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The Dual Elements of Function Sets and Fefferman–Stein Decomposition of Triebel–Lizorkin Functions via Wavelets

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Abstract

Let $D \in \mathbb{N}$, $q \in [2, \infty)$ and $(\mathbb{R}^D, |\cdot|, dx)$ be the Euclidean space equipped with the D-dimensional Lebesgue measure. In this article, we establish the Fefferman–Stein decomposition of Triebel–Lizorkin spaces $\dot{F}^0_{\infty, q'}(\mathbb{R}^D)$ with the help of the dual on function sets which have special topological structure. A function in Triebel–Lizorkin spaces $\dot{F}^0_{\infty, q'}(\mathbb{R}^D)$ can be written as a specific combination of $D+1$ functions in $\dot{F}^0_{\infty, q'}(\mathbb{R}^D) \cap L^\infty(\mathbb{R}^D)$. To get such a decomposition, first, some auxiliary function spaces $\mathrm{WE}^{1,q}(\mathbb{R}^D)$ and $\mathrm{WE}^{\infty,q'}(\mathbb{R}^D)$ are defined via wavelet expansions. It is shown that

$$\dot{F}^0_{1,q}(\mathbb{R}^D) \subsetneqq L^1(\mathbb{R}^D) \cup \dot{F}^0_{1,q}(\mathbb{R}^D) \subset \mathrm{WE}^{1,q}(\mathbb{R}^D) \subset L^1(\mathbb{R}^D) + \dot{F}^0_{1,q}(\mathbb{R}^D)$$

and $\mathrm{WE}^{\infty,q'}(\mathbb{R}^D)$ is strictly contained in $\dot{F}^0_{\infty, q'}(\mathbb{R}^D)$. Next, the Riesz transform characterization of Triebel–Lizorkin spaces $\dot{F}^0_{1,q}(\mathbb{R}^D)$ by the function set $\mathrm{WE}^{1,q}(\mathbb{R}^D)$ is established. Then the dual of $\mathrm{WE}^{1,q}(\mathbb{R}^D)$ is considered. As a consequence of the above results, a Riesz transform characterization of Triebel–Lizorkin spaces $\dot{F}^0_{1,q}(\mathbb{R}^D)$ by Banach space $L^1(\mathbb{R}^D) + \dot{F}^0_{1,q}(\mathbb{R}^D)$ is obtained. Although Fefferman–Stein type decompositions when $D = 1$ was obtained by Lin et al. (Mich Math J 62:691–703,

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2013), as was pointed out by Lin et al., the approach used in the case $D = 1$ cannot be applied to the cases $D \geq 2$. In the latter cases, some new skills related to Riesz transforms are to be developed.

Keywords Riesz transform · Fefferman–Stein decomposition · Triebel–Lizorkin space

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1 Introduction and Main Results

The Riesz transforms on $\mathbb{R}^D$, $D \geq 2$, which are natural generalizations of the Hilbert transform on $\mathbb{R}$, may be the most typical examples of Calderón–Zygmund operators (see, for example, [8,16,17] and references therein). The Riesz transform characterization of Hardy spaces plays important roles in the real variable theory of Hardy spaces (see, for example, [3,16]). Via this Riesz transform characterization of the Hardy space $H^1(\mathbb{R}^D)$ and the duality between $H^1(\mathbb{R}^D)$ and the space of functions with bounded mean oscillation, $\mathrm{BMO}(\mathbb{R}^D)$, Fefferman and Stein [3] further obtained the so-called Fefferman–Stein decomposition of $\mathrm{BMO}(\mathbb{R}^D)$. Later, Uchiyama [23] gave a constructive proof of the Fefferman–Stein decomposition of $\mathrm{BMO}(\mathbb{R}^D)$. Since then, many articles have focussed on the classical Riesz transform characterization and the Fefferman–Stein decomposition of different variants of Hardy spaces and BMO spaces; see, for example, [1,2,7,10,25] and references therein. Recently, Lin et al. [11] established the Hilbert transform characterization of Triebel–Lizorkin spaces $\dot{F}^0_{1,q}(\mathbb{R})$ and the Fefferman–Stein decomposition of Triebel–Lizorkin spaces $\dot{F}^0_{\infty,q'}(\mathbb{R})$ for each $q \in [2,\infty)$. Yang et al. [26] obtained the Fefferman–Stein decomposition for Q-spaces $Q_\alpha(\mathbb{R}^D)$ and the Riesz transform characterization of $P^\alpha(\mathbb{R}^D)$, the predual of $Q_\alpha(\mathbb{R}^D)$, for any $\alpha \in [0,\infty)$.

As was pointed out by Lin et al. in [11, Rem. 1.4], the approach used in [11] for the Hilbert transform characterization of Triebel–Lizorkin spaces $\dot{F}^0_{1,q}(\mathbb{R})$ cannot be applied to $\dot{F}^0_{1,q}(\mathbb{R}^D)$ when $D \geq 2$. Hence, new techniques have to be developed (see also "organization of this article" at the end of this section). In this article, motivated by some ideas from [11,26], we establish the Riesz transform characterization of Triebel–Lizorkin spaces $\dot{F}^0_{1,q}(\mathbb{R}^D)$ and the Fefferman–Stein decomposition of Triebel–Lizorkin spaces $\dot{F}^0_{\infty,q'}(\mathbb{R}^D)$ for all $D \in \mathbb{N} := \{1, 2, \ldots\}$ and $q \in [2,\infty)$.

In order to state the main results of this article, we now recall the definition of the Triebel–Lizorkin space $\dot{F}^0_{1,q}(\mathbb{R}^D)$ from [19]; see also [5,20–22]. Let $\mathscr{S}(\mathbb{R}^D)$ and $\mathscr{S}'(\mathbb{R}^D)$ be the *Schwartz space* and its *dual* respectively, and $\mathscr{P}(\mathbb{R}^D)$ the *class of all polynomials on* $\mathbb{R}^D$. Following [19], we also let

$$\mathscr{S}_\infty(\mathbb{R}^D) := \left\{\varphi \in \mathscr{S}(\mathbb{R}^D) : \int_{\mathbb{R}^D} \varphi(x)x^\alpha\, dx = 0 \text{ for all } \alpha \in \mathbb{Z}^D_+\right\}$$

and $\mathscr{S}'_\infty(\mathbb{R}^D)$ be its dual. Here and hereafter, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, $\mathbb{Z}^D_+ := (\mathbb{Z}_+)^D$ and, for any $\alpha := (\alpha_1, \ldots, \alpha_D) \in \mathbb{Z}^D_+$ and $x := (x_1, \ldots, x_D) \in \mathbb{R}^D$, $x^\alpha := x_1^{\alpha_1} \cdots x_D^{\alpha_D}$.

