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Fourier Spectrum Characterizations of H^p Spaces on Tubes Over Cones for $1 \leq p \leq \infty$

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Abstract We give Fourier spectrum characterizations of functions in the Hardy H^p spaces on tubes for $1 \leq p \leq \infty$. For $F \in L^p(\mathbb{R}^n)$, we show that F is the non-tangential boundary limit of a function in a Hardy space, $H^p(T_\Gamma)$, where Γ is an open cone of $\mathbb{R}^n$ and T_Γ is the related tube in $\mathbb{C}^n$, if and only if the classical or the distributional Fourier transform of F is supported in Γ^*, where Γ^* is the dual cone of Γ. This generalizes the results of Stein and Weiss for $p = 2$ in the same context, as well as those of Qian et al. in one complex variable for $1 \leq p \leq \infty$. Furthermore, we extend the Poisson and Cauchy integral representation formulas to the H^p spaces on tubes for $p \in [1, \infty]$ and $p \in [1, \infty)$, with, respectively, the two types of representations.

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1 Introduction

The Paley–Wiener Theorem in relation to the Hardy $H^2(\mathbb{C}^+)$ asserts that for $f \in L^2(\mathbb{R})$, f is the non-tangential boundary limit of some function in the Hardy space $H^2(\mathbb{C}^+)$ if and only if $\operatorname{supp}\hat{f} \subset [0, \infty)$ (see, for instance, [24]). We recall it in the following.

Theorem A (Paley–Wiener) [13], ([24]) *$F \in H^2(\mathbb{C}^+)$ if and only if there exists a function $f \in L^2(0, \infty)$, such that*

$$F(z) = \int_0^\infty f(t)e^{2\pi itz}dt,$$

for $z \in \mathbb{C}^+$, and, furthermore,

$$\int_0^\infty |f(t)|^2 dt = \|F\|^2_{H^2}.$$

Related generalizations were subsequently obtained in [17], [9], [20] and [21]. The mentioned literature proves the following Fourier spectrum characterization result: For $f \in L^p(\mathbb{R}),\ 1 \le p \le \infty,\ f$ is the non-tangential boundary limit of a function in the Hardy $H^p(\mathbb{C}^+)$ if and only if, in the pointwise or distributional sense, $\operatorname{supp}\hat{f} \subset [0, \infty)$. Especially, Qian et al. obtained the following Theorems B and C.

Theorem B ([20]) *Let $f \in H^p(\mathbb{C}^+),\ 1 \le p \le \infty$. Then, as a tempered distribution, $\hat{f}$ is supported in $[0, \infty)$, denoted as d-supp $\hat{f} \subset [0, \infty)$. That amounts to*

$$(\hat{f}, \varphi) = 0,$$

for all $\varphi \in S(\mathbb{R})$ with supp $\hat{\varphi} \subset (-\infty, 0]$. Moreover, in the range $1 \le p \le 2$ there holds $\hat{f}(x) = 0$ for almost all $x \in (-\infty, 0]$.

In 2009, Qian et al. proved the converse result of Theorem B (see [21]) as follows.

Theorem C ([21]) *For $1 \le p \le \infty$, if $f \in L^p(\mathbb{R})$ and d-supp$\hat{f} \subset [0, \infty)$, then f is the boundary limit of a function in $H^p(\mathbb{C}_+)$.*

It is certainly a natural question to ask whether in higher dimensional spaces, for instance, involving several complex variables, analogous results hold. To the authors' knowledge, in the Euclidean space $\mathbb{R}^n$ context, there have been no general Fourier spectrum characterization results for all range of the index p. In [26] of Stein and Weiss, it is shown that the characterization result is valid for the Hardy spaces $H^2(T_\Gamma)$ on tubes over open cones Γ : They show that a square-integrable function is the non-tangential boundary limit of a function in $H^2(T_\Gamma)$ if and only if its Fourier transform vanishes outside Γ^*, the latter being the dual cone of Γ. The following related result of Stein and Weiss [26] is restricted to $p = 2$.

Theorem D ([26]) *Let B be an open connected subset of $\mathbb{R}^n$. F belongs to $H^2(T_B)$ if and only if it has the form*

$$F(z) = \int_{\mathbb{R}^n} e^{2\pi i z\cdot t} f(t)\, dt, \quad z \in T_\Gamma,$$

where f is a function satisfying

$$\sup_{y\in B} \int_{\mathbb{R}^n} e^{-2\pi y\cdot t} |f(t)|^2\, dt \le A^2 < \infty.$$

When the base B is an open cone one obtains the following representation theorem of H^2−functions sharper than that is obtained in the above Theorem D.

Theorem E ([26]) *Let Γ be a regular open cone in $\mathbb{R}^n$. $F \in H^2(T_\Gamma)$ if and only if*

$$F(z) = \int_{\Gamma^*} e^{2\pi i z\cdot t} f(t)\, dt, \quad z \in T_\Gamma,$$

where $f(t)\mathcal{X}_{\Gamma^*}(t)$ is the Fourier transform of

$$F(\xi) = \lim_{\eta\to 0,\ \eta\in\Gamma} F(\xi + i\eta),$$

$\mathcal{X}_{\Gamma^*}(t)$ is the characteristic function of Γ, that is, $\mathcal{X}_{\Gamma^*}(t) = 1$, for $t \in \Gamma^*$; and otherwise $\mathcal{X}_{\Gamma^*}(t) = 0$. Moreover,

$$\|F\|_{H^2} = \|F\|_2 = \left(\int_{\Gamma^*} |f(t)|^2\, dt\right)^{1/2}.$$

Moreover, for all range p, Stein and Weiss [26] obtained the fundamental result as given in the following Theorem F which is an important result of the theory of Hardy spaces on tubes.

Theorem F ([26]) *Let Γ be a regular open cone in $\mathbb{R}^n$. Suppose that $F \in H^p(T_\Gamma)$. Then* ([26], Section 5, Chapter 3)*,*

(a) *for $1 \le p \le \infty$,*

$$\lim_{y\in\Gamma,\ y\to 0} F(x+iy) = F(x)$$

holds for almost all $x \in \mathbb{R}^n$. $F(x)$ is called the restricted (non-tangential) boundary limit *of $F(z)$;*

(b) *for $1 \le p < \infty$,*

$$\lim_{y\in\Gamma,\ y\to 0} \|F(x+iy) - F(x)\|_p = 0;$$

(c) *for* $1 \leq p \leq \infty$, ([26], page 120)

$$F(z) = \int_{\mathbb{R}^n} P(x-t, y)F(t)dt.$$

In [15] *the author proved the one-way result that for* $1 \leq p < \infty$ *and* $f \in H^p(T_\Omega)$, *where* Ω *is an irreducible symmetric cone, there holds* supp $\hat{f} \subset \overline{\Omega}$. *In* [18] *Hörmander proved some results corresponding to the Paley–Wiener Theorem for bandlimited functions involving entire functions in several complex variables. The complete collection of Fourier characterization results for Hardy space functions like what have been established in one-dimension is missing.*

The aim of this paper is to prove the analogous Fourier spectrum characterizations (*two-way results*)*, as well as the related Poisson and Cauchy integral representation formulas for Hardy* $H^p(T_\Gamma)$ *spaces on tubes over a wider class of bases* Γ *and for all the indices* p *in* $[1, \infty]$. *For the classical one dimensional case, the following results related integral representation formulas of Hardy spaces on the half plane are well known.*

Theorem G ([7], [13]) *Let* $F(x) \in L^p(\mathbb{R}),\ 1 < p < \infty$, *and*

$$C(F)(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(t)}{t-z}\, dt, \quad \operatorname{Im} z > 0.$$

Then $C(F)(z) \in H^p(\mathbb{C}^+)$, *and there exists a constant* $C_p < \infty$ *such that*

$$\int_{-\infty}^{\infty} |C(F)(x+iy)|^p\, dx \leq C_p \int_{-\infty}^{\infty} |F(x)|^p\, dx.$$

Theorem H ([7], [13]) *If* $F(z) \in H^p(\mathbb{C}^+),\ 1 \leq p < \infty$, *then* $F(z)$ *is the Cauchy integral of its boundary limit* $F(x)$, *that is,*

$$F(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(t)}{t-z}\, dt, \quad \operatorname{Im} z > 0.$$

In order to better understand the present paper, we introduce notation and terminology as follows. Let B *be an open subset of* $\mathbb{R}^n$. *Then the tube* T_B *in* $\mathbb{R}^n \times i\mathbb{R}^n = \mathbb{C}^n$ *with base* B *is the subset*

$$T_B = \left\{z = x + iy \in \mathbb{C}^n :\ x \in \mathbb{R}^n,\ y \in B\right\}.$$

For example, for $n = 1$, *the classical upper-half plane* $\mathbb{C}^+$ *and the lower-half plane* $\mathbb{C}^-$ *are the tubes in* $\mathbb{R} \times i\mathbb{R} = \mathbb{C}$ *with the bases* $B_+ = \{y \in \mathbb{R};\ y > 0\}$ *and* $B_- = \{y \in \mathbb{R};\ y < 0\}$, *respectively, i.e.,* $\mathbb{C}^+ = \{z = x + iy :\ x \in \mathbb{R},\ y > 0\}$ *and* $\mathbb{C}^- = \{z = x + iy :\ x \in \mathbb{R},\ y < 0\}$. *Obviously, tubes* T_B *are generalizations of* $\mathbb{C}^+$ *and* $\mathbb{C}^-$.

