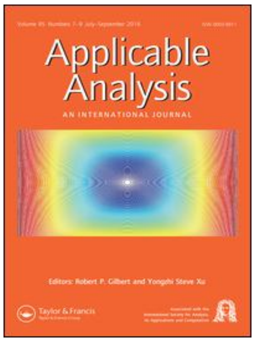

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Reconstruction of analytic signal in Sobolev space by framelet sampling approximation

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ABSTRACT
Based on dual framelets, we construct the sampling approximation for the whole Sobolev space $\mathbf{H}^s(\mathbb{R})$ where $s > 1/2$. In particular, the sampling system has adjustable shift parameters. By the B-spline sampling system, we construct the approximation of Hilbert transform of any function of $\mathbf{H}^s(\mathbb{R})$. Combining the approximation of the function $\mathbf{H}^s(\mathbb{R})$ and that of its Hilbert transform, we establish a reconstruction method for the analytic signal. Particularly, the reconstruction series converges exponentially with respect to the scale level. Moreover, the numerical singularity emerging in computation of Hilbert transform can be removed by adjusting the shift parameters. That is, the method of reconstruction of analytic signal is numerically and L^2-stable. Several numerical experiments are carried out to check the efficiency of our reconstruction method.

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1. Introduction

For a real function $f \in L^2(\mathbb{R})$, its analytic signal is defined to be $f + iHf$, where i is the imaginary unit, and Hf is the Hilbert transform (HT) of f, defined by the Cauchy principal value integral as follow:

$$Hf(t) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x)}{t-x}\mathrm{d}x.$$

By analytic signal, many mathematical manipulations in signal processing can be facilitated. For example, when the negative frequency components of the Fourier transform of f are superfluous, then it can be discarded by analytic signal without any loss of information since

$$\widehat{f+iHf}(\xi) = \widehat{f}(\xi)\chi_{[0,\infty)}(\xi),$$

where χ_I is the characteristic function of the set I, and the Fourier transform $\widehat{g}$ of any tempered distribution g is defined by $\widehat{g}(\,\cdot\,) = \int_{\mathbb{R}} g(x)e^{-ix\cdot}\mathrm{d}x$. In addition to the discard of negative frequency, analytic signal is a commonly accepted tool of defining instantaneous features such as instantaneous amplitude (IA), instantaneous phase (IP) and instantaneous frequency (IF).[1,2] Specifically,

$$f + iHf = \rho(\,\cdot\,)e^{i\theta(\cdot)},$$

where ρ, θ and its derivative θ' are referred to as IA, IP and IF, respectively.

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On the other hand, the data we acquire in practice are usually the samples of f on $\mathbb{R}$. Then, we naturally ask how to reconstruct the analytic signal $f + iHf$ by the samples. The problem is important for discarding negative frequency components or extracting instantaneous features mentioned previously.[2] Sobolev space $\mathbf{H}^s(\mathbb{R}), s > 1/2$, defined by

$$\mathbf{H}^s(\mathbb{R}) = \left\{ f : \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 (1 + \xi^2)^s \mathrm{d}\xi < \infty \right\}, \tag{1.1}$$

has a lot of applications in signal analysis.[3–7] But to the best of our knowledge, it has not been solved: how to reconstruct the analytic signal in $\mathbf{H}^s(\mathbb{R})$. In this paper, we will solve the problem by the framelet sampling method. From the definition of analytic signal, this problem can be split into the reconstructions of f and its Hilbert transform. Before proceeding further, let us introduce some related developments on this topic.

Sampling theorem is a key tool for the conversion between an analogue signal and its digital form (A/D). The most classical sampling theorem in $\mathbf{H}^s(\mathbb{R})$ is the Shannon sampling,[8,9] by which a bandlimited signal f can be perfectly reconstructed. Since the space of bandlimited signals is just a subspace of $\mathbf{H}^s(\mathbb{R})$, Shannon sampling theory has been extended to other subspaces.[10–13] In [14], using a special pair of dual framelets in Sobolev space, Li and Yang established the sampling theorem holding for all functions in $\mathbf{H}^s(\mathbb{R})$. Based on [14], Li constructed the framelet sampling approximation for the functions in $\mathbf{H}^s(\mathbb{R})$, of which the Fourier transforms pointwisely decay at the exponential rate.[15] An unsolved problem is how to establish the sampling approximation for all the functions in $\mathbf{H}^s(\mathbb{R})$. We shall continue the work in [14–16] and construct the framelet sampling approximation holding for the whole $\mathbf{H}^s(\mathbb{R})$. Using the isometry of HT in $L^2(\mathbb{R})$, a reconstruction method of HT will be developed.

Our method of reconstructing the functions in $\mathbf{H}^s(\mathbb{R})$ has its roots in the theory of dual framelets in the Sobolev space. Before proceeding further, we shall give necessary background knowledge on framelets in Sobolev space. Readers are referred to [3,4,10,14–18] for more details and developments on this topic.

Sobolev space $\mathbf{H}^s(\mathbb{R})$ for any $s \in \mathbb{R}$ is defined by (1.1). $\mathbf{H}^s(\mathbb{R})$ is equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathbf{H}^s(\mathbb{R})}$ defined by

$$\langle f, g \rangle_{\mathbf{H}^s(\mathbb{R})} = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}(1 + \xi^2)^s \mathrm{d}\xi, \quad \forall f, g \in \mathbf{H}^s(\mathbb{R}).$$

Straightforward observation on (1.1) gives that $\mathbf{H}^{s_1}(\mathbb{R}) \supseteq \mathbf{H}^{s_2}(\mathbb{R})$ if and only if $s_1 \leq s_2$. When $s = 0$, $\mathbf{H}^0(\mathbb{R}) = L^2(\mathbb{R})$ and $\sqrt{\langle f, f \rangle_{\mathbf{H}^0(\mathbb{R})}} = ||f||_2$. For any $f \in L^2(\mathbb{R})$, define

$$\nu_2(f) := \sup\{s : f \in \mathbf{H}^s(\mathbb{R})\}, \tag{1.2}$$

referred to as the Sobolev smoothness exponent of f. Define the bracket product $[\cdot, \cdot]_\gamma$ with respect to exponent $\gamma \in \mathbb{R}$, for $f, g : \mathbb{R} \longrightarrow \mathbb{C}$, by

$$[f, g]_\gamma(\xi) = \sum_{k \in \mathbb{Z}} \widehat{f}(\xi + 2k\pi)\overline{\widehat{g}(\xi + 2k\pi)}(1 + |\xi + 2k\pi|^2)^\gamma, \forall \xi \in \mathbb{R}.$$

A function $f \in L^2(\mathbb{R})$ has κ **vanishing moments** if $\widehat{f}^{(j)}(0) = 0$ where $\kappa \in \mathbb{N}$ and $j = 0, 1, \ldots, \kappa - 1$.

Let a 2-refinable function $\phi \in \mathbf{H}^s(\mathbb{R}), s \in \mathbb{R}$, be given by

$$\phi = 2 \sum_{k \in \mathbb{Z}} a[k]\phi(2 \cdot -k) \tag{1.3}$$

for a finitely supported sequence $a := \{a[k]\}_{k\in\mathbb{Z}}$, referred to as the mask of ϕ. Implementing the Fourier transform to both sides of (1.3) leads to

$$\widehat{\phi}(2\,\cdot\,) = \widehat{a}(\,\cdot\,)\widehat{\phi}(\,\cdot\,), \tag{1.4}$$

where $\widehat{a}(\,\cdot\,) := \sum_{k\in\mathbb{Z}} a[k]e^{ik\cdot}$ is the mask symbol of a. We say that ϕ has $\kappa + 1$ *sum rules* if there exists a 2π-periodic trigonometric polynomial $\widehat{Y}$ with $\widehat{Y}(0) \neq 0$ such that $\widehat{a}$ satisfies

$$\widehat{Y}(2\,\cdot\,)\widehat{a}(\,\cdot\,+2\pi\gamma) = \Delta_\gamma \widehat{Y}(\,\cdot\,) + \mathcal{O}(|\,\cdot\,|^{\kappa+1}), \quad \forall\gamma \in \{0, 1/2\},$$

where $\{\Delta_j\}$ is the Dirac sequence, i.e. $\Delta_0 = 1$ and $\Delta_j = 0$ for $j \neq 0$. $\{\psi^\ell\}_{\ell=1}^L$ is assumed to be a set of wavelet functions defined by

$$\widehat{\psi^\ell}(2\,\cdot\,) = \widehat{b^\ell}(\,\cdot\,)\widehat{\phi}(\,\cdot\,), \tag{1.5}$$