Let $\varphi \in \mathscr{S}(\mathbb{R}^D)$ satisfy $\operatorname{supp}(\widehat{\varphi}) \subset \{\xi \in \mathbb{R}^D : 1/2 \le |\xi| \le 2\}$, $|\widehat{\varphi}(\xi)| \ge c > 0$ if $3/5 \le |\xi| \le 5/3$, and $\sum_{j\in\mathbb{Z}} |\widehat{\varphi}(2^j\xi)| = 1$ if $\xi \ne 0$, where c is a positive constant. Write $\varphi_j(\cdot) := 2^{Dj}\varphi(2^j\cdot)$ for any $j \in \mathbb{Z}$. Such φ_j have been used to define the *homogeneous Triebel–Lizorkin space* $\dot{F}^0_{p,q}$, $\mathbb{R}^D$, $0 < p \le \infty$, $0 < q < \infty$. Further, such definitions are independent of the choices of φ. See [5,19–22]. We consider the case where $p = 1$ and ∞.

Definition 1.1 Let $q \in (1,\infty)$. Then the *homogeneous Triebel–Lizorkin space* $\dot{F}^0_{1,q}(\mathbb{R}^D)$ is defined to be the set of all $f \in \mathscr{S}'_\infty(\mathbb{R}^D)$ such that

$$\|f\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} := \left\| \left\{ \sum_{j\in\mathbb{Z}} \left|\varphi_j * f\right|^q \right\}^{1/q} \right\|_{L^1(\mathbb{R}^D)} < \infty.$$

Remark 1.2 (i) It is well known that $\mathscr{S}'_\infty(\mathbb{R}^D) = \mathscr{S}'(\mathbb{R}^D)\backslash\mathcal{P}(\mathbb{R}^D)$ with equivalent topologies; see for example, [9, Thm. 24.0.4], [13, Thm. 28], [14, Prop. 35.4, Prop. 35.5] and [27, Prop. 8.1] for an exact proof.

(ii) From [5, p. 42], it follows that $\dot{F}^0_{1,2}(\mathbb{R}^D) = H^1(\mathbb{R}^D)$ with equivalent norms. Obviously, for any $q \in [2,\infty)$, $\|f\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \lesssim \|f\|_{H^1(\mathbb{R}^D)}$. Hence $H^1(\mathbb{R}^D) \subset \dot{F}^0_{1,q}(\mathbb{R}^D)$.

Now we recall the definition of $\dot{F}^0_{\infty,q}(\mathbb{R}^D)$.

Definition 1.3 Let $q \in (1,\infty)$. Then the *homogeneous Triebel–Lizorkin space* $\dot{F}^0_{\infty,q}(\mathbb{R}^D)$ is defined to be the set of all $f \in \mathscr{S}'_\infty(\mathbb{R}^D)$ such that

$$\|f\|_{\dot{F}^0_{\infty,q}(\mathbb{R}^D)} := \sup_{\{Q:\ \text{dyadic cube}\}} \left\{ \frac{1}{|Q|} \int_Q \sum_{j=-\log_2 \ell(Q)}^{\infty} \left|\varphi_j * f(x)\right|^q dx \right\}^{1/q} < \infty,$$

where the supremum is taken over all dyadic cubes Q in $\mathbb{R}^D$ and $\ell(Q)$ denotes the *side length* of Q.

Remark 1.4 (i) From [5, p. 42], it follows that $\dot{F}^0_{\infty,2}(\mathbb{R}^D) = \mathrm{BMO}(\mathbb{R}^\mathrm{D})$ with equivalent norms.

(ii) It was shown in [4, (5.2)] that, for each $q \in (1,\infty)$, $\dot{F}^0_{\infty,q'}(\mathbb{R}^D)$ is the dual space of $\dot{F}^0_{1,q}(\mathbb{R}^D)$. In particular, $\mathrm{BMO}(\mathbb{R}^\mathrm{D})$ is the dual space of $H^1(\mathbb{R}^D)$, which was shown before in [3].

Next we recall the definition of the 1-dimensional Meyer wavelets from [24]; see also [6,11,12,15] for a different version. Let $\Phi \in C^\infty(\mathbb{R})$, the *space of all infinitely differentiable functions on* $\mathbb{R}$, satisfy $0 \le \Phi(\xi) \le 1/\sqrt{2\pi}$ for any $\xi \in \mathbb{R}$, $\Phi(\xi) = 1/\sqrt{2\pi}$ for any $\xi \in [-2\pi/3, 2\pi/3]$, $[\Phi(\xi)]^2 + [\Phi(\xi - 2\pi)]^2 = 1/(2\pi)$ for any $\xi \in [0, 2\pi]$. Further,

$$\Phi(\xi) = \Phi(-\xi) \quad \text{for any } \xi \in \mathbb{R}, \tag{1.1}$$

and

$$\Phi(\xi) = 0 \quad \text{for any } \xi \in (-\infty, 4\pi/3] \cup [4\pi/3, \infty). \tag{1.2}$$

In what follows, the *Fourier transform* and the *reverse Fourier transform* of a suitable function f on $\mathbb{R}^D$ are defined by

$$\widehat{f}(\xi) := (2\pi)^{-D/2} \int_{\mathbb{R}^D} e^{-i\xi x} f(x)\, dx \quad \text{for any } \xi \in \mathbb{R}^D,$$

and

$$\check{f}(x) := (2\pi)^{-D/2} \int_{\mathbb{R}^D} e^{ix\xi} f(\xi)\, d\xi \quad \text{for any } x \in \mathbb{R}^D,$$

respectively.

