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Adaptive Fourier decomposition in $\mathbb{H}^p$

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In this paper, we study decomposition of functions in Hardy spaces $\mathbb{H}^p(\mathbb{T})(1 < p < \infty)$. First, we will give a direct application of adaptive Fourier decomposition (AFD) of $\mathbb{H}^2(\mathbb{T})$ to functions in $\mathbb{H}^p(\mathbb{T})$. Then, we study adaptive decomposition by the system

$$\mathfrak{D} := \left\{ e_a(z) = \frac{A_{a,p}}{1 - \bar{a}z}, a \in \mathbb{D} \right\}, \tag{1}$$

where $A_{a,p}$ is the normalization factor making $e_a(z)$ to be of unit p-norm. Under the proposed decomposition procedure, we show that every $f \in \mathbb{H}^p(\mathbb{T})$ can be effectively expressed by a linear combination of $\{e_{a_n}(z)\}_{n=1}^{+\infty}$. We give a maximal selection principle of e_{a_n} at the nth step and prove the convergence.

KEYWORDS
adaptive fourier decomposition, Hardy space, supporting functional, Szegö kernel

1 | INTRODUCTION

Let X be a Banach space. We say $\mathfrak{D} = \{g_\alpha, \alpha \in \Gamma\} \subset X$ is a dictionary of X, if it satisfies $\|g_\alpha\|_X = 1$, and Span$\{\mathfrak{D}\}$ is dense in X. By the terminology adaptive Fourier decomposition (AFD), we refer to approximations in the $\mathbb{H}^2(\mathbb{T})$ space by using linear combinations of the analytic Szegö kernels. The earlier and main studies on AFD were presented in previous studies[1,2] when studying the problem of signal decomposition into mono-components.[3,4] In the same spirit but restricted to Hilbert spaces, the adaptive approximation under a dictionary can be found in the study of projection pursuit regression and neural network training.[5] In the series studies, S. Mallat gave a matching pursuit algorithm in real Hilbert spaces and applied it to signal decomposition with a so-called time-frequency dictionary.[6,7] The Hilbert space matching pursuit algorithm is also called pure greedy algorithm (PGA) being referred to Temlyakov's work.[8] Based on PGA, similar algorithms such as weak greedy algorithm (WGA), Chebyshev threshold greedy algorithm (TGA), and relaxed greedy algorithm (RGA) in Hilbert spaces and in real Banach spaces with smoothness were studied.[9-12]

The present paper gives two AFD-type approximations to functions in the $\mathbb{H}^p(\mathbb{T})$ spaces in the unit disc for $1 < p < \infty$ other than $p = 2$. In the sequel, we use $\mathbb{D}$ and $\mathbb{T}$ for the unit disc and the unit circle, respectively. Below, we give a quick summary of the AFD algorithm for the Hardy space $\mathbb{H}^2(\mathbb{T})$, where the dictionary consists of the normalized Szegö kernel of the context, namely

$$\mathfrak{D} := \left\{ e_a(z) = \frac{\sqrt{1 - |a|^2}}{1 - \bar{a}z}, a \in \mathbb{D} \right\}. \tag{2}$$

If a sequence $\{a_n\}_{n=1}^{+\infty}$ is given where repetitions are allowed, applying G-S procedure to the multiple Szegö kernels $\{\tilde{e}_{a_n}\}_{n=1}^{+\infty}$, we can get an orthogonal system

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$$\mathbf{B} := \left\{ B_n(z) = \frac{\sqrt{1-|a_n|^2}}{1-\bar{a}_n z} \prod_{k=1}^{n-1} \frac{z-a_k}{1-\bar{a}_k z}, n = 1, 2... \right\}, \tag{3}$$

which is called T-M system.[13] Traditional studies are based on the hyperbolic nonseparability condition

$$\sum_{n=1}^{+\infty} (1-|a_n|) = +\infty, \tag{4}$$

which makes the system to be a basis. In AFD, for any given $f \in \mathbb{H}^2(\mathbb{T})$, we adaptively select the parameter a_n based on a maximal selection principle,[1] and the algorithm is given by

