J Fourier Anal Appl
DOI 10.1007/s00041-016-9508-4

Adaptative Decomposition: The Case of the Drury–Arveson Space

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Received: 14 February 2016

Abstract The maximum selection principle allows to give expansions, in an adaptive way, of functions in the Hardy space $\mathbf{H}_2$ of the disk in terms of Blaschke products. The expansion is specific to the given function. Blaschke factors and products have counterparts in the unit ball of $\mathbb{C}^N$, and this fact allows us to extend in the present paper the maximum selection principle to the case of functions in the Drury–Arveson space of functions analytic in the unit ball of $\mathbb{C}^N$. This will give rise to an algorithm which is a variation in this higher dimensional case of the greedy algorithm. We also introduce infinite Blaschke products in this setting and study their convergence.

Keywords Drury–Arveson space · Adaptative decomposition · Blaschke products

Mathematics Subject Classification 47A56 · 41A20

Communicated by Dorin Ervin Dutkay.

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Published online: 12 October 2016

Birkhäuser

1 Introduction

In [15] Qian and Wang introduced an algorithm based on the maximum selection principle, to decompose a given function of the Hardy space $\mathbf{H}_2(\mathbb{D})$ of the unit disk into intrinsic components which correspond to modified Blaschke products

$$B_n(z) = \frac{\sqrt{1-|a_n|^2}}{1-z\overline{a_n}} \prod_{k=1}^{n-1} \frac{z-a_k}{1-z\overline{a_k}}, \qquad n = 1, 2, \dots \tag{1.1}$$

where the points $a_n \in \mathbb{D}$ are adaptively chosen according to the given function. These points a_n do not necessarily satisfy the so-called hyperbolic non-separability condition

$$\sum_{n=1}^{\infty} 1 - |a_n| = \infty, \tag{1.2}$$

and so the functions $B_n(z)$ do not necessarily form a complete system in $\mathbf{H}_2(\mathbb{D})$. This decomposition may be obtained in an adaptive way, see [14], making the algorithm more efficient than the greedy algorithm of which it is a variation.

In [7] the above algorithm is extended to the matrix-valued case and the choice of a point and of a projection is based at each step on the maximal selection principle. The extension is possible because of the existence of matrix-valued Blaschke factors and is based on the existence of solutions of interpolation problems in the matrix-valued Hardy space of the disk.

When leaving the realm of one complex variable, a number of possibilities occur, and in particular the unit ball $\mathbb{B}_N$ of $\mathbb{C}^N$ and the polydisk. The polydisk case and the case of Hilbert space valued coefficients will be studied in future publications. In this paper we focus on the case of the unit ball. For the present purposes, it is more convenient to consider the Drury–Arveson space rather than the Hardy space of the ball, and we extend some of the results of [15] and [7] to the setting of the Drury–Arveson space, denoted here $\mathbf{H}(\mathbb{B}_N)$. This is the space with reproducing kernel

$$\frac{1}{1-\langle z, w\rangle}, \quad z, w \in \mathbb{B}_N,$$

with

$$\langle z, w\rangle = \sum_{u=1}^{N} z_u \overline{w_u} = zw^*,$$

where $z = (z_1, \dots, z_N)$ and $w = (w_1, \dots, w_N)$ belong to $\mathbb{B}_N$. This space has a long history (see for instance [1,2,8,10,12]) and is used in the proof of a von Neumann inequality for row contractions. We refer in particular to the recent survey paper [17] for more information on applications of the Drury–Arveson space to modern operator theory. Interpolation inside the space $\mathbf{H}(\mathbb{B}_N)$ was done in [6]. A key tool in [6] was the existence in the ball of the counterpart of a Blaschke factor (appearing in [16]; see (2.5) below). The existence of these Blaschke factors and the fact that one can solve interpolation problems in $\mathbf{H}(\mathbb{B}_N)$ allow us to develop the asserted extension.

The approach in [6] is based on the solution of Gleason's problem. For completeness we recall that given a space, say $\mathcal{F}$, of functions analytic in $\Omega \subset \mathbb{C}^N$, Gleason's problem consists in finding for every $f \in \mathcal{F}$ and $a = (a_1, \ldots, a_N) \in \Omega$, functions $g_1(z,a), \ldots, g_N(z,a) \in \mathcal{F}$ and such that

$$f(z) - f(a) = \sum_{u=1}^{N} (z_u - a_u) g_u(z,a), \quad z \in \Omega. \tag{1.3}$$

Using power series, one sees that there always exist analytic functions satisfying (1.3). The requirement is that one can choose them in $\mathcal{F}$.

The paper consists of five sections besides the introduction. In Sects. 2 and 3 we review some basic facts on the Drury–Arveson space, and on the interpolation in it. The latter will be necessary to prove the maximum selection principle. This principle is proved in Sect. 4. In Sect. 5 we prove the convergence of the algorithm. In the last section, which is of independent interest, we consider infinite Blaschke products. When $N > 1$ the a_n in (1.2) are vectors in $\mathbb{B}_N$ and condition (1.2) is replaced by the requirement

$$\sum_{n=1}^{\infty} \sqrt{1 - a_n a_n^*} < \infty.$$

We note that most of the analysis presented here still holds for general complete Nevanlinna–Pick kernels, that is kernels of the form

$$\frac{1}{c(z)\overline{c(w)} - \langle d(z), d(w) \rangle_{\mathcal{H}}},$$

where c is scalar and d is $\mathcal{H}$-valued where $\mathcal{H}$ is some Hilbert space or more generally, in some reproducing kernel Hilbert spaces in which Gleason's problem is solvable with bounded operators; see [5] for the latter.

2 The Drury–Arveson Space

We use the multi-index notations

$$z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}, \quad \text{and} \quad \alpha! = \alpha_1! \cdots \alpha_N!,$$

with $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ and $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$. For $z, w \in \mathbb{B}_N$ we have

$$\frac{1}{1 - zw^*} = \sum_{\alpha \in \mathbb{N}_0^N} \frac{|\alpha|!}{\alpha!} z^\alpha \overline{w}^\alpha, \tag{2.1}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_N$. The function (2.1) is thus positive definite in $\mathbb{B}_N$.