From [24, Prop. 3.2], it follows that $\phi := \check{\Phi}$ (the *"father" wavelet*) is a scaling function of a *multiresolution analysis* defined as in [24, Def. 2.2]. The *corresponding function* m_ϕ of ϕ, satisfying $\widehat{\phi}(2\cdot) = m_\phi(\cdot)\widehat{\phi}(\cdot)$, is a 2π-periodic function which equals $\sqrt{2\pi}\,\Phi(2\cdot)$ on the interval $[-\pi, \pi)$.

Furthermore, by [24, Thm. 2.20], we construct a 1-dimensional wavelet ψ (the *"mother" wavelet*) by setting $\widehat{\psi}(\xi) := e^{i\xi/2} m_\phi(\xi/2 + \pi)\Phi(\xi/2)$ for any $\xi \in \mathbb{R}$. It was shown in [24, Prop. 3.3] that ψ is a real-valued $C^\infty(\mathbb{R})$ function, $\psi(-1/2 - x) = \psi(-1/2 + x)$ for all $x \in \mathbb{R}$, and

$$\operatorname{supp}\left(\widehat{\psi}\right) \subset [-8\pi/3, -2\pi/3] \cup [2\pi/3, 8\pi/3]. \tag{1.3}$$

Such a wavelet ψ is called a 1*-dimensional Meyer wavelet* and $\psi(0) \neq 0$.

Let $D \in \mathbb{N} \cap [2, \infty)$ and $\vec{0} := (\overbrace{0, \ldots, 0}^{D \text{ times}})$. The D-dimensional Meyer wavelets are constructed by tensor products as follows. Let $x := (x_1, \ldots, x_D) \in \mathbb{R}^D$, $E_D := \{0, 1\}^D \backslash \{\vec{0}\}$ and, for any $\lambda := (\lambda_1, \ldots, \lambda_D) \in E_D$, define

$$\psi^\lambda(x) := \phi^{\lambda_1}(x_1) \cdots \phi^{\lambda_D}(x_D),$$

with $\phi^{\lambda_j}(x_j) := \phi(x_j)$ if $\lambda_j = 0$ and $\phi^{\lambda_j}(x_j) := \psi(x_j)$ if $\lambda_j = 1$. As in [24], for any $(\lambda, j, k) \in \Lambda_D := \{(\lambda, j, k) : \lambda \in E_D,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^D\}$ and $x \in \mathbb{R}^D$, we let $\psi^\lambda_{j,k}(x) := 2^{Dj}\psi^\lambda(2^j x - k)$ and, for $\lambda = \vec{0}$ and any $k := (k_1, \ldots, k_D)$, let $\psi^{\vec{0}}_{j,k}(x) := 2^{Dj}\phi(2^j x_1 - k_1) \cdots \phi(2^j x_D - k_D)$ and $\psi^{\vec{0}}(x) := \phi(x_1) \cdots \phi(x_D)$.

By [24, Prop. 3.1] and arguments of tensor products, we know that, for any $(\lambda, j, k) \in \Lambda_D$, $\psi^\lambda_{j,k} \in \mathscr{S}_\infty(\mathbb{R}^D)$. Thus, for any $(\lambda, j, k) \in \Lambda_D$ and any $f \in \mathscr{S}'_\infty(\mathbb{R}^D)$, let $a^\lambda_{j,k}(f) := \langle f, \psi^\lambda_{j,k} \rangle$, where $\langle \cdot, \cdot \rangle$ represents the duality between $\mathscr{S}'_\infty(\mathbb{R}^D)$ and $\mathscr{S}_\infty(\mathbb{R}^D)$. From the proof of [5, Thm. (7.20)], it follows that, for any $f \in \mathscr{S}'_\infty(\mathbb{R}^D)$,

$$f = \sum_{\lambda \in E_D} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^D} a^\lambda_{j,k}(f)\psi^\lambda_{j,k} \quad \text{in} \quad \mathscr{S}'_\infty(\mathbb{R}^D). \tag{1.4}$$

Moreover, by [24, Prop. 5.2], we know that $\{\psi^{\lambda}_{j,k}\}_{(\lambda,j,k)\in\Lambda_D}$ is an orthonormal basis of $L^2(\mathbb{R}^D)$.

For any $\ell \in \{1,\ldots,D\}$ and any $f \in \mathscr{S}(\mathbb{R}^D)$, denote by $R_\ell(f)$ the ℓth-*Riesz transform* of f, which is defined by setting

$$\widehat{R_\ell(f)}(\xi) := -i\frac{\xi_\ell}{|\xi|}\widehat{f}(\xi) \quad \text{for any } \xi \in \mathbb{R}^D \text{ and } \xi \neq 0.$$

Since (1.2) and (1.3) hold true, by [26, (5.2)], we know that, for any $\ell \in \{1,\ldots,D\}$, (λ, j, k), $(\widetilde{\lambda}, \widetilde{j}, \widetilde{k}) \in \Lambda_D$ and $|j-\widetilde{j}| \geq 2$, we have

$$\left(R_\ell\left(\psi^{\lambda}_{j,k}\right), \psi^{\widetilde{\lambda}}_{\widetilde{j},\widetilde{k}}\right) = 0, \tag{1.5}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\mathbb{R}^D)$. Note that (1.5) implies that

$$R_\ell\left(\psi^{\lambda}_{j,k}\right) = \sum_{(\widetilde{\lambda},\widetilde{j},\widetilde{k})\in\Lambda_D: j-1\leq\widetilde{j}\leq j+1} b^{\lambda,j,k}_{\widetilde{\lambda},\widetilde{j},\widetilde{k}}\psi^{\widetilde{\lambda}}_{\widetilde{j},\widetilde{k}}$$

for some coefficients

$$b^{\lambda,j,k}_{\widetilde{\lambda},\widetilde{j},\widetilde{k}} = \left(R_\ell\left(\psi^{\lambda}_{j,k}\right), \psi^{\widetilde{\lambda}}_{\widetilde{j},\widetilde{k}}\right).$$