$$| < \left(\prod_{k=1}^{n-1} \frac{1-\bar{a}_k z}{z-a_k} \right) f_{n-1}, e_{a_n} > | = \sup_{a \in \mathbb{D}} | < \left(\prod_{k=1}^{n-1} \frac{1-\bar{a}_k z}{z-a_k} \right) f_{n-1}, e_a > |,$$

$$f_n = f - \sum_{k=1}^{n} < f, B_k > B_k.$$

We proved that the sup is always attainable in the open unit disc and that under the maximal principle selection, the series converges to the original function whether the condition (4) holds or not. Later, we gave a convergence rate of the AFD algorithm in Qian and Wang's study.[14]

2 | APPLICATION OF AFD TO $\mathbb{H}^p(\mathbb{T})$ SPACES

As mentioned above, the system **B** becomes a basis if and only if the condition (4) holds, explicitly normalized orthogonal basis for $\mathbb{H}^2(\mathbb{T})$ and Schauder basis for $\mathbb{H}^p(\mathbb{T})$.[15] Compared with a basis, AFD adaptively decomposes functions in $\mathbb{H}^2(\mathbb{T})$ space. Empirically, a small number of terms can approximate well.[16] If we directly apply AFD to $\mathbb{H}^p(\mathbb{T})$ spaces, we can get

Theorem 2.1. *Given $f \in \mathbb{H}^p(\mathbb{T})$, $p, p_0 \in (1, +\infty)$, and $p \in (2, p_0)$ or $p \in (p_0, 2)$, then for $\forall \varepsilon > 0$, there exists a finite combination of T-M system as an approximation such that the error is less than $\varepsilon + C^{1-\alpha}\varepsilon^{\alpha}$, where C and α are constants depending on p, p_0.*

Proof. Since p is between 2 and p_0, the space $\mathbb{H}^2(\mathbb{T}) \bigcap \mathbb{H}^{p_0}(\mathbb{T})$ is dense in $\mathbb{H}^p(\mathbb{T})$. Given $f \in \mathbb{H}^p(\mathbb{T})$, for $\forall \varepsilon > 0$, there must exist $g \in \mathbb{H}^2(\mathbb{T}) \bigcap \mathbb{H}^{p_0}(\mathbb{T})$ such that $\|f - g\|_p < \varepsilon$. Apply AFD to g, we can get a finite T-M system $\{B_k\}_{k=1}^n$ such that

$$\|g - \sum_{k=1}^{n} < g, B_k > B_k\|_2 < \varepsilon. \tag{5}$$

By *Hölder* inequality, for some $\alpha \in (0, 1)$, we have

$$\|g - \sum_{k=1}^{n} < g, B_k > B_k\|_p \le \|g - \sum_{k=1}^{n} < g, B_k > B_k\|_2^{\alpha} \|g - \sum_{k=1}^{n} < g, B_k > B_k\|_{p_0}^{1-\alpha}. \tag{6}$$

On the other hand, as an equivalent condition of the Schauder basis property of the T-M system in $\mathbb{H}^p(\mathbb{T})$,[15] there exists a constant C_{p_0}, which is regardless of n, such that

$$\|g - \sum_{k=1}^{n} < g, B_k > B_k\|_{p_0} \le C_{p_0} \|g\|_{p_0}. \tag{7}$$

From inequalities (5), (6), and (7), we have

$$\begin{aligned}\|f-\sum_{k=1}^{n}<g,B_k>B_k\|_p &\le \|f-g\|_p+\|g-\sum_{k=1}^{n}<g,B_k>B_k\|_p,\\ &< \varepsilon + C^{1-\alpha}\varepsilon^{\alpha}\end{aligned} \tag{8}$$

where C and α depend on p, p_0. □

Remark 1. The function g in the above theorem can have simple choices, for instance, a partial sum of the Fourier expansion of f, or, a Poisson integral of f. The difference between this result and the partial sum convergence of $\sum_{k=1}^{+\infty} < f, B_k > B_k$ with respect to a Schauder basis $\{B_k\}_{k=1}^{+\infty}$ is that we use effective AFD in place of a basis, and the coefficients of two representations are different.

