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A novel 2D partial unwinding adaptive Fourier decomposition method with application to frequency domain system identification

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This paper proposes a two-dimensional (2D) partial unwinding adaptive Fourier decomposition method to identify 2D system functions. Starting from Coifman in 2000, one-dimensional (1D) unwinding adaptive Fourier decomposition and later a type called unwinding AFD have been being studied. They are based on the Nevanlinna factorization and a maximal selection. This method provides fast-converging rational approximations to 1D system functions. However, in the 2D case, there is no genuine unwinding decomposition. This paper proposes a 2D partial unwinding adaptive Fourier decomposition algorithm that is based on algebraic transforms reducing a 2D case to the 1D case. The proposed algorithm enables rational approximations of real coefficients to 2D system functions of real coefficients. Its fast convergence offers efficient system identification. Numerical experiments are provided, and the advantages of the proposed method are demonstrated.

KEYWORDS

Hardy space, maximal selection principle, Nevanlinna factorization, system identification, unwinding adaptive Fourier decomposition

1 | INTRODUCTION

System identification is to establish mathematical models using the input-output measurements. The models built are usually approximations to the original systems. System identification plays a key role in various fields such as model-based control,[1-3] signal and image analysis, [4-7] texture synthesis,[8] image filtering,[9,10] and restoration.[11,12] In recent years, the problem of two-dimensional (2D) system identification has attracted the interest of researchers. To our knowledge, a number of 2D system identification methods including neural network[13] and subspace identification[14,15] were proposed as generalizations of the corresponding one-dimensional (1D) methods.

For the 1D system identification, there have been a lot of studies focusing on using rational functions to approximate system functions.[16-20] Among the various 1D methods, 1D adaptive Fourier decomposition (1D-AFD)[20] has promising approximation effect that depends on the maximal selection principle to adaptively select rational functions. Recently, 1D unwinding adaptive Fourier decomposition (1D-UAFD)[21] was proposed, which combines maximal selection and Nevanlinna factorization. Among different kinds of rational approximation algorithms,[20-26] 1D-UAFD outperforms the other types in fast signal reconstruction.[22]

Inspired by 1D-UAFD, it raises a natural question whether one can develop this method in 2D system identification. There is no genuine 2D-UAFD analogous to the 1D case, for there exist essential differences between 1D and 2D complex analysis. In the one complex variable case, if f is analytic at z_0 and $f(z_0) = 0$, then there exists a function g analytic at z_0, and $f(z) = (z - z_0)g(z)$. In higher dimension, the analogous factorization result does not hold. This implies that the unwinding (factorization) process cannot be implemented in higher dimensions. Due to difficulty in estimating true poles of 2D system functions,[27,28] we instead to obtain fast-converging rational approximations to 2D system functions.

The aim of this paper is to introduce what we call 2D partial UAFD (2D-PUAFD), in which 1D-UAFD is applied to the 2D case through elementary algebraic operations. The proposed algorithm gives rise to fast convergence. It provides rational approximations of real coefficients to 2D transfer functions. The necessity of real coefficients is due to the real-valued impulse response property of the systems under study.

The contributions of this paper include the following:

- The theory of 2D-PUAFD is proposed. The convergence of the algorithm is proved. The computational complexity of the proposed algorithm is calculated.
- We design a two-step procedure for 2D frequency domain system identification by using the tensor type Cauchy integral and the proposed 2D-PUAFD algorithm. It yields rational approximations of real coefficients to 2D transfer functions.
- Two examples of 2D system identification are presented. The experimental results show that the proposed algorithm outperforms 2D Fourier series (2D-FS) and the method in Valenzuela and Salvia.[29]

This paper is organized as follows. Section 2 gives the problem setting. Section 3 presents the mathematical foundation of 2D-PUAFD. Section 4 proposes a 2D-PUAFD algorithm and designs a two-step procedure for 2D frequency domain system identification. Experimental results are presented in Section 5. In Section 6, conclusions are drawn.

2 | PROBLEM SETTING

We consider a state-space model for a 2D discrete linear time-invariant (LTI) system, namely, the second Fornasini-Marchesini model (F-MM II).[30] It is given by

$$\begin{aligned} x(j+1,k+1) &= A_0x(j,k+1) + A_1x(j+1,k) + B_0w(j,k+1) + B_1w(j+1,k), \\ y(j,k) &= A_2x(j,k) + B_2w(j,k), \end{aligned} \tag{1}$$

where $j,k \in \mathbb{N}, x(j,k) \in \mathbb{R}^p$ is the real local state vector at (j,k), $w(j,k) \in \mathbb{R}^q$ is the real input vector at (j,k), $y(j,k) \in \mathbb{R}^m$ is the real output vector at (j,k), and A_i and B_i, $i = 0,1,2$, are real constant matrices of appropriate dimensions. The transfer function of F-MM II is a proper rational function matrix

$$T(z_1,z_2) = \frac{Y(z_1,z_2)}{W(z_1,z_2)} = \frac{\sum_{l,n=0}^{q} a_{ln} z_1^{-l} z_2^{-n}}{\sum_{l,n=0}^{p} b_{ln} z_1^{-l} z_2^{-n}} = \sum_{l,n=0}^{\infty} d_{ln} z_1^{-l} z_2^{-n}, \tag{2}$$

where $Y(z_1,z_2)$ and $W(z_1,z_2)$ are the 2D Z-transforms of $y(j,k)$ and $w(j,k)$, respectively, and d_{ln} is the impulse response. Such a system is referred to as a quadrant-causal system.[31]

Herein, we focus on discrete LTI quadrant-causal single-input single-output (SISO) systems. Besides, we assume that the transfer function $T(z_1,z_2)$ of real coefficients has no poles in $\mathbb{D}^c \times \mathbb{D}^c = \{(z_1,z_2) : |z_1| \geq 1, |z_2| \geq 1\}$.