Now we recall the wavelet characterization of $\dot{F}^0_{1,q}(\mathbb{R}^D)$ and $\dot{F}^0_{\infty,q}(\mathbb{R}^D)$ (see, for example, [5, Thm. (7.20)]). For $j \in \mathbb{Z}$ and $k = (k_1,\ldots,k_D) \in \mathbb{Z}^D$, denote

$$Q_{j,k} = \prod_{l=1}^{D}[2^{-j}k_l, 2^{-j}(1+k_l)[.$$

Theorem 1.5 *Let $q \in (1,\infty)$. Then*

(i) $f \in \dot{F}^0_{1,q}(\mathbb{R}^D)$ *if and only if* $f \in \mathscr{S}'_\infty(\mathbb{R}^D)$ *and*

$$\mathcal{J}_f := \left\| \left\{ \sum_{(\lambda,j,k)\in\Lambda_D} \left[2^{Dj}\left|a^{\lambda}_{j,k}(f)\right|\chi\left(2^j x - k\right)\right]^q \right\}^{1/q} \right\|_{L^1(\mathbb{R}^D)} < \infty,$$

where χ denotes the characteristic function of the cube $[0,1)^D$. Moreover, there exists a positive constant C such that, for all $f \in \dot{F}^0_{1,q}(\mathbb{R}^D)$,

$$\frac{1}{C}\|f\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \leq \mathcal{J}_f \leq C\|f\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}.$$

(ii) $f \in \dot{F}^{0}_{\infty,q}(\mathbb{R}^D)$ *if and only if* $f \in \mathscr{S}'_{\infty}(\mathbb{R}^D)$ *and there exists* $C > 0$ *such that for all dyadic cube* Q*,*

$$\sum_{(\lambda,j,k)\in\Lambda_D, Q_{j,k}\subset Q} 2^{(q-1)jD}|a^{\lambda}_{j,k}(f)|^q \le C|Q|.$$

Remark 1.6 By Remark 1.2 and Theorem 1.5, we also obtain the wavelet characterization of $H^1(\mathbb{R}^D)$ as in [12, p. 143].

To consider Fefferman–Stein type decomposition for $\dot{F}^{0}_{\infty,q}(\mathbb{R}^D)$, we need to study some properties relative to frequency. Hence we use Meyer wavelets to introduce the auxiliary function sets $\mathrm{WE}^{1,q}(\mathbb{R}^D)$. We consider the linear functional on these function sets and consider some exchangeability of Riesz transform and some sums of orthogonal projector operator defined by Meyer wavelets.

Let $q \in (1, \infty)$ and $f \in \mathscr{S}'_{\infty}(\mathbb{R}^D)$. For any $s \in \mathbb{Z}$, $N \in \mathbb{N}$ and $t \in \{0, \ldots, N+1\}$, let $P_{s,N}$ be the projection onto the subspace generated by the wavelets at the scales j living in the band $[s-N, s]$, namely

$$P_{s,N} f := \sum_{\{(\lambda,j,k)\in\Lambda_D:\, s-N\le j\le s\}} a^{\lambda}_{j,k}(f)\psi^{\lambda}_{j,k} \quad \text{in} \quad \mathscr{S}'_{\infty}(\mathbb{R}^D). \tag{1.6}$$

For each $t \in \{0, \ldots, N+1\}$, we further restrict the projections to the sub-bands corresponding to the scale intervals $[s-t+1, s]$ and $[s-N, s-t]$,

$$T^{(1)}_{s,t,N}(f) := \begin{cases} 0, & t = 0, \\ \sum_{\{(\lambda,j,k)\in\Lambda_D:\, s-t+1\le j\le s\}} a^{\lambda}_{j,k}(f)\psi^{\lambda}_{j,k}, & t \in \{1, \ldots, N+1\} \end{cases} \tag{1.7}$$

and

$$T^{(2)}_{s,t,N}(f) := \begin{cases} \sum_{\{(\lambda,j,k)\in\Lambda_D:\, s-N\le j\le s-t\}} a^{\lambda}_{j,k}(f)\psi^{\lambda}_{j,k}, & t \in \{0, \ldots, N\}, \\ 0, & t = N+1. \end{cases} \tag{1.8}$$

Observe that $P_{s,N} = T^{(1)}_{s,t,N} + T^{(2)}_{s,t,N}$ and the operators $T^{(i)}_{s,t,N}$ are also orthogonal projections. More precisely, $T^{(1)}_{s,t,N} = P_{s,t-1}$ and $T^{(2)}_{s,t,N} = P_{s-t,N-t}$. When f belongs to the projection space, then the projection operator acts like the identity, a fact that will be used repeatedly. The operators $T^{(1)}_{s,t,N}(f)$ and $T^{(2)}_{s,t,N}(f)$ in the above Eqs. (1.7) and (1.8) are quite important. They are adapted to the Eq. (1.5) where the frequencies does not change much when considering the action of Riesz operators on Meyer wavelets.

Definition 1.7 For $1 < q \le \infty$, the *space* $\mathrm{WE}^{\infty,q}(\mathbb{R}^D)$ is defined to be the space of all $f \in \mathscr{S}'_{\infty}(\mathbb{R}^D)$ such that

$$\|f\|_{\mathrm{WE}^{\infty,q}(\mathbb{R}^D)} := \sup_{\{s\in\mathbb{N},\, N\in\mathbb{N}\}} \sup_{t\in\{0,\dots,N+1\}} \left[\left\|T^{(1)}_{s,t,N}(f)\right\|_{\dot F^0_{\infty,q}(\mathbb{R}^D)} + \left\|T^{(2)}_{s,t,N}(f)\right\|_{L^\infty(\mathbb{R}^D)}\right] < \infty.$$

It is easy to see that

Proposition 1.8 *For* $1 < q \le \infty$, $\mathrm{WE}^{\infty,q}(\mathbb{R}^D) = L^\infty(\mathbb{R}^D) \cap \dot F^0_{\infty,q}(\mathbb{R}^D)$ *are Banach spaces.*