3 | ALGORITHM IN $\mathbb{H}^p(\mathbb{T}), 1 < p < +\infty$

Given dictionary $\mathfrak{D} := \left\{e_a(z) = \frac{A_{a,p}}{1-\bar{a}z}, a \in \mathbb{D}\right\}$, where $A_{a,p}$ is the normalization constant with respect to a and p, making $\|e_a(z)\|_p = 1$. For any given $f(e^{it}) \in \mathbb{H}^p(\mathbb{T})$, we construct a sequence of $\{e_{a_n}\}_{n=1}^{+\infty}$ and a sequence of complex numbers $\{c_n\}_{n=1}^{+\infty}$ inductively by

$$|F_{f_{n-1}}(e_{a_n})| = \sup_{a\in\mathbb{D}} |F_{f_{n-1}}(e_a)| \quad (if\ attainable), \tag{9}$$

$$\|f_{n-1} - c_n e_{a_n}\|_p = \min_{c\in\mathbb{C}} \|f_{n-1} - c e_{a_n}\|_p, \tag{10}$$

$$f_n = f_{n-1} - c_n e_{a_n}, \tag{11}$$

where $F_{f_{n-1}}$ is the unique complex supporting functional of the remainder f_{n-1} satisfying $\|F_{f_{n-1}}\| = 1$ and $F_{f_{n-1}}(f_{n-1}) = \|f_{n-1}\|_p$. When e_{a_n} is fixed through (9), we obtain the best approximation to f_{n-1} from Span$\{e_{a_n}\}$. From convexity of $\mathbb{H}^p(\mathbb{T})$ and Span$\{e_{a_n}\}$ is closed, the best approximation at the nth step always exists and unique. In the following, we prove that the sup in (9) can be attained.

Lemma 3.1. ([17]). *If $f(z) \in \mathbb{H}^p(\mathbb{D})$, then*

$$|f(z)| \le \left(\frac{1}{1-|z|^2}\right)^{\frac{1}{p}} \|f\|_p,$$

and the derivatives $f^{(n)}(z)$ satisfy

$$|f^{(n)}(z)| \le C_{n,p}\frac{1}{(1-|z|^2)^{n+\frac{1}{p}}}\|f\|_p.$$

With this lemma, we can prove the sup in (9) is attainable.

Theorem 3.2. *For any nonzero function $f(e^{it}) \in \mathbb{H}^p(\mathbb{T})$, there exists $\tilde{a} \in \mathbb{D}$ such that*

$$|F_f(e_{\tilde{a}})| = \sup_{a\in\mathbb{D}} |F_f(e_a)|.$$

Proof. It suffices to show that

$$\lim_{|a|\to 1} |F_f(e_a)| = 0.$$

From the Riesz representation theorem,

$$F_f(e_a) = \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(e^{it})|^{p-1}}{\|f\|_p^{p-1}}\overline{\mathrm{sgn} f(e^{it})} e_a(e^{it})dt. \tag{12}$$

Denote $\frac{|f(e^{it})|^{p-1}}{\|f\|_p^{p-1}}\overline{\mathrm{sgn} f(e^{it})} = h(e^{it}) \in \mathbb{L}^q(\mathbb{T})$, where q denotes the usual conjugate exponent satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Write $h(e^{it})$ as

$$h(e^{it}) = [h(e^{it}) - \mathrm{P}_r(e^{it}) * h(e^{it})] + \mathrm{P}_r(e^{it}) * h(e^{it}), \tag{13}$$

where $\mathrm{P}_r * h$ is the Poisson integral of h, then

$$\begin{aligned}|F_f(e_a)| = \left|\frac{1}{2\pi}\int_0^{2\pi} h(e^{it})e_a(e^{it})dt\right| &\leq \left|\frac{1}{2\pi}\int_0^{2\pi} [h(e^{it}) - \mathrm{P}_r(e^{it}) * h(e^{it})]e_a(e^{it})dt\right| \\ &+ \left|\frac{1}{2\pi}\int_0^{2\pi} [\mathrm{P}_r(e^{it}) * h(e^{it})]e_a(e^{it})dt\right|\end{aligned}.$$