Definition 2.1 The reproducing kernel Hilbert space $\mathbf{H}(\mathbb{B}_N)$ with reproducing kernel (2.1) is called the Drury–Arveson space.

The Drury–Arveson space can be characterized as

$$\mathbf{H}(\mathbb{B}_N) = \left\{ f(z) = \sum_{\alpha \in \mathbb{N}_0^N} z^\alpha f_\alpha \ : \ \|f\|^2_{\mathbf{H}(\mathbb{B}_N)} = \sum_{\alpha \in \mathbb{N}_0^N} \frac{\alpha!}{|\alpha|!} |f_\alpha|^2 < \infty \right\}. \tag{2.2}$$

For $N > 1$ the Drury–Arveson space is contractively included in, but different from, the Hardy space of the ball. The latter has reproducing kernel

$$\frac{1}{(1 - \langle z, w \rangle)^N}, \quad z, w \in \mathbb{B}_N.$$

See [4] for an expression for the inner product (not in terms of a surface integral).

We define $(\mathbf{H}(\mathbb{B}_N))^{n\times m}$ as in (2.2), but with now $f_\alpha, g_\alpha \in \mathbb{C}^{n\times m}$ and define for $f, g \in (\mathbf{H}(\mathbb{B}_N))^{n\times m}$, with $g(z) = \sum_{\alpha \in \mathbb{N}_0^N} z^\alpha g_\alpha$,

$$[f, g]_{(\mathbf{H}(\mathbb{B}_N))^{n\times m}} = \sum_{\alpha \in \mathbb{N}_0^N} g_\alpha^* f_\alpha, \quad \text{and} \tag{2.3}$$

$$\langle f, g \rangle_{(\mathbf{H}(\mathbb{B}_N))^{n\times m}} = \mathrm{Tr}\,[f, g]_{(\mathbf{H}(\mathbb{B}_N))^{n\times m}}. \tag{2.4}$$

In the sequel we will not write anymore explicitly the space in these forms. We will write sometimes $\mathbb{C}^N$ instead of $\mathbb{C}^{1\times N}$.

For $a \in \mathbb{B}_N$ we will use the notations e_a and b_a for the normalized Cauchy kernel and the $\mathbb{C}^N$-valued Blaschke factor at the point a respectively, that is:

$$e_a(z) = \frac{\sqrt{1 - \|a\|^2}}{1 - \langle z, a \rangle} \quad \text{and} \quad b_a(z) = \frac{(1 - \|a\|^2)^{1/2}}{1 - \langle z, a \rangle}(z - a)(I_N - a^* a)^{-1/2}. \tag{2.5}$$

Let $w \in \mathbb{B}_N$. Then (see [16]; another more analytic and maybe easier proof can be found in [6]):

$$\frac{1 - b_a(z) b_a(w)^*}{1 - z w^*} = \frac{1 - a a^*}{(1 - z a^*)(1 - w^* a)}, \quad z, w \in \mathbb{B}_N. \tag{2.6}$$

Gleason's problem is solvable in the Drury–Arveson space and in the Hardy space; see [5].

For $a = 0$ and by setting $g_u(z, 0) = g_u(z)$, a solution is given by

$$g_u(z, 0) = \int_0^1 \frac{\partial}{\partial z_u} f(tz) dt = \sum_{\alpha \in \mathbb{N}_0^N} \frac{\alpha_u}{|\alpha|} z^{\alpha - \epsilon_u},$$

where ϵ_u is the N-index with all the other entries equal to 0, but the u-th one equal to 1, and with the understanding that

$$\frac{\alpha_u}{|\alpha|} z^{\alpha - \epsilon_u} = 0$$

if $\alpha_u = 0$. We set $(R_u f)(z) = \int_0^1 \frac{\partial}{\partial z_u} f(tz)dt$. We thus have

$$f(z) - f(0) = \sum_{u=1}^{N} z_u (R_u f)(z).$$

When $N = 1$, then R_1 reduces to the classical backward-shift operator which to f associates the function $\frac{f(z)-f(0)}{z}$ for $z \neq 0$ and $f'(0)$ for $z = 0$.

3 Interpolation in the Drury–Arveson Space

This section is based on [6] and reviews the tools necessary to develop the maximum selection principle and the convergence result in the next section. We provide the proofs for completeness.

Proposition 3.1 *Let* $0 \neq c \in \mathbb{C}^{n\times 1}$, $a \in \mathbb{B}_N$, *and let* $f \in \mathbf{H}(\mathbb{B}_N)^{n\times 1}$. *Then*

$$c^* f(a) = 0 \iff f(z) = B(z)g(z),$$

where B *is given by*

$$B(z) = U \begin{pmatrix} b_a(z) & 0_{1\times(n-1)} \\ 0_{(n-1)\times N} & I_{n-1} \end{pmatrix}, \tag{3.1}$$

where $b_a(z) \in \mathbb{C}^{1\times N}$, $U \in \mathbb{C}^{n\times n}$ *is a unitary matrix with the first column equal to* $\frac{c}{c^*c}$, *and* g *is an arbitrary element of* $\mathbf{H}(\mathbb{B}_N)^{(N+n-1)\times 1}$.

Proof We recall the proof of the proposition; see [6, Proposition 4.5, p. 15]. We note that

$$c^* U = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix},$$

and hence

$$c^* B(a) = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} b_a(z) & 0_{1\times(n-1)} \\ 0_{(n-1)\times N} & I_{n-1} \end{pmatrix} = 0_{1\times(N+(n-1))}.$$

and so every function of the form Bg with $g \in \mathbf{H}(\mathbb{B}_N)^{(N+n-1)\times 1}$ is a solution of the interpolation problem. To prove the converse statement, we first remark that

$$\frac{I_n - B(z)B(w)^*}{1 - zw^*} = \frac{cc^*}{c^*c} \frac{1 - aa^*}{(1 - za^*)(1 - w^*a)}.$$