Definition 1.9 For $1 < q \le \infty$, the relative *space* $\mathrm{WE}^{1,q}(\mathbb{R}^D)$ is defined to be the space of all $f \in \mathscr{S}'_\infty(\mathbb{R}^D)$ such that

$$\|f\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} := \sup_{\{s\in\mathbb{N},\, N\in\mathbb{N}\}} \min_{t\in\{0,\dots,N+1\}} \times \left[\left\|T^{(1)}_{s,t,N}(f)\right\|_{\dot F^0_{1,q}(\mathbb{R}^D)} + \left\|T^{(2)}_{s,t,N}(f)\right\|_{L^1(\mathbb{R}^D)}\right] < \infty.$$

Further, for $f \in L^1(\mathbb{R}^D) \cup \dot F^0_{1,q}(\mathbb{R}^D)$, we define

$$\|f\|_{\{1,q\}} := \min(\|f\|_{L^1}, \|f\|_{\dot F^0_{1,q}}).$$

For $f \in L^1(\mathbb{R}^D) + \dot F^0_{1,q}(\mathbb{R}^D)$, we define

$$\|f\|_{1,q} := \inf_{f=h+g\in L^1+\dot F^0_{1,q}} \{\|h\|_{L^1} + \|g\|_{\dot F^0_{1,q}}\}. \tag{1.9}$$

Remark 1.10 For $D = 1$ the spaces $\mathrm{WE}^{1,q}(\mathbb{R}^D)$ and $\mathrm{WE}^{\infty,q}(\mathbb{R}^D)$, $q \in (1,\infty)$, have been introduced by Lin et al. in [11, p. 693] and [11, p. 694], respectively, and were denoted by $L^{1,q}(\mathbb{R})$, and $L^{\infty,q}(\mathbb{R})$, respectively. To distinguish these spaces from the well-known Lorentz spaces, we use the notation $\mathrm{WE}^{1,q}(\mathbb{R}^D)$ and $\mathrm{WE}^{\infty,q}(\mathbb{R}^D)$ which indicate that these spaces are defined via wavelet expansions. Recall also that the space $\mathrm{WE}^{1,q}(\mathbb{R}^D)$ was also called the relative L^1 space in [11, p. 693].

We know that the function

$$\tilde P_N f(x) = \sum_{(\epsilon,j,k)\in\Lambda_D,|j|+|k|\le 2^N} a^\lambda_{j,k}(f)\psi^\lambda_{j,k}(x)$$

belongs to $\mathscr{S}_\infty(\mathbb{R}^D)$ for all $N \ge 1$. Set $A = \mathrm{WE}^{1,q}(\mathbb{R}^D)$ or $L^1(\mathbb{R}^D) \cup \dot F^0_{1,q}(\mathbb{R}^D)$ or $L^1(\mathbb{R}^D) + \dot F^0_{1,q}(\mathbb{R}^D)$. If $f \in A$, then $\tilde P_N f \in A$. It is easy to see that

Proposition 1.11 *For* $1 \le q < \infty$,

(i) $\mathrm{WE}^{1,q}(\mathbb{R}^D)$ *is complete with the above induced norm.*
(ii) *The set* $\mathscr{S}_\infty(\mathbb{R}^D)$ *is dense in* A.

Remark 1.12 Let $q \in [2, \infty)$. It was shown in [20, p. 239] that the dual space of $\dot{F}^0_{1,q}(\mathbb{R}^D)$ is $\dot{F}^0_{\infty,q'}(\mathbb{R}^D)$. Further $\dot{F}^0_{\infty,2}(\mathbb{R}^D) = \mathrm{BMO}(\mathbb{R}^D)$, $\dot{F}^0_{1,2}(\mathbb{R}^D) = H^1(\mathbb{R}^D) \subset L^1(\mathbb{R}^D)$. Hence for $q = 2$,

$$L^1(\mathbb{R}^D) \cup \dot{F}^0_{1,2}(\mathbb{R}^D) = \mathrm{WE}^{1,2}(\mathbb{R}^D) = L^1(\mathbb{R}^D) + \dot{F}^0_{1,2}(\mathbb{R}^D) = L^1(\mathbb{R}^D). \quad (1.10)$$

Let $2 < q < \infty$. $L^1(\mathbb{R}^D) + \dot{F}^0_{1,q}(\mathbb{R}^D)$ are Banach spaces. $WE^{1,q}(\mathbb{R}^D)$ are function sets, not Banach spaces. Moreover, the following equalities are **not** true

$$L^1(\mathbb{R}^D) \cup \dot{F}^0_{1,q}(\mathbb{R}^D) = \mathrm{WE}^{1,q}(\mathbb{R}^D) = L^1(\mathbb{R}^D) + \dot{F}^0_{1,q}(\mathbb{R}^D).$$

In the following Theorem 1.17, the above two equal signs both have been changed to the inclusion sign "$\subset$". i.e.

$$L^1(\mathbb{R}^D) \cup \dot{F}^0_{1,q}(\mathbb{R}^D) \subset \mathrm{WE}^{1,q}(\mathbb{R}^D) \subset L^1(\mathbb{R}^D) + \dot{F}^0_{1,q}(\mathbb{R}^D). \quad (1.11)$$

For A, we can use distributions to define their dual elements.

Definition 1.13 For $1 \leq q < \infty$ and a function set $A \subset L^1(\mathbb{R}^D) + \dot{F}^0_{1,q}(\mathbb{R}^D)$, we call l to be a dual element of A, if $l \in \mathscr{S}'_\infty(\mathbb{R}^D)$ and

$$\sup_{f \in \mathscr{S}_\infty(\mathbb{R}^D), \|f\|_A \leq 1} |\langle l, f\rangle| < \infty.$$

We write $l \in A'$.