A function $F(z)$ holomorphic in the tube T_B is said to belong to the space $H^p(T_B),\ 0 < p < \infty$, if

$$\|F\|_{H^p} = \sup\left\{\left(\int_{\mathbb{R}^n} |F(x+iy)|^p dx\right)^{\frac{1}{p}} : y \in B\right\} < \infty.$$

For $1 \le p < \infty$, the number $\|F\|_{H^p}$ is defined to be the H^p-norm of the function $F(z)$. Under this norm $H^p(T_B),\ 1 \le p < \infty$, is a Banach space. For $0 < p < 1$, under the distance function $d(F,G) = \|F-G\|^p_{H^p}$ the space $H^p(T_B)$ becomes a complete metric space.

When $p = \infty$, the Hardy space $H^\infty(T_B)$, as a Banach space, is defined by

$$H^\infty(T_B) = \{F : F \text{ holomorphic on } T_B \text{ and } \|F\|_{H^\infty} = \sup_{z \in T_B} |F(z)| < \infty\}.$$

For any open and connected subset B of $\mathbb{R}^n$, Stein and Weiss ([12,25,26,28]) *studied the space $H^2(T_B)$ and obtained a fundamental representation theorem* ([26]) *restricted to only $H^2(T_B)$. They commented that the theory of $H^2(T_B)$ becomes richer if the bases B are with more restrictions. In this paper, base on the existing results, we will continue to study the theory of the spaces $H^p(T_\Gamma),\ 1 \le p \le \infty$, on tubes over open cones Γ. What we call* open cones *are those satisfying the following two conditions:*

(1) 0 *does not belong to* Γ*;*
(2) *For any $x,\ y \in \Gamma$, and any $\alpha,\ \beta > 0$, there holds $\alpha x + \beta y \in \Gamma$.*

We note that a cone Γ is a convex set. In this paper, we will mainly study the $H^p(T_\Gamma)$ spaces on tubes over open cones.

The dual cone of Γ*, denoted by Γ^*, is defined as*

$$\Gamma^* = \{y \in \mathbb{R}^n : y \cdot x \ge 0,\ \text{for any } x \in \Gamma\}.$$

A cone Γ is said to be regular *if the interior of its dual cone Γ^* is nonempty. For instance, when $n = 1$, there are only two open cones, viz., the open half-lines $\{x \in \mathbb{R} :\ x > 0\}$ and $\{x \in \mathbb{R} :\ x < 0\}$. Their dual cones are the closed half-lines $\{x \in \mathbb{R} :\ x \ge 0\}$ and $\{x \in \mathbb{R} :\ x \ge 0\}$, both having non-empty interiors. When $n = 2$, the open cones are the angular regions between two rays meeting at the origin. Such a cone is regular if and only if the corresponding angle is strictly less than π. The first octant of $\mathbb{R}^n$ is a particular open convex regular cone whose dual cone is the closure of itself. There are two particular types of regular cones, that is,* polygonal cones *and* circular cones. *They, too, receive great attentions from researchers* ([3–5]). *Part of the results on symmetric cones can be found in the book of* [10] *and the lecture notes* [1].

An open subset Γ of $\mathbb{R}^n$ is called a *polygonal cone* if it is the interior of the convex hull of a finite number of rays meeting at the origin among which one can find at least n sides that are linearly independent. We call a polygonal cone an *n-sided polygonal cone* if the minimum number of linearly independent rays convex-spanning Γ is exactly n.

The set

$$\Gamma_c = \left\{(x, x_n) \in \mathbb{R}^n : |x| \leq cx_n, \ x \in \mathbb{R}^{n-1}, \ c > 0\right\},$$

is called a *canonical circular cone*. If through a rotation Γ can become identical with a canonical circular cone, then Γ is called a *circular cone*.

There are two important kernels related to the Hardy spaces $H^p(T_\Gamma)$, viz., the Cauchy kernels and the Poisson kernels ([2], [5], [26], and [14]). Below we recall the definitions and main properties of the Cauchy and the Poisson kernels.

If Γ is a regular open convex cone of $\mathbb{R}^n$, then the Cauchy kernel and the Poisson kernel of T_Γ are, respectively, the functions defined by

$$K(z) = \int_{\Gamma^*} e^{2\pi i z\cdot t}\, dt \quad \text{and} \quad P(x, y) = |K(z)|^2/K(2iy), \tag{1}$$

where $z = x + iy \in T_\Gamma$ ([26]).

As mentioned previously, if Γ is the first octant of $\mathbb{R}^n$, then its dual cone is the closure of Γ, namely, $\Gamma^* = \overline{\Gamma}$. The Cauchy kernel ([26]) associated with the first octant of $\mathbb{R}^n$ can be computed explicitly, being identical with the tensor product of the one-dimensional Cauchy kernel as follows,

$$K(z) = \prod_{j=1}^{n} \frac{-1}{2\pi i z_j}, \quad z \in T_\Gamma, \tag{2}$$

and the Poisson kernel of the tube T_Γ is the tensor product of n copies of the one-dimensional Poisson kernel in the upper-half plane:

$$P(x, y) = |K(z)|^2/K(2iy) = \prod_{j=1}^{n} \frac{1}{\pi} \frac{y_j}{x_j^2 + y_j^2}, \quad z \in T_\Gamma. \tag{3}$$

Let $F \in (S(\mathbb{R}^n))'$ be a tempered distribution. If there exists a function $f(x + iy)$ analytic in T_Γ such that for any φ in the Schwarz class $S(\mathbb{R}^n)$,

$$F(\varphi) = (f, \varphi) = \lim_{y\in\Gamma, \ y\to 0} \int_{\mathbb{R}^n} f(x + iy)\varphi(x)dx,$$

then we say that F is a *holomorphic distribution* and $f(x + iy)$ is an *analytic representation* of F. We note that here and below the self-explanatory notation $\lim\limits_{y\in\Gamma, \ y\to 0}$ means the restricted limit along Γ.

Let $f \in L^1(\mathbb{R}^n)$. The Fourier transform of f, denoted $\hat{f}$, is defined by

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(t)e^{-2\pi i x\cdot t} dt$$

for all $x \in \mathbb{R}^n$.

Let T be any tempered distribution. If $(T, \varphi) = 0$, for any $\varphi \in S(\mathbb{R}^n)$ with $\text{supp}\,\varphi \subset C$, where C is a closed subset of $\mathbb{R}^n$, then we say that *the distributional support of* T, denoted as d-supp T, *is contained in* $\overline{\mathbb{R}^n \backslash C}$, or, d-supp $T \subset \overline{\mathbb{R}^n \backslash C}$.

In what following, the second section of this paper states our main results. The third section contains main technical Lemmas. The last section provides proofs of the main theorems.

2 Main Results

We note that the following results have been announced, without proofs, in our recent survey paper [8]. Theorems 2.1–2.4 given below are higher-dimensional generalizations of the Paley–Wiener Theorem to $p \neq 2$.

Theorem 2.1 *Let* Γ *be a regular open cone in* $\mathbb{R}^n$ *and* $F(x) \in L^p(\mathbb{R}^n),\ 1 \leq p \leq 2$. *Then* $F(x)$ *is the boundary limit function of some* $F(x+iy) \in H^p(T_\Gamma)$ *if and only if* $\text{supp}\hat{F} \subset \Gamma^*$. *Moreover, if the condition is satisfied, then*

$$F(z) = \int_{\mathbb{R}^n} \mathcal{X}_{\Gamma^*}(t) e^{2\pi i z\cdot t} \hat{F}(t)\, dt = \int_{\mathbb{R}^n} F(t) K(z-t)\, dt = \int_{\mathbb{R}^n} P(x-t, y) F(t)\, dt, \tag{4}$$

where $K(z)$ *and* $P(x, y)$ *are, respectively, the Cauchy kernel and the Poisson kernel for the tube* T_Γ.

Theorem 2.2 *Let* Γ *be a regular open cone in* $\mathbb{R}^n$, *and* $F(z) \in H^p(T_\Gamma),\ 2 < p \leq \infty$. *Then* $F(x)$, *as the boundary limit of* $F(z)$, *is the tempered holomorphic distribution represented by the function* $F(z)$. *Moreover, d*-$\text{supp}\hat{F} \subset \Gamma^*$, *that is* $(\hat{F}, \varphi) = 0$ *for all* $\varphi \in S(\mathbb{R}^n)$ *with* $\text{supp}\varphi \subset \overline{\mathbb{R}^n \backslash \Gamma^*}$.

In order to prove the main Theorem 2.4, we have to prove first Theorem 2.3, which itself is an important result in the Hardy space theory.

Theorem 2.3 *Let* Γ *be a regular open cone in* $\mathbb{R}^n$. *Suppose that* $F(z) \in H^p(T_\Gamma),\ 1 \leq p \leq \infty$, *and* $F(x)$ *is the boundary limit function of the function* $F(z)$. *If* $F(x) \in L^q(\mathbb{R}^n),\ 1 \leq q \leq \infty$, *then* $F(z) \in H^q(T_\Gamma)$.