for a $2\pi\mathbb{Z}$-periodic trigonometric polynomial $\widehat{b^\ell}(\,\cdot\,)$. Now a wavelet system $X^s(\phi; \psi^1, \dots, \psi^L)$ in $\mathbf{H}^s(\mathbb{R})$ is defined as

$$\begin{aligned} X^s(\phi; \psi^1, \dots, \psi^L) \; &:= \{\phi_{0,k} : k \in \mathbb{Z}\} \\ &\cup \{\psi_{j,k}^{\ell,s} : k \in \mathbb{Z}, j \in \mathbb{N}_0, \ell = 1, \dots, L\}, \end{aligned} \tag{1.6}$$

where $\phi_{0,k} = \phi(\,\cdot\,-k)$, $\psi_{j,k}^{\ell,s} = 2^{j(1/2-s)}\psi^\ell(2^j\,\cdot\,-k)$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If, for any $f \in \mathbf{H}^s(\mathbb{R})$, there exist two positive constants C_1 and C_2 such that

$$\begin{aligned} C_1||f||^2_{\mathbf{H}^s(\mathbb{R})} \; &\leq \sum_{k\in\mathbb{Z}} |\langle f, \phi_{0,k}\rangle_{\mathbf{H}^s(\mathbb{R})}|^2 + \sum_{\ell=1}^{L}\sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}^{\ell,s}\rangle_{\mathbf{H}^s(\mathbb{R})}|^2 \\ &\leq C_2||f||^2_{\mathbf{H}^s(\mathbb{R})}, \end{aligned} \tag{1.7}$$

then we say that $X^s(\phi; \psi^1, \dots, \psi^L)$ is a 2-framelet in $\mathbf{H}^s(\mathbb{R})$. Furthermore, if there exists another 2-framelet $X^{-s}(\widetilde{\phi}; \widetilde{\psi}^1, \dots, \widetilde{\psi}^L)$ in $\mathbf{H}^{-s}(\mathbb{R})$, which is related to a 2-refinable function $\widetilde{\phi} \in \mathbf{H}^{-s}(\mathbb{R})$ and a set of wavelet function $\{\widetilde{\psi}_\ell\}_{\ell=1}^L$ given by

$$\widehat{\widetilde{\phi}}(2\,\cdot\,) = \widehat{\widetilde{a}}(\,\cdot\,)\widehat{\widetilde{\phi}}(\,\cdot\,),\; \widehat{\widetilde{\psi}_\ell}(2\,\cdot\,) = \widehat{\widetilde{b}^\ell}(\,\cdot\,)\widehat{\widetilde{\phi}}(\,\cdot\,), \tag{1.8}$$

such that for any $f \in \mathbf{H}^s(\mathbb{R})$ and $g \in \mathbf{H}^{-s}(\mathbb{R})$, there holds

$$\langle f, g\rangle = \sum_{k\in\mathbb{Z}} \langle \phi_{0,k}, g\rangle\langle f, \widetilde{\phi}_{0,k}\rangle + \sum_{\ell=1}^{L}\sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{Z}} \langle \psi_{j,k}^{\ell,s}, g\rangle\langle f, \widetilde{\psi}_{j,k}^{\ell,-s}\rangle, \tag{1.9}$$

then we say that $X^s(\phi; \psi^1, \dots, \psi^L)$ and $X^{-s}(\widetilde{\phi}; \widetilde{\psi}^1, \dots, \widetilde{\psi}^L)$ are a pair of dual 2-framelets in $(\mathbf{H}^s(\mathbb{R}), \mathbf{H}^{-s}(\mathbb{R}))$. It follows directly from (1.9) that

$$f = \sum_{k\in\mathbb{Z}} \langle f, \widetilde{\phi}_{0,k}\rangle\phi_{0,k} + \sum_{\ell=1}^{L}\sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{Z}} \langle f, \widetilde{\psi}_{j,k}^{\ell,-s}\rangle\psi_{j,k}^{\ell,s} \tag{1.10}$$

and

$$g = \sum_{k\in\mathbb{Z}} \langle g, \phi_{0,k}\rangle \widetilde{\phi}_{0,k} + \sum_{\ell=1}^{L} \sum_{j\in\mathbb{N}_0} \sum_{k\in\mathbb{Z}} \langle g, \psi_{j,k}^{\ell,s}\rangle \widetilde{\psi}_{j,k}^{\ell,-s}.$$

2. Adjustable B-spline sampling system in Sobolev space

It will be witnessed in (3.2) that the HT of a spline can be expressed explicitly. As such, the B-spline framelet sampling system will be used to reconstruct the HT of any function in $\mathbf{H}^s(\mathbb{R})$. We shall prove that the reconstruction is L^2-stable. However, it will follow from (3.1) and (3.2) that the HT of B-spline has numerical singularity at the knots. Recall that if a function is α-Hölder continuous with $0 < \alpha < 1$, then its HT is continuous on $\mathbb{R}$ [19, Chapter 6]. Thus, in addition to being L^2-stable, the reconstruction method of reconstructing HT should be numerically stable as well. The multiscale cardinal B-spline sampling system to be established is adjustable, and therefore the deduced reconstruction of HT satisfies the two requirements above.

2.1. Two lemmas on convergence of framelet series in Sobolev space

Lemma 2.1: *Let $\widetilde{\phi} \in \boldsymbol{H}^{-t}(\mathbb{R})$, $t > 0$, be 2-refinable. Construct a wavelet function $\widetilde{\psi}$ by*

$$\widehat{\widetilde{\psi}}(2\,\cdot\,) = \widehat{\widetilde{b}}(\,\cdot\,)\widehat{\widetilde{\phi}}(\,\cdot\,)$$

such that $\widetilde{\psi}$ has $\kappa + 1$ vanishing moments, where $\widehat{\widetilde{b}}$ is a 2π-periodic trigonometric polynomial, $\kappa \in \mathbb{N}_0$ and $\kappa + 1 > t$. Then, there exists a positive constant $C(\kappa, s, s^*)$ such that for any $N \in \mathbb{N}$ and any $f \in \boldsymbol{H}^s(\mathbb{R})$ with $s \in (t, \kappa + 1)$, it holds

$$\sum_{j=N}^{\infty} \sum_{k\in\mathbb{Z}} |\langle f, \widetilde{\psi}_{j,k}^{s^*}\rangle|^2 \le C(\kappa, s, s^*) 2^{-2N\eta_{\kappa+1}(s,s^*)} ||f||_{H^s}^2, \tag{2.1}$$

where s^* is any fixed number in the interval (t, s), and

$$\eta_{\kappa+1}(s, s^*) := (\kappa + 1 - s^*)(s - s^*)/(\kappa + 1 - s^* + s). \tag{2.2}$$

***Proof*:** See Appendix section. □

Given a pair of dual framelets $X^s(\phi; \psi^1, \ldots, \psi^L)$ and $X^{-s}(\widetilde{\phi}; \widetilde{\psi}^1, \ldots, \widetilde{\psi}^L)$ in $(\mathbf{H}^s(\mathbb{R}), \mathbf{H}^{-s}(\mathbb{R}))$, any $f \in \mathbf{H}^s(\mathbb{R})$ can be reconstructed by (1.10). Clearly, the scale level j in (1.10) goes from 0 to infinite. It is necessary to truncate it for practical computation. Now we define the multiscale operator $\mathcal{S}_\phi^N$, a truncation form of (1.10), by

$$\mathcal{S}_\phi^N f = \sum_{k\in\mathbb{Z}} \langle f, \widetilde{\phi}_{0,k}\rangle \phi_{0,k} + \sum_{\ell=1}^{L} \sum_{j=0}^{N-1} \sum_{k\in\mathbb{Z}} \langle f, \widetilde{\psi}_{j,k}^{\ell,-s}\rangle \psi_{j,k}^{\ell,s}, \tag{2.3}$$

where $N \in \mathbb{N}$. The truncation error $||(I - \mathcal{S}_\phi^N)f||$ will be estimated in the following Lemma 2.2, where I is the identity operator. Straightforward observation gives us that for any $s^* \in (0, s)$ and $\widetilde{\phi} \in \mathbf{H}^{-s^*}(\mathbb{R})$, every term of the second part of the series in (2.3) satisfies

$$\langle f, \widetilde{\psi}_{j,k}^{\ell,-s}\rangle \psi_{j,k}^{\ell,s} = \langle f, \widetilde{\psi}_{j,k}^{\ell,-s^*}\rangle \psi_{j,k}^{\ell,s^*}.$$

That is, the operator $\mathcal{S}_\phi^N$ is independent of Sobolev smoothness, which will be affirmed by (2.9) that Sobolev smoothness does not affect sampling data. However, as will be witnessed in Theorem 2.1 and Note 2.1, Sobolev smoothness is useful for estimating the approximation error.