A' is a linear space. In fact, for $\alpha, \beta \in \mathbb{C}$ and $l_1, l_2 \in A'$, we know that $\alpha l_1 + \beta l_2 \in A'$. Further, $L^1(\mathbb{R}^D) + \dot{F}^0_{1,q}(\mathbb{R}^D)$ is the linearization function space of the set $L^1(\mathbb{R}^D) \cup \dot{F}^0_{1,q}(\mathbb{R}^D)$ or the set $\mathrm{WE}^{1,q}(\mathbb{R}^D)$. The dual elements on the set $L^1(\mathbb{R}^D) \cup \dot{F}^0_{1,q}(\mathbb{R}^D)$ or on the set $\mathrm{WE}^{1,q}(\mathbb{R}^D)$ are the same as those on the linear space $L^1(\mathbb{R}^D) + \dot{F}^0_{1,q}(\mathbb{R}^D)$. Now we are ready to state the first main auxiliary result of this paper.

Theorem 1.14 *For $q \in [2, \infty)$, we have*

$$\left(L^1(\mathbb{R}^D) \cup \dot{F}^0_{1,q}(\mathbb{R}^D)\right)' = \left(\mathrm{WE}^{1,q}(\mathbb{R}^D)\right)' = \left(L^1(\mathbb{R}^D) + \dot{F}^0_{1,q}(\mathbb{R}^D)\right)'$$
$$= L^\infty(\mathbb{R}^D) \cap \dot{F}^0_{\infty,q'}(\mathbb{R}^D).$$

For $q = 2$, due to the Eq. (1.10), the above Theorem 1.14 is evident. For general q, this theorem is new and its proof will be given in the final section.

Let $R_0 :=$ Id denote the *identity operator*. We state the second main auxiliary result which will be needed in the proof of our Fefferman–Stein type decomposition. We will use certain exchangeability of Meyer wavelets and Riesz transform to prove Theorem 1.15 in Sect. 2.

Theorem 1.15 *Let $D \in \mathbb{N}$ and $q \in [2, \infty)$. Then $f \in \mathscr{S}'_\infty(\mathbb{R}^D)$ belongs to $\dot{F}^0_{1,q}(\mathbb{R}^D)$ if and only if $f \in \mathrm{WE}^{1,q}(\mathbb{R}^D)$ and $\{R_\ell(f)\}_{\ell=1}^D \subset \mathrm{WE}^{1,q}(\mathbb{R}^D)$. Moreover, there exists a positive constant C such that, for all $f \in \dot{F}^0_{1,q}(\mathbb{R}^D)$,*

$$\frac{1}{C}\|f\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \leq \sum_{\ell=0}^{D} \|R_\ell(f)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} \leq C\|f\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}.$$

Theorem 1.15 provides a Riesz transform characterization of the homogeneous Triebel–Lizorkin spaces $\dot{F}^0_{1,q}(\mathbb{R}^D)$ via the wavelet expansion sets $\mathrm{WE}^{1,q}(\mathbb{R}^D)$ for $q \geq 2$.

Remark 1.16 If $D = 1$, Theorem 1.15 is just [11, Thm. 1.3].

Fefferman–Stein decomposition says, for some function space A, there exists some space B satisfying $B \nsubseteq A$ such that, for $f \in A$, there exist $f_l \in B$ such that

$$f = \sum_{l=0}^{D} R_l f_l.$$

The functions in B have better properties than those in A. But a function in A has been written as a linear combination of a function in B and the D images of the D Riesz transformations acting on the D functions in B respectively. Such slightly better properties can improve certain results in PDE and in harmonic analysis. The following Theorems 1.17 and 1.18 tell us that we have also Fefferman–Stein decomposition for $\dot{F}^0_{\infty,q}(\mathbb{R}^D)$.

By Remark 1.12, we know that, for any $q \in [2, \infty)$, $\dot{F}^0_{1,q}(\mathbb{R}^D) \subset \mathrm{WE}^{1,q}(\mathbb{R}^D)$ and $\mathrm{WE}^{\infty,q'}(\mathbb{R}^D) \subset \dot{F}^0_{\infty,q'}(\mathbb{R}^D)$. The above inclusions of sets are proper. The facts $A \subsetneq B$ means the functions in A have better properties than the functions in B. Fefferman–Stein decomposition needs such facts. The following conclusions are extensions of [11, Rem. 1.8]. The proof of Theorem 1.17 will be given at Sect. 3.

Theorem 1.17 *Let $D \in \mathbb{N}$ and $q \in [2, \infty)$. Then*

(i) $\dot{F}^0_{1,q}(\mathbb{R}^D) \subsetneqq L^1(\mathbb{R}^D) \cup \dot{F}^0_{1,q}(\mathbb{R}^D) \subset \mathrm{WE}^{1,q}(\mathbb{R}^D) \subset L^1(\mathbb{R}^D) + \dot{F}^0_{1,q}(\mathbb{R}^D)$*;*
(ii) $\mathrm{WE}^{\infty,q'}(\mathbb{R}^D) \subsetneqq \dot{F}^0_{\infty,q'}(\mathbb{R}^D)$.

Combining Theorems 1.15 and 1.17 and some arguments analogous to those used in the proof of [11, Thm. 1.7], we obtain the following Fefferman–Stein decomposition of $\dot{F}^0_{\infty,q}(\mathbb{R}^D)$, the proof will be given in the final section.

Theorem 1.18 *Let $D \in \mathbb{N}$ and $q \in (1, 2]$. Then $f \in \dot{F}^0_{\infty,q}(\mathbb{R}^D)$ if and only if there exist $\{f_\ell\}_{\ell=0}^D \in \mathrm{WE}^{\infty,q}(\mathbb{R}^D)$ such that $f = f_0 + \sum_{\ell=1}^D R_\ell(f_\ell)$.*

By Theorems 1.14 and 1.18, we obtain the following Riesz transform characterization of the homogeneous Triebel–Lizorkin spaces $\dot{F}^0_{1,q}(\mathbb{R}^D)$.