It follows that the one dimensional subspace $\mathcal{H}_1$ of $\mathbf{H}(\mathbb{B}_N)^{n\times 1}$ spanned by the vector $\frac{\frac{c}{c^*c}}{1-za^*}$ has reproducing kernel $\frac{I_n - B(z)B(w)^*}{1-zw^*}$. Thus the decomposition of kernels

$$\frac{I_n}{1-zw^*} = \frac{I_n - B(z)B(w)^*}{1-zw^*} + \frac{B(z)B(w)^*}{1-zw^*}$$

leads to an orthogonal decomposition of the space $\mathbf{H}(\mathbb{B}_N)^{n\times 1}$ as

$$\mathbf{H}(\mathbb{B}_N)^{n\times 1} = \mathcal{H}_1 \oplus \mathcal{H}_1^{\perp},$$

where $\mathcal{H}_1^{\perp}$ is the subspace of $\mathbf{H}(\mathbb{B}_N)^{n\times 1}$ consisting of functions g such that $c^*g(a) = 0$. Since the reproducing kernel of $\mathcal{H}_1^{\perp}$ is $\frac{B(z)B(w)^*}{1-zw^*}$ we have

$$\mathcal{H}_1^{\perp} = \left\{ Bg\, ;\ g \in \mathbf{H}(\mathbb{B}_N)^{(N+n-1)\times 1} \right\},$$

with norm

$$\|Bg\|_{\mathbf{H}(\mathbb{B}_N)^{n\times 1}} = \inf_{g\in\mathbf{H}(\mathbb{B}_N)^{(N+n-1)\times 1}} \|g\|_{\mathbf{H}(\mathbb{B}_N)^{(N+n-1)\times 1}}.$$

□

We note that we do not write the dependence of B on a and c.

Definition 3.2 The $\mathbb{C}^{n\times(N+n-1)}$-valued function B is an elementary Blaschke factor. A (possibly infinite) Blaschke product is a product of terms of the form (3.1) of compatible (growing) sizes.

Remark 3.3 Let $\mathcal{B}$ be a $\mathbb{C}^{n\times m}$-valued Blaschke product (or taking values operators from $\mathbb{C}^n$ into ℓ_2 if $m = \infty$). Then $\mathcal{B}$ is a Schur multiplier, meaning that the kernel $\frac{I_n - \mathcal{B}(z)\mathcal{B}(w)^*}{1-\langle z,w\rangle}$ is positive definite in $\mathbb{B}_N$. When $N > 1$, the family of Schur multipliers is strictly included in the family of fucntions analytic and contractive in the unit ball. For the realization theory of Schur multipliers, see for instance [9,10].

More generally than (3.1) we have (see [6, Theorem 5.2, p. 17]):

Theorem 3.4 *Given $a_1, \ldots, a_M \in \mathbb{B}_N$ and vectors $c_1, \ldots, c_M \in \mathbb{C}^{n\times 1}$ different from $0_{n\times 1}$, a function $f \in \mathbf{H}(\mathbb{B}_N)^{n\times 1}$ satisfies*

$$c_j^* f(a_j) = 0, \quad j = 1, \ldots, M$$

if and only if it is of the form $f(z) = B(z)u(z)$, where $B(z)$ is a fixed rational $\mathbb{C}^{n\times(n+k(N-1))}$-valued function, for some integer $k \le M$, taking coisometric values on the boundary of $\mathbb{B}_N$, depending on the interpolation data, and u is an arbitrary element in $\mathbf{H}(\mathbb{B}_N)^{(n+k(N-1))\times 1}$.

Proof Indeed, starting with $j = 1$ we have that $f = B_1 g_1$, where B_1 is given by (3.1) with $a = a_1$ (and an appropriately constructed matrix U) and $g_1 \in \mathbf{H}(\mathbb{B}_N)^{(N+n-1)\times 1}$. The interpolation condition $c_2^* f(a_2) = 0$ becomes

$$c_2^* B_1(a_2) g_1(a_2) = 0. \tag{3.2}$$

If $c_2^* B_1(a_2) = 0_{1\times(N+n-1)}$, any g_1 will be a solution. Otherwise, we solve (3.2) using Proposition 3.1 and get

$$g_1(z) = B_2(z) g_2(z),$$

where B_2 is $\mathbb{C}^{(n+(N-1))\times(n+2(N-1))}$-valued and obtained from (3.1) with $a = a_2$ and an appropriately constructed matrix U. Iterating this proceduce we obtain the result. The fact that k may be strictly smaller than M comes from the possibility that conditions as (3.2) occur. This will not happen when $N = 1$ and when all the a_j chosen are different. □

4 The Maximum Selection Principle

The proof is similar to the one in the original paper [15] and in [7], but one relevant difference is the use of orthogonal projections in $\mathbb{C}^{n\times n}$ of fixed rank. The fact that the set of such projections is compact in $\mathbb{C}^{n\times n}$ ensures the existence of a maximum. Besides the use of the normalized Cauchy kernel, the possibility of approximating by polynomials is a key tool in the proof.

Proposition 4.1 *Let B be a $\mathbb{C}^{u\times n}$-valued rational function of the variables $z_1, \ldots, z_N$, analytic in an neighborhood of the closed unit ball $\overline{\mathbb{B}_N}$, and taking co-isometric values on the unit sphere, let $r_0 \in \{1, \ldots, n\}$, and let $F \in \mathbf{H}(\mathbb{B}_N)^{n\times m}$. There exists $w_0 \in \mathbb{B}_N$ and a $\mathbb{C}^{n\times n}$-valued orthogonal projection P_0 of rank r_0 such that*

$$(1 - \|w_0\|^2)\,(\mathrm{Tr}\,[B(w_0)P_0F(w_0), B(w_0)P_0F(w_0)]) \quad \textit{is maximal.}$$

Proof We first recall that for $f \in \mathbf{H}(\mathbb{B}_N)$ (that is, $n = m = 1$), with power series $f(z) = \sum_{\alpha\in\mathbb{N}_0^N} f_\alpha z^\alpha$, and for $w \in \mathbb{B}_N$, we have

$$\sqrt{1 - \|w\|^2}\,|f(w)| = |[f, e_w]| \leq \|f\|. \tag{4.1}$$

Let $F = (f_{ij}) \in \mathbf{H}(\mathbb{B}_N)^{n\times m}$, where the entries $f_{ij} \in \mathbf{H}(\mathbb{B}_N)$ ($i = 1, \ldots, n$ and $j = 1, \ldots m$), and let P denote a projection of rank r_0. Then:

$$\begin{aligned}\mathrm{Tr}\ F(w)^* P B(w)^* B(w) P F(w) &\leq \mathrm{Tr}\ F(w)^* F(w)\\ &\qquad \text{(since } B(w) \text{ is contractive inside the sphere)}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} |f_{ij}(w)|^2.\end{aligned}$$