Through the mappings $\frac{1}{z_1} \to z$ and $\frac{1}{z_2} \to w$, $T(z_1,z_2)$ is converted to a function $f(z,w)$ in $H^2(\mathbb{D}^2)$ that can be holomorphically continued to outside the closed unit poly-disc $\mathbb{D}^2 := \mathbb{D} \times \mathbb{D} = \{(z,w) : |z| < 1, |w| < 1\}$. $H^2(\mathbb{D}^2)$ is the class of complex holomorphic functions in the poly-disc $\mathbb{D} \times \mathbb{D}$ satisfying

$$\sup_{0 \leq \rho_1,\rho_2 < 1} \int_0^{2\pi} \int_0^{2\pi} |f(\rho_1 e^{it}, \rho_2 e^{is})|^2 dtds < \infty. \tag{3}$$

It is a Hilbert space equipped with the inner product defined by

$$\langle f, g \rangle = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f\left(e^{it}, e^{is}\right) \bar{g}\left(e^{it}, e^{is}\right) dtds. \tag{4}$$

Because of the assumption of $T(z_1, z_2)$, the transformation function $f(z, w)$ enjoys the property that $f(\bar{z}, \bar{w}) = \overline{f(z, w)}$. It is noted that $f\left(e^{it}, e^{is}\right) = T\left(e^{-it}, e^{-is}\right)$ and $\|f\| = \|T\|_{L^2}$, where $t, s \in [0, 2\pi)$, $\|T\|_{L^2} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} |T\left(e^{it}, e^{is}\right)|^2 dtds$, and $\|\cdot\|$ is the H^2 norm.

The problem of 2D frequency domain system identification is set as follows. Given frequency response measurements $\{E^{J,M}_{j,m}\}_{j=1,2,\dots J, m=1,2,\dots M}$ from a 2D system

$$E^{J,M}_{j,m} = f\left(e^{it_j}, e^{is_m}\right), \tag{5}$$

where $f\left(e^{it_j}, e^{is_m}\right) = T\left(e^{-it_j}, e^{-is_m}\right)$, $t_j = \frac{2\pi(j-1)}{J}$, $s_m = \frac{2\pi(m-1)}{M}$, J and M are even, find rational functions f^N of real coefficients to f such that in the H^2-norm sense

$$\lim_{N\to\infty} f^N = f. \tag{6}$$

3 | PRELIMINARIES

As the proposed real coefficients rational approximation method depends on 1D-UAFD, we provide a brief introduction to 1D-UAFD.[21]

Let $\mathbb{D}$ denote the unit disc. The Hardy $H^2(\mathbb{D})$ space is defined as

$$H^2(\mathbb{D}) = \left\{ f(z) : f \text{ is analytic in } \mathbb{D}, \quad \text{and } \sup_{0\le r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2 dt < \infty \right\}. \tag{7}$$

The Szegö kernel in $H^2(\mathbb{D})$, namely,

$$e_a(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}, \quad a \in \mathbb{D}, \tag{8}$$

enjoys the reproducing property, viz,

$$\langle f, e_a\rangle = \sqrt{1-|a|^2} f(a), \quad f \in H^2(\mathbb{D}). \tag{9}$$

Let $a_1, a_2, \dots, a_n, \dots$ be a sequence of complex numbers in $\mathbb{D}$. Applying the Gram-Schmit orthogonalization process to $e_{a_1}, e_{a_2}, \dots, e_{a_n}, \dots$, the associated rational orthonormal system, known as the Takenaka-Malmquist system, is obtained as

$$B_k(z) = \frac{\sqrt{1-|a_k|^2}}{1-\overline{a_k}z}\prod_{j=1}^{k-1}\frac{z-a_j}{1-\bar{a}_j z}. \tag{10}$$

Let $f \in H^2(\mathbb{D})$. Through the Nevanlinna factorization,[32] there holds

$$f(z) = I(z)O(z), \tag{11}$$

where

$$O(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log|f\left(e^{it}\right)|dt\right\} \tag{12}$$

is the outer function and $I(z)$ is the inner function. Furthermore, $I(z) = B(z)S(z)$, where

$$B(z) = z^n \prod_{z_k\neq 0}\frac{|z_k|}{z_k}\frac{z_k - z}{1-\bar{z}_k z} \tag{13}$$

is the Blaschke product part that collects all the zeros of $f(z)$, and $S(z)$ is the singular inner function part given by a regular Borel measure on the circle singular to the Lebesgue measure.

Let $f_1 = f \in H^2(\mathbb{D})$. Taking Nevanlinna factorization[32] into consideration, we have $f_1(z) = O_1(z)I_1(z)$. To further decompose the outer function $O_1(z)$ with rapid convergence, we apply the *maximal selection principle*.[21] Because $O_1 \in H^2(\mathbb{D})$, there exists $a_1 \in \mathbb{D}$ satisfying

$$a_1 = \arg\max_{a\in\mathbb{D}} |\langle O_1, e_a\rangle|. \tag{14}$$

Accordingly, $f(z)$ can be further decomposed into

$$f(z) = I_1(z)\langle O_1, e_{a_1}\rangle e_{a_1}(z) + I_1(z)\frac{z-a_1}{1-\overline{a_1}z} f_2(z), \tag{15}$$

where $f_2(z) = \frac{O_1(z)-\langle O_1, e_{a_1}\rangle e_{a_1}(z)}{\frac{z-a_1}{1-\overline{a_1}z}}$. Next, repeat the same process to decompose f_2, and so on. After N steps, we have

$$f(z) = \sum_{k=1}^{N} \langle O_k, e_{a_k}\rangle I^{(k)}(z)B_k(z) + I^{(N)}(z)\prod_{k=1}^{N}\frac{z-a_k}{1-\overline{a_k}z} f_{N+1}(z), \tag{16}$$

where $I^{(k)}(z) = \prod_{j=1}^{k} I_j(z)$, $f_k(z) = I_k(z)O_k(z)$, and $f_{k+1}(z) = \frac{O_k(z)-\langle O_k, e_{a_k}\rangle e_{a_k}(z)}{\frac{z-a_k}{1-\overline{a_k}z}}$. Apart from the Nevanlinna factorization, the key step in the 1D-UAFD is that each selection of a_k satisfies the maximal selection principle

$$a_k = \arg\max_{a\in\mathbb{D}} |\langle O_k, e_a\rangle|. \tag{17}$$

It was proved in Qian[21] that the above decomposition (16) is orthogonal and

$$f(z) = \sum_{k=1}^{\infty} \langle O_k, e_{a_k}\rangle I^{(k)}(z)B_k(z). \tag{18}$$

Thus, through performing the Nevanlinna factorization and the maximal selection principle, 1D-UAFD generates an adaptive orthonormal system.