Theorem 1.19 *Let $D \in \mathbb{N}$ and $q \in [2, \infty)$. Then $f \in \mathscr{S}'_\infty(\mathbb{R}^D)$ belongs to $\dot{F}^0_{1,q}(\mathbb{R}^D)$ if and only if $f \in L^1(\mathbb{R}^D) + \dot{F}^0_{1,q}(\mathbb{R}^D)$ and $\{R_\ell(f)\}_{\ell=1}^D \subset L^1(\mathbb{R}^D) + \dot{F}^0_{1,q}(\mathbb{R}^D)$. Moreover, there exists a positive constant C such that, for all $f \in \dot{F}^0_{1,q}(\mathbb{R}^D)$,*

$$\frac{1}{C}\|f\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \leq \sum_{\ell=0}^{D} \|R_\ell(f)\|_{L^1(\mathbb{R}^D)+\dot{F}^0_{1,q}(\mathbb{R}^D)} \leq C\|f\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}.$$

Remark 1.20 (i) Theorem 1.18 when $D = 1$ is just [11, Thm. 1.7].
(ii) For $q = 2$, $\dot{F}^0_{1,q}(\mathbb{R}^D)$ is the Hardy space, $L^1(\mathbb{R}^D) + \dot{F}^0_{1,q}(\mathbb{R}^D) = L^1(\mathbb{R}^D)$, this theorem becomes the well-known characterization of Hardy space by L^1. For $2 < q < \infty$, the conclusion of Theorem 1.19 is new.

The organization of this article is as follows.

In Sect. 2, via the definition of the space $\mathrm{WE}^{1,q}(\mathbb{R}^D)$, the boundedness of Riesz transforms on $\dot{F}^0_{1,q}(\mathbb{R}^D)$, the Riesz transform characterization of $H^1(\mathbb{R}^D)$ and some ideas from [11,26], we prove Theorem 1.15, namely, establish the Riesz transform characterization of Triebel–Lizorkin spaces $\dot{F}^0_{1,q}(\mathbb{R}^D)$. Denote $f_{s\,N} := P_{s,N} f$. Comparing with the corresponding proof of [26, Sect. 6.2], the main innovation of this proof is that we regard the corresponding parts of the norms of Riesz transforms $\{R_\ell(f_{s_1,N_1})\}_{\ell=1}^D$ in $\mathrm{WE}^{1,q}(\mathbb{R}^D)$ as a whole to choose only one $t^1_{s,N} \in \{0, \dots, N+1\}$ such that (2.6) below holds true as one did in the corresponding proof of [26, (6.6)]. For $D \geq 2$, we do not choose D numbers $t^\ell_{s,N} \in \{0, \dots, N+1\}$ for each $\ell \in \{1, \dots, D\}$ separately. Using this technique, we successfully overcome those difficulties described in [11, Rem. 1.4].

In Sect. 3, we prove Theorem 1.17. In Sect. 4, we give the proof of Theorems 1.14, 1.18 and 1.19.

Finally, we state some conventions on notation. Throughout the whole paper, C stands for a *positive constant* which is independent of the main parameters, but it may vary from line to line. If, for two real functions f and g, $f \leq Cg$, we then write $f \lesssim g$; if $f \lesssim g \lesssim f$, we then write $f \sim g$. For $q \in (1, \infty)$, let q' be the *conjugate number* of q defined by $1/q + 1/q' = 1$. Let $\mathbb{C}$ be the set of complex numbers and $\mathbb{N} := \{1, 2, \dots\}$. Furthermore, $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ represent the duality relation, and the $L^2(\mathbb{R}^D)$ inner product respectively.

2 Proof of Theorem 1.15

In this section, we prove Theorem 1.15. To this end, we need to recall some well known results. The following conclusion is taken from [5, Cor. (8.21)].

Theorem 2.1 *Let $D \in \mathbb{N}$ and $q \in (1, \infty)$. Then the Riesz transform R_ℓ for each $\ell \in \{1, \dots, D\}$ is bounded on $\dot{F}^0_{1,q}(\mathbb{R}^D)$.*

Remark 2.2 By Remark 1.2, $\dot{F}^0_{1,2}(\mathbb{R}^D) = H^1(\mathbb{R}^D)$. Theorem 2.1 says that the Riesz transform R_ℓ for each $\ell \in \{1, \dots, D\}$ is bounded on $H^1(\mathbb{R}^D)$.

The Riesz transform characterization of $H^1(\mathbb{R}^D)$ can be found in [16, p. 221].

Theorem 2.3 *Let $D \in \mathbb{N}$. The space $H^1(\mathbb{R}^D)$ is isomorphic to the space of all functions $f \in L^1(\mathbb{R}^D)$ such that $\{R_\ell(f)\}_{\ell=1}^D \subset L^1(\mathbb{R}^D)$. Moreover, there exists a positive constant C such that, for all $f \in H^1(\mathbb{R}^D)$,*

$$\frac{1}{C}\|f\|_{H^1(\mathbb{R}^D)} \le \|f\|_{L^1(\mathbb{R}^D)} + \sum_{\ell=1}^{D} \|R_\ell(f)\|_{L^1(\mathbb{R}^D)} \le C\|f\|_{H^1(\mathbb{R}^D)}.$$

The following lemma is completely analogous to [11, Lem. 2.2], the details are omitted.

Lemma 2.4 *Let $D \in \mathbb{N}$ and $q \in [2, \infty)$. If $f \in \mathrm{WE}^{1,q}(\mathbb{R}^D)$, then, for any $j \in \mathbb{Z}$, $Q_j(f) \in H^1(\mathbb{R}^D)$, where $Q_j(f) := \sum_{(\lambda,k)\in E_D\times\mathbb{Z}^D} a_{j,k}^\lambda(f)\psi_{j,k}^\lambda$. Moreover, there exists a positive constant C such that, for all $j \in \mathbb{Z}$ and $f \in \mathrm{WE}^{1,q}(\mathbb{R}^D)$,*

$$\left\|Q_j(f)\right\|_{H^1(\mathbb{R}^D)} \le C\|f\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}.$$

2.1 Proof of Theorem 1.15

Proof of Theorem 1.15 We first show the necessity of Theorem 1.15. By (1.11) in Remark 1.12 and Theorem 2.1, we have

$$\sum_{\ell=0}^{D} \|R_\ell(f)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} \lesssim \sum_{\ell=0}^{D} \|R_\ell(f)\|_{\dot F^0_{1,q}(\mathbb{R}^D)} \lesssim \|f\|_{\dot F^0_{1,q}(\mathbb{R}^D)},$$

which completes the proof of the necessity of Theorem 1.15.