We turn to an equivalent but more concise form of (2.3). Suppose that $X^s(\phi;\psi^1,\dots,\psi^L)$ and $X^{-s}(\widetilde{\phi};\widetilde{\psi}^1,\dots,\widetilde{\psi}^L)$ are constructed by the mixed extension principle (MEP), namely: the mask symbols $\widehat{a},\widehat{b^1},\dots,\widehat{b^L}$ and $\widehat{\widetilde{a}},\widehat{\widetilde{b}^1},\dots,\widehat{\widetilde{b}^L}$ in (1.4), (1.5) and (1.8) satisfy

$$\begin{bmatrix} \overline{\widehat{b^1}(\cdot)} & \overline{\widehat{b^2}(\cdot)} & \dots & \overline{\widehat{b^L}(\cdot)} \\ \overline{\widehat{b^1}(\cdot+\pi)} & \overline{\widehat{b^2}(\cdot+\pi)} & \dots & \overline{\widehat{b^L}(\cdot+\pi)} \end{bmatrix} \begin{bmatrix} \widehat{\widetilde{b}^1}(\cdot) \\ \widehat{\widetilde{b}^2}(\cdot) \\ \vdots \\ \widehat{\widetilde{b}^L}(\cdot) \end{bmatrix} = \begin{bmatrix} 1-\overline{\widehat{a}}(\cdot)\widehat{\widetilde{a}}(\cdot) \\ -\overline{\widehat{a}}(\cdot+\pi)\widehat{\widetilde{a}}(\cdot) \end{bmatrix}. \tag{2.4}$$

Then, by [14, (4.5)], the multiscale operator $\mathcal{S}_\phi^N$ in (2.3) can be equivalently written by

$$\mathcal{S}_\phi^N f = \sum_{k\in\mathbb{Z}} \langle f, \widetilde{\phi}_{N,k}^{-s}\rangle \phi_{N,k}^s, \tag{2.5}$$

depending only on the refinable functions ϕ and $\widetilde{\phi}$.

Lemma 2.2: *Let $s,t\in\mathbb{R}^+$ and $\kappa\in\mathbb{N}_0$ such that $t<\min\{\kappa+1,s\}$. Assume that $\phi\in \boldsymbol{H}^s(\mathbb{R}),\widetilde{\phi}\in \boldsymbol{H}^{-t}(\mathbb{R}):\mathbb{R}\longrightarrow\mathbb{C}$ be the pair of 2-refinable functions defined in (1.4) and (1.8), respectively. Moreover, ϕ has $\kappa+1$ sum rules. Let s^* be any fixed number in $(t,\min\{\kappa+1,s\})$. Then, for any $f\in \boldsymbol{H}^s(\mathbb{R})$, there exists a positive number $g(s,s^*,t)$ such that*

$$||(I-\mathcal{S}_\phi^N)f||_{\boldsymbol{H}^{s^*}} \le g(s,s^*,t)||f||_{\boldsymbol{H}^s} 2^{-N\eta_{\kappa+1}(s^*,s)}, \tag{2.6}$$

where $\eta_{\kappa+1}$ is defined in (2.2).

***Proof*:** See Appendix section. □

2.2. Adjustable multiscale sampling system in Sobolev space

2.2.1. Multiscale sampling approximation and its iterative sampling form

Based on the approximation accuracy given in (2.6), we are ready to establish the multiscale sampling approximation and its iterative form for $\mathbf{H}^s(\mathbb{R})$, where $s>1/2$. The following multiscale sampling operator to be defined in (2.7) is the special form of $\mathcal{S}_\phi^N$ in (2.5). As such, from now on, we use the same denotation $\mathcal{S}_\phi^N$ to mean the multiscale sampling operator in (2.7).

Theorem 2.1: *Suppose that $\phi\in \boldsymbol{H}^s(\mathbb{R}):\mathbb{R}\longrightarrow\mathbb{C}$ is a 2-refinable function defined in (1.4) and has $\kappa+1$ sum rules, where $s>1/2$ and $\kappa\in\mathbb{N}_0$. Let t and s^* be any two fixed numbers such that $1/2<t<s$ and $t<s^*<\min\{s,\kappa+1\}$. Then, for any scale level N, any $f\in \boldsymbol{H}^s(\mathbb{R})$ can be approximated by*

$$\mathcal{S}_\phi^N f := \sum_{k\in\mathbb{Z}} f(2^{-N}k)\phi(2^N\cdot -k) \tag{2.7}$$

with the approximation error $||(I-\mathcal{S}_\phi^N)f||_2$ estimated by

$$||(I-\mathcal{S}_\phi^N)f||_2 \le ||(I-\mathcal{S}_\phi^N)f||_{\boldsymbol{H}^{s^*}} \le g(s,s^*,t)||f||_{\boldsymbol{H}^s} 2^{-N\eta_{\kappa+1}(s,s^*)}, \tag{2.8}$$

where $\eta_{\kappa+1}$ is defined in (2.2), and $g(s,s^*,t)$ is as in Lemma 2.2.

***Proof*:** Let $\widetilde{\phi}$ in Lemma 2.2 be δ, the Delta distribution on $\mathbb{R}$. It follows from $\widehat{\delta}\equiv 1$ that $\widetilde{\phi}$ is 2-refinable and $\nu_2(\widetilde{\phi})=-1/2$. By (2.5) and the integral property $\langle f,\delta\rangle=f(0)$, the operator $\mathcal{S}_\phi^N$ acting on f substantially takes the form of N-scale sampling approximation due to that

$$\begin{aligned}\mathcal{S}_\phi^N f &= \sum_{k\in\mathbb{Z}} \langle f, 2^{N(1/2+s)}\delta(2^N\cdot -k)\rangle 2^{N(1/2-s)}\phi(2^N\cdot -k)\\ &= \sum_{k\in\mathbb{Z}} f(2^{-N}k)\phi(2^N\cdot -k).\end{aligned} \tag{2.9}$$

Now it follows directly from (2.9) and (2.6) that any $f\in\mathbf{H}^s(\mathbb{R})$ can be approximated by $\sum_{k\in\mathbb{Z}} f(2^{-N}k)$ $\phi(2^N\cdot -k)$, of which the approximation error is estimated by (2.8) and the quantity $g(s,s^*,t)$ is given in (A12) with $\widetilde{\phi}$ being replaced by δ. □

Note 2.1: *The upper bound in (2.8) implies that the approximation scheme in (2.7) is L^2-stable. Specifically, the series in (2.7) converges exponentially to f when N tends to ∞. For fixed $t(>1/2)$, s^* is a free variable in the interval $(t,\min\{s,\kappa+1\})$. On the upper bound $g(s^*,t)2^{-N\eta_{\kappa+1}(s,\,s^*)}$ in (2.8), a problem is its optimal (minimum) value. It follows from*

$$\begin{aligned}\eta_{\kappa+1}(s,s^*) &= \frac{(\kappa+1-s^*)(s-s^*)}{(\kappa+1-s^*+s)}\\ &= \frac{(s-s^*)}{1+s/(\kappa+1-s^*)}\end{aligned} \tag{2.10}$$

that $\eta_{\kappa+1}(s,s^*)$ increases when s^* tends to t. Consequently, $2^{-N\eta_{\kappa+1}(s,s^*)}\geq 2^{-N\eta_{\kappa+1}(s,t)}>0$. On the other hand, from (A11) and (A12), we arrive at $\lim_{s^*\to t} g(s,s^*,t)=\infty$. Therefore,

$$s_{op} := \arg_{s^*\in(t,\min\{s,\kappa+1\})}\min g(s,s^*,t)2^{-N\eta_{\kappa+1}(s,\,s^*)} \tag{2.11}$$

exists, and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}s^*}\left[g(s,s^*,t)2^{-N\eta_{\kappa+1}(s,\,s^*)}\right]|_{s^*=\,s_{op}}=0. \tag{2.12}$$

It follows from Theorem 2.1 that the sampling operator $\mathcal{S}_\phi^N$ defined in (2.7) maps $\mathbf{H}^s(\mathbb{R})$ to $\mathbf{H}^{s^*}(\mathbb{R})$, and the error or residue $(I-\mathcal{S}_\phi^N)f$ is estimated in (2.8). The following theorem is on the iterative sampling scheme. It will be witnessed in the numerical experiments of Subsection 4.2 that the iterative sampling may perform better than (2.9), a non-iterative form.