4 | ADAPTIVE RATIONAL APPROXIMATION TO 2D FREQUENCY DOMAIN SYSTEM IDENTIFICATION

In this section, given measurements $\left\{E_{j,m}^{J,M}\right\}$ of $f(z,w) \in H^2(\mathbb{D}^2)$ on the unit poly-circle $\partial\mathbb{D} \times \partial\mathbb{D}$, we give a two-step procedure to reconstruct f. We first construct a function $\tilde{f}$ in $H^2(\mathbb{D}^2)$ to approximate f by using the given measurements. Then we apply 2D-PUAFD to obtain rational approximations of real coefficients to $\tilde{f}$.

4.1 | First-step procedure

The first step is to construct a function $\tilde{f}(z,w)$ in $H^2(\mathbb{D}^2)$ as an approximation to the true function $f(z,w)$. Through using the tensor type Cauchy integral, $\tilde{f}$ is computed through the formula

$$\tilde{f}(z,w) = -\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{\sum_{j=1}^{J}\sum_{m=1}^{M} E_{j,m}^{J,M}\chi_{j,m}(t,s)}{(e^{it}-z)(e^{is}-w)} de^{it}de^{is}, \tag{19}$$

where $\chi_{j,m}(t,s)$ is the 2D step function defined as

$$\begin{cases} \chi_{[t_j,t_{j+1})\times[s_m,s_{m+1})}(t,s), & j \in \left\{1, \dots, \frac{J}{2}\right\}, m \in \left\{1, \dots, \frac{M}{2}\right\} \\ \chi_{[t_j,t_{j+1})\times(s_{m-1},s_m]}(t,s), & j \in \left\{1, \dots, \frac{J}{2}\right\}, m \in \left\{\frac{M}{2}+2, \dots, M+1\right\} \\ \chi_{(t_{j-1},t_j]\times[s_m,s_{m+1})}(t,s), & j \in \left\{\frac{J}{2}+2, \dots, J+1\right\}, m \in \left\{1, \dots, \frac{M}{2}\right\} \\ \chi_{(t_{j-1},t_j]\times(s_{m-1},s_m]}(t,s), & j \in \left\{\frac{J}{2}+2, \dots, J+1\right\}, m \in \left\{\frac{M}{2}+2, \dots, M+1\right\} \end{cases}$$

and $\{E_{j,m}^{J,M}\}$ is given by (5). When J and M are large enough, $\sum_{j=1}^{J}\sum_{m=1}^{M} E_{j,m}^{J,M}\chi_{j,m}(t,s)$ is an approximation of $f(e^{it}, e^{is})$. In addition, Hardy space theory implies that $\tilde{f}(z,w)$ is in $H^2(\mathbb{D}^2)$ and

$$\|\tilde{f} - f\| \le C\left\|\sum_{j=1}^{J}\sum_{m=1}^{M} E_{j,m}^{J,M}\chi_{j,m} - f\right\|_{L^2}, \tag{20}$$

where C is a constant. Equation 19 shows that

$$\tilde{f}(z,w)=-\frac{1}{4\pi^2}\left[\sum_{j=1}^{\frac{J}{2}}\sum_{m=1}^{\frac{M}{2}}\ln\left(\frac{e^{it_{j+1}}-z}{e^{it_j}-z}\right)\ln\left(\frac{e^{is_{m+1}}-w}{e^{is_m}-w}\right)E_{j,m}^{J,M}+\sum_{j=\frac{J}{2}+2}^{J+1}\sum_{m=\frac{M}{2}+2}^{M+1}\ln\left(\frac{e^{it_j}-z}{e^{it_{j-1}}-z}\right)\ln\left(\frac{e^{is_m}-w}{e^{is_{m-1}}-w}\right)E_{j,m}^{J,M}\right.$$
$$\left.+\sum_{j=1}^{\frac{J}{2}}\sum_{m=\frac{M}{2}+2}^{M+1}\ln\left(\frac{e^{it_{j+1}}-z}{e^{it_j}-z}\right)\ln\left(\frac{e^{is_m}-w}{e^{is_{m-1}}-w}\right)E_{j,m}^{J,M}+\sum_{j=\frac{J}{2}+2}^{J+1}\sum_{m=1}^{\frac{M}{2}}\ln\left(\frac{e^{it_j}-z}{e^{it_{j-1}}-z}\right)\ln\left(\frac{e^{is_{m+1}}-w}{e^{is_m}-w}\right)E_{j,m}^{J,M}\right].$$

Since $E_{j,m}^{J,M}=f(e^{it_j},e^{is_m})$ and $f(e^{it_j},e^{is_m})=\overline{f(e^{-it_j},e^{-is_m})}$, we have

$$\tilde{f}(z,w)=\overline{\tilde{f}(\bar{z},\bar{w})}$$
$$=-\frac{1}{4\pi^2}\left[\sum_{j=1}^{\frac{J}{2}}\sum_{m=1}^{\frac{M}{2}}\ln\left(\frac{e^{it_{j+1}}-z}{e^{it_j}-z}\right)\ln\left(\frac{e^{is_{m+1}}-w}{e^{is_m}-w}\right)E_{j,m}^{J,M}+\sum_{j=1}^{\frac{J}{2}}\sum_{m=1}^{\frac{M}{2}}\ln\left(\frac{e^{-it_{j+1}}-z}{e^{-it_j}-z}\right)\ln\left(\frac{e^{-is_{m+1}}-w}{e^{-is_m}-w}\right)\overline{E_{j,m}^{J,M}}\right.$$
$$\left.+\sum_{j=1}^{\frac{J}{2}}\sum_{m=\frac{M}{2}+2}^{M+1}\ln\left(\frac{e^{it_{j+1}}-z}{e^{it_j}-z}\right)\ln\left(\frac{e^{is_m}-w}{e^{is_{m-1}}-w}\right)E_{j,m}^{J,M}+\sum_{j=1}^{\frac{J}{2}}\sum_{m=\frac{M}{2}+2}^{M+1}\ln\left(\frac{e^{-it_{j+1}}-z}{e^{-it_j}-z}\right)\ln\left(\frac{e^{-is_m}-w}{e^{-is_{m-1}}-w}\right)\overline{E_{j,m}^{J,M}}\right].$$

4.2 | Rational approximation of real coefficients

The second step is to obtain 2D rational functions of real coefficients approximating to $\tilde{f}$. We propose a 2D-PUAFD algorithm to achieve rational approximations of real coefficients. As the proposed algorithm depends on 1D-UAFD, we first modify 1D-UAFD so that it offers 1D rational approximations of real coefficients.