The following converse result of Theorem 2.2 holds.

Theorem 2.4 *Let* Γ *be a regular open cone in* $\mathbb{R}^n$. *If* $F \in L^p(\mathbb{R}^n),\ 2 < p \leq \infty$, *and* $(\hat{F}, \varphi) = 0$, *for all* $\varphi \in S(\mathbb{R}^n)$ *with* $\text{supp}\varphi \subset \overline{\mathbb{R}^n \backslash \Gamma^*}$, *then* F *is the boundary value of a function in* $H^p(T_\Gamma)$.

Theorems 2.2 and 2.4 show that we also have the spectrum characterization for functions in the Hardy spaces $H^p(T_\Gamma),\ 2 < p \leq \infty$.

We note that the proofs of Theorem 2.4 and the L^1-results of Theorem 2.1 use some methodologies different from those for the analogous results in the one-dimension case given in [21]. In the latter one uses the Calderón-Zygmund decomposition for L^1-functions, while in the present paper we take advantage of the Poisson integral representation together with the "lifting up" method for the Hardy space functions (see [26] and [7]).

Up to this point, we have considered $H^p(T_\Gamma)$ spaces on tubes over open cones. In what follows, we will restrict ourselves to consider tubes over the first octant of $\mathbb{R}^n$, and more generally, tubes over the polygonal cones, and obtain more delicate results. In particular, for the first octant and the polygonal cones the counterparts of Theorems G and H are obtained for higher dimensions.

Theorem 2.5 *Let $K(z)$ be the Cauchy kernel associated with the tube T_Γ, where Γ is the first octant of $\mathbb{R}^n$, $F(x) \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, and*

$$C(F)(z) = \int_{\mathbb{R}^n} F(t)K(z-t)\,dt.$$

Then $C(F)(z) \in H^p(T_\Gamma)$, and there exists a finite constant C_p, depending only on p and the dimension n such that

$$\int_{\mathbb{R}^n} |C(F)(x+iy)|^p\,dx \le C_p \int_{\mathbb{R}^n} |F(x)|^p\,dx.$$

Theorem 2.5 has a counterpart for tubes over the polygonal cones, as given in the following Theorem 2.5*.

Theorem 2.5* *Let $K(z)$ be the Cauchy kernel for the tube domain T_Γ, where Γ is a polygonal cone in $\mathbb{R}^n$, $F(x) \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, and*

$$C(F)(z) = \int_{\mathbb{R}^n} F(t)K(z-t)\,dt.$$

Then $C(F)(z) \in H^p(T_\Gamma)$, and there exists a finite number N depending on Γ, and a constant C_p, the same as in Theorem 2.5, such that

$$\int_{\mathbb{R}^n} |C(F)(x+iy)|^p\,dx \le NC_p \int_{\mathbb{R}^n} |F(x)|^p\,dx.$$

Theorem 2.5 does not hold for general cones except for $p = 2$. We recall the famous result of Fefferman ([11]) asserting that the characteristic function of the unit ball in $\mathbb{R}^n$ is not a bounded Fourier multiplier for $L^p(\mathbb{R}^n)$ when $p \neq 2$. This result, combined with the loose theorem of Stein ([27]) and one by de Leeuw ([6], [19]), implies that the counterpart result of Theorem 2.5 in the circular-cone case do not hold for $p \neq 2$.

Next, we have the following result.

Theorem 2.6 *Let Γ be the fist octant of $\mathbb{R}^n$. If $F(z) \in H^p(T_\Gamma)$, $1 \le p < \infty$, then $F(z)$ is the Cauchy integral of its boundary limit $F(x)$, that is,*

$$F(z) = \int_{\mathbb{R}^n} F(t)K(z-t)\,dt, \quad z \in T_\Gamma, \tag{5}$$

where $K(z)$ is the Cauchy kernel of the tube T_Γ.

Like Theorem 2.5*, Theorem 2.6 may be extended to the polygonal case:

Theorem 2.6* *Let* Γ *be a polygonal cone in* $\mathbb{R}^n$. If $F(z) \in H^p(T_\Gamma),\ 1 \le p < \infty$, *then* $F(z)$ *is the Cauchy integral of its boundary limit* $F(x)$, *that is,*

$$F(z) = \int_{\mathbb{R}^n} F(t)K(z-t)\,dt, \quad z \in T_\Gamma, \tag{6}$$

where $K(z)$ *is the Cauchy kernel of the tube* T_Γ.

We note that, when $1 \le p \le 2$, as concluded in Theorem 2.1, the Cauchy integral representation formulas of the Hardy space functions given in Theorem 2.6, in fact, are valid for more general cones other than the polygonal ones, since they, in fact, are valid for at least all regular open cones. In the case $2 < p < \infty$, however, we do not know whether the same representation formula holds for a general regular open cone. We are aware of some Cauchy integral representation results for holomorphic functions in several complex variables ([16,22]). For the Hardy space functions our results for the polygonal cones and the range $2 < p < \infty$ are new.

The results of this study are summarized as follows.

Theorem 2.7 *Let* Γ *be a regular open cone in* $\mathbb{R}^n$, *and* $F(x) \in L^p(\mathbb{R}^n),\ 1 \le p \le \infty$. *Then* $F(x)$ *is the restricted boundary limit function of some* $F(x+iy) \in H^p(T_\Gamma)$ *if and only if* d-supp $\widehat{F} \subset \Gamma^*$. *In such case, we have*

(1) *for* $1 \le p \le 2$, $\hat{F}$ *is locally integrable, and*

$$F(z) = \int_{\mathbb{R}^n} \mathcal{X}_{\Gamma^*}(t)e^{2\pi i z\cdot t}\hat{F}(t)\,dt = \int_{\mathbb{R}^n} F(t)K(z-t)\,dt;$$

(2) *for* $1 \le p \le \infty$,

$$F(z) = \int_{\mathbb{R}^n} P(x-t, y)F(t)\,dt.$$

Moreover, if Γ *is the first octant of* $\mathbb{R}^n$, *or a polygonal cone, we have*

(3) *for* $2 < p < \infty$,

$$F(z) = \int_{\mathbb{R}^n} F(t)K(z-t)\,dt,$$

where $K(z)$ *and* $P(x, y)$ *are, respectively, the Cauchy and the Poisson kernels associated with the tube* T_Γ.

Theorem 2.8 *Let* $K(z)$ *be the Cauchy kernel associated with the tube* T_Γ, *where* Γ *is a polygonal cone in* $\mathbb{R}^n$. *Then* $F(z) \in H^p(T_\Gamma),\ 1 < p < \infty$, *if and only of there exists* $G(x) \in L^p(\mathbb{R}^n)$ *such that*

$$F(z) = C(G)(z) = \int_{\mathbb{R}^n} G(t)K(z-t)\,dt,$$

and there exists a finite constant C_p, depending only on p and the dimension n such that

$$\int_{\mathbb{R}^n} |F(x+iy)|^p \, dx \le C_p \int_{\mathbb{R}^n} |G(x)|^p \, dx.$$

In the "only if" part, the range of p can be extended to $1 \le p < \infty$, and $G(x)$ can be taken as the restricted boundary limit $F(x)$ of the Hardy space function $F(z)$.

3 Main Technical Lemmas

In order to prove our theorems, we need the following lemmas.

Lemma 3.1 ([1], p. 103) *Let Γ be an open convex cone. Suppose $F \in H^p(T_\Gamma)$, $0 < p \le \infty$, and $\Gamma_0 \subset \Gamma$ satisfies $d(\Gamma_0, \Gamma^c) = \inf\{|y_1 - y_2|;\ y_1 \in \Gamma_0,\ y_2 \in \Gamma^c\} \ge \varepsilon > 0$, then there exists a constant $C = C(\varepsilon)$, depending on (ε, n) but not on F, such that*

$$\sup_{z \in T_{\Gamma_0}} |F(z)| \le C \|F\|_{H^p}.$$

Lemma 3.2 ([1], p. 105) *If $1 \le q \le \infty$, then $P_y(x) = P(x, y) \in L^q(\mathbb{R}^n)$ for each $y \in \Gamma$, where Γ is an open convex cone and $P(x, y)$ is the Poisson kernel associated with the tube T_Γ.*

Lemma 3.3 *If $f \in L^p(\mathbb{R}^n)$, $\varphi \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, $1 \le p \le 2$, then*

$$\int_{\mathbb{R}^n} f(x)\hat{\varphi}(x) \, dx = \int_{\mathbb{R}^n} \hat{f}(x)\varphi(x) \, dx.$$

As generalization of a corresponding result in [26], we have

Lemma 3.4 *Let $F(z) \in H^p(T_{\Gamma_n})$, $1 \le p \le \infty$, where Γ_n denotes the first octant of $\mathbb{R}^n$. Then, for any $n = n_1 + n_2$, $1 \le n_1, n_2 < n$, and fixed $(z_{n_1+1}, \ldots, z_n) \in T_{\Gamma_{n_2}}$, we have*

$$F(\ldots, z_{n_1+1}, \ldots, z_n) \in H^p(T_{\Gamma_{n_1}}).$$

Lemma 3.5 ([5]) *Let Γ be a polygonal cone, $K(z)$ the Cauchy kernel associated with T_Γ. Then $K_y(x) = K(x+iy) \in L^p(\mathbb{R}^n)$, $p > 1$, for any $y \in \Gamma$, and there holds:*

$$\|K_y\|_p \le C(n, p, \Gamma) K(iy)^{1-1/p}.$$

Lemma 3.6 (1) *If Γ is an $n-$sided polygonal cone in $\mathbb{R}^n$, then its dual cone Γ^* is an $n-$sided polygonal cone in $\mathbb{R}^n$.*

(2) *If* Γ *is a polygonal cone in* $\mathbb{R}^n$, *then* $\overline{\Gamma} = \bigcup_{j=1}^{N} \overline{\Gamma}_j,\ \Gamma_j \cap \Gamma_k = \varnothing,\ j \neq k$, *where each* Γ_j *is an* n*-sided polygonal cone,* $j = 1, 2, \ldots, N$. *The dual cone* Γ^* *is also a polygonal cone in* $\mathbb{R}^n$. *Moreover,* $\Gamma^* = \bigcap_{j=1}^{N} \Gamma_j^*$, *where* Γ_j^* *is the dual cone of* $\Gamma_j,\ j = 1, 2, \ldots, N$.