Theorem 2.2: *Let a 2-refinable function $\phi\in\boldsymbol{H}^s(\mathbb{R})$ satisfy (1.4) and have $\kappa+1$ sum rules, where $s>1/2$ and $\kappa\in\mathbb{N}_0$. Then, for any $L\in\mathbb{N}$ and any function $f\in\boldsymbol{H}^s(\mathbb{R})$, there exists a sequence $\{c_{N,L}[k]\}_{k\in\mathbb{Z}}$, depending only on N-scale samples $\{f(2^{-N}k)\}_{k\in\mathbb{Z}}$, such that*

$$[I-(I-\mathcal{S}_\phi^N)^L]f=\sum_{k\in\mathbb{Z}} c_{N,L}[k]\phi(2^N\cdot -k). \tag{2.13}$$

Proof: What we need to prove is that the expression in (2.13) can be completed only by the N-scale samples $\{f(2^{-N}k)\}_{k\in\mathbb{Z}}$. For this, direct computation leads to a decomposition of the operator $[I-(I-\mathcal{S}_\phi^N)^L]$ as

$$[I-(I-\mathcal{S}_\phi^N)^L]=\mathcal{P}_L+\mathcal{P}_{L-1}+\ldots+\mathcal{P}_1 \tag{2.14}$$

where $\mathcal{P}_L=\mathcal{S}_\phi^N$ and $\mathcal{P}_{L-j}=\mathcal{P}_L(I-\sum_{k=0}^{j-1}\mathcal{P}_{L-k})$ for $j=1,2,\ldots,L-1$. Recalling that (2.9) implies $\mathcal{P}_L$ acting on any function $f\in\mathbf{H}^s(\mathbb{R})$ just requires the samples $\{f(2^{-N}k)\}_{k\in\mathbb{Z}}$. Hence, the induction method gives us that there exists a sequence $\{c_{N,L}[k]\}_{k\in\mathbb{Z}}$, depending on $\{f(2^{-N}k)\}_{k\in\mathbb{Z}}$ only, such that (2.13) holds. □

The following proposition, which can be proved by (2.14), is on the computation of $\{c_{N,L}[k]\}_{k\in\mathbb{Z}}$ in (2.13).

Proposition 2.1: *The sequence $\{c_{N,L}[k]\}_{k\in\mathbb{Z}}$ in (2.13) can be iteratively computed by*

$$c_{N,1}[k] := f(2^{-N}k), c_{N,L}[k] := c_{N,L-1}[k] + c_{N,1}[k] - \sum_{\ell\in\mathbb{Z}} c_{N,L-1}[\ell]\phi(k-\ell), \tag{2.15}$$

where $L \geq 2$.

For any $L \in \mathbb{N}$ and any strictly increasing sequence $\{s_k^*\}_{k=0}^L$ in (t,s), it can be proved by the mathematical induction method that the approximation error $||f - [I - (I - \mathcal{S}_\phi^N)^L]f||_2 = ||(I - \mathcal{S}_\phi^N)^L f||_2$ can be estimated by

$$||(I - \mathcal{S}_\phi^N)^L f||_2 \leq ||f||_{\mathbf{H}^s} \left(\prod_{k=0}^{L} g(s, s_{k+1}^*, s_k^*)\right) 2^{-N(\eta_{\kappa+1}(s,s_L^*)+\sum_{j=0}^{L-1}\eta_{\kappa+1}(s_{j+1}^*,s_j^*))}. \tag{2.16}$$

An application of [16, Theorem 3.2] and (2.15) leads to that the sequence $\{c_{N,L}[k]\}_{k\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$ for any $L \in \mathbb{N}$. Recalling that ϕ has $\kappa + 1$ sum rules, then it has approximation order $\kappa + 1$, namely:

$$||f - \mathbb{P}_N f||_2 = \mathcal{O}(2^{-N(\kappa+1)})$$

where $\mathbb{P}_N f$ is the best approximation of f in the shift-invariant space

$$\left\{\sum_{k\in\mathbb{Z}} c_k \phi(2^N - k) : \sum_{k\in\mathbb{Z}} |c_k|^2 < \infty\right\}.$$

Therefore, there exists the optimal L, denoted by L_{op}, such that

$$||f - [I - (I - \mathcal{S}_\phi^N)^{L_{op}}]f||_2 = ||(I - \mathcal{S}_\phi^N)^{L_{op}} f||_2 = \min_{L\in\mathbb{N}} ||(I - \mathcal{S}_\phi^N)^L f||_2. \tag{2.17}$$

In practical computation, L_{op} can be estimated by solving the problem above on a finite subset of $\mathbb{R}$.

2.2.2. Adjustable cardinal B-spline multiscale sampling approximation

The cardinal B-spline of order $m(\in \mathbb{N})$ is defined by

$$B_m := \overbrace{\chi_{[0,1)} * \ldots * \chi_{[0,1)}}^{m \text{ copies}}, \tag{2.18}$$

where $\chi_\mathcal{I}$ is the characteristic function of an interval $\mathcal{I}$, and $*$ the convolution operation. A useful fact about B_m is that $\widehat{B_m}(\cdot) = e^{-im\cdot/2}(\frac{\sin\cdot/2}{\cdot/2})^m$ and thus $\nu_2(B_m) = m - 1/2$. B_m is 2-refinable, concretely,

$$\widehat{B_m}(2\cdot) = \left(\frac{1+e^{i\cdot}}{2}\right)^m \widehat{B_m}(\cdot),$$

which also implies that B_m has m sum rules.

As to be pointed in Lemma 3.1, HB_m has numerical singularity on $[0, m]$. As such, to remove the numerical singularity arising in reconstruction of analytic signal, it is necessary to construct an adjustable sampling system. The following definition is necessary for the adjustable multiscale sampling system.

Definition 2.1: Let $\phi \in \mathbf{H}^s(\mathbb{R})$ be 2-refinable. The set of functions $\mathcal{X}_{N,\theta}(\phi)$ defined by

$$\mathcal{X}_{N,\theta}(\phi) := \{\phi(2^N \cdot -k - \theta_k)\}_{k\in\mathbb{Z}} \tag{2.19}$$

is referred to as an N-scale **adjustable function system** (AFS) of ϕ, where the shift parameter sequence $\theta := \{\theta_k\} \subseteq (-1, 1)$. Following (2.13), define the **adjustable sampling operator** (ASO) $\mathcal{A}^{N,L}_{\phi,\theta}$ by

$$\mathcal{A}^{N,L}_{\phi,\theta}(f) = \sum_{k\in\mathbb{Z}} c_{N,L}[k]\phi(2^N \cdot -k - \theta_k), \forall f \in \mathbf{H}^s(\mathbb{R}), \tag{2.20}$$

where the coefficient sequence $\{c_{N,L}[k]\}_{k\in\mathbb{Z}}$ can be computed by (2.15). By straightforward observation, $\mathcal{A}^{N,L}_{\phi,\theta}$ is substantially the shift perturbation of $I - (I - \mathcal{S}^N_\phi)^L$ in (2.13).

As $I - (I - \mathcal{S}^N_{B_m})^L$, the following theorem asserts that $\mathcal{A}^{N,L}_{B_m,\theta}$ is L^2-stable as well.