4.2.1 | Modify 1D-UAFD

Mi and Qian[20] give 1D rational approximations of real coefficients by modifying core 1D-AFD.[24] As given below, a key lemma in Mi and Qian[20] promotes the realization of rational approximations of real coefficients.

Lemma 4.1. *Assume that $f\in H^2(\mathbb{D})$ satisfies $f(\bar{z})=\overline{f(z)}$. When the chosen parameters $\{a_k\}_{k=1}^N$ appear as either real numbers or complex conjugate pairs, the N-partial sum of core 1D-AFD is a rational function of real coefficients in $H^2(\mathbb{D})$.*

Inspired by the above lemma, we propose the modified 1D-UAFD to allow it to provide 1D rational approximations of real coefficients. Let $F_1=f\in H^2(\mathbb{D})$. As shown in Section 3, we have $F_1(z)=O_1(z)I_1(z)$ by the Nevanlinna factorization theorem. To further decompose $O_1(z)$ rapidly, we use the *modified maximal selection principle*. Similar to the proof of *maximal selection principle* in Qian,[21] there indeed exists $a_1\in\mathbb{D}$ satisfying

$$a_1=\arg\max_{a\in\mathbb{D}}\left(|\langle O_1,e_a\rangle|^2+|\langle O_1,B_{\mathbf{a}}\rangle|^2\right), \tag{21}$$

where $B_{\mathbf{a}}(z)=e_{\bar{a}}(z)\frac{z-a}{1-\bar{a}z}=\frac{\sqrt{1-|\bar{a}|^2}}{1-az}\frac{z-a}{1-\bar{a}z}$. Thus, $O_1(z)$ can be decomposed into

$$O_1(z)=\langle O_1,e_{a_1}\rangle e_{a_1}(z)+\langle O_1,B_{\mathbf{a_1}}\rangle B_{\mathbf{a_1}}(z)+F_2(z)\frac{z-\overline{a_1}}{1-a_1z}\frac{z-a_1}{1-\overline{a_1}z},$$

where $B_{\mathbf{a_1}}(z)=\frac{\sqrt{1-|\overline{a_1}|^2}}{1-a_1z}\frac{z-a_1}{1-\overline{a_1}z}$ and $F_2(z)=\frac{O_1(z)-\langle O_1,e_{a_1}\rangle e_{a_1}(z)-\langle O_1,B_{\mathbf{a_1}}\rangle B_{\mathbf{a_1}}(z)}{\frac{z-\overline{a_1}}{1-a_1z}\frac{z-a_1}{1-\overline{a_1}z}}\in H^2(\mathbb{D})$. Accordingly,

$$f(z)=I_1(z)\left[\langle O_1,e_{a_1}\rangle e_{a_1}(z)+\langle O_1,B_{\mathbf{a_1}}\rangle B_{\mathbf{a_1}}(z)\right]+I_1(z)F_2(z)\frac{z-\overline{a_1}}{1-a_1z}\frac{z-a_1}{1-\overline{a_1}z}.$$

Next, repeat the same process for $F_2(z)$ as for $F_1(z)$, and so on. After N steps, we obtain

$$f(z)=\sum_{k=1}^{N}I^{(k)}(z)\left[\langle O_k,e_{a_k}\rangle e_{a_k}(z)+\langle O_k,B_{\mathbf{a_k}}\rangle B_{\mathbf{a_k}}(z)\right]\prod_{j=0}^{k-1}A_{\mathbf{a_j}}(z)+I^{(N)}(z)F_{N+1}(z)\prod_{j=1}^{N}A_{\mathbf{a_j}}(z), \tag{22}$$

where

$$F_k(z)=I_k(z)O_k(z), \tag{23}$$

$$a_k = \arg\max_{a\in\mathbb{D}} \left(|\langle O_k, e_a\rangle|^2 + |\langle O_k, B_{\mathbf{a}}\rangle|^2\right), \tag{24}$$

$$A_{\mathbf{a_j}}(z) = \begin{cases} 1, & j = 0 \\ \frac{z-\overline{a_j}}{1-a_j z}\frac{z-a_j}{1-\overline{a_j}z}, & j \geq 1 \end{cases}, \tag{25}$$

$$I^{(k)}(z) = \prod_{j=1}^{k} I_j(z), B_{\mathbf{a_k}}(z) = \frac{\sqrt{1-|\overline{a_k}|^2}}{1-a_k z}\frac{z-a_k}{1-\overline{a_k}z}, \tag{26}$$

and

$$F_{k+1}(z) = \frac{O_k - \langle O_k, e_{a_k}\rangle e_{a_k}(z) - \langle O_k, B_{\mathbf{a_k}}\rangle B_{\mathbf{a_k}}(z)}{A_{\mathbf{a_k}}(z)}. \tag{27}$$

Denote the N-partial sum by

$$f^N(z) = \sum_{k=1}^{N} I^{(k)}(z)\left[\langle O_k, e_{a_k}\rangle e_{a_k}(z) + \langle O_k, B_{\mathbf{a_k}}\rangle B_{\mathbf{a_k}}(z)\right]\prod_{j=0}^{k-1} A_{\mathbf{a_j}}(z). \tag{28}$$

We note that in (28) all the decomposing terms of $f(z)$ are mutually orthogonal. This can be proved by using Cauchy's theorem when calculating the inner product between any two of the above decomposing terms. On the other hand, the modified 1D-UAFD is convergent. The proof of its convergence is similar to that of the convergence of 1D-UAFD. We omit the proof here.