From Hölder's inequality and the approximation to identity property of the Poisson integral, for $\forall \epsilon > 0$, there exists r such that

$$\left|\frac{1}{2\pi}\int_0^{2\pi} [h(e^{it}) - \mathrm{P}_r(e^{it}) * h(e^{it})]e_a(e^{it})dt\right| \leq \|h - \mathrm{P}_r * h\|_q \|e_a\|_p \leq \epsilon, \tag{14}$$

and

$$\left|\frac{1}{2\pi}\int_0^{2\pi} [\mathrm{P}_r(e^{it}) * h(e^{it})]e_a(e^{it})dt\right| \leq \|h\|_q \|\mathrm{P}_r * e_a\|_p, \tag{15}$$

where

$$\|\mathrm{P}_r * e_a\|_p^p = \frac{1}{2\pi}\int_0^{2\pi} |e_a(re^{it})|^p dt, \tag{16}$$

$$= |A_{a,p}|^p \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1}{1 - \bar{a}re^{it}}\right|^p dt, \tag{17}$$

$$= \frac{\|\frac{1}{1-\bar{a}rz}\|_p^p}{\|\frac{1}{1-\bar{a}z}\|_p^p}, \tag{18}$$

$$\leq \frac{(1 - |a|^2)^{p-1}}{(1 - |a|r)^p}. \tag{19}$$

The last inequality holds because of the fact that the numerator in (18) is not larger than $\frac{1}{(1-|a|r)^p}$, and the denominator is not less than $\frac{1}{(1-|a|^2)^{p-1}}$, which is deduced from Lemma 3.1. With the fixed r, when $|a|$ is sufficiently close to 1, $\|\mathrm{P}_r * e_a\|_p \leq \epsilon$. We thus conclude $\lim_{|a|\to 1} |F_f(e_a)| = 0$. □

Remark 2. This theorem gives a principle of element selection. It ensures that the algorithm can continue step by step. If at the nth step, $\sup_{a\in\mathbb{D}} |F_{f_{n-1}}(e_a)| = 0$, the algorithm automatically stops and f can be expressed as a finite linear combination of n terms. However, the proved boundary vanishing property does not hold for general dictionary. When the sup can not be attained, a so-called weak form is used, ie, for any fixed $0 < c < 1$, $|F_f(\tilde{g})| \geq c \sup_{g\in\mathfrak{D}} |F_f(g)|$.

4 | CONVERGENCE

By the proposed algorithm, for any given f, we can get an approximation $\sum_{k=1}^{n} c_k e_{a_k}$ after n steps with the residual f_n. The algorithm is convergent if $\lim_{n\to+\infty} f_n = 0$. The proof is separated into two cases: $\{c_n\} \in l_1$ or not. We first have

Theorem 4.1. *Let f be any nonzero function in* $\mathbb{H}^p(\mathbb{T})$, $\{e_{a_n}\}_{n=1}^{+\infty}$, $\{c_n\}_{n=1}^{+\infty}$ *are sequences from (9) and (10). If* $\sum_{n=1}^{+\infty}|c_n| < +\infty$, *then* $f = \sum_{n=1}^{+\infty} c_n e_{a_n}$.

Proof. Since $\sum_{n=1}^{+\infty}|c_n| < +\infty$, the sequence $\{f_n\}$ is a cauchy sequence. Denote

$$g = f - \sum_{k=1}^{+\infty} c_k e_{a_k} = \left(f - \sum_{k=1}^{n} c_k e_{a_k}\right) + \left(-\sum_{k=n+1}^{+\infty} c_k e_{a_k}\right) = f_n + h_n \quad .$$