Hence, using (4.1) for every f_{ij}, we obtain

$$(1-\|w\|^2)\,(\mathrm{Tr}\,[B(w)PF(w),\,B(w)PF(w)]) \le \sum_{i=1}^{n}\sum_{j=1}^{m}\|f_{ij}\|^2 = \|F\|^2. \tag{4.2}$$

Let $\epsilon > 0$. In view of the power series expansion characterization (2.2) of the elements of the Drury–Arveson space, there exists a $\mathbb{C}^{n\times m}$-valued polynomial p in $z_1,\ldots,z_N$ such that $\|F-p\| \le \epsilon$. We have

$$\begin{aligned}
&(1-\|w\|^2)(\mathrm{Tr}\,[B(w)PF(w),\,B(w)PF(w)])\\
&\quad\le (1-\|w\|^2)\,(\mathrm{Tr}\,[F(w),\,F(w)])\\
&\quad= (1-\|w\|^2)\|(F-p)(w)+p(w)\|^2\\
&\quad\le 2(1-\|w\|^2)\,(\|(F-p)(w)\|+\|p(w)\|)^2\\
&\quad\le 2(1-\|w\|^2)\|(F-p)(w)\|^2+2(1-\|w\|^2)\|p(w)\|^2\\
&\quad\le 2\|F-p\|^2+2(1-\|w\|^2)\|p(w)\|^2 \quad \text{(where we have used (4.1))}\\
&\quad\le 2\epsilon^2+2(1-\|w\|^2)\|p(w)\|^2.
\end{aligned}$$

Since $(1-\|w\|^2)\|p(w)\|^2$ tends to 0 as w approaches the unit sphere, the expression $(1-\|w\|^2)\,(\mathrm{Tr}\,[B(w)PF(w),\,B(w)PF(w)])$ can be made arbitrary small, uniformly with respect to P, as w approaches the unit sphere. Thus,

$$(1-\|w\|^2)\,(\mathrm{Tr}\,[B(w)PF(w),\,B(w)PF(w)]) \tag{4.3}$$

is uniformly bounded as $w \in \mathbb{B}_N$ and P runs through the projections of rank r_0, and goes uniformly to 0 as w tends to the boundary. It has therefore a finite supremum, which is in fact a maximum and is in $\mathbb{B}_N$ (and not on the boundary because the function (4.3) goes uniformly to 0 there), as is seen by taking a subsequence tending to this supremum, and this ends the proof. □

Let us rewrite $F(z)$ as

$$F(z) = P_0F(w_0)e_{w_0}(z)\sqrt{1-\|w_0\|^2}+F(z)-P_0F(w_0)e_{w_0}(z)\sqrt{1-\|w_0\|^2}. \tag{4.4}$$

We now show that (4.4) gives an orthogonal decomposition of F, which is the first step in the expansion of F that we are looking for (see (5.2) for a more precise way of writing the decomposition) and for the algorithm that will arise repeating this construction.

Lemma 4.2 *Let*

$$\begin{aligned}
H(z) &= F(z)-P_0F(w_0)e_{w_0}(z)\sqrt{1-\|w_0\|^2}\\
H_0(z) &= P_0F(w_0)e_{w_0}(z)\sqrt{1-\|w_0\|^2},
\end{aligned}$$

where w_0, P_0 *are as in Proposition* 4.1. *It holds that*

$$P_0 H(w_0) = 0 \tag{4.5}$$

and

$$[F, F] = [H_0, H_0] + [H, H].$$

Proof First we have (4.5) since

$$P_0 H(w_0) = P_0 F(w_0) - P_0 F(w_0) e_{w_0}(w_0)\sqrt{1 - \|w_0\|^2} = 0.$$

Using (4.5) we have

$$[H, P_0 F(w_0) e_{w_0}(z)\sqrt{1 - \|w_0\|^2}] = F(w_0)^* P_0 H(w_0)(1 - \|w_0\|^2) = 0.$$

So, $[H, H_0] = 0$ and

$$[F, F] = [H_0 + H, H_0 + H] = [H_0, H_0] + [H, H].$$

□

5 The Algorithm

To proceed and take care of the condition (4.5) (that is, in the scalar case, to divide by a Blaschke factor) we use a factor of the form (3.1). Then, we use Theorem 3.4 to find a $\mathbb{C}^{n\times(n+r_0'(N-1))}$-valued rational function B_{w_0,P_0} with $r_0' \le r_0$ and such that

$$\text{ran } P_0 e_{w_0} = \mathbf{H}(\mathbb{B}_N)^{n\times m} \ominus B_{w_0,P_0}(\mathbf{H}(\mathbb{B}_N))^{(n+r_0'(N-1))\times m},$$

and so

$$\begin{aligned}\mathbf{H}(\mathbb{B}_N)^{n\times m} &= \left(\mathbf{H}(\mathbb{B}_N)^{n\times m} \ominus B_{w_0,P_0}(\mathbf{H}(\mathbb{B}_N))^{(n+r_0'(N-1))\times m}\right)\\ &\quad \oplus B_{w_0,P_0}\mathbf{H}(\mathbb{B}_N))^{(n+r_0'(N-1))\times m}.\end{aligned} \tag{5.1}$$