Theorem 4.1. *For an arbitrary function $f \in H^2(\mathbb{D})$ under the modified maximal selection principle (24), in the H^2-norm sense, we have*

$$\lim_{N\to\infty} f^N = f. \tag{29}$$

Furthermore, if $f \in H^2(\mathbb{D})$ satisfies the property $f(\bar{z}) = \overline{f(z)}$ and can be analytically continued to outside the closed unit disc, Hardy space theory implies that the singular inner function $S(z)$ of f is constant and that its Blaschke product has only a finite number of zeros that are either real numbers or complex conjugate pairs. These facts imply that the inner function part $I^{(k)}(z)$ in (22) is merely a rational function of real coefficients. Meanwhile, Lemma 4.1 indicates that $\langle O_k, e_{a_k}\rangle e_{a_k}(z) + \langle O_k, B_{\mathbf{a_k}}\rangle B_{\mathbf{a_k}}(z)$ is also a rational function of real coefficients. All these mean that the following theorem holds.

Theorem 4.2. *Suppose that $f \in H^2(\mathbb{D})$ satisfies the property $f(\bar{z}) = \overline{f(z)}$ and can be analytically continued to outside the closed unit disc. In the process of modified 1D-UAFD, the N-partial sum $f^N(z)$ is a rational function of real coefficients.*

For the algorithm design of 1D-UAFD, Mai et al[33] proposed first factorizing out a finite Blaschke product $B(z)$ by finding a finite number of zeros of f and then obtaining the outer function $O(z)$ from $O(z) = f(z)/B(z)$. Based on the theory of the modified 1D-UAFD, we modify the algorithm of Mai et al to obtain Algorithm 1.

Algorithm 1 Modified 1D-UAFD

Input: A function f in $H^2(\mathbb{D})$ and decomposition level N.
Output: Approximation result f^N.

Initialize $F_1 = f$, $I^{(0)} = 1$, $A^{(\mathbf{a}_0)} = 1$, and $f^N = 0$.
for $k = 1$ to N **do**
Get zeros of F_k from reference[33];
Get I_k from (13) and $O_k = F_k/I_k$ from (23);
Get $a_k = \arg\max_{a\in\mathbb{D}}(|\langle O_k, e_a\rangle|^2 + |\langle O_k, B_{\mathbf{a}}\rangle|^2)$ from (24);
Get $A_{\mathbf{a}_k}$ from (25), $I^{(k)} = I^{(k-1)}I_k$ and $B_{\mathbf{a}_k}$ from (26);
Get $G_k = \langle O_k, e_{a_k}\rangle e_{a_k} + \langle O_k, B_{\mathbf{a}_k}\rangle B_{\mathbf{a}_k}$ and $A^{(\mathbf{a}_k)} = A^{(\mathbf{a}_{k-1})}A_{\mathbf{a}_k}$;
Get $f^N = f^N + I^{(k)}G_kA^{(\mathbf{a}_k)}$ from (28);
Get $F_{k+1} = (O_k - G_k)/A_{\mathbf{a}_k}$ from (27);
end for
return f^N.

Remark 4.1. The 1D-UAFD algorithm of a Hardy $H^2(\mathbb{D})$ space signal f involves calculating its inner and outer functions. There are various algorithms first to calculate the outer function $O(z)$. In general, computing the outer function $O(z)$ involves computing the Hilbert transform of $\log|f(e^{it})|$. Due to the reason that $f(e^{it})$ may approach zero, the computation of $O(z)$ may be unstable. A lot of methods have been proposed to make the computation of Hilbert transform stable. Some methods are to regularize $\log|f(e^{it})|$ by adding a small positive number[23] or small pure sinusoid.[34] However, adding a small positive number or small pure sinusoid in each iteration may result in big errors after iterations. To avoid this deficiency[35] gave a mechanical quadrature algorithm to show better stability when calculating the Hilbert transform of $\log|f(e^{it})|$. Due to the instability of the computation of Hilbert transform, Tan and Qian[36] found a way to directly extract the outer function $O(z)$ without computing the Hilbert transform of $\log|f(e^{it})|$. This method is effective for rational functions, but it does not work well for general Hardy space functions. Therefore, we adopt the method of Mai et al[33] first to calculate the inner function $B(z)$. They showed that the method of unwinding only a finite Blaschke product part guarantees the applicability of 1D-UAFD.

4.2.2 | 2D-PUAFD

For an arbitrary function $f(z,w)$ in $H^2(\mathbb{D}^2)$, it holds that $f(z,w)=\sum_{l\geq 0,n\geq 0} c_{ln}z^l w^n$, where $\sum_{l\geq 0,n\geq 0}|c_{ln}|^2<\infty$. In this subsection, we do not assume $c_{ln}=0$ for $l=0$ or $n=0$. By setting $f_1=f$ and

$$h(z,w)\triangleq f_1(z,w)-f_1(0,w)-f_1(z,0)+f_1(0,0), \tag{30}$$

we have $h(z,w)=0$ for $z=0$ and any $w\in\mathbb{D}$, and $h(z,w)=0$ for $w=0$ and any $z\in\mathbb{D}$. There then follows

$$h(z,w)=zwf_2(z,w), \tag{31}$$

where f_2 belongs to $H^2(\mathbb{D}^2)$. We thus obtain

$$f(z,w)=zwf_2(z,w)+f_1(0,w)+f_1(z,0)-f_1(0,0), \tag{32}$$

where

$$f_2(z,w)=\frac{f_1(z,w)-f_1(0,w)-f_1(z,0)+f_1(0,0)}{zw}. \tag{33}$$

Next, by repeating the same process on $f_2(z,w)$ as on $f_1(z,w)$, we get

$$f(z,w)=(zw)^2f_3(z,w)+zw\left[f_2(0,w)+f_2(z,0)-f_2(0,0)\right]+f_1(0,w)+f_1(z,0)-f_1(0,0),$$

where

$$f_3(z,w)=\frac{f_2(z,w)-f_2(0,w)-f_2(z,0)+f_2(0,0)}{zw}. \tag{34}$$

Repeating the process for N times, we obtain

$$f(z,w)=\sum_{k=1}^{N}(zw)^{k-1}\left[f_k(0,w)+f_k(z,0)-f_k(0,0)\right]+(zw)^N f_{N+1}(z,w), \tag{35}$$

where

$$f_{k+1}(z,w)=\frac{f_k(z,w)-f_k(0,w)-f_k(z,0)+f_k(0,0)}{zw}. \tag{36}$$

Denoting

$$S_N(f)(z,w)=\sum_{k=1}^{N}(zw)^{k-1}\left[f_k(0,w)+f_k(z,0)-f_k(0,0)\right] \tag{37}$$