Obviously, we have $\lim_{n\to+\infty} f_n = g$, $\lim_{n\to+\infty} h_n = 0$. Suppose $g \neq 0$, from Theorem 3.2, there exists $\tilde{a} \in \mathbb{D}$ such that $|F_g(e_{\tilde{a}})| = \sup_{a\in\mathbb{D}} |F_g(e_a)| \neq 0$. For this $e_{\tilde{a}}$, there exists a unique $\tilde{c} \in \mathbb{C}$, such that $\|g - \tilde{c}e_{\tilde{a}}\|_p = \min_{c\in\mathbb{C}} \|g - ce_{\tilde{a}}\|_p$. Hence,

$$\begin{aligned} \|g\|_p &\geq \|g - \tilde{c}e_{\tilde{a}}\|_p \\ &= \|f_n + h_n - \tilde{c}e_{\tilde{a}}\|_p \\ &\geq \|f_n - \tilde{c}e_{\tilde{a}}\|_p - \|h_n\|_p \\ &\geq \|f_n - d_n e_{\tilde{a}}\|_p - \|h_n\|_p \end{aligned} , \tag{20}$$

where $\|f_n - d_n e_{\tilde{a}}\|_p = \min_{d\in\mathbb{C}} \|f_n - de_{\tilde{a}}\|_p$.

On the other hand, the Frechet differentiability of p-norm and strong convergence of sequence $\{f_n\}_{n=1}^{+\infty}$ show that $\lim_{n\to+\infty} \|F_{f_n} - F_{f_{n+1}}\| = 0$. Further, from definition of f_{n+1}, we have $F_{f_{n+1}}(e_{a_{n+1}}) = 0$, which indicates $\lim_{n\to+\infty} |F_{f_n}(e_{a_{n+1}})| = \lim_{n\to+\infty} \sup_{a\in\mathbb{D}} |F_{f_n}(e_a)| = 0$. As a result, $\lim_{n\to+\infty} |F_{f_n}(e_{\tilde{a}})| = 0$. From the definition of d_n, the sequence $\{d_n\}_{n=1}^{+\infty}$ is uniformly bounded. Hence, we actually have $\lim_{n\to+\infty} |F_{f_n}(d_n e_{\tilde{a}})| = 0$. The residual f_n satisfies

$$\begin{aligned} 0 &\leq \|f_n\|_p - \|f_n - d_n e_{\tilde{a}}\|_p \\ &\leq \|f_n\|_p - |F_{f_n}(f_n - d_n e_{\tilde{a}})| \\ &\leq |F_{f_n}(f_n) - F_{f_n}(f_n - d_n e_{\tilde{a}})| \\ &= |F_{f_n}(d_n e_{\tilde{a}})| \end{aligned} . \tag{21}$$

That is,

$$\lim_{n\to+\infty} \|f_n - d_n e_{\tilde{a}}\|_p = \lim_{n\to+\infty} \|f_n\|_p = \|g\|_p. \tag{22}$$

Combining (20) and (22), we have $\|g - \tilde{c}e_{\tilde{a}}\|_p = \min_{c\in\mathbb{C}} \|g - ce_{\tilde{a}}\|_p = \|g\|_p$, which means that $|F_g(e_{\tilde{a}})| = \sup_{a\in\mathbb{D}} |F_g(e_a)| = 0$. The density of Span$\{\mathfrak{D}\}$ shows $g = 0$, which contradicts with the assumption $g \neq 0$. □

In the following, we will prove convergence for the other case $\{c_n\} \notin l_1$.

Lemma 4.2. *For any complex numbers* $a, b \in \mathbb{C}$, *we have*

$$|a + b|^p - |a|^p - p|a|^{p-1}|b|Resgn(\bar{a}b) \geq 0, \tag{23}$$

where Resgn is the real part of signum function.

Proof. Set $a = |a|e^{i\theta_1}, b = |b|e^{i\theta_2}$. If $a = 0$, the inequality obviously holds.