Theorem 2.3: *Let $m \geq 2$. Suppose that $\mathcal{X}_{N,\theta}(B_m)$ and $\mathcal{A}^{N,L}_{B_m,\theta}$ are defined by (2.19) and (2.20) with ϕ being replaced by B_m, respectively. Moreover, let the shift parameter sequence θ satisfy $||\theta||_\infty < \frac{1}{4}$, and $J_0 = \lfloor ||\theta||_\infty^{-\frac{1}{m+1}} \rfloor$. Then, for any function $f \in \boldsymbol{H}^s(\mathbb{R})$, $s > 1/2$, there exists a positive number $C(N, L, J_0)$ such that*

$$||(I - \mathcal{A}^{N,L}_{B_m,\theta})f||_2 \leq ||(I - \mathcal{S}^N_{B_m})^L f||_2 + C(N, L, J_0)||\theta||_\infty^{\frac{2m-1}{2m+2}}. \tag{2.21}$$

***Proof*:** Triangle inequality gives us that

$$||\mathcal{A}^{N,L}_{B_m,\theta} f - f||_2 \leq ||(I - \mathcal{S}^N_{B_m})^L f||_2 + ||\mathcal{A}^{N,L}_{B_m,\theta} f - [I - (I - \mathcal{S}^N_{B_m})^L](f)||_2. \tag{2.22}$$

Next, we compute the second term of the inequality above as follows,

$$\begin{aligned}
&||\mathcal{A}^{N,L}_{B_m,\theta} f - [I - (I - \mathcal{S}^N_{B_m})^L](f)||_2^2 \\
&= \int_{\mathbb{R}} |\sum_{k\in\mathbb{Z}} c_{N,L}[k](B_m(2^N t - k - \theta_k) - B_m(2^N t - k))|^2 \mathrm{d}t \\
&= 2^{-N} \int_{\mathbb{R}} |\sum_{k\in\mathbb{Z}} c_{N,L}[k](B_m(x - k - \theta_k) - B_m(x - k))|^2 \mathrm{d}x \\
&= \frac{2^{-N}}{2\pi} \int_{\mathbb{R}} |\widehat{B_m}(\xi)|^2 |\sum_{k\in\mathbb{Z}} c_{N,L}[k]e^{-ik\xi}(1 - e^{-i\theta_k\xi})|^2 \mathrm{d}\xi \\
&= \frac{2^{-N}}{2\pi} \sum_{j=-\infty}^{\infty} \int_{2j\pi}^{2(j+1)\pi} |\widehat{B_m}(\xi)|^2 |\sum_{k\in\mathbb{Z}} c_{N,L}[k]e^{-ik\xi}(1 - e^{-i\theta_k\xi})|^2 \mathrm{d}\xi \\
&= \frac{2^{-N}}{2\pi} \int_0^{2\pi} \sum_{j=-\infty}^{\infty} |\widehat{B_m}(\xi + 2j\pi)|^2 |\sum_{k\in\mathbb{Z}} c_{N,L}[k]e^{-ik\xi}(1 - e^{-i\theta_k(\xi+2j\pi)})|^2 \mathrm{d}\xi \\
&= I_1(J) + I_2(J),
\end{aligned} \tag{2.23}$$

where

$$I_1(J) = \frac{2^{-N}}{2\pi} \int_0^{2\pi} \sum_{|j|\geq J+1} |\widehat{B_m}(\xi + 2j\pi)|^2 |\sum_{k\in\mathbb{Z}} c_{N,L}[k]e^{-ik\xi}(1 - e^{-i\theta_k(\xi+2j\pi)})|^2 \mathrm{d}\xi$$

and

$$I_2(J) = \frac{2^{-N}}{2\pi} \int_0^{2\pi} \sum_{j=-J}^{J} |\widehat{B_m}(\xi + 2j\pi)|^2 |\sum_{k\in\mathbb{Z}} c_{N,L}[k]e^{-ik\xi}(1 - e^{-i\theta_k(\xi+2j\pi)})|^2 \mathrm{d}\xi$$

with $J(\in \mathbb{N})$ to be optimally selected. The two terms $I_1(J)$ and $I_2(J)$ are estimated as follows,

$$\begin{aligned} I_1(J) &\le \frac{2^{-N+2}}{2\pi}||\{c_{N,L}[k]\}||_2^2\int_0^{2\pi}\sum_{|j|\ge J+1}|\widehat{B_m}(\xi+2j\pi)|^2\mathrm{d}\xi \\ &\le 2^{-N+2}(2\pi)^{-2m}||\{c_{N,L}[k]\}||_2^2\sum_{|j|\ge J+1}\frac{1}{(j-1)^{2m}} \\ &\le 2^{-N+2}(2\pi)^{-2m}||\{c_{N,L}[k]\}||_2^2\int_J^{\infty}\frac{1}{x^{2m}}\mathrm{d}x \\ &= 2^{-N+2}(2\pi)^{-2m}||\{c_{N,L}[k]\}||_2^2(2m+1)^{-1}J^{1-2m} \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} I_2(J) &\le \frac{2^{-N}}{2\pi}||\widehat{B_m}||_{L^\infty(\mathbb{R})}^2\sum_{j=-J}^{J}\int_0^{2\pi}|\sum_{k\in\mathbb{Z}}c_{N,L}[k]e^{-ik\xi}(1-e^{-i\theta_k(\xi+2j\pi)})|^2\mathrm{d}\xi \\ &\le \frac{2^{-N+1}}{2\pi}\sum_{j=-J}^{J}\int_0^{2\pi}|\sum_{k\in\mathbb{Z}}c_{N,L}[k]e^{-ik\xi}(1-e^{-i\theta_k\xi})|^2 \\ &\qquad +|\sum_{k\in\mathbb{Z}}c_{N,L}[k](1-e^{-i2\theta_k j\pi})e^{-i(k+\theta_k)\xi}|^2\mathrm{d}\xi. \end{aligned} \tag{2.25}$$

By Kadec's $\frac{1}{4}$-Theorem [20, Theorem 14], we have

$$\begin{aligned} \int_0^{2\pi}|\sum_{k\in\mathbb{Z}}c_{N,L}[k]e^{-ik\xi}(1-e^{-i\theta_k\xi})|^2\mathrm{d}\xi &\le ||\{c_{N,L}[k]\}||_2^2(1-\cos\pi||\theta||_\infty+\sin\pi||\theta||_\infty)^2 \\ &\le 2\pi^2||\{c_{N,L}[k]\}||_2^2||\theta||_\infty^2. \end{aligned} \tag{2.26}$$

On the other hand,

$$\begin{aligned} &\int_0^{2\pi}|\sum_{k\in\mathbb{Z}}c_{N,L}[k](1-e^{-i2\theta_k j\pi})e^{-i(k+\theta_k)\xi}|^2\mathrm{d}\xi \\ &\le 2\int_0^{2\pi}|\sum_{k\in\mathbb{Z}}c_{N,L}[k](1-e^{-i2\theta_k j\pi})e^{-ik\xi}|^2\mathrm{d}\xi \\ &\quad +2\int_0^{2\pi}|\sum_{k\in\mathbb{Z}}c_{N,L}[k](1-e^{-i2\theta_k j\pi})e^{-ik\xi}(1-e^{-i\theta_k\xi})|^2\mathrm{d}\xi \\ &\le 4\pi||\{c_{N,L}[k](1-e^{-i2\theta_k j\pi})\}||_2^2+4\pi^2||\{c_{N,L}[k](1-e^{-i2\theta_k j\pi})\}||_2^2||\theta||_\infty^2 \\ &\le 16\pi^{\mu+1}||\theta||_\infty^{\mu}|j|^{\mu}||\{c_{N,L}[k]\}||_2^2+16\pi^2||\{c_{N,L}[k]\}||_2^2||\theta||_\infty^2, \end{aligned} \tag{2.27}$$

where the parameter $\mu\in(0,2]$ will be determined optimally. From (2.25), (2.26) and (2.27), we arrive at

$$\begin{aligned} I_2(J) &\le \frac{2^{-N+1}}{\pi}(2J+1)||\widehat{B_m}||_{L^\infty(\mathbb{R})}^2||\{c_{N,L}[k]\}||_2^2||\theta||_\infty^{\mu}J^{\mu}\Big[8\pi^{\mu+1}+9\pi^2||\theta||_\infty^{2-\mu}\Big] \\ &\le \frac{2^{-N+2}}{\pi}(J+1)^{\mu+1}||\{c_{N,L}[k]\}||_2^2||\theta||_\infty^{\mu}\Big[8\pi^{\mu+1}+9\pi^2||\theta||_\infty^{2-\mu}\Big] \\ &\le \frac{2^{-N+3+\mu}}{\pi}J^{\mu+1}||\{c_{N,L}[k]\}||_2^2||\theta||_\infty^{\mu}\Big[8\pi^{\mu+1}+9\pi^2||\theta||_\infty^{2-\mu}\Big]. \end{aligned} \tag{2.28}$$

It is easy to check that when $\mu=2$ and $J=\lfloor||\theta||_\infty^{-\frac{1}{m+1}}\rfloor$, the order of the bound of $I_1(J)+I_2(J)$ is optimal. Specifically,

$$I_1(J)+I_2(J)=\mathrm{O}\Big(||\theta||_\infty^{\frac{2m-1}{m+1}}\Big).$$

Now we select

$$C(N,L,J_0) = ||\{c_{N,L}[k]\}||_2 \Big[2^{-N+2}(2\pi)^{-2m}(2m+1)^{-1} + \frac{2^{-N+5}}{\pi}(8\pi^3+9) \Big]^{1/2}$$

to conclude the proof. □

3. Stable reconstruction of Hilbert transform by adjustable cardinal B-spline sampling system

By the adjustable cardinal B-spline sampling system, we shall establish a numerically and L^2-stable reconstruction method of the HT.