Proofs of the lemmas are easy exercises ([10]).

4 Proofs of Theorems

4.1 Proof of Theorem 2.1

Proof To prove the "only if" part of Theorem 2.1, we need to show that, if $F(x+iy) \in H^p(T_\Gamma)$, then d-supp$\hat{F} \subset \Gamma^*$.

Let $1 \leq p \leq 2$. For any fixed $y_0 \in \Gamma$,

$$F_{y_0}(z) = F(z + iy_0),\ \Gamma_0 = \{y + y_0 :\ y \in \Gamma\} \subset \Gamma.$$

Obviously, $\Gamma_0 \subset \Gamma$, and, there exists $\epsilon > 0$ such that

$$d(\Gamma_0,\ \Gamma^c) = \inf\{|y_1 - y_2|;\ y_1 \in \Gamma_0,\ y_2 \in \Gamma^c\} \geq \varepsilon > 0.$$

Then by Lemma 3.1, there exists a positive constant $C = C(y_0,\ n)$ such that

$$\sup_{z \in T_\Gamma} |F_{y_0}(z)| \leq C\|F\|_{H^p} < \infty. \tag{7}$$

From the last inequality, we have, for any $y \in \Gamma$,

$$\begin{aligned}\int_{\mathbb{R}^n} |F_{y_0}(x+iy)|^2\, dx &= \int_{\mathbb{R}^n} |F_{y_0}(x+iy)|^p |F_{y_0}(x+iy)|^{2-p}\, dx \\ &\leq \int_{\mathbb{R}^n} |F_{y_0}(x+iy)|^p (C\|F\|_{H^p})^{2-p}\, dx \\ &\leq (C\|F\|_{H^p})^{2-p} \|F_{y_0}\|_p^p < \infty,\end{aligned}$$

so that,

$$F_{y_0}(z) \in H^2(T_\Gamma).$$

The Paley–Wiener Theorem for the H^2-functions (Theorem E) implies that

$$\text{supp}\hat{F}_{y_0} \subset \Gamma^*. \tag{8}$$

When $p = 1$, the relation for L^1 functions holds

$$\|\hat{F}\|_\infty \leq \|F\|_1.$$

When $1 < p \leq 2$, by Hausdorff–Young's inequality, for $\frac{1}{q} + \frac{1}{p} = 1$, we have

$$||\hat{F}||_q \leq ||F||_p.$$

Hence, for $1 \leq p \leq 2$, by invoking Theorem E and the above two inequalities, we have

$$||\hat{F}_{y_0} - \hat{F}||_q \leq ||F_{y_0} - F||_p \to 0, \quad y_0 \to 0, \tag{9}$$

where $\frac{1}{q} + \frac{1}{p} = 1$.

Due to (8) and (9), for any $\varphi \in S(\mathbb{R}^n)$ satisfying supp$\varphi \subset \overline{\mathbb{R}^n \backslash \Gamma^*}$, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \hat{F}(x)\varphi(x)\, dx \right| &= \left| \int_{\mathbb{R}^n} (\hat{F}(x) - \hat{F}_{y_0}(x))\varphi(x)\, dx + \int_{\mathbb{R}^n} \hat{F}_{y_0}(x)\varphi(x)\, dx \right| \\ &= \left| \int_{\mathbb{R}^n} (\hat{F}(x) - \hat{F}_{y_0}(x))\varphi(x)\, dx \right| \\ &\leq ||\hat{F} - \hat{F}_{y_0}||_q ||\varphi||_p \to 0, \quad y_0 \to 0. \end{aligned}$$

This implies that d-supp$\hat{F} \subset \Gamma^*$.

Now we turn to prove the "if part" of Theorem 2.1. We have to show that if $F \in L^p$ and d-supp$\widehat{F} \subset \Gamma^*$, then the relation (4) holds, and, as consequence, $F(x + iy) \in H^p(T_\Gamma)$.

Let $F(x) \in L^p$, $1 \leq p \leq 2$, and d-supp$\hat{F} \subset \Gamma^*$. First, we can easily show that for any fixed $y \in \Gamma$ there exists a $\delta = \delta_y > 0$ such that $y \cdot t \geq \delta|t|$ for all t in Γ^* (see [26]).

As consequence, for any fixed $y \in \Gamma$,

$$\begin{aligned} |\mathcal{X}_{\Gamma^*}(t)e^{2\pi i z\cdot t}\hat{F}(t)| &= \mathcal{X}_{\Gamma^*}(t)|\hat{F}(t)|e^{-2\pi y\cdot t} \\ &\leq \mathcal{X}_{\Gamma^*}(t)|\hat{F}(t)|e^{-2\pi\delta|t|} \in L^1(\mathbb{R}^n). \end{aligned}$$

Restricted to a neighborhood of $y \in \Gamma$, the Lebesgue Dominated Convergence Theorem can be used to conclude that the function

$$G(z) = \int_{\mathbb{R}^n} e^{2\pi i z\cdot t}\hat{F}(t)\, dt = \int_{\mathbb{R}^n} \mathcal{X}_{\Gamma^*}(t)e^{2\pi i z\cdot t}\hat{F}(t)\, dt \tag{10}$$

is holomorphic at y. Therefore, G is holomorphic in the whole T_Γ.

Next, we show $G(z) \in H^p(T_\Gamma)$. Fix $z \in T_\Gamma$. Let

$$g_z(t) = \mathcal{X}_{\Gamma^*}(t)e^{2\pi i z\cdot t}, \quad \widetilde{g}_z(t) = g_z(-t).$$

Then $g_z \in L^1(\mathbb{R}^n) \bigcap L^2(\mathbb{R}^n)$. By (1), $\hat{g}_z(s) = K(z - s)$.

For $z,\ w \in T_\Gamma$, let

$$I(z, w) = \int_{\mathbb{R}^n} K(z - t)F(t)K(t + w)\, dt. \tag{11}$$

Then

$$I(z,w)=\int_{\mathbb{R}^n}\hat{g}_z(t)F(t)\hat{g}_w(-t)\,dt.$$

Since, for $z,\ w\in T_\Gamma$, $\hat{g}_z(t)\hat{g}_w(-t)=\widehat{g_z*\widetilde{g}_w}(t)$, where

$$\begin{aligned}(g_z*\widetilde{g}_w)(t)&=\int_{\mathbb{R}^n}g_z(\xi)\widetilde{g}_w(t-\xi)d\xi=\int_{\mathbb{R}^n}g_z(\xi)g_w(\xi-t)\,d\xi\\&=\int_{\mathbb{R}^n}\mathcal{X}_{\Gamma^*}(\xi)e^{2\pi iz\cdot\xi}\mathcal{X}_{\Gamma^*}(\xi-t)e^{2\pi iw\cdot(\xi-t)}\,d\xi,\end{aligned}$$

we have

$$\begin{aligned}I(z,w)&=\int_{\mathbb{R}^n}\hat{F}(s)\mathcal{X}_\Gamma(s)(g_z*\widetilde{g}_w)(s)\,ds\\&=\int_{\mathbb{R}^n}\hat{F}(s)\mathcal{X}_{\Gamma^*}(s)\int_{\mathbb{R}^n}\mathcal{X}_{\Gamma^*}(\xi)e^{2\pi iz\cdot\xi}\mathcal{X}_{\Gamma^*}(\xi-s)e^{2\pi iw\cdot(\xi-s)}\,d\xi ds.\end{aligned}$$