Suppose $|a| > 0$, set $\theta_2 - \theta_1 = \theta$ and $\frac{|b|}{|a|} = c$, (23) can be written as

$$\begin{aligned}
&\left||a|e^{i\theta_1} + |b|e^{i\theta_2}\right|^p - |a|^p - p|a|^{p-1}|b|Resgn(e^{i(\theta_2-\theta_1)}) \\
&= \left||a| + |b|e^{i\theta}\right|^p - |a|^p - p|a|^{p-1}|b|\cos\theta \\
&= |a|^p[\left|1 + \frac{|b|}{|a|}e^{i\theta}\right|^p - 1 - p\frac{|b|}{|a|}\cos\theta] \\
&= |a|^p(|1 + ce^{i\theta}|^p - 1 - pc\cos\theta)
\end{aligned}.$$

Obviously, we have $|1 + ce^{i\theta}|^p \geq |1 + c\cos\theta|^p \geq 1 + pc\cos\theta$. □

Lemma 4.3. *For any complex numbers $a, b, e^{i\beta} \in \mathbb{C}$, there exists a positive constant C_p such that the following inequality holds:*

$$\begin{aligned}
&|a + b|^{p-1}|bResgn[(\bar{a}b + |b|^2)e^{i\beta}] - |a|^{p-1}|b|Resgn(\bar{a}be^{i\beta}) \\
&\leq C_p[|a + b|^p - |a|^p - p|a|^{p-1}|b|Resgn(\bar{a}b)].
\end{aligned} \tag{24}$$

Proof. When $a = 0$ or $b = 0$, it obviously holds. To prove the lemma, we only need to prove it for $a = e^{i\theta}$, because we can factorize $|a|^p$ from the both sides. Further, we only need to prove it for $b \in \mathbb{R}^+$, because we can put the phase of b on the phase of $a = e^{i\theta}$. From above, we only need to prove it for $a = e^{i\theta}, b \in \mathbb{R}^+$.

Denote

$$\begin{aligned}
\Psi(b) &= \frac{|e^{i\theta} + b|^{p-1}bResgn[(e^{-i\theta}b + b^2)e^{i\beta}] - bResgn(e^{-i\theta}be^{i\beta})}{|e^{i\theta} + b|^p - 1 - pbResgn(e^{-i\theta}b)} \\
&= \frac{(1 + b^2 + 2b\cos\theta)^{\frac{p}{2}-1}[b^2\cos\beta + b\cos(\beta - \theta)] - b\cos(\beta - \theta)]}{(1 + b^2 + 2b\cos\theta)^{\frac{p}{2}} - 1 - pb\cos\theta}
\end{aligned},$$

which is a continuous function with respect to b, and

$$\begin{aligned}
&\lim_{b\to+\infty} \Psi(b) = \cos\beta \leq 1 \\
&\lim_{b\to 0+} \Psi(b) = \frac{\left(\frac{p}{2} - 1\right)\cos\theta\cos(\beta - \theta) + \cos(\beta - \theta) + 2\cos\beta}{p[1 + (p - 2)\cos^2\theta]} \\
&\leq \begin{cases} \frac{|\frac{p}{2}-1|+3}{p}, p \geq 2 \\ \frac{|\frac{p}{2}-1|+3}{p(p-1)}, 1 < p < 2 \end{cases}
\end{aligned}. \tag{25}$$

Hence, we can always find a constant $C_p \geq 1$ such that (24) holds. □

Lemma 4.4. *For the sequences $\{f_n\}_{n=1}^{+\infty}, \{c_n\}_{n=1}^{+\infty}, \{F_{f_{n-1}}(e_{a_n})\}_{n=1}^{+\infty}$ in the proposed algorithm, there exists $0 < \gamma_p < 1$, such that*

$$\frac{\|f_{n-1}\|_p - \|f_n\|_p}{|c_n|} \geq \gamma_p|F_{f_{n-1}}(e_{a_n})|. \tag{26}$$

Proof. In the following, we sometimes omit the variable e^{it} without confusion. $F_{f_{n-1}}(c_ne_{a_n})$ is a complex number, there must exist a β_n such that

$$|F_{f_{n-1}}(c_ne_{a_n})| = F_{f_{n-1}}(c_ne_{a_n}e^{i\beta_n}) = ReF_{f_{n-1}}(c_ne_{a_n}e^{i\beta_n}). \tag{27}$$