3.1. Stable cardinal B-spline method for Hilbert transform

It is not difficult to prove that

$$HB_1(\,\cdot\,) = \frac{1}{\pi} \ln \Big| \frac{\cdot}{\cdot - 1} \Big|. \tag{3.1}$$

Clearly, 0 and 1 are the two singular points of HB_1. For $m \geq 2$, the value of HB_m at $x \in \mathbb{R}$ can be recursively computed by

$$HB_m(x) = \frac{x}{m-1} HB_{m-1}(x) + \frac{m-x}{m-1} HB_{m-1}(x-1). \tag{3.2}$$

On the numerical singularity of HB_m, the following lemma can be proved by the implementing mathematical induction on (3.2).

Lemma 3.1: *For any $m \in \mathbb{N}$, the singular points of $HB_m(\,\cdot\,)$ are $0, 1, \ldots, m$.*

Now with the multiscale sampling approximation (2.21) at hand, we shall give an approximation of the HT. The approximation error, induced by truncating the scale level, will imply that the approximation of HT is L^2-stable.

Theorem 3.1: *Let $\mathcal{X}_{N,\theta}(B_m)$ and $\mathcal{A}^{N,L}_{B_m,\theta}$ be referred to in Theorem 2.3 where $m \geq 2$. Then, for any function $f \in \boldsymbol{H}^s(\mathbb{R}), s > 1/2$, there holds*

$$||H\mathcal{A}^{N,L}_{B_m,\theta} f - Hf||_2 \leq ||(I - \mathcal{S}^N_{B_m})^L f||_2 + C(N,L,J_0) J_0^{-(2m-1)/2}, \tag{3.3}$$

where J_0 and $C(N,L,J_0)$ are defined in Theorem 2.3.

***Proof*:** Theorem 2.3, together with the isometry property of the HT, namely: $||Hf||_2 = ||f||_2$ for any $f \in L^2(\mathbb{R})$, leads to (3.3). □

The following theorem is on the approximation error induced by truncating the shift of $H\mathcal{A}^{N,L}_{B_m,\theta} f$ in (3.3). It asserts that for any function in $\mathbf{H}^s(\mathbb{R}), s > 1/2$, even if it is not compactly supported, its HT can be approximated by a finite series, provided that the scale level N and the amount of sample are sufficiently large.

Theorem 3.2: *For any $f \in \boldsymbol{H}^s(\mathbb{R}), s > 1/2$, define the coefficient sequence $\{c_{N,L}[k]\}_{k\in\mathbb{Z}}$ by (2.15). Then, for any fixed $\mathcal{K} \in \mathbb{N}$, it holds that*

$$\begin{aligned} &\Big\| \sum_{k=-\mathcal{K}}^{\mathcal{K}} c_{N,L}[k] HB_m(2^N \cdot - k - \theta_k) - Hf \Big\|_2 \\ &\quad = \sqrt{\Big\| \sum_{k=-\mathcal{K}}^{\mathcal{K}} c_{N,L}[k] B_{m,\mathcal{I}_1}(2^N \cdot - k - \theta_k) - f_{\mathcal{I}_1} \Big\|_2^2 + \Big\| f_{\mathcal{I}_2} \Big\|_2^2}, \end{aligned} \tag{3.4}$$

where $g_{\mathcal{I}_i}$ denotes the restriction of g on $\mathcal{I}_i$, $i = 1, 2$,

$$\mathcal{I}_1 := \left[\frac{-\mathcal{K}-1}{2^N}, \frac{m+\mathcal{K}+1}{2^N}\right], \quad and \quad \mathcal{I}_2 := \mathbb{R} \setminus \mathcal{I}_1. \tag{3.5}$$

Moreover,

$$\begin{aligned} &\left\| \sum_{k=-\mathcal{K}}^{\mathcal{K}} c_{N,L}[k] HB_m(2^N \cdot -k - \theta_k) - Hf \right\|_2 \\ &\quad \le \sqrt{(\|(I - \mathcal{S}_{B_m}^N)^L f\|_2 + C(N, L, J_0)\|\theta\|_\infty^{\frac{2m-1}{2m+2}})^2 + \|f_{\mathcal{I}_2}\|_2^2}, \end{aligned} \tag{3.6}$$

where C and J_0 are as in Theorem 2.3.

Proof: By the isometry property of the HT and support$(B_m) = [0, m]$, it is easy to check that

$$\begin{aligned} &\left\| \sum_{k=-\mathcal{K}}^{\mathcal{K}} c_{N,L}[k] HB_m(2^N \cdot -k - \theta_k) - Hf \right\|_2 \\ &\quad = \left\| \sum_{k=-\mathcal{K}}^{\mathcal{K}} c_{N,L}[k] B_m(2^N \cdot -k - \theta_k) - f \right\|_2 \\ &\quad = \sqrt{\left\| \sum_{k=-\mathcal{K}}^{\mathcal{K}} c_{N,L}[k] B_{m,\mathcal{I}_1}(2^N \cdot -k - \theta_k) - f_{\mathcal{I}_1} \right\|_2^2 + \|f_{\mathcal{I}_2}\|_2^2}. \end{aligned}$$

Since $\sum_{k=-\mathcal{K}}^{\mathcal{K}} c_{N,L}[k] B_{m,\mathcal{I}_1}(2^N \cdot -k - \theta_k) - f_{\mathcal{I}_1}$ is the restriction of $(I - \mathcal{S}_{B_m}^N)^L f$ on $\mathcal{I}_1$, we have

$$\left\| \sum_{k=-\mathcal{K}}^{\mathcal{K}} c_{N,L}[k] B_{m,\mathcal{I}_1}(2^N \cdot -k - \theta_k) - f_{\mathcal{I}_1} \right\|_2 \le \|(I - \mathcal{S}_{B_m}^N)^L f\|_2 + C(N, L, J_0)\|\theta\|_\infty^{\frac{2m-1}{2m+2}}.$$

Then, the proof of (3.6) is concluded. □

Remark 3.1: A direct observation on (3.5) leads to that $\mathcal{I}_2$ has nothing to do with L, and therefore $f_{\mathcal{I}_2}$ is independent of L. Therefore, for fixed N and $\mathcal{K}$, the error given in (3.4) actually gives us a strategy for choosing L. Specifically, the smaller

$$\left\| \sum_{k=-\mathcal{K}}^{\mathcal{K}} c_{N,L}[k] B_{m,\mathcal{I}_1}(2^N \cdot -k - \theta_k) - f_{\mathcal{I}_1} \right\|_2 \tag{3.7}$$

is, the better L is. On the other hand, $\|f_{\mathcal{I}_2}\|_2$ in (3.4) tends to 0 when $\mathcal{K} \to \infty$. Therefore, by (3.6), Hf can be well approximated by

$$\sum_{k=-\mathcal{K}}^{\mathcal{K}} c_{N,L}[k] HB_m(2^N \cdot -k - \theta_k) \tag{3.8}$$

when the scale level N is sufficiently large and the sample amount $(2\mathcal{K} + 1) \gg 2^N$. Note that, for any $k \in \mathbb{Z}$, directly utilising Lemma 3.1 gives us that the set of singular points of $HB_m(2^N \cdot -k)$ is $\{\frac{k+j}{2^N} : j = 0, 1, 2, \ldots, m\}$. Therefore, when making use of (3.4) to compute $Hf(x)$, it is necessary to adaptively select a sequence $\theta_x := \{\theta_{k,x}\}$ such that

$$\{2^N x - k - \theta_{k,x} : k = -\mathcal{K}, \ldots, \mathcal{K}\} \cap \{0, 1, 2, \ldots, m\} = \emptyset. \tag{3.9}$$

To avoid destroying approximation accuracy, it follows from Theorem 3.2 that $||\theta_x||_\infty$ should be sufficiently small. As a result of (3.9), the numerical singularity is adaptively removed when using (3.8) to compute HT of f at the point x. On the other hand, the approximation errors induced by scale truncation and shift truncation can be estimated by (3.3) and (3.6). Above all, our method of computing the HT is L^2-stable and numerically stable, and the numerical singularity can be removed.