and

$$R_N(f)(z,w)=(zw)^N f_{N+1}(z,w), \tag{38}$$

where f_{N+1} belongs to $H^2(\mathbb{D}^2)$, we have

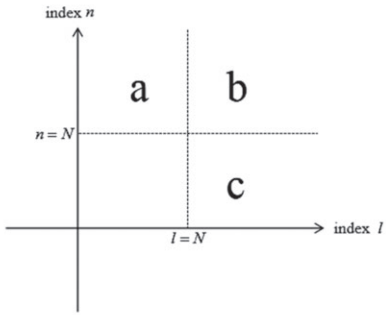

FIGURE 1 Energy distribution regions of $R_N(f)$ and $\tilde{R}_N(f)$

$$f(z,w) = S_N(f)(z,w) + R_N(f)(z,w). \tag{39}$$

Denote the classical 2D Fourier series N-partial sum by

$$f(z,w) = \tilde{S}_N(f)(z,w) + \tilde{R}_N(f)(z,w), \tag{40}$$

where $\tilde{S}_N(f)(z,w) = \sum_{0\le l,n\le N-1} c_{ln} z^l w^n$. Fourier analysis theory implies that $\|\tilde{R}_N(f)\|^2 = \sum_{l \text{ or } n\ge N} |c_{ln}|^2 \to 0$. Figure 1 shows that $\widetilde{R}_N(f)$ spreads over the region $\mathbf{a} \cup \mathbf{b} \cup \mathbf{c}$, whereas $R_N(f)$ spreads over only the region $\mathbf{b}$. We further show that $S_N(f)$ converges rapidly to f in the H^2-norm. Moreover, if f can be holomorphically continued to outside the closed unit poly-disc, an exponential decay rate can be achieved.

Theorem 4.3. *Suppose $f(z,w) \in H^2(\mathbb{D}^2)$. It holds that*

$$\|f - S_N(f)\| \to 0 \quad for \quad N \to \infty. \tag{41}$$

Furthermore, if f can be holomorphically continued to $(1+\sigma_1)\mathbb{D} \times (1+\sigma_2)\mathbb{D} = \{(z,w) | \, |z| < 1+\sigma_1 \text{ and } |w| < 1+\sigma_2\}$, $\sigma_i > 0, i = 1,2$, we have the H^2-norm of $R_N(f)$ decays exponentially.

Proof. Because of the uniqueness of power series expansion of a holomorphic function, $R_N(f)$ is equal to the sum of the power series entries $c_{ln}z^l w^n$ with both $l \ge N$ and $n \ge N$. The energy of $R_N(f)$ is the square sum of the norms of the Fourier coefficients indexed by the integer pairs in the region $\mathbf{b}$ of Figure 1, that is, $\|f - S_N(f)\|^2 = \sum_{l\ge N, n\ge N} |c_{ln}|^2 \to 0, \quad N \to \infty$. In addition, let $f(z,w) \in H^2(\mathbb{D}^2)$ be holomorphically continued to $(1+\sigma_1)\mathbb{D}\times(1+\sigma_2)\mathbb{D}$, $\sigma_i > 0, i = 1,2$. By letting $\delta_1 = 1 + \frac{\sigma_1}{2}$ and $\delta_2 = 1 + \frac{\sigma_2}{2}$, we have $c_{ln}\delta_1^l \delta_2^n \to 0$ for either $l \to \infty$ or $n \to \infty$. Thus, we can find a positive number C_1 such that $|c_{ln}| < \frac{C_1}{\delta_1^l \delta_2^n}$ for any $l \ge 0$ and $n \ge 0$. This yields $\|f - S_N(f)\|^2 \le \sum_{l\ge N, n\ge N} \frac{C_1^2}{\delta_1^{2l}\delta_2^{2n}} = C_1^2 \frac{1}{\delta_1^{2N}\delta_2^{2N}} \frac{\delta_2^2}{\delta_2^2-1} \frac{\delta_1^2}{\delta_1^2-1}$. So we have the desired result $\|f - S_N(f)\| \le C_2 a_1^N$, where $C_2 = \frac{C_1\delta_1\delta_2}{\sqrt{(\delta_1^2-1)(\delta_2^2-1)}}$ and $a_1 = \frac{1}{\delta_1\delta_2} < 1$.

□

The iterative process (35) for a Hardy $H^2(\mathbb{D}^2)$ space function f that can be holomorphically continued to outside the closed unit poly-disc shows that $f_k(z,w)$ belongs to $H^2(\mathbb{D}^2)$ for any $k \ge 1$ and can be holomorphically continued to outside the closed unit poly-disc. From this, we can show readily that the univariate functions $f_k(0,w)$ and $f_k(z,0)$ are both in $H^2(\mathbb{D})$ and can be analytically continued to outside the closed unit disc. In addition, since $f(\bar{z},\overline{w}) = \overline{f(z,w)}$, $f_k(0,w)$ and $f_k(z,0)$ enjoy the properties that $f_k(0,\bar{w}) = \overline{f_k(0,w)}$ and $f_k(\bar{z},0) = \overline{f_k(z,0)}$, respectively. We note from Theorem 4.2 that both $f_k(0,w)$ and $f_k(z,0)$ can be approximated by 1D rational functions of real coefficients using the modified 1D-UAFD. Further combining (22) and (35), we obtain modified 1D-UAFD–based 2D-PUAFD. Algorithm 2 illustrates how the proposed 2D-PUAFD is implemented. Such the algorithm achieves 2D rational approximations of real coefficients to transfer functions in the F-MM II model.