Let $t=\xi-s$. By (1), as well as the relation $\mathcal{X}_{\Gamma^*}(t)\mathcal{X}_{\Gamma^*}(t+s)\mathcal{X}_{\Gamma^*}(s)=\mathcal{X}_{\Gamma^*}(t)\mathcal{X}_{\Gamma^*}(s)$, the last expression for $I(z,w)$ coincides with

$$\begin{aligned}I(z,w)&=\int_{\mathbb{R}^n}\hat{F}(s)\mathcal{X}_{\Gamma^*}(s)\int_{\mathbb{R}^n}\mathcal{X}_\Gamma(t)\mathcal{X}_\Gamma(t+s)e^{2\pi iz\cdot(s+t)}e^{2\pi iw\cdot t}\,dtds\\&=\int_{\mathbb{R}^n}\hat{F}(s)\mathcal{X}_{\Gamma^*}(s)e^{2\pi iz\cdot s}\int_{\mathbb{R}^n}\mathcal{X}_{\Gamma^*}(t)e^{2\pi i(z+w)\cdot t}\,dtds\\&=G(z)K(z+w).\end{aligned}$$

For $z\in T_\Gamma$, we have $-\bar{z}\in T_\Gamma$, and

$$I(z,-\bar{z})=G(z)K(z-\bar{z})=G(z)K(2iy).\tag{12}$$

Hence, by the formulas (11) and (12), we have

$$G(z)=\int_{\mathbb{R}^n}F(t)\frac{K(z-t)K(t-\bar{z})}{K(2iy)}\,dt=\int_{\mathbb{R}^n}F(t)P(x-t,y)\,dt,\tag{13}$$

where $P_y=P(\cdot,y)$ is the Poisson kernel associated with the tube T_Γ. Due to (13), by invoking the corresponding result in [26] (Theorem 2.1 in Chapter 1) we have

$$\sup_{y\in\Gamma}\int_{\mathbb{R}^n}|G(x+iy)|^p\,dx<\infty.$$

$G(z)$ proven before, we have $G(x+iy)\in H^p(T_\Gamma)$.

By Theorem E, the above formula (13) assures that the limit of $G(z)$ as $y \to 0$ is $F(x)$. Defining $F(z) = G(z)$, we therefore have a function in the Hardy space $H^p(T_\Gamma)$ having $F(x)$ as its restricted boundary limit.

Moreover, by Lemma 3.3 and (10), we have

$$
\begin{aligned}
F(z) = G(z) &= \int_{\mathbb{R}^n} \mathcal{X}_{\Gamma^*}(t) e^{2\pi i z\cdot t} \hat{F}(t)\, dt \\
&= \int_{\mathbb{R}^n} \left(\mathcal{X}_{\Gamma^*}(\cdot) e^{2\pi i z\cdot(\cdot)}\right)\hat{}(t) F(t)\, dt \\
&= \int_{\mathbb{R}^n} F(t) K(z-t)\, dt.
\end{aligned}
$$

The proof of Theorem 2.1 is complete. □

4.2 Proof of Theorem 2.2

Proof We are to prove that if $F(z) \in H^p(T_\Gamma)$, $2 < p \le \infty$, then $F(x)$, as the boundary limit of function $F(z)$, is the tempered holomorphic distribution represented by $F(z)$, and, moreover, d-supp$\hat{F} \subset \Gamma^*$.

For $2 < p < \infty$, by Hölder's inequality and (b) of Theorem F, for any $\varphi \in S(\mathbb{R}^n)$, we have

$$
\begin{aligned}
|\langle F_y, \varphi\rangle - \langle F, \varphi\rangle| &= |\int_{\mathbb{R}^n} (F(x+iy) - F(x))\varphi(x)\, dx| \\
&\le ||F_y - F||_p ||\varphi||_q \to 0, \quad y \to 0,
\end{aligned}
$$

where $F_y(x) = F(x+iy)$, $\frac{1}{p} + \frac{1}{q} = 1$.

For $p = \infty$, under the same notation, by (a) and (c) of Theorem F and the Lebesgue Dominated Convergence Theorem, we have the same result. Therefore, $F(x)$ is the tempered holomorphic distribution represented by $F(z) \in H^p(T_\Gamma)$.

By using the same method as in the proof of the inequality (7), we have

$$
|F_{y_0}(z)| \le C\|F\|_p < \infty, \tag{14}
$$

where C is a constant only depending on n and y_0.

Let ϕ be a continuous function with compact support in Γ^* satisfying $\int_{\mathbb{R}^n} \phi(t)\, dt = 1$. Set

$$
\psi(z) = \int_{\mathbb{R}^n} e^{2\pi i z\cdot t} \phi(t)\, dt,
$$

and $\psi_\delta(z) = \psi(\delta z)$, $\delta > 0$.

By the definition of $\psi(z)$, and taking into account supp$\phi \subset \Gamma^*$, Theorem E assures that $\psi \in H^2(T_\Gamma)$.

Consequently, due to the boundedness of F_{y_0} given by (14), we have, for all $\delta > 0$,

$$
F_{y_0}(z)\psi_\delta(z) \in H^2(T_\Gamma). \tag{15}
$$

For $z \in T_\Gamma$, the Lebesgue Dominated Convergence Theorem implies

$$\lim_{\delta\to 0} \psi_\delta(z) = \lim_{\delta\to 0} \int_{\mathbb{R}^n} e^{2\pi i\delta z\cdot t}\phi(t)\,dt = 1.$$

Moreover, $|\psi_\delta(z)| \le M_1$, where M_1 is a constant. In fact, for $y \in \Gamma$, by supp $\phi \subset \Gamma^*$, we have

$$\begin{aligned} |\psi_\delta(z)| &\le \int_{\mathbb{R}^n} |e^{2\pi i\delta z\cdot t}\phi(t)|\,dt = \int_{\mathbb{R}^n} e^{-2\pi\delta y\cdot t}|\phi(t)|\,dt \\ &\le \int_{\mathbb{R}^n} |\phi(t)|\,dt \le M_1 < \infty. \end{aligned} \tag{16}$$

Together with (14), this gives

$$|\psi_\delta(z)F_{y_0}(z)\hat{\varphi}(t)| \le M|\hat{\varphi}(t)| \in L^1,$$

where $M = M_1C\|f\|_p$ is a constant, and $\varphi \in S(\mathbb{R}^n)$ with supp$\varphi \subset \overline{\mathbb{R}^n\backslash\Gamma^*}$. The relation

$$\lim_{\delta\to 0} \psi_\delta(z)F_{y_0}(z)\hat{\varphi}(t) = F_{y_0}(z)\hat{\varphi}(t) \tag{17}$$

implies, by invoking the Lebesgue Dominated Convergence Theorem,

$$\lim_{\delta\to 0} \int_{\mathbb{R}^n} \psi_\delta(t+iy_0)F_{y_0}(t)\hat{\varphi}(t)\,dt = \int_{\mathbb{R}^n} F_{y_0}(t)\hat{\varphi}(t)\,dt.$$

Therefore,

$$\begin{aligned} (\hat{F},\varphi) = (F,\hat{\varphi}) &= \lim_{y_0\to 0} \int_{\mathbb{R}^n} F(t+iy_0)\hat{\varphi}(t)\,dt \\ &= \lim_{y_0\to 0}\lim_{\delta\to 0} \int_{\mathbb{R}^n} \psi_\delta(t+iy_0)F_{y_0}(t)\hat{\varphi}(t)\,dt \\ &= \lim_{y_0\to 0}\lim_{\delta\to 0} \int_{\mathbb{R}^n} (\psi_\delta(\cdot+iy_0)F_{y_0}(\cdot))\hat{}(t)\varphi(t)\,dt. \end{aligned} \tag{18}$$

For $y_0 \in \Gamma$, $\psi_\delta(x)F_{y_0}(x)$ is the boundary limit of $F_{y_0}(z)\psi_\delta(z) \in H^2(T_\Gamma)$ [by formula (15)]. Theorem D then assures that the distributional support of the Fourier transform of $\psi_\delta(x)F_{y_0}(x)$ is contained in Γ^*, that is,

$$\text{supp}(\psi_\delta(\cdot)F_{y_0}(\cdot))\hat{} \subset \Gamma^*. \tag{19}$$

From (18), (19), and the condition supp$\varphi \subset \overline{\mathbb{R}^n\backslash\Gamma^*}$, we obtain

$$(\hat{F},\varphi) = 0.$$

The proof of Theorem 2.2 is complete. □

4.3 Proof of Theorem 2.3

Proof To prove Theorem 2.3 is to show that if $F(z) \in H^p(T_\Gamma),\ 1 \le p \le \infty$, and $F(x) \in L^q(\mathbb{R}^n),\ 1 \le q \le \infty$, where $F(x)$ is the boundary limit of $F(z)$, then $F(z) \in H^q(T_\Gamma)$.

Let $F(z) \in H^p(T_\Gamma),\ 1 \le p \le \infty$. By Theorem E, we have

$$F(z) = \int_{\mathbb{R}^n} P(x-t, y)F(t)\, dt = (F * P_y)(x),$$

for all $z = x + iy \in T_\Gamma$, where $F(x) = \lim_{y\to 0,\ y\in\Gamma} F(x+iy)$, $P_y(x) = P(x, y)$ is the Poisson kernel of the tube T_Γ.