Let $F \in (\mathbf{H}(\mathbb{B}_N))^{n\times m}$. We choose $w_0 \in \mathbb{B}_N$ and $r_0 \in \{1, \ldots, n\}$. Using the maximum selection principle with $B(z) = I_n$ we get a decomposition of the form (5.1). We rewrite (4.4) as

$$F(z) = P_0 F(w_0) e_{w_0}(z)\sqrt{1 - \|w_0\|^2} + B_{w_0,P_0}(z) F_1(z), \tag{5.2}$$

where $F_1 \in (\mathbf{H}(\mathbb{B}_N))^{(n+r_0'(N-1))\times m}$. We now select $w_1 \in \mathbb{B}_N$ and $r_1 \in \left\{1, \ldots, n + r_0'(N-1)\right\}$, and apply the maximum selection principle to the pair $(B_{w_0,P_0}(z), F_1(z))$. We have then

$$F_1(z) = P_1 F_1(w_1) e_{w_1}(z) \sqrt{1 - \|w_1\|^2} + B_{w_1,P_1}(z) F_2(z), \tag{5.3}$$

where $F_2 \in (\mathbf{H}(\mathbb{B}_N))^{(n+(r_0'+r_1')(N-1))\times m}$ (with $r_1' \le r_1$) is not uniquely defined when $N > 1$. So

$$\begin{aligned} F(z) =& P_0 F(w_0) e_{w_0}(z) \sqrt{1 - \|w_0\|^2} + B_{w_0,P_0}(z) P_1 F_1(w_1) e_{w_1}(z) \sqrt{1 - \|w_1\|^2} \\ & + B_{w_0,P_0}(z) B_{w_1,P_1}(z) F_2(z). \end{aligned}$$

We iterate the procedure with the pair $(B_{w_0,P_0}(z) B_{w_1,P_1}(z), F_2(z))$ and observe the appearance of the Blaschke product

$$\mathcal{B}_k(z) = B_{w_0,P_0} B_{w_1,P_1} B_{w_2,P_2} \cdots B_{w_{k-1},P_{k-1}}, \text{ for } k \ge 1,$$

which will be $\mathbb{C}^{n\times(1+s_k(N-1))}$-valued for some $s_k \le \sum_{j=0}^{k-1} r_j$. We set

$$M_k = F_k(w_k) \in \mathbb{C}^{s_k \times m}, \tag{5.4}$$

and

$$\begin{aligned} &\mathfrak{B}_k(z) \\ &= \begin{cases} \sqrt{1 - \|w_0\|^2} e_{w_0}(z) & \text{for } k = 0, \\ \sqrt{1 - \|w_k\|^2} e_{w_k}(z) B_{w_0,P_0}(z) B_{w_1,P_1}(z) B_{w_2,P_2}(z) \cdots B_{w_{k-1},P_{k-1}}(z) & \text{for } k \ge 1. \end{cases} \end{aligned}$$

Note that

$$\mathfrak{B}_k(w_k) = \mathcal{B}_k(w_k), \quad k \ge 1. \tag{5.5}$$

We have

$$F(z) = \sum_{k=0}^{u} \mathfrak{B}_k(z) M_k + \mathcal{B}_{u+1}(z) F_{u+1}(z). \tag{5.6}$$

Moreover,

$$\langle \mathfrak{B}_k M_k \,,\, \mathfrak{B}_\ell M_\ell \rangle_{\mathbf{H}(\mathbb{B}_N)} = 0 \quad \text{for} \quad k \neq \ell \tag{5.7}$$

and we have by the orthogonality of the decomposition that

$$\|F\|^2_{\mathbf{H}(\mathbb{B}_N)} = \sum_{k=0}^{u} \|\mathfrak{B}_k M_k\|^2_{\mathbf{H}(\mathbb{B}_N)} + \|\mathcal{B}_{u+1}(z) F_{u+1}\|^2_{\mathbf{H}(\mathbb{B}_N)}. \tag{5.8}$$

This recursive procedure gives, at the k-th step, the best approximation. However we have to ensure that when k tends to infinity the algorithm converges. This is guaranteed by virtue of the next result.

Theorem 5.1 *Suppose that in* (5.6) *at each step one selects* w_k *and* P_k *according to the maximum selection principle applied to* $(\mathcal{B}_{w_k}(z), F_k(z))$. *Then the algorithm converges, meaning that*

$$F(z) = \sum_{k=0}^{\infty} \mathfrak{B}_k(z) M_k$$

in the norm of the Drury–Arveson space.

Proof We follow the arguments of [15] and [7]. We set

$$R_u(z) = F(z) - \sum_{k=0}^{u} \mathfrak{B}_k(z) M_k = \mathcal{B}_{w_{u+1}}(z) F_{u+1}(z) \tag{5.9}$$

(where F_{u+1} is not uniquely defined when $N > 1$) and

$$S_u(z) = \sum_{k=u+1}^{\infty} \mathfrak{B}_k(z) M_k.$$

In view of (5.7)–(5.8) the sum $\sum_{k=0}^{\infty} \mathfrak{B}_k(z) M_k$ converges in the Drury–Arveson space. Let G be its limit, and assume that $G \neq F$. Thus there exists $w \in \mathbb{B}_N$ such that $G(w) \neq F(w)$. We now proceed in a number of steps to obtain a contradiction.

STEP 1: There exists $u_0 \in \mathbb{N}$ such that for $u \geq u_0$

$$\sqrt{1 - \|w\|^2} \cdot \|R_u(w)\| > \sup_{\substack{c \in \mathbb{C}^n,\ \|c\|=1 \\ d \in \mathbb{C}^m,\ \|d\|=1}} \frac{|\langle (F - G)d\, ,\, c e_w \rangle_{(\mathbf{H}(\mathbb{B}_N))^n}|}{2}. \tag{5.10}$$

Indeed, S_u tends to 0 in norm in $(\mathbf{H}(\mathbb{B}_N))^{n \times m}$. Since in a reproducing kernel Hilbert space convergence in norm implies pointwise convergence, we have $\lim_{u \to \infty} S_u(w) = 0_{n \times m}$ in the norm of $\mathbb{C}^{n \times m}$, and there exists $u_0 \in \mathbb{N}$ such that

$$u \geq u_0 \implies \|S_u(w)\| < \frac{\|F(w) - G(w)\|}{2}.$$

Thus

$$\|R_u(w)\| + \frac{\|F(w) - G(w)\|}{2} > \|R_u(w)\| + \|S_u(w)\| \geq \|F(w) - G(w)\|,$$

and so

$$\|R_u(w)\| > \frac{\|F(w) - G(w)\|}{2},$$

which can be rewritten as (5.10).