4. Reconstruction of analytic signal

4.1. Reconstruction method of analytic signal

For any $f \in \mathbf{H}^s(\mathbb{R}), s > 1/2$, its analytic signal $f + iHf$ can be approximated by

$$\sum_{k\in\mathbb{Z}} c_{N,L}[k]\Big(B_m + iHB_m\Big)(2^N \cdot -k - \theta_k), \tag{4.1}$$

where N is sufficiently large, the sequence $\{c_{N,L}[k]\}_{k\in\mathbb{Z}}$ can be computed by (2.15) and $B_m + iHB_m$ is the analytic signal of B_m. By Theorem 2.3 and Theorem 3.1, the approximation error is given by

$$\begin{aligned}||f + iHf - \sum_{k\in\mathbb{Z}} c_{N,L}[k]\Big(B_m + iHB_m\Big)(2^N \cdot -k - \theta_k)||_2 \\ \leq 2||(I - \mathcal{S}_{B_m}^N)^L f||_2 + 2C(N, L, J_0)J_0^{-(2m-1)/2}.\end{aligned} \tag{4.2}$$

For numerical computation, the series in (4.2) needs to be truncated and $(f + iHf)(x)$ is reconstructed by the finite series as follows,

$$(f + iHf)(x) \approx \sum_{k=-\mathcal{K}}^{\mathcal{K}} c_{N,L}[k]\Big(B_m + iHB_m\Big)(2^N x - k - \theta_k), \tag{4.3}$$

where $\mathcal{K} \gg 2^N$, and θ satisfies (3.9). By Theorem 3.2,

$$\begin{aligned}&\left\|f + iHf - \sum_{k=-\mathcal{K}}^{\mathcal{K}} c_{N,L}[k]\Big(B_m + iHB_m\Big)(2^N \cdot -k - \theta_k)\right\|_2 \\ &\leq 2\sqrt{(||(I - \mathcal{S}_{B_m}^N)^L f||_2 + C(N, L, J_0)||\theta||_\infty^{\frac{2m-1}{2m+2}})^2 + ||f_{\mathcal{I}_2}||_2^2},\end{aligned} \tag{4.4}$$

where C, J_0 and $\mathcal{I}_2$ are as in Theorem 3.2.

4.2. Numerical experiments of reconstruction of analytic signal

4.2.1. The first numerical experiment

The Hilbert transform of $f(x) := \frac{1}{1+x^2}$ is $\frac{x}{1+x^2}$. Therefore, the analytic signal of f is

$$\frac{1}{1+x^2} + i\frac{x}{1+x^2}.$$

In this experiment, we use the method in (4.3) to reconstruct the analytic signal on $[-10, 10]$. The efficiency of reconstructing $f + iHf$ is shown in Table 1. The graphs of $f + iHf$ and its approximation $\mathcal{A}_{B_2,\theta}^{5,3}f + iH\mathcal{A}_{B_2,\theta}^{5,3}f$ are shown in Figure 1.

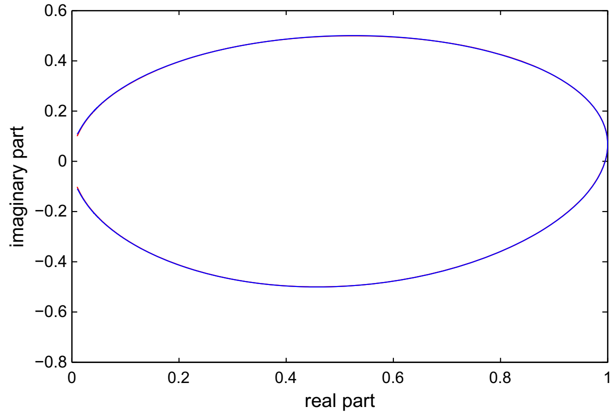

Figure 1. The graph of $f + iHf$ is red while that of $\mathcal{A}^{5,3}_{B_2,\theta} f + iH\mathcal{A}^{5,3}_{B_2,\theta} f$ is blue.

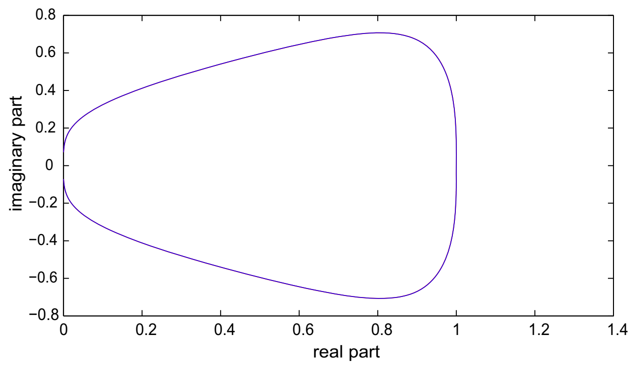

Figure 2. The graph of $f + iHf$ is red while that of $\mathcal{A}^{6,3}_{B_2,\theta} f + iH\mathcal{A}^{6,3}_{B_2,\theta} f$ is blue.

Table 1. Error=$||f + iHf - \sum_{k=-\mathcal{K}}^{\mathcal{K}} c_{N,L}[k]\Big(B_2 + iHB_2\Big)(2^N \cdot -k - \theta_k)||_2/||f + iHf||_2$.

N	L	θ_k	Error	$\mathcal{K}$
4	2	0.000053	0.0047	1000
4	3	–	0.00088653	–
5	2	–	0.0012	–
5	3	–	0.00021091	–

Table 2. Error=$||f + iHf - \sum_{k=-\mathcal{K}}^{\mathcal{K}} c_{N,L}[k]\Big(B_2 + iHB_2\Big)(2^N \cdot -k - \theta_k)||_2/||f + iHf||_2$.

N	L	θ_k	Error	$\mathcal{K}$
5	3	0.00153	0.00029939	1023
5	4	–	0.00021277	–
6	2	–	0.00044703	–
6	3	0.00000153	0.000051793	–
6	3	0.00000053	0.000051802	876

4.2.2. The second numerical experiment

The Hilbert transform of $f(x) := \frac{1}{1+x^4}$ is $\frac{x(1+x^2)}{\sqrt{2}(1+x^4)}$; then, the analytic signal of f is

$$\frac{\sin x}{1+x^4} + i\frac{x(1+x^2)}{\sqrt{2}(1+x^4)}.$$

In this experiment, we use (4.3) to reconstruct the analytic signal. See Table 2 for efficiency. The graphs of $f + iHf$ and its approximation $\mathcal{A}^{5,3}_{B_2,\theta}f + iH\mathcal{A}^{5,3}_{B_2,\theta}f$ are shown in Figure 2.

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Appendix 1. Appendix: proofs of Lemmas 2.1 and 2.2

A.1. Proof of Lemma 2.1

Since $\widetilde{\psi}$ has $\kappa+1$ vanishing moments, there exists a positive constant g_0 such that $|\widehat{\widetilde{b}}(\xi)| \leq g_0|\xi|^{\kappa+1}$ for any $\xi \in \mathbb{R}$. By the similar procedures as [16, Theorem 2.2], we have

$$\begin{aligned}
&\sum_{j=N}^{\infty}\sum_{k\in\mathbb{Z}}|\langle f,\widetilde{\psi}_{j,k}^{-s^*}\rangle|^2 \\
&\leq \frac{1}{\pi}||[\widehat{\widetilde{\phi}},\widehat{\widetilde{\phi}}]_{-t}||_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|\widehat{f}(\xi)|^2(1+|\xi|^2)^{s^*}\sum_{j=N}^{\infty}|\widehat{\widetilde{b}}(2^{-j-1}\xi)|^2 2^{2js^*}(1+|\xi|^2)^{-s^*}(1+2^{-2(j+1)}|\xi|^2)^t d\xi \qquad \text{(A1)}\\
&\leq \frac{g_0^2}{\pi}||[\widehat{\widetilde{\phi}},\widehat{\widetilde{\phi}}]_{-t}||_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|\widehat{f}(\xi)|^2(1+|\xi|^2)^{s^*}\sum_{j=N}^{\infty}|2^{-j-1}\xi|^{2\kappa+2}2^{2js^*}\frac{(1+2^{-2(j+1)}|\xi|^2)^t}{(1+|\xi|^2)^{s^*}}d\xi \\
&= C_1(I_1+I_2),
\end{aligned}$$

where $C_1=\frac{g_0^2}{\pi}||[\widehat{\widetilde{\phi}},\widehat{\widetilde{\phi}}]_{-t}||_{L^\infty(\mathbb{R})}$,

$$I_1=\sum_{j=N}^{\infty}\int_{|\xi|\leq 2^{j\nu}}|\widehat{f}(\xi)|^2(1+|\xi|^2)^{s^*}|2^{-j-1}\xi|^{2\kappa+2}2^{2js^*}\frac{(1+2^{-2(j+1)}|\xi|^2)^t}{(1+|\xi|^2)^{s^*}}d\xi \qquad \text{(A2)}$$