Direct computation gives

$$\|f_{n-1}\|_p^{p-1}|F_{f_{n-1}}(c_n e_{a_n})| = \|f_{n-1}\|_p^{p-1} ReF_{f_{n-1}}(c_n e_{a_n} e^{i\beta_n})$$

$$= Re\frac{1}{2\pi}\int_0^{2\pi} |f_{n-1}|^{p-1}\overline{\text{sgn} f_{n-1}} c_n e_{a_n} e^{i\beta_n} dt$$

$$= Re\frac{1}{2\pi}\int_0^{2\pi} |f_n + c_n e_{a_n}|^{p-1}|c_n e_{a_n}|\text{sgn}(\bar{f}_n + \overline{c_n e_{a_n}})\text{sgn}(c_n e_{a_n})\text{sgn}(e^{i\beta_n})dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} |f_n + c_n e_{a_n}|^{p-1}|c_n e_{a_n}|Re\text{sgn}[(\bar{f}_n c_n e_{a_n} + |c_n e_{a_n}|^2)e^{i\beta_n}]dt$$

From Lemma 4.3,

$$\begin{aligned} &|f_n + c_n e_{a_n}|^{p-1}|c_n e_{a_n}|Re\text{sgn}[(\bar{f}_n c_n e_{a_n} + |c_n e_{a_n}|^2)e^{i\beta_n}] \\ &\le C_p(|f_n + c_n e_{a_n}|^p - |f_n|^p) + |f_n|^{p-1}|c_n e_{a_n}|Re\text{sgn}(\bar{f}_n c_n e_{a_n} e^{i\beta_n}) \\ &- pC_p|f_n|^{p-1}||c_n e_{a_n}|Re\text{sgn}(\bar{f}_n c_n e_{a_n}) \end{aligned} \tag{28}$$

Notice that

$$F_{f_n}(e_{a_n}) = F_{f_n}(c_n e_{a_n}) = F_{f_n}(c_n e_{a_n} e^{i\beta_n}) = 0. \tag{29}$$

From (29), we can respectively get

$$\int_0^{2\pi} |f_n|^{p-1}|c_n e_{a_n}|Re\text{sgn}(\bar{f}_n c_n e_{a_n})dt = 0 \tag{30}$$

and

$$\int_0^{2\pi} |f_n|^{p-1}|c_n e_{a_n} Re\text{sgn}(\bar{f}_n c_n e_{a_n} e^{i\beta_n})dt = 0. \tag{31}$$

Integrating the both sides of (28),

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} |f_n + c_n e_{a_n}|^{p-1}|c_n e_{a_n}|Re\text{sgn}[(\bar{f}_n c_n e_{a_n} + |c_n e_{a_n}|^2)e^{i\beta_n}]dt \\ &\le C_p\left(\frac{1}{2\pi}\int_0^{2\pi} |f_n + c_n e_{a_n}|^p dt - \frac{1}{2\pi}\int_0^{2\pi} |f_n|^p dt\right) \\ &= C_p(\|f_{n-1}\|_p^p - \|f_n\|_p^p) \\ &\le pC_p\|f_{n-1}\|_p^{p-1}(\|f_{n-1}\|_p - \|f_n\|_p) \end{aligned}$$

From all above,

$$|F_{f_{n-1}}(c_n e_{a_n})| \le pC_p(\|f_{n-1}\|_p - \|f_n\|_p), \tag{32}$$

hence $\gamma_p = \frac{1}{pC_p} \in (0, 1)$ satisfies

$$\gamma_p|F_{f_{n-1}}(e_{a_n})| \le \frac{\|f_{n-1}\|_p - \|f_n\|_p}{|c_n|}. \tag{33}$$

Now, we are ready to prove the convergence. □

Theorem 4.5. *Let f be a nonzero function in* $\mathbb{H}^p(\mathbb{T})$, $\{e_{a_n}\}, \{c_n\}$ *are sequences from (9) and (10). If* $\sum_{n=1}^{+\infty}|c_n| = +\infty$, *then* $f = \sum_{n=1}^{+\infty} c_n e_{a_n}$.