STEP 2: It holds that

$$\lim_{k \to \infty} (1 - \|w_k\|^2) \|\mathfrak{B}_k(w_k) M_k\|^2 = 0 \tag{5.11}$$

Indeed, from the convergence of $\sum_{k=0}^{\infty} \mathfrak{B}_k M_k$ we have

$$\lim_{k\to\infty} \|\mathfrak{B}_k M_k\|_{(\mathbf{H}(\mathbb{B}_N))^{n\times m}} = 0.$$

Thus, with $c \in \mathbb{C}^m$ and $d \in \mathbb{C}^n$, we have:

$$\begin{aligned} |\langle \mathfrak{B}_k(w_k) M_k c, d\rangle_{(\mathbf{H}(\mathbb{B}_N))^{n\times 1}}| &= |\langle \mathfrak{B}_k M_k c, \frac{d}{1-\langle \cdot, w_k\rangle}\rangle_{(\mathbf{H}(\mathbb{B}_N))^{n\times 1}}| \\ &\le \|\mathfrak{B}_k M_k c\|_{(\mathbf{H}(\mathbb{B}_N))^{n\times 1}} \cdot \frac{\|d\|}{\sqrt{1-\|w_k\|^2}} \\ &\le \|\mathfrak{B}_k M_k\|_{(\mathbf{H}(\mathbb{B}_N))^{n\times m}} \cdot \|c\| \cdot \frac{\|d\|}{\sqrt{1-\|w_k\|^2}}, \end{aligned}$$

where we have used the Cauchy-Schwarz inequality. So, after taking supremum on c and d,

$$\|\sqrt{1-\|w_k\|^2}\mathfrak{B}_k(w_k)M_k\| \le \|\mathfrak{B}_k M_k\|_{(\mathbf{H}(\mathbb{B}_N))^{n\times m}} \longrightarrow 0 \text{ as} n \to \infty,$$

and so (5.11) holds in view of (5.5).

STEP 3: We conclude the proof.

Let $u \ge u_0$, where u_0 is as in Step 1. Since $R_u(z) = \mathcal{B}_{w_{u+1}}(z)F_{u+1}(z)$ and since w is such that $F(w) \ne G(w)$ we have

$$\sqrt{1-\|w\|^2} \cdot \|\mathcal{B}_{w_{u+1}}(w)F_{u+1}(w)\| > \sup_{\substack{c\in\mathbb{C}^n,\ \|c\|=1 \\ d\in\mathbb{C}^m,\ \|d\|=1}} \frac{|\langle (F-G)d\ ,\ ce_w\rangle_{(\mathbf{H}(\mathbb{B}_N))^{n\times 1}}|}{2}. \tag{5.12}$$

By definition of w_{u+1} we have

$$\begin{aligned} &\sqrt{1-\|w_{u+1}\|^2} \cdot \|\mathcal{B}_{w_{u+1}}(w_{u+1})F_{u+1}(w_{u+1})\| \\ &< \sup_{\substack{c\in\mathbb{C}^n,\ \|c\|=1 \\ d\in\mathbb{C}^m,\ \|d\|=1}} \frac{|\langle (F-G)d\ ,\ ce_w\rangle_{(\mathbf{H}(\mathbb{B}_N))^{n\times 1}}|}{2}, \end{aligned}$$

and using (5.5) we contradict (5.11). □

6 Infinite Blaschke Products

In the previous sections appeared the counterpart of finite Blaschke products in the setting of the ball. We now consider the case of infinite products.

Let $a \in \mathbb{B}_N$, and let $b_a(z)$ be a $\mathbb{C}^{1\times N}$-valued Blaschke factor. We use the formula

$$b_a(z) = \frac{a - \frac{za^*}{aa^*}a - \sqrt{1-aa^*}\left(z - \frac{za^*}{aa^*}a\right)}{1 - za^*}, \tag{6.1}$$

from [16, (2), p. 25] rather than the formula in (2.5). See [6, Lemma 4.2, p. 13] for the equality between the two expressions.

We first prove a technical lemma useful in the proof of the convergence of an infinite Blaschke product.

Lemma 6.1 *Let* $\alpha = -\frac{a}{\sqrt{aa^*}} \in \partial\mathbb{B}_N$. *Then,*

$$b_a(z) - b_a(\alpha) = \frac{(z-\alpha)\left(a^*a\left(\frac{1-\sqrt{1-aa^*}}{aa^*}\right) - I_N\right) + z(\alpha a^*) - \alpha(za^*)}{(1-za^*)(1+\sqrt{aa^*})} \cdot \sqrt{1-aa^*} \tag{6.2}$$

and

$$\|b_a(z) - b_a(\alpha)\| \le \frac{4\sqrt{1-aa^*}}{1-\|z\|}. \tag{6.3}$$

Proof We write $b_a(z) - b_a(\alpha) = \frac{\Delta}{(1-za^*)(1-\alpha a^*)}$, where the numerator

$$\Delta = \left(a - \frac{za^*}{aa^*}a - \sqrt{1-aa^*}\left(z - \frac{za^*}{aa^*}a\right)\right)(1-\alpha a^*) - \left(a - \frac{\alpha a^*}{aa^*}a - \sqrt{1-aa^*}\left(\alpha - \frac{\alpha a^*}{aa^*}a\right)\right)(1-za^*)$$

has 16 terms. Out of there, a and $-a$ cancel each other, and

$$\frac{za^*}{aa^*}a(\alpha a^*) = \frac{\alpha a^*}{aa^*}a(za^*)$$

and

$$\sqrt{1-aa^*}\frac{za^*}{aa^*}a(\alpha a^*) = \sqrt{1-aa^*}\frac{\alpha a^*}{aa^*}a(za^*).$$

We are thus left with 10 terms, which can be rewritten as:

$$\Delta = (z-\alpha)\left(\left(-\frac{a^*a}{aa^*} - \sqrt{1-aa^*}I_N + \sqrt{1-aa^*}\frac{a^*a}{aa^*} + a^*a\right) + \sqrt{1-aa^*}\left(z(\alpha a^*) - \alpha(za^*)\right)\right).$$

Note that $z(\alpha a^*) - \alpha(za^*)$ does not vanish when $N > 1$. Therefore

$$b_a(z) - b_a(\alpha) = \frac{(z-\alpha)\left(\left(-\frac{a^*a}{aa^*} - \sqrt{1-aa^*}I_N + \sqrt{1-aa^*}\frac{a^*a}{aa^*} + a^*a\right) + \sqrt{1-aa^*}\,(z(\alpha a^*) - \alpha(za^*))\right)}{(1-za^*)(1+\sqrt{aa^*})}. \tag{6.4}$$

□

Remark 6.2 We note that

$$\|a^*a\left(\frac{1-\sqrt{1-aa^*}}{aa^*}\right) - I_N\| = \sqrt{1-aa^*},$$

as can be seen by computing the eigenvalues of the matrix in the left hand side.