Given processes (22) and (35), we get the 2D partial unwinding adaptive rational system consisting of $\left\{(zw)^{m-1}\right\}_{m=1}^{\infty}$, $\left\{(zw)^{j-1}I^{(k)}(z)e_{a_k}(z)\prod_{p=0}^{k-1}A_{\mathbf{a_p}}(z)\right\}_{j,k=1}^{\infty}$, $\left\{(zw)^{j-1}I^{(k)}(z)B_{\mathbf{a_k}}(z)\prod_{p=0}^{k-1}A_{\mathbf{a_p}}(z)\right\}_{j,k=1}^{\infty}$, $\left\{(zw)^{l-1}I^{(n)}(w)e_{a_n}(w)\prod_{q=0}^{n-1}A_{\mathbf{a_q}}(w)\right\}_{l,n=1}^{\infty}$, and $\left\{(zw)^{l-1}I^{(n)}(w)B_{\mathbf{a_n}}(w)\prod_{q=0}^{n-1}A_{\mathbf{a_q}}(w)\right\}_{l,n=1}^{\infty}$. The adaptivity of the above system is due to that of the modified 1D-UAFD.

Algorithm 2 2D-PUAFD

Input: 2D signal $f(e^{it}, e^{is})$, $t, s \in [0, 2\pi)$, and decomposition level N.
Output: Approximation result f^N.

1: Initialize $f_1 = f$ and $f^N = 0$.
2: **for** $k = 1$ to N **do**
3: Get $f_k(0, e^{is})$, $f_k(e^{it}, 0)$, and $f_k(0, 0)$ from (35);
4: Get $T_k = (e^{it}e^{is})^{k-1}$ and choose the modified 1D-UAFD level n;
5: Get the 1D rational approximations of real coefficients $f_k^n(0, e^{is})$ of $f_k(0, e^{is})$ and $f_k^n(e^{it}, 0)$ of $f_k(e^{it}, 0)$ by Algorithm 1;
6: Get $f^N = f^N + T_k[f_k^n(0, e^{is}) + f_k^n(e^{it}, 0) - f_k(0, 0)]$ from (37);
7: Get $f_{k+1} = [f_k - f_k(0, e^{is}) - f_k(e^{it}, 0) + f_k(0, 0)]/(e^{it}e^{is})$ from (36);
8: **end for**
9: **return** f^N.

The computational complexity of Algorithm 2 is computed below. An input 2D discrete signal f is assumed to be of size $K \times K$.

- The complexity of calculating the zeros of the finite Blaschke product in step 5 is $\mathcal{O}(MK)$,[37] where M is the number of the discrete points in $\mathbb{D}$.
- The complexity of choosing parameters through the modified maximal selection principle in step 5 is $\mathcal{O}(MK^2)$.[38]
- The computational complexity of step 6 is $\mathcal{O}(K^2)$.

Therefore, the computational complexity of the 2D-PUAFD algorithm is $\mathcal{O}(K^2)$.

5 | EXPERIMENTAL RESULTS

We use the proposed 2D-PUAFD to approximate the transfer functions given in Valenzuela and Salvia.[29] In Valenzuela and Salvia,[29] Valenzuela and Salvia identify the transfer functions by directly computing the coefficients of the polynomials of two complex variables in the numerator and denominator. As well as comparing the proposed algorithm with the method in Valenzuela and Salvia,[29] we will also compare it with 2D-FS. In fact, 2D-FS is the generalization of the FIR model in the 2D case.

We use the relative error (RE) and color graph to test the accuracy of the approximation function. Given the measurements $E_{j,m}^{J,M} = f(e^{it_j}, e^{is_m})$, $j = 1, 2, \ldots J$, $m = 1, 2, \ldots M$, the RE between f and its approximation f^N is defined as

$$\mathrm{RE} = \frac{\sum_{j=1}^{J}\sum_{m=1}^{M}\left|f^N\left(e^{it_j}, e^{is_m}\right) - f\left(e^{it_j}, e^{is_m}\right)\right|^2}{\sum_{j=1}^{J}\sum_{m=1}^{M}\left|f\left(e^{it_j}, e^{is_m}\right)\right|^2}, \tag{42}$$

where $t_j = \frac{2\pi(j-1)}{J}$, $s_m = \frac{2\pi(m-1)}{M}$, and N is the decomposition level. Color graph is drawn to show $\log_{10}\left|f^N\left(e^{it_j}, e^{is_m}\right)\right|^2$ that reflects the details of f^N at the frequency $\left(e^{it_j}, e^{is_m}\right)$.

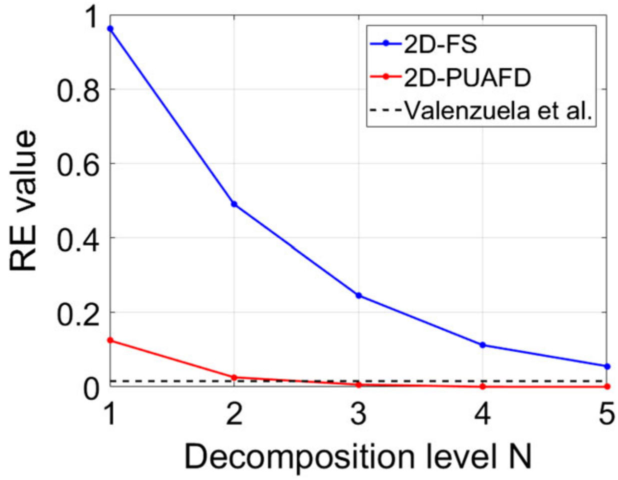

FIGURE 2 Comparison of relative errors (REs) between different methods. 2D-FS, two-dimensional Fourier series; 2D-PUAFD, two-dimensional partial unwinding adaptive Fourier decomposition [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 1 Comparison of REs between 2D-FS and 2D-PUAFD at decomposition level $N = 3, 4, 5$

N	2D-FS	2D-PUAFD
3	2.46e-01	**6.10e-03**
4	1.12e-01	**2.48e-04**
5	5.57e-02	**1.01e-04**

Abbreviations: 2D-FS, two-dimensional Fourier series; 2D-PUAFD, two-dimensional partial unwinding adaptive Fourier decomposition; RE, relative error.