and

$$I_2=\sum_{j=N}^{\infty}\int_{|\xi|>2^{j\nu}}|\widehat{f}(\xi)|^2(1+|\xi|^2)^{s^*}\sum_{j=N}^{\infty}|\widehat{\widetilde{b}}(2^{-j-1}\xi)|^2 2^{2js^*}(1+|\xi|^2)^{-s^*}(1+2^{-2(j+1)}|\xi|^2)^t d\xi \qquad \text{(A3)}$$

with $\nu\in\mathbb{R}^+$ to be optimally determined. I_1 is estimated as follows,

$$\begin{aligned}
I_1 &\leq 2^t\sum_{j=N}^{\infty}2^{-2(j+1)(\kappa+1)}2^{j2(\kappa+1)\nu}2^{2js^*}2^{-2s^*j\nu}\int_{|\xi|\leq 2^{j\nu}}|\widehat{f}(\xi)|^2(1+|\xi|^2)^{s^*}d\xi \\
&\leq 2^{t-2(\kappa+1)}\sum_{j=N}^{\infty}2^{-2j[(\kappa+1)(1-\nu)-s^*]}||f||_{H^{s^*}}^2 \qquad \text{(A4)}\\
&\leq \frac{2^{t-2(\kappa+1)}}{1-2^{-2((\kappa+1)(1-\nu)-s^*)}}2^{-2N((\kappa+1)(1-\nu)-s^*)}||f||_{H^s}^2.
\end{aligned}$$

By [16, Lemma 2.2],

$$\sum_{j=N}^{\infty}|\widehat{\widetilde{b}}(2^{-j-1}\xi)|^2 2^{2js^*}(1+|\xi|^2)^{-s^*}(1+2^{-2(j+1)}|\xi|^2)^t \leq \frac{||\widehat{b}(\xi)||_{L^\infty(\mathbb{R}^d)}^2 2^t}{2^{2(s^*-t)}-1}+\frac{g_0^2}{1-2^{-2(\kappa+1-s^*)}} \qquad \text{(A5)}$$

for any $\xi\in\mathbb{R}$. Denote the upper bound in (A5) by $C_2(\kappa,s^*,t)$. Then,

$$\begin{aligned}
I_2 &\leq C_2(\kappa,s^*,t)\sum_{j=N}^{\infty}\int_{|\xi|>2^{j\nu}}|\widehat{f}(\xi)|^2(1+|\xi|^2)^{s}(1+|\xi|^2)^{s^*-s}d\xi \qquad \text{(A6)}\\
&\leq \frac{C_2(\kappa,s^*,t)}{1-2^{-2\nu(s-s^*)}}2^{-2N\nu(s-s^*)}||f||_{H^s}^2.
\end{aligned}$$

It follows from (A1), (A4) and (A6) that

$$\begin{aligned}
\sum_{j=N}^{\infty}\sum_{k\in\mathbb{Z}}|\langle f,\widetilde{\psi}_{j,k}^{-s^*}\rangle|^2 \leq C_1\Big[&\frac{2^{t-2(\kappa+1)}}{1-2^{-2((\kappa+1)(1-\nu)-s^*)}}2^{-2N((\kappa+1)(1-\nu)-s^*)}+ \\
&\frac{C_2(\kappa,s^*,t)}{1-2^{-2\nu(s-s^*)}}2^{-2N\nu(s-s^*)}\Big]||f||_{H^s}^2. \qquad \text{(A7)}
\end{aligned}$$

When selecting $\nu = (\kappa + 1 - s^*)/(\kappa + 1 + s - s^*)$, the approximation order reaches the optimal value. Specifically,

$$\sum_{j=N}^{\infty}\sum_{k\in\mathbb{Z}} |\langle f, \widetilde{\psi}_{j,k}^{-s^*}\rangle|^2 \leq C(\kappa, s, s^*)2^{-2N(s-s^*)(\kappa+1-s^*)/(\kappa+1+s-s^*)}||f||_{H^s}^2, \tag{A8}$$

where

$$C(\kappa, s, s^*) := C_1\left[\frac{2^{t-2(\kappa+1)}}{1-2^{-2(s-s^*)(\kappa+1-s^*)/(\kappa+1+s-s^*)}} + \frac{C_2}{1-2^{-2(\kappa+1-s^*)(s-s^*)/(\kappa+1+s-s^*)}}\right]. \tag{A9}$$

A.2. Proof of Lemma 2.2

By the mixed extension principle,[14,16] we can construct a pair of dual 2-framelets $X^{s^*}(\phi; \psi^1, \psi^2)$ and $X^{-s^*}(\widetilde{\phi}; \widetilde{\psi}^1, \widetilde{\psi}^2)$ in $(\mathbf{H}^{s^*}(\mathbb{R}), \mathbf{H}^{-s^*}(\mathbb{R}))$ such that both $\widetilde{\psi}^1$ and $\widetilde{\psi}^2$ have $\kappa + 1$ vanishing moments. Let ℓ^2 be the space of square summable sequences. For $X^{s^*}(\phi; \psi^1, \psi^2)$, we define a pre-frame operator $\mathbb{P} : \mathbf{H}^{s^*}(\mathbb{R}) \longrightarrow \ell^2$, of which the elements of $\mathbb{P}f$ are

$$\langle f, \phi_{0,k}\rangle_{\mathbf{H}^{s^*}(\mathbb{R})}, \langle f, \psi_{j,k}^{\ell,s}\rangle_{\mathbf{H}^{s^*}(\mathbb{R})}, j \in \mathbb{N}_0, k \in \mathbb{Z}, \ell = 1, 2,$$

where f is any function in $\mathbf{H}^{s^*}(\mathbb{R})$. By (1.7), it is evident that $\mathbb{P}f \in \ell^2$. Moreover, operator norm $||\mathbb{P}||$ can be estimated by [14, Theorem 2.2]. Specifically,

$$||\mathbb{P}|| < \infty, \tag{A10}$$

namely: $\mathbb{P}$ is bounded. By [21, p.57–58] , $\mathbb{P}^* : \ell^2 \longrightarrow \mathbf{H}^{s^*}(\mathbb{R})$, the adjoint operator of $\mathbb{P}$ is defined by

$$\mathbb{P}^*(c) = \sum_{k\in\mathbb{Z}} c_k\phi_{0,k} + \sum_{\ell=1}^{2}\sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{Z}} c_{j,k}^{\ell,-s}\psi_{j,k}^{\ell,s},$$

where $c \in \ell^2$ and its elements are $c_{n;0,k}$ and $c_{j,k}^{\ell,-s}$. Recalling $||\mathbb{P}^*|| = ||\mathbb{P}||$ gives us

$$||\mathbb{P}^*(c)||_{H^{s^*}(\mathbb{R})} \leq ||\mathbb{P}||||c||_{\ell_2}.$$

Now for $X^{-s}(\widetilde{\phi}; \widetilde{\psi}^1, \widetilde{\psi}^2)$ being a Bessel 2-wavelet sequence in $H^{-s}(\mathbb{R})$, it follows from (2.1) that

$$\begin{aligned}\left\|\sum_{\ell=1}^{2}\sum_{j=N}^{\infty}\sum_{k\in\mathbb{Z}}\langle f, \widetilde{\psi}_{j,k}^{\ell,-s}\rangle\psi_{j,k}^{\ell,s}\right\|_{H^s} &\leq ||\mathbb{P}||\left(\sum_{\ell=1}^{2}\sum_{j=N}^{\infty}\sum_{k\in\mathbb{Z}}|\langle f, \widetilde{\psi}_{j,k}^{\ell,-s}\rangle|^2\right)^{1/2}\\ &\leq ||\mathbb{P}||\sqrt{G(s^*, t)}2^{-\eta_{\kappa+1}(s^*,s)N},\end{aligned}$$

where

$$G(s^*, t) = \frac{1}{\pi}||[\widehat{\widetilde{\phi}}, \widehat{\widetilde{\phi}}]_{-t}||_\infty \sum_{\ell=1}^{2}\left\|\frac{\widehat{b_\ell}(\xi)}{\xi}\right\|_\infty\left[2^{t+1} + 2^{2t}\left(\frac{||\widehat{b_\ell}||_\infty}{2^{2(s^*-t)}-1} + \frac{||\frac{\widehat{b_\ell}(\xi)}{\xi^{\kappa+1}}||_\infty}{1-2^{-2(\kappa+1-s^*)}}\right)\right]. \tag{A11}$$

Now we select

$$g(s, s^*, t) = ||\mathbb{P}||\sqrt{G(s^*, t)} \tag{A12}$$

to conclude this proof.