We now consider a term of the form (3.1) and write (where $\alpha = -\frac{a}{\sqrt{aa^*}}$ and W is a unitary matrix to be determined)

$$\begin{aligned} B_a(z) &= B(z)W \\ &= \left(U\begin{pmatrix} b_a(\alpha) & 0_{1\times(n-1)} \\ 0_{(n-1)\times N} & I_{n-1}\end{pmatrix} + U\begin{pmatrix} (b_a(z) - b_a(\alpha) & 0_{1\times(n-1)} \\ 0_{(n-1)\times N} & I_{n-1}\end{pmatrix}\right) W, \end{aligned} \tag{6.5}$$

where we do not stress the dependence on the matrices U and W. Since $b_a(\alpha)$ is a unit vector, the matrix

$$U\begin{pmatrix} b_a(\alpha) & 0_{1\times(n-1)} \\ 0_{(n-1)\times N} & I_{n-1}\end{pmatrix}$$

is coisometric, and we can complete the columns of its adjoint to a unitary matrix W. Then we have

$$U\begin{pmatrix} b_a(\alpha) & 0_{1\times(n-1)} \\ 0_{(n-1)\times N} & I_{n-1}\end{pmatrix} W = \begin{pmatrix} I & 0\end{pmatrix} \tag{6.6}$$

and show that the corresponding infinite product will converge when $\sum_{n=1}^{\infty}\sqrt{1-a_na_n^*}$ converges.

In Theorem 6.3 below we imbed $\mathbb{C}^m$ inside ℓ_2 via the formula:

$$i_m(z_1, \ldots, z_m) = (z_1, \ldots, z_m, 0, 0, \ldots). \tag{6.7}$$

We also need some notation and introduce the matrices

$$E_k = \begin{pmatrix} 1 & 0_{1\times k(N-1)}\end{pmatrix} \quad (= 1 \text{ when } N = 1),$$

$$F_k = \begin{pmatrix} I_{1+(k-1)(N-1)} & 0_{(1+(k-1)(N-1))\times(N-1)}\end{pmatrix} \in \mathbb{C}^{(1+(k-1)(N-1))\times(N+(k-1)(N-1))},$$

and note that $E_1 = F_1$ and

$$E_k = E_1 F_2 \cdots F_k \quad \text{and} \quad E_{k+1} = E_k F_{k+1}. \tag{6.8}$$

We also note that multiplication by F_k on the right imbeds $\mathbb{C}^{1+(k-1)(N-1)}$ into $\mathbb{C}^{1+k(N-1)}$. It will be useful to use the notation

$$F_{m_1}^{m_2} = \overset{\curvearrowright}{\prod_{k=m_1+1}^{m_2}} E_k. \tag{6.9}$$

Theorem 6.3 *The infinite product $b_{w_0}(z)B_{w_1}(z)B_{w_2}(z)\cdots B_{w_{k-1}}(z)\cdots$ where the factors are normalized as in (6.6) converges pointwise for $z \in \mathbb{B}_N$ to a non-identically vanishing ℓ_2-valued function analytic in $\mathbb{B}_N$ if*

$$\sum_{k=0}^{\infty} \sqrt{1 - a_k a_k^*} < \infty. \tag{6.10}$$

Proof The idea is to follow the proof for the scalar case appearing in sources such as [11, 13] and reproduced in [3, pp. 104-105]. We consider the product

$$\overset{\curvearrowright}{\prod_{k=1}^{m}} (F_k + A_k(z))$$

with

$$\begin{aligned} A_k(z) &= U_k \begin{pmatrix} b_{a_k}(z) - b_{a_k}(\alpha) & 0_{1\times(k-1)(N-1)} \\ 0_{(k-1)(N-1)\times N} & I_{(k-1)(N-1)} \end{pmatrix} W_k \\ &\in \mathbb{C}^{(1+(k-1)(N-1))\times(N+(k-1)(N-1))}, \end{aligned}$$

and note that, in view of (6.3),

$$\|A_k(z)\| \le \frac{4\sqrt{1 - a_k a_k^*}}{1 - \|z\|}. \tag{6.11}$$

Following the classical proof we now prove the convergence in a number of steps and use [3, pp. 104–105] as a source.

Note that, to ease the notation, in Steps 1–3 we do not stress the dependence of A_k on the variable z.

STEP 1: It holds that

$$\| \overset{\curvearrowright}{\prod_{k=1}^{m}} (F_k + A_k) - E_m \| \le \prod_{k=1}^{m} (1 + \|A_k\|) - 1, \quad m \in \mathbb{N}. \tag{6.12}$$

We proceed by induction, the case $m = 1$ being trivial since $E_1 = F_1$. We have

$$
\begin{aligned}
\| \overset{\curvearrowright}{\prod_{k=1}^{m+1}} (F_k + A_k) - E_{m+1}\| &= \| \left(\prod_{k=1}^{m} (F_k + A_k) \right) (F_{m+1} + A_{m+1}) - E_{m+1}\| \\
&= \| \left(\overset{\curvearrowright}{\prod_{k=1}^{m}} (F_k + A_k) \right) (F_{m+1} + A_{m+1}) - E_m F_{m+1}\| \\
&\le \| \left(\left(\overset{\curvearrowright}{\prod_{k=1}^{m}} (F_k + A_k) \right) - E_m \right) F_{m+1})\| + \\
&\quad + \|A_{m+1}\| \left(\prod_{k=1}^{m} (1 + \|A_k\|) \right) \\
&\le \left(\left(\prod_{k=1}^{n} (1 + \|A_k\|) \right) - 1 \right) + \|A_{m+1}\| \left(\prod_{k=1}^{m} (1 + \|A_k\|) \right) \\
&= \left(\prod_{k=1}^{m+1} (1 + \|A_k\|) \right) - 1,
\end{aligned}
$$

where we have used the induction hypothesis to go from the third to the fourth line.