With the notation $F_y(x) = F(x+iy)$, we have

$$\|F_y\|_q = \|F * P_y\|_q \le \|F\|_q \|P_y\|_1 = \|F\|_q < \infty,$$

for all $y \in \Gamma$. Thus,

$$\sup_{y\in\Gamma} \int_{\mathbb{R}^n} |F(x+iy)|^q\, dx < \infty.$$

Since $F(z)$ is analytic in T_Γ, we have $F(z) \in H^q(T_\Gamma)$. □

4.4 Proof of Theorem 2.4

Proof We have to show that if $F \in L^p(\mathbb{R}^n),\ 2 < p \le \infty$, and $(\hat{F}, \varphi) = 0$ for all $\varphi \in S(\mathbb{R}^n)$ with supp$\varphi \subset \overline{\mathbb{R}^n \backslash \Gamma^*}$, then $F(x)$ is the boundary limit of a function $F(z)$ in $H^p(T_\Gamma)$.

We recall that if $F \in L^p(\mathbb{R}^n),\ 2 < p \le \infty$, then F is a tempered distribution T_F defined through

$$(F, \varphi) = \int_{\mathbb{R}^n} F(t)\varphi(t)\, dt$$

for φ in the Schwartz class $S(\mathbb{R}^n)$. The Fourier transform of F viewed as a tempered distribution is also a tempered distribution, $\hat{F}$, given by

$$(\hat{F}, \varphi) = (F, \hat{\varphi})$$

for φ in the Schwartz class $S(\mathbb{R}^n)$.

Let $\phi \in S(\mathbb{R}^n)$ be a continuous function with compact support in Γ^* that satisfies $\int_{\mathbb{R}^n} \phi(t)dt = 1$. Set

$$\psi(z) = \int_{\mathbb{R}^n} e^{2\pi i z\cdot t}\phi(t)\, dt.$$

Then $\psi(x) \in S(\mathbb{R}^n)$ and $\widehat{\phi}(-x) = \psi(x)$. For $\delta > 0$, let $\psi^\delta(z) = \psi(\delta z)$. Theorem D assures that

$$\psi^\delta(z) \in H^2(T_\Gamma). \tag{20}$$

Moreover, $\psi^\delta(x) = \hat{\phi}(-\delta x) \in S(\mathbb{R}^n)$ and

$$\psi^\delta(x) \in L^q(\mathbb{R}^n) \tag{21}$$

for all $q \geq 1$. Therefore, for $2 \leq p < \infty$, by Hölder's inequality, for $\frac{1}{p/2} + \frac{1}{q'} = 1$, we have

$$\int_{\mathbb{R}^n} |F(x)\psi^\delta(x)|^2 dx \leq \left(\int_{\mathbb{R}^n} (|F(x)|^2)^{p/2}\, dx \right)^{2/p} \|\psi^\delta\|_{2q'}^2 < \infty.$$

Hence,

$$F(x)\psi^\delta(x) \in L^2(\mathbb{R}^n).$$

For $p = \infty$, the facts that $F(x) \in L^\infty$ and $\psi^\delta(x) \in L^q(\mathbb{R}^n)$ for all $q \geq 1$ show that

$$F(x)\psi^\delta(x) \in L^2(\mathbb{R}^n).$$

Under the simplified notation $h(x) = \psi^\delta(x)$, we have

$$\hat{h}(-s) = \frac{1}{\delta^n}\phi\left(\frac{-s}{\delta}\right) = \frac{1}{\delta^n}\phi\left(\frac{-s}{\delta}\right)\mathcal{X}_{\Gamma^*}(-s).$$

Now,

$$\begin{aligned}
(\widehat{Fh}, \varphi) = (Fh, \hat{\varphi}) &= \int_{\mathbb{R}^n} F(x)h(x)\hat{\varphi}(x)\, dx \\
&= \int_{\mathbb{R}^n} F(x)\widehat{\check{h} * \varphi}(x)\, dx \\
&= (F, \widehat{\check{h} * \varphi}) = (\hat{F}, \check{h} * \varphi),
\end{aligned} \tag{22}$$

where

$$(\check{h} * \varphi)(t) = \int_{\mathbb{R}^n} \hat{h}(-s)\varphi(t-s)\mathcal{X}_{\Gamma^*}(-s)\mathcal{X}_{\overline{\mathbb{R}^n \setminus \Gamma^*}}(t-s)\, ds. \tag{23}$$

Noting that $t \in \Gamma^*$ and $-s \in \Gamma^*$ imply that $t - s \in \Gamma^*$, the relation (23) gives that $t \in \Gamma^*$ implies $(\check{h} * \varphi)(t) = 0$, viz.,

$$\mathrm{supp}(\check{h} * \varphi) \subset \overline{\mathbb{R}^n \setminus \Gamma^*}. \tag{24}$$

The condition d-supp$\hat{F} \subset \Gamma^*$, being combined with (22), (24), implies

$$(\widehat{Fh}, \varphi) = 0.$$

Due to the proven fact that $F(t)h(t) \in L^2(\mathbb{R}^n)$, Theorem 2.1 forces that $F(t)h(t) = F(t)\psi^\delta(t)$ be the boundary limit of a function, say, $G^\delta(z)$, in $H^2(T_\Gamma)$, viz.,

$$G^\delta(x) = \lim_{y\in\Gamma,\ y\to 0} G^\delta(x+iy) = F(x)\psi^\delta(x),$$

where, in particularly, $G^\delta(z)$ can be written as

$$G^\delta(z) = \int_{\mathbb{R}^n} P(x-t, y)F(t)\psi^\delta(t)\, dt.$$

Since $\psi^\delta \in L^\infty(\mathbb{R}^n)$, the assumption $F(x) \in L^p(\mathbb{R}^n)$ implies that

$$G^\delta(x) = F(x)\psi^\delta(x) \in L^p(\mathbb{R}^n).$$

Therefore, from Theorem 2.3,

$$G^\delta(z) \in H^p(T_\Gamma).$$

Let

$$G(z) = \int_{\mathbb{R}^n} P(x-t, y)F(t)\, dt,$$

which implies that

$$\sup_{y\in\Gamma} \int_{\mathbb{R}^n} |G(x+iy)|^p\, dt \le \|F\|_p^p.$$

We further show that $G(z)$ is holomorphic in T_Γ. Once we achieve that, we can conclude that $G(z) \in H^p(T_\Gamma)$. Now, since $G(z)$ is harmonic, we denote the left-hand-side of the above inequality by $\|G\|_{H^p}$.

Let $z_0 = x_0 + iy_0 \in T_{\Gamma_0}$ be a fixed point, where $\Gamma_0 \subset \Gamma$, satisfying

$$d(\Gamma_0,\ \Gamma^c) = \inf\{|y_1 - y_2|;\ y_1 \in \Gamma_0,\ y_2 \in \Gamma^c\} \ge \varepsilon > 0.$$

let $S_\varepsilon = \{z \in \mathbb{C}^n;\ |z-z_0| < \varepsilon\}$. If $\Sigma_\varepsilon = \{z \in \mathbb{R}^n;\ |y-y_0| < \varepsilon\}$, then $S_\varepsilon \subset T_{\Sigma_\varepsilon} \subset T_\Gamma$. Since $G^\delta(z)$ and $G(z)$ both are harmonic functions, $|G^\delta(z) - G(z)|^p$ is subharmonic as a function of $2n$ real variables ([26]). Therefore, we have

$$\begin{aligned} |G^\delta(z_0) - G(z_0)|^p &\le \frac{1}{\Omega_{2n}\varepsilon^{2n}} \int_{S_\varepsilon} |G^\delta(x+iy) - G(x+iy)|^p\, dxdy \\ &\le \frac{1}{\Omega_{2n}\varepsilon^{2n}} \int_{T_{\Sigma_\varepsilon}} |G^\delta(x+iy) - G(x+iy)|^p\, dxdy \end{aligned}$$

$$\leq \frac{1}{\Omega_{2n}\varepsilon^{2n}}\int_{\Sigma_\varepsilon}\int_{\mathbb{R}^n}|G^\delta(x+iy)-G(x+iy)|^p\,dxdy$$
$$\leq \frac{\Omega_n}{\Omega_{2n}\varepsilon^n}\|G^\delta - G\|_{h^p} \leq \frac{\Omega_n}{\Omega_{2n}\varepsilon^n}\|F(\psi^\delta-1)\|_{L^p}^p. \tag{25}$$

From the above inequality, and together with $\|F(\psi^\delta-1)\|_{L^p}\to 0,\ \ \delta\to 0$, we have

$$|G^\delta(z_0)-G(z_0)|^p\to 0,\quad \delta\to 0. \tag{26}$$

The above estimate (25), in fact, is valid for z in a neighborhood of z_0. Hence the limit (26) is uniform in a neighborhood of z_0. Since G^δ is holomorphic in T_Γ, we conclude that $G(z)$ is holomorphic in T_Γ. The proof is complete. □

4.5 Proof of Theorem 2.5

Proof The proof of Theorem 2.5 amounts to showing that the Cauchy integral of a function in $L^p(\mathbb{R}^n)$ is a function in $H^p(T_\Gamma),\ 1<p<\infty$.

Let $n=n_1+n_2,\ 1\leq n_1,\ n_2<n$, and fix $(t_{n_1+1},\ldots,t_n)\in\mathbb{R}^{n_2}$. Now we consider the function f of n_1 real variables defined by

$$f(x_1,x_2\ldots,x_{n_1})=F(x_1,\ldots,x_{n_1},t_{n_1+1},\ldots,t_n),$$

where $(x_1,x_2\ldots,x_{n_1})\in\mathbb{R}^{n_1}$.