Example 5.1. The transfer function is

$$T(z_1, z_2) = \frac{\left(1 + z_1^{-1}\right) + \left(3 + z_1^{-1}\right) z_2^{-1}}{\left(1 + .6z_1^{-1} + .36z_1^{-2} + .048z_1^{-3}\right)\left(1 + .7z_2^{-1}\right)}.$$

Through the mappings $\frac{1}{z_1} \to z$ and $\frac{1}{z_2} \to w$, $T(z_1, z_2)$ is transformed into a function $f(z, w)$ in $H^2(\mathbb{D}^2)$ that can be holomorphically continued to outside the closed unit poly-disc. We apply the proposed method and 2D-FS to f and select $l = m = 256$ discrete points in (42).[29] gives the RE of 0.0154. Figure 2 compares the REs between 2D-FS and the proposed 2D-PUAFD at different decomposition levels and method in Valenzuela and Salvia.[29] Table 1 gives the comparison values of RE between 2D-FS and 2D-PUAFD at decomposition level $N = 3, 4, 5$. The comparison results of color graphs between 2D-FS and the proposed 2D-PUAFD at $N = 5$ and method in Valenzuela and Salvia[29] are displayed in Figure 3. Figure 2 shows that when N increases the REs of 2D-FS and 2D-PUAFD decrease. Besides, the RE of 2D-PUAFD is significantly smaller than that of 2D-FS at the same decomposition level. Meanwhile, 2D-PUAFD achieves smaller RE starting from $N = 3$ than the method in Valenzuela and Salvia.[29] We further see from Table 1 that when $N = 5$, the RE of 2D-PUAFD is two orders of magnitude smaller than that of 2D-FS and method in Valenzuela and Salvia,[29] respectively. Figure 3 shows that the proposed 2D-PUAFD at $N = 5$ demonstrates best detail effect at each frequency between the tested methods.

Valenzuela and Salvia[29] give estimation of the same degree as the transfer function in below

$$\hat{T}(z_1, z_2) = \frac{1 + 1.0364z_1^{-1} + 3.2713z_2^{-1} + 1.0485z_1^{-1}z_2^{-1}}{\left(1 - .6027z_1^{-1} + .3897z_1^{-2} - .0551z_1^{-3}\right)\left(1 + .6978z_2^{-1}\right)}.$$

Although the proposed method cannot give the same degree estimation, its fast convergence makes it better to approximate the transfer function.

Example 5.2. The transfer function is

$$T(z_1, z_2) = \frac{\left(2 + z_1^{-1}\right) + \left(3 - .5z_1^{-1}\right) z_2^{-1}}{\left(1 - 1.6z_1^{-1} + 1.4z_1^{-2} - .48z_1^{-3}\right)\left(1 - .6z_2^{-1} + .25z_2^{-2}\right)}.$$

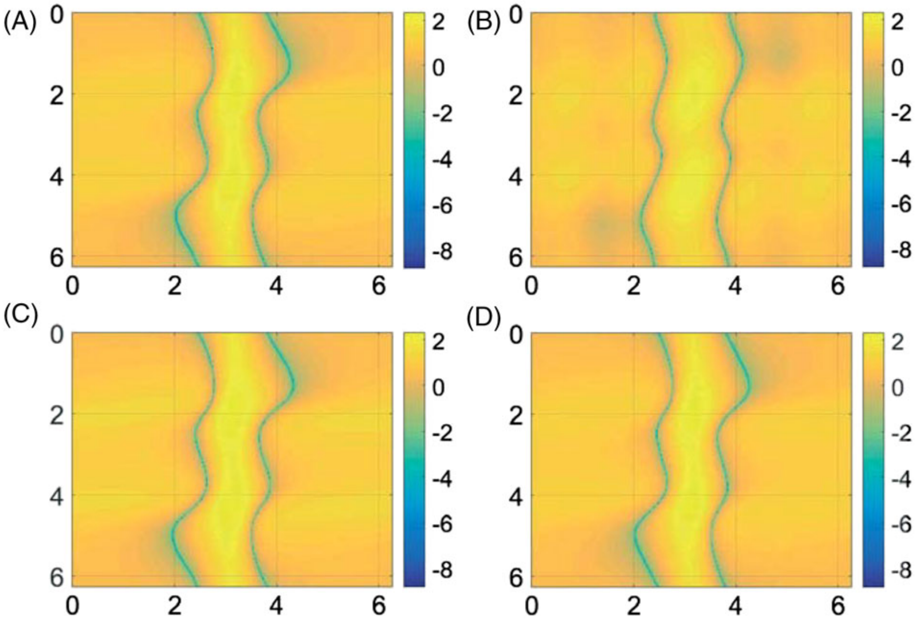

FIGURE 3 Color graph comparison of three methods. A, Original transfer function. B, 2D-FS at $N = 5$. C, Method in Valenzuela and Salvia.[29] D, 2D-PUAFD at $N = 5$. 2D-FS, two-dimensional Fourier series; 2D-PUAFD, two-dimensional partial unwinding adaptive Fourier decomposition

TABLE 2 Comparison of REs between 2D-FS and 2D-PUAFD at different decomposition levels

N	2D-FS	2D-PUAFD
1	9.85e-01	**7.41e-01**
2	1.36e-01	**8.80e-02**
3	5.72e-02	**3.80e-02**
4	1.87e-02	**3.20e-03**
5	1.23e-02	**8.98e-04**

Abbreviations: 2D-FS, two-dimensional Fourier series; 2D-PUAFD, two-dimensional partial unwinding adaptive Fourier decomposition; RE, relative error

Similar to the transformation of the transfer function in the above example, we apply the proposed method to the transformation function *f*. Here, we also choose $l = m = 256$ discrete points. Table 2 compares the REs between 2D-FS and 2D-PUAFD at different decomposition levels. The RE given in Valenzuela and Salvia[29] is 0.0027. We omit the color graphs for this example.

As given in Table 2, the REs of 2D-FS and 2D-PUAFD become increasingly smaller with the increase of *N*. Meanwhile, 2D-PUAFD achieves smaller RE compared to 2D-FS at the same decomposition level. Furthermore, we note that the RE of 2D-PUAFD is smaller than 10^{-3} from $N = 5$. Therefore, the proposed algorithm achieves the best rational approximations of real coefficients among the tested three methods.

6 | CONCLUSIONS

In this paper, we propose the novel 2D partial unwinding adaptive Fourier decomposition 2D-PUAFD algorithm to solve 2D system identification. The proposed algorithm is based on the modified 1D-UAFD to adaptively choose parameters. It provides rational approximations of real coefficients to transfer functions. Its fast convergence offers efficient system identification. Further study on the system identification with noise will be explored in future work.

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CONFLICTS OF INTEREST

The authors declare that they have no conflict of interest.

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