Replacing A_k by A_{k+m_1} we have for $m_2 > m_1$:

$$
\left\| \left(\overset{\curvearrowright}{\prod_{k=m_1+1}^{m_2}} (E_k + A_k) \right) - \overset{\curvearrowright}{\prod_{k=m_1+1}^{m_2}} E_{m_2} \right\| \le \left(\prod_{k=m_1+1}^{m_2} (1 + \|A_k\|) \right) - 1. \tag{6.13}
$$

STEP 2: Let $Z_m = \overset{\curvearrowright}{\prod}{}_{k=1}^{m} (F_k + A_k)$. Then,

$$
\|Z_m\| \le e^{\sum_{k=1}^{m} \|A_k\|} < \infty
$$

Indeed,

$$
\begin{aligned}
\|Z_m\| &\le \prod_{k=1}^{m} \|F_k + A_k\| \\
&\le \prod_{k=1}^{m} (1 + \|A_k\|) \\
&\le \prod_{k=1}^{m} e^{\|A_k\|} \le e^{\sum_{k=1}^{\infty} \|A_k\|} < \infty,
\end{aligned}
$$

in view of (6.10) and (6.11).

STEP 3: Let i_m be defined by (6.7). Then, $(i_m(Z_m))_{m\in\mathbb{N}}$ is a Cauchy sequence in ℓ_2.

For $m_2 > m_1$ and using (6.13), we have

$$\begin{aligned}
\|i_{m_2}(Z_{m_2}) - i_{m_1}(Z_{m_1})\|_{\ell_2} &= \|Z_{m_2} - i_{m_2}\cdots i_{m_1+1}(Z_{m_1})\|_{\mathbb{C}^{1\times(1+(m_2+1)(N-1))}}\\
&= \left(\overset{\curvearrowright}{\prod_{k=1}^{m_1}}(F_k + A_k)\right)\cdot\left(\overset{\curvearrowright}{\prod_{k=m_1+1}^{m_2}}(F_k + A_k) - F_{m_1+1}^{m_2}\right)\\
&\le \left(\prod_{k=1}^{m_1}(1+\|A_k\|)\right)\cdot\|\overset{\curvearrowright}{\prod_{k=m_1+1}^{m_2}}(F_k + A_k) - F_{m_1+1}^{m_2}\|\\
&\le e^K\left\{\left(\prod_{k=m_1+1}^{m_2}(1+\|A_k\|)\right) - 1\right\}\\
&\le e^K\left\{\left(\prod_{k=m_1+1}^{m_2}e^{\|A_k\|}\right) - 1\right\}\\
&\le \left(\sum_{k=m_1+1}^{m_2}\|A_k\|\right)e^{2K},
\end{aligned}\tag{6.14}$$

with $K = \sum_{k=1}^{\infty}\|A_k\|$ (which is finite, thanks to (6.10) and (6.11)), and using inequality

$$e^x \le 1 + xe^x, \quad x \ge 0,$$

with $x = \sum_{k=m_1+1}^{m_2}\|A_k\|$.

STEP 4: The ℓ_2-valued function $Z(z) = \lim_{m\to\infty} Z_m(z)$ does not vanish identically in $\mathbb{B}_N$.

We first assume that $\sum_{k=1}^{\infty}\|A_k(z)\| < \frac{1}{2}$ and prove by induction that

$$\|Z_m(z)\| \ge 1 - \sum_{k=1}^{m}\|A_k(z)\|.\tag{6.15}$$

The claim $Z \neq 0$ will then follow by letting $m \to \infty$. For $m = 1$ the claim is trivial. Assume that (6.15) holds for m. We then have:

$$\begin{aligned}
\|Z_{m+1}(z)\| &= \|Z_m(z)(F_{m+1} + A_{m+1}(z))\|\\
&\ge \|Z_m(z)F_{m+1}\| - \|Z_m(z)A_{m+1}(z)\| \quad (\text{since}\|Z_m(z)F_{m+1}\| = \|Z_m(z)\|)\\
&\ge \|Z_m(z)\| - \|Z_m(z)\|\|A_{m+1}(z)\| \quad (\text{since}\|Z_m(z)A_{m+1}(z)\|\\
&\le \|Z_m(z)\|\|A_{m+1}(z)\|)\\
&= \|Z_m(z)\|\cdot(1 - \|A_{m+1}(z)\|)
\end{aligned}$$

$$\geq \left(1 - \sum_{k=1}^{m} \|A_k(z)\|\right)(1 - \|A_{m+1}(z)\|)$$
$$\geq \left(1 - \sum_{k=1}^{m+1} \|A_k(z)\|\right).$$

Let $M \in \mathbb{N}$ (depending on z) be such that $\sum_{k=M}^{\infty} \|A_k(z)\| < \frac{1}{2}$. Then the same inequality holds in an open neighborhood V of z in view of (6.11), and so the same M can be taken for $z \in V$. Let

$$Z_{M-1}(z) = \overset{\curvearrowright}{\prod}_{u=1}^{M-1} (F_u + A_u(z)) \in \mathbb{C}^{1\times(1+(M-2)(N-1))},$$

where

$$\widetilde{Z_M}(z) = \overset{\curvearrowright}{\prod}_{u=M}^{\infty} (F_u + A_u(z)).$$

We can patch together all the $Z_{M-1}(z)\widetilde{Z_M}(z)$ to a common function defined in $\mathbb{B}_N$. Assume that $Z_{M-1}(z)\widetilde{Z_M}(z) \equiv 0$ in one of the neighborhoods V. Then the infinite product vanishes identically in $\mathbb{B}_N$. Letting z go to the boundary we get a contradiction since $Z_{M-1}(z)\widetilde{Z_M}(z)$ takes coisometric values on $\partial\mathbb{B}_N$.

STEP 5: Using (6.12) and (6.14), we obtain the bound:

$$\left\| \overset{\curvearrowright}{\prod_{k=1}^{m}} (F_k + A_k(z)) - Z \right\| \leq e^{2K} \left(\sum_{k=m+1}^{\infty} \|A_k(z)\| \right). \tag{6.16}$$

□

It is worthwhile to note that the above theorem allows to further extending the results of [6] to the case of an infinite number of points.

Acknowledgments The authors thank the Macao Government FDCT 098/2012/A3 and D. Alpay thanks the Earl Katz family for endowing the chair which supported his research.

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