (1982)

[4] Christensen, O.: An Introduction to Frames and Riesz Bases. Birkhäuser, Boston (2002)

[5] Christensen, O., Rahimi, A.: Frame properties of wave packet systems in $L_2(\mathbb{R}^d)$. Adv. Comput. Math. **29**(2), 101–111 (2008)

[6] Dahlke, S., Lorenz, D., Maass, P., Sagiv, C., Teschke, G.: The canonical coherent states associated with quotients of the affine Weyl–Heisenberg group. J. Appl. Funct. Anal. **3**(2), 215–232 (2008)

[7] Daubechies, I., Grossmann, A., Meyer, Y.: Painless nonorthogonal expansions. J. Math. Phys. **27**, 1271–1283 (1986)

[8] Daubechies, I.: The wavelet transform, time—frequency localization and signal analysis. IEEE Trans. Inf. Theor. **36**(5), 961–1005 (1990)

[9] Daubechies, I.: Ten Lectures on Wavelets, CBMS, vol. 61. SIAM, Philadelphia (1992)

[10] Daubechies, I., Han, B.: Pairs of dual wavelet frames from any two refinable functions. Constr. Approx. **20**(3), 325–352 (2004)

[11] Delanghe, R., Sommen, F., Soucek, V.: Clifford Algebras and Spinor Valued Functions: A Function Theory for Dirac Operator. Kluwer Academic, Dordrecht (1992)

[12] Duffin, R.J., Schaeffer, A.C.: A class of nonharmonic Fourier series. Trans. Am. Math. Soc. **72**, 341–366 (1952)

[13] Epstein, B.: Orthogonal Families of Analytic Functions. Macmillan, New York (1965)

[14] Feichtinger, H.G., Gröchenig, K.H.: Gabor frames and time-frequency analysis of distributions. J. Funct. Anal. **146**, 464–495 (1997)

[15] Feichtinger, H.G., Sun, W.C.: Sufficient conditions for irregular Gabor frames. Adv. Comput. Math. **26**(4), 403–430 (2007)

[16] Gröchenig, K.H., Ron, A.: Tight compactly supported wavelet frames of arbitrarily high smoothness. Proc. Am. Math. Soc. **126**(4), 1101–1107 (1998)

[17] Gröuser, K.H.: Foundations of Time-Frequency Analysis. Birkhauser, Boston (2000)

[18] Han, B.: Dual multiwavelet frames with high balancing order and compact fast frame transform. Appl. Comput. Harmon. Anal. **26**, 14–42 (2009)

[19] Han, B., Shen, Z.: Dual wavelet frames and Riesz bases in Sobolev spaces. Constr. Approx. **29**(3), 369–406 (2009)

[20] Heil, C.: A Basis Theory Primer, Expanded edn. Birkhäuser, Boston (2011)

[21] Heil, C., Walnut, D.: Continuous and discrete wavelet transforms. SIAM Rev. **31**, 628–666 (1989)

[22] Hoffman, K.: Banach Spaces of Analytic Functions. Prentice-Hall, Englewood Cliffs (1962)

[23] Li, C., McIntosh, A., Qian, T.: Clifford algebras, Fourier transforms, and singular convolution operators on Lipschitz surfaces. Rev. Mat. Iberoam. **10**, 665–721 (1994)

[24] Mitrea, M.: Clifford wavelets, singular integrals, and Hardy spaces. In: Lecture Notes in Mathematics, vol. 1575. Springer, Berlin (1994)

[25] Qian, T., Wang, Y.: Adaptive Fourier series—a variation of Greedy algorithm. Adv. Comput. Math. **34**(3), 279–293 (2011)

[26] Ron, A., Shen, Z.W.: Frames and stable bases for shift-invariant subspaces of $L^2(\mathbb{R}^d)$. Can. J. Math. **47**, 1051–1094 (1995)

[27] Stein, E.M., Weiss, G.: Introduce to Fourier Analysis on Euclidean Spaces. Princeton University Press, Princeton (1971)

[28] Takenaka, S.: On the orthogonal functions and a new formula of interpolation. Jpn. J. Math. **II**, 129–145 (1925)

[29] Wong, M.W.: Wavelet transforms and localization operators. In: Operator Theory: Advances and Applications, vol. 136. Birkhäuser, Basel (2002)

[30] Young, R.: An Introduction to Nonharmonic Fourier Series. Academic Press, New York (1980)

Qiuhui Chen
Department of Mathematics
College of Mathematics and Informatics
South China Agricultural University
Guangzhou
China
e-mail: chenqiuhui@hotmail.com

Pei Dang
Faculty of Information Technology
Macau University of Science and Technology
Taipa Macau
China
e-mail: pdang@must.edu.mo

Tao Qian
Department of Mathematics
Faculty of Science and Technology
University of Macau
Taipa Macau
China
e-mail: fsttq@umac.mo

Received: April 8, 2016.
Accepted: October 27, 2016.