Proof. Denote $S_n = \sum_{k=1}^{n}|c_k|$, then $\sum_{n=2}^{+\infty} ln\frac{S_n}{S_{n-1}} = +\infty$ deduces to $\sum_{n=1}^{+\infty}\frac{|c_n|}{S_n} = +\infty$. From the convergence of series $\sum_{n=1}^{+\infty}(\|f_{n-1}\|_p - \|f_n\|_p)$, there must exist a subsequence $\{n_k\}$ such that

$$\lim_{k\to+\infty}(\|f_{n_k-1}\|_p - \|f_{n_k}\|_p)\frac{S_{n_k}}{|c_{n_k}|} = 0. \tag{34}$$

Because if not, there exists a positive number r_0, such that the sequence $(\|f_{n-1}\|_p - \|f_n\|_p)\frac{S_n}{|c_n|} \geq r_0$ holds except finitely many terms, which leads to the convergence of series $\sum_{n=1}^{+\infty}\frac{|c_n|}{S_n}$. From Lemma 4.4, we can get

$$\begin{aligned} S_{n_k-1}|F_{f_{n_k-1}}(e_{a_{n_k}})| &\leq S_{n_k}|F_{f_{n_k-1}}(e_{a_{n_k}})| \\ &\leq S_{n_k}\frac{1}{\gamma_p}\frac{\|f_{n_k-1}\|_p - \|f_{n_k}\|_p}{|c_{n_k}|} \end{aligned}.$$

Combined with (34) and $\lim_{k\to+\infty} S_{n_k-1} = +\infty$, it shows that

$$\lim_{k\to+\infty}|F_{f_{n_k-1}}(e_{a_{n_k}})| = 0. \tag{35}$$

Suppose $\lim_{n\to+\infty}\|f_n\|_p = R > 0$, then for $l < n_k - 1$,

$$\begin{aligned} \left||F_{f_{n_k-1}}(f_l)| - \|f_{n_k-1}\|_p\right| &\leq |F_{f_{n_k-1}}(f_l) - F_{f_{n_k-1}}(f_{n_k-1})| \\ &\leq \left|F_{f_{n_k-1}}\left(\sum_{j=l+1}^{n_k-1} c_j e_{a_j}\right)\right| \\ &\leq \sum_{j=l+1}^{n_k-1}|c_j||F_{f_{n_k-1}}(e_{a_j})| \\ &\leq \sum_{j=1}^{n_k-1}|c_j||F_{f_{n_k-1}}(e_{a_{n_k}})| \end{aligned},$$

which indicates

$$\lim_{k\to+\infty}|F_{f_{n_k-1}}(f_l)| = R. \tag{36}$$

Since $\mathbb{H}^p(\mathbb{T})$ is separable, the closed unit ball of its dual space is weak-star sequentially compact. Let F be a weak-star cluster point of $F_{f_{n_k-1}}$, then (36) indicates the F to be a nonzero functional. For this F, $\sup_{a\in\mathbb{D}}|F(e_a)| = \delta > 0$, so, for sufficiently large k, $|F_{f_{n_k-1}}(e_{a_{n_k}})| \geq \frac{\delta}{2}$, which is a contradiction with (35). □

5 | CONCLUSION

In this paper, we study adaptive decomposition of functions in Hardy spaces under the Cauchy-kernel dictionary. The adaptivity is due to the parameter sequence $\{a_n\}$ being selected according to the given function f. The selection principle is to maximize the absolute value of applying the supporting functional of remainder f_{n-1} to elements in the dictionary.

With this dictionary, the maximizer can be attained. The coefficient sequence $\{c_n\}$ is to minimize norm of the residual f_n step by step. We prove strong convergence of the proposed algorithm. Also, we can directly apply AFD algorithm in $\mathbb{H}^2(\mathbb{T})$ to functions in $\mathbb{H}^p(\mathbb{T})$ instead of a basis.

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