The Fubini Theorem ensures that $f(x_1,x_2\ldots,x_{n_1})$ belongs to $L^p(\mathbb{R}^{n_1})$ for almost all $(t_{n_1+1},\ldots,t_n)\in\mathbb{R}^{n_2}$. Below we will prove Theorem 2.5 by mathematical induction. When $n=1$, by Theorem G, Theorem 2.5 holds.

Next, we assume $n>1$. The induction hypothesis is that Theorem 2.5 holds for $n-1$. Take $n_1=n-1$. Since $f(x_1,x_2\ldots,x_{n-1})\in L^p(\mathbb{R}^{n-1})$, we have, as the $n-1$ dimensional Cauchy integral,

$$\mathcal{C}_{n-1}(f)(z_1,\ldots,z_{n-1})$$
$$=\frac{1}{(2\pi i)^{n-1}}\int_{\mathbb{R}^{n-1}}\frac{f(t_1,\ldots,t_{n-1})}{\prod_{j=1}^{n-1}(t_j-z_j)}\,dt_1\cdots dt_{n-1}\in H^p(T_{\Gamma_{n-1}}),$$

where Γ_{n-1} is the first octant of $\mathbb{R}^{n-1}$. In an alternative notation, we have,

$$\mathcal{C}_{n-1}(F)(z_1,\ldots,z_{n-1},t_n)$$
$$=\frac{1}{(2\pi i)^{n-1}}\int_{\mathbb{R}^{n-1}}\frac{F(t_1,\ldots,t_n)}{\prod_{j=1}^{n-1}(t_j-z_j)}\,dt_1\cdots dt_{n-1}\in H^p(T_{\Gamma_{n-1}}), \tag{27}$$

for $(z_1,\ldots,z_{n-1})\in T_{\Gamma_{n-1}}$ and a fixed $t_n\in\mathbb{R}$.

By the induction hypothesis, there exists a finite constant C_{n-1}, depending only on $n-1$ and p, such that

$$\int_{\mathbb{R}^{n-1}} |\mathcal{C}_{n-1}(F)(x_1+iy_1,\ldots,x_{n-1}+iy_{n-1},t_n)|^p\,dx_1\cdots dx_{n-1} \\ \leq C_{n-1}\int_{\mathbb{R}^{n-1}} |F(x_1,\ldots,x_{n-1},t_n)|^p\,dx_1\cdots dx_{n-1}. \tag{28}$$

Since $F(x)\in L^p(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^{n}} |\mathcal{C}_{n-1}(F)(x_1+iy_i,\ldots,x_{n-1}+iy_{n-1},t_n)|^p\,dx_1\cdots dx_{n-1}dt_n \\ \leq C_{n-1}\int_{\mathbb{R}^{n}} |F(x_1,\ldots,x_{n-1},t_n)|^p\,dx_1\cdots dx_{n-1}dt_n<\infty.$$

This implies that $\mathcal{C}_{n-1}(F)(x_1+iy_1,\ldots,x_{n-1}+iy_{n-1},t_n)\in L^p(\mathbb{R}^n)$. As consequence, $\mathcal{C}_{n-1}(F)(x_1+iy_1,\ldots,x_{n-1}+iy_{n-1},t_n)\in L^p(\mathbb{R})$ as a function of one variable t_n for almost all $(x_1+iy_1,\ldots,x_{n-1}+iy_{n-1})\in T_{\Gamma_{n-1}}$.

Using Theorem G again, we obtain, by fixing $(z_1,\ldots,z_{n-1})\in T_{\Gamma_{n-1}}$,

$$\begin{aligned}\mathcal{C}(F)(z_1,\ldots,z_n) &= \mathcal{C}_1(\mathcal{C}_{n-1}F)(z_1,\ldots,z_{n-1},z_n)\\ &= \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\mathcal{C}_{n-1}(F)(z_1,\ldots,z_{n-1},t_n)}{t_n-z_n}\,dt_n\\ &= \frac{1}{(2\pi i)^n}\int_{\mathbb{R}^n}\frac{F(t_1,\ldots,t_n)}{\prod_{j=1}^n(t_j-z_j)}\,dt_1\cdots dt_n\\ &= \int_{\mathbb{R}^n}F(t)K(z-t)\,dt\in H^p(\mathbb{C}^+)\end{aligned} \tag{29}$$

for $z_n\in\mathbb{C}^+$, and

$$\int_{\mathbb{R}} |\mathcal{C}_1(\mathcal{C}_{n-1}F)(z_1,\ldots,z_{n-1},x_n+iy_n)|^p\,dx_n \\ \leq C_1\int_{\mathbb{R}} |\mathcal{C}_{n-1}(F)(z_1,\ldots,z_{n-1},x_n)|^p\,dx_n. \tag{30}$$

Therefore, by (28) and (30), we have

$$\begin{aligned}&\int_{\mathbb{R}^n}|\mathcal{C}(F)(x+iy)|^p\,dx\\ &= \int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}}|\mathcal{C}_1(\mathcal{C}_{n-1}F)(x_1+iy_1,\ldots,x_n+iy_n)|^p\,dx_n\right)dx_1\cdots dx_{n-1}\\ &\leq C_1\int_{\mathbb{R}}\left(\int_{\mathbb{R}^{n-1}}|\mathcal{C}_{n-1}(F)(z_1,\ldots,z_{n-1},x_n)|^p\,dx_1\cdots dx_{n-1}\right)dx_n\end{aligned}$$

$$\leq C_1 \int_{\mathbb{R}} \left(C_{n-1} \int_{\mathbb{R}^{n-1}} |F(x_1, \ldots, x_{n-1}, x_n)|^p \, dx_1 \cdots dx_{n-1} \right) dx_n$$
$$= C_n \|F\| p < \infty, \tag{31}$$

where $C_n = C_1 C_{n-1}$ depending only on n and p. Thus, $\mathcal{C}(F)(z_1, \ldots, z_n) \in H^p(T_\Gamma)$. The proof of Theorem 2.5 is complete. □

4.6 Proof of Theorem 2.5*

Proof If Γ is a polygonal cone in $\mathbb{R}^n$, then, according to Lemma 3.6, the dual cone Γ^* is also a polygonal cone in $\mathbb{R}^n$. Applying Lemma 3.6 (2) to the polygonal cone Γ^*, Γ^* can be decomposed into the union of a finite number of n-sided polygonal cones Γ_j^*, $j = 1, 2, \ldots, N$, with disjoint interior parts. Using such decomposition, we denote

$$K_\Gamma(z) = \sum_{j=1}^{N} K_{\Gamma_j}(z),$$

where Γ_j is the interior of $(\Gamma_j^*)^*$, $j = 1, 2, \ldots, N$, being also $n-$sided polygonal cones (by Lemma 3.6). K_Γ, in fact, is the Cauchy kernel associated with the tube T_Γ. This can be seen from the general definition of the Cauchy kernel:

$$\begin{aligned} K_\Gamma(z) &= \int_{\Gamma^*} e^{2\pi i z\cdot t} \, dt = \int_{\bigcup_{j=1}^N \Gamma_j^*} e^{2\pi i z\cdot t} \, dt \\ &= \sum_{j=1}^{N} \int_{\Gamma_j^*} e^{2\pi i z\cdot t} \, dt = \sum_{j=1}^{N} K_{\Gamma_j}(z). \end{aligned} \tag{32}$$

Moreover, each Γ_j, $j = 1, 2, \ldots, N$, can be mapped onto the first octant by a linear transformation ([26], [23]). Now we work out the linear transformation. Each Γ_j is the interior of the convex hull of n linearly independent rays meeting at the origin. By selecting vectors $\eta_{j_1}, \eta_{j_2}, \ldots, \eta_{j_n}$ as the unit vectors in the directions of those rays, the set Γ_j is expressed as

$$\Gamma_j = \{\lambda_1 \eta_{j_1} + \lambda_2 \eta_{j_2} + \cdots + \lambda_n \eta_{j_n} \in \mathbb{R}^n;\ \lambda_1 \geq 0, \ldots, \lambda_n \geq 0\}.$$

For each j, consider the linear transformations A_j, $j = 1, 2, \ldots, N$, that maps the standard basis vectors e_k onto η_{j_k}, $k = 1, 2, \ldots, n$. Then A_j maps the first octant Γ_0 onto the cone Γ_j. That is, $\Gamma_j = A_j \Gamma_0$. We further extend the mapping A_j to T_{Γ_0} by defining $A_j^{-1}(z) = x + i A_j^{-1} y \in T_{\Gamma_0}$, where $z = x + iy$, $y \in \Gamma_j$, $j = 1, 2, \ldots, N$.

Let $\{\eta_{j_1}^*, \ldots, \eta_{j_n}^*\}$ be the dual base of $\{\eta_{j_1}, \ldots, \eta_{j_n}\}$ such that $(\eta_{j_l}, \eta_{j_k}^*) = \delta_{kl}$, where $\delta_{kl} = 1$ if $l = k$; and otherwise zero. This implies that $\{\eta_{j_1}^*, \ldots, \eta_{j_n}^*\}$ is the base of Γ_j^*, the dual cone of Γ_j. Let B_j be the linear transformation mapping the standard basis vectors e_k onto η^*_{jk}, $k = 1, 2, \ldots, n$, or $\Gamma^*_j = B_j \Gamma_0$, $j = 1, 2, \ldots, N$. Then we have

$$(\eta_{jl}, \eta^*_{jk}) = (A_j e_l, B_j e_k) = (A_j e_l)^T (B_j e_k) = e_l^T A_j^T (B_j e_k) = \delta_{kl},$$

where δ_{kl} is the same function as above. Therefore, $A_j^T B_j = I$, or $B_j = (A_j^T)^{-1}$. Now, together with (32), we have

$$\begin{aligned} K_\Gamma(z) &= \sum_{j=1}^N K_{\Gamma_j}(z) = \sum_{j=1}^N \int_{\Gamma_j^*} e^{2\pi i z\cdot t}\, dt = \sum_{j=1}^N \int_{B_j\Gamma_0} e^{2\pi i z\cdot t}\, dt \\ &= \sum_{j=1}^N \int_{\Gamma_0} e^{2\pi i z\cdot B_j s} |\det B_j|\, ds = \sum_{j=1}^N \int_{\Gamma_0} e^{2\pi i z\cdot (A_j^T)^{-1} s} \frac{1}{|\det A_j|}\, ds \\ &= \sum_{j=1}^N \int_{\Gamma_0} e^{2\pi i (A_j)^{-1} z\cdot s} \frac{1}{|\det A_j|}\, ds \\ &= \sum_{j=1}^N \frac{1}{|\det A_j|} K_{\Gamma_0}(A_j^{-1} z). \end{aligned}$$

Thus, for any $z \in T_\Gamma$,

$$\begin{aligned} C(F)(z) &= \int_{\mathbb{R}^n} F(t) K_\Gamma(z-t)\, dt = \int_{\mathbb{R}^n} F(t) \sum_{j=1}^N \frac{1}{|\det A_j|} K_{\Gamma_0}(A_j^{-1} z - A_j^{-1} t)\, dt \\ &= \sum_{j=1}^N \int_{\mathbb{R}^n} F(A_j s) K_{\Gamma_0}(A_j^{-1} z - s)\, ds = \sum_{j=1}^N \int_{\mathbb{R}^n} F_j(s) K_{\Gamma_0}(A_j^{-1} z - s)\, ds \\ &= \sum_{j=1}^N C_0(F_j)(A_j^{-1} z) := \sum_{j=1}^N C_0(F_j)(\zeta), \end{aligned} \tag{33}$$

where $F_j(x) = F(A_j x) \in L^p(\mathbb{R}^n)$, $\zeta = A_j^{-1} z \in T_{\Gamma_0}$, and

$$C_0(F_j)(\zeta) = \int_{\mathbb{R}^n} F_j(t) K_{\Gamma_0}(\zeta - t)\, dt.$$

By Theorem 2.5, $C_0(F_j)(\zeta) \in H^p(T_{\Gamma_0})$, and there exists a finite constant C_p, depending only on p and the dimension n such that, for any $\zeta = \xi + i\eta = A_j^{-1} z \in T_{\Gamma_0}$

$$\int_{\mathbb{R}^n} |C_0(F_j)(\xi + i\eta)|^p\, d\xi \le C_p \|F_j\|_p^p.$$

Those imply that the function $C_0(F_j)(\zeta) = C_0(F_j)(A_j^{-1}z)$ as a function of $z \in T_{\Gamma_j}$ is holomorphic in tube T_{Γ_j}, $j = 1, 2, \dots, N$, and

$$\int_{\mathbb{R}^n} |C_0(F_j)(\xi + iA_j^{-1}y)|^p \, d\xi \le C_p \|F_j\|_p^p.$$

Therefore, for any $z \in T_\Gamma$, together with the relation $\Gamma^* = \bigcup_{j=1}^N \Gamma_j^*$, which implies that $\overline{\Gamma} = \cap_{j=1}^N \Gamma_j$ (by Lemma 3.6), we get that $C(F)(z)$ is holomorphic in tube T_Γ. Moreover, by the inequality and (33), we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} |C(F)(x+iy)|^p \, dx\right)^{1/p} \\
&= \left(\int_{\mathbb{R}^n} \left|\sum_{j=1}^N C_0(F_j)\left(A_j^{-1}(x+iy)\right)\right|^p dx\right)^{1/p} \\
&\le \sum_{j=1}^N \left(\int_{\mathbb{R}^n} |C_0(F_j)\left(A_j^{-1}(x+iy)\right)|^p \, dx\right)^{1/p} \\
&= \sum_{j=1}^N \left(\int_{\mathbb{R}^n} \left|C_0(F_j)\left(\xi + iA_j^{-1}y\right)\right|^p |\det A_j| \, d\xi\right)^{1/p} \\
&\le \sum_{j=1}^N |\det A_j|^{1/p} C_p \left(\int_{\mathbb{R}^n} |F_j(\xi)|^p \, d\xi\right)^{1/p} \\
&= \sum_{j=1}^N C_p \left(\int_{\mathbb{R}^n} |F(A_j\xi)|^p |\det A_j| \, d\xi\right)^{1/p} \\
&= \sum_{j=1}^N C_p \left(\int_{\mathbb{R}^n} |F(x)|^p \, dx\right)^{1/p} = NC_p \|F\|_{L^p}.
\end{aligned}$$

Thus,

$$\sup_{y\in\Gamma} \int_{\mathbb{R}^n} |C(F)(x+iy)|^p \, dx \le NC_p \|F\|_{L^p}^p < \infty.$$

Hence, the proof of Theorem 2.5* is complete. □

4.7 Proof of Theorem 2.6

The Theorem 2.6 asserts that any function in $H^p(T_\Gamma)$ can be represented as the Cauchy integral of its restricted boundary limit. Below we provide a proof using a density argument combined with Theorem 2.5.

Proof The assertion for the case $1 \le p \le 2$ is proved in Theorem 2.1. We only need to prove the case $2 < p < \infty$.

Let $F(z) \in H^p(T_\Gamma)$, $2 < p < \infty$, $F(x) \in L^p(\mathbb{R}^n)$ be the boundary limit of function $F(z)$. We adopt the function $\psi^\delta(z)$ in the proof of Theorem 2.4. We will show

$$F(z)\psi^\delta(z) \in H^2(T_\Gamma).$$

In fact, on one hand, $F(z) \in H^p(T_\Gamma)$, and (20) imply that $F_(z)\psi^\delta(z)$ is holomorphic in T_Γ. On the other hand, by (20) and (21), Theorem 2.3 shows that $\psi^\delta(z) \in H^{2r}(T_\Gamma)$, $2r > 1$. Therefore, for $2 \le p < \infty$, by Hölder's inequality, for $\frac{1}{p/2} + \frac{1}{r} = 1$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |F(x+iy)\psi^\delta(x+iy)|^2 dx \\
&\le \left(\int_{\mathbb{R}^n} (|F(x+iy)|^2)^{p/2}\, dx\right)^{2/p} \left(\int_{\mathbb{R}^n} |\psi^\delta(x+iy)|^{2r}\, dx\right)^{1/r} \\
&\le \|F\|_{H_p}^2 \|\psi^\delta\|_{H_{2r}}^2.
\end{aligned} \tag{34}$$

Therefore,

$$\sup_{y\in T_\Gamma} \int_{\mathbb{R}^n} |F(x+iy)\psi^\delta(x+iy)|^2 dx \le \|F\|_{H_p}^2 \|\psi^\delta\|_{H_{2r}}^2 < \infty.$$

Due to the Cauchy integral representation in the Theorem 2.1, we have

$$F(z)\psi^\delta(z) = \int_{\mathbb{R}^n} F(t)\psi^\delta(t)K(z-t)\, dt.$$

By Theorem E, we have

$$\lim_{\delta\to 0} F(z)\psi^\delta(z) = F(z), z \in T_\Gamma. \tag{35}$$

Let, for any $z \in T_\Gamma$,

$$\mathcal{C}(F)(z) = \int_{\mathbb{R}^n} F(t)K(z-t)\, dt,$$

where $K(z)$ is the Cauchy kernel associated with the tube T_Γ. Then, by Theorem 2.5, we have $\mathcal{C}(F)(z) \in H^p(T_\Gamma)$.

Let q satisfy the equation $\frac{1}{p}+\frac{1}{q}=1$. For any fixed z, by Lemma 3.5, we have

$$\begin{aligned}&|F(z)\psi^{\delta}(z)-\mathcal{C}(F)(z)|\\&\leq\int_{\mathbb{R}^n}|(F(t)\psi^{\delta}(t)-F(t))K(z-t)|\,dt\\&\leq\|F\psi^{\delta}-F\|_p\|K(z-\cdot)\|_q\to 0,\quad \delta\to 0.\end{aligned}\tag{36}$$

(35) and (36) conclude that $G(z)=\mathcal{C}(F)(z)$. This completes the proof. □

The proof of Theorem 2.6* is similar to the proof of Theorem 2.6.

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