Now we show the sufficiency of Theorem 1.15. To this end, for any $f \in \mathrm{WE}^{1,q}(\mathbb{R}^D)$ such that $\{R_\ell(f)\}_{\ell=1}^D \subset \mathrm{WE}^{1,q}(\mathbb{R}^D)$, it suffices to show that, for any $s_1 \in \mathbb{Z}$, $N_1 \in \mathbb{N}$ and $f_{s_1,N_1} := P_{s_1,N_1} f$ defined as in (1.6), we have

$$\left\|f_{s_1,N_1}\right\|_{\dot F^0_{1,q}(\mathbb{R}^D)} \lesssim \sum_{\ell=0}^{D} \left\|R_\ell\left(f_{s_1,N_1}\right)\right\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}, \tag{2.1}$$

where the implicit constant is independent of s_1, N_1 and f. Recall that with the notation introduced in (1.7) and (1.8), we have that $f_{s_1,N_1} = T^{(1)}_{s_1,t_1,N_1}(f) + T^{(2)}_{s_1,t_1,N_1}(f)$.

Indeed, assume that (2.1) holds true for the time being. Owing to (1.5), for any $\ell \in \{1, \dots, D\}$, there exists a sequence $\{f_{j,k}^{\lambda,\ell}\}_{(\lambda,j,k)\in\Lambda_D} \subset \mathbb{C}$ such that

$$R_\ell\left(f_{s_1,N_1}\right) := \sum_{\{(\lambda,j,k)\in\Lambda_D:\ s_1-N_1-1\le j\le s_1+1\}} f_{j,k}^{\lambda,\ell}\psi_{j,k}^\lambda \quad \text{in} \quad \mathscr{S}'_\infty(\mathbb{R}^D).$$

By this and the orthogonality of $\{\psi^{\lambda}_{j,k}\}_{(\lambda,j,k)\in\Lambda_D}$, with the notation introduced in (1.7) and (1.8), we know that, for each $\ell \in \{1,\ldots,D\}$,

$$\begin{aligned}
&\left\|R_\ell\left(f_{s_1,N_1}\right)\right\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}\\
&= \sup_{\{\widetilde{s}\in\mathbb{Z},\,\widetilde{N}\in\mathbb{N}\}} \min_{t\in\{0,\ldots,\widetilde{N}+1\}} \left[\left\|T^{(1)}_{\widetilde{s},t,\widetilde{N}} R_\ell\left(f_{s_1,N_1}\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}\right.\\
&\quad \left.+\left\|T^{(2)}_{\widetilde{s},t,\widetilde{N}} R_\ell\left(f_{s_1,N_1}\right)\right\|_{L^1(\mathbb{R}^D)}\right]\\
&= \sup_{\substack{\widetilde{s}\in\mathbb{Z},\,\widetilde{N}\in\mathbb{N}\\ \widetilde{s}\le s_1+1,\,\widetilde{s}-\widetilde{N}\ge s_1-N_1-1}} \min_{t\in\{0,\ldots,\widetilde{N}+1\}} \left[\left\|T^{(1)}_{\widetilde{s},t,\widetilde{N}} R_\ell\left(f_{s_1,N_1}\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}\right.\\
&\quad \left.+\left\|T^{(2)}_{\widetilde{s},t,\widetilde{N}} R_\ell\left(f_{s_1,N_1}\right)\right\|_{L^1(\mathbb{R}^D)}\right]\\
&= \sup_{\substack{\widetilde{s}\in\mathbb{Z},\,\widetilde{N}\in\mathbb{N}\\ \widetilde{s}\le s_1+1,\,\widetilde{s}-\widetilde{N}\ge s_1-N_1-1}} \min_{t\in\{0,\ldots,\widetilde{N}+1\}} \left[\left\|T^{(1)}_{\widetilde{s},t,\widetilde{N}} R_\ell(f)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}+\left\|T^{(2)}_{\widetilde{s},t,\widetilde{N}} R_\ell(f)\right\|_{L^1(\mathbb{R}^D)}\right]\\
&\le \|R_\ell(f)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} < \infty.
\end{aligned} \tag{2.2}$$

A similar conclusion holds for $f_{s_1,N_1} = R_0\left(f_{s_1,N_1}\right)$. We choose to write out the process, for there is a distinction on the ranges.

$$\begin{aligned}
&\left\|f_{s_1,N_1}\right\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}\\
&= \sup_{\substack{\widetilde{s}\in\mathbb{Z},\,\widetilde{N}\in\mathbb{N}\\ \widetilde{s}\le s_1+1,\,\widetilde{s}-\widetilde{N}\ge s_1-N_1-1}} \min_{t\in\{0,\ldots,\widetilde{N}+1\}} \left[\left\|T^{(1)}_{\widetilde{s},t,\widetilde{N}} \left(f_{s_1,N_1}\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}\right.\\
&\quad \left.+\left\|T^{(2)}_{\widetilde{s},t,\widetilde{N}} \left(f_{s_1,N_1}\right)\right\|_{L^1(\mathbb{R}^D)}\right]\\
&= \sup_{\substack{\widetilde{s}\in\mathbb{Z},\,\widetilde{N}\in\mathbb{N}\\ \widetilde{s}\le s_1,\,\widetilde{s}-\widetilde{N}\ge s_1-N_1}} \min_{t\in\{0,\ldots,\widetilde{N}+1\}} \left[\left\|T^{(1)}_{\widetilde{s},t,\widetilde{N}}(f_{s_1,N_1})\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}+\left\|T^{(2)}_{\widetilde{s},t,\widetilde{N}}(f_{s_1,N_1})\right\|_{L^1(\mathbb{R}^D)}\right]\\
&= \sup_{\substack{\widetilde{s}\in\mathbb{Z},\,\widetilde{N}\in\mathbb{N}\\ \widetilde{s}\le s_1,\,\widetilde{s}-\widetilde{N}\ge s_1-N_1}} \min_{t\in\{0,\ldots,\widetilde{N}+1\}} \left[\left\|T^{(1)}_{\widetilde{s},t,\widetilde{N}}(f)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}+\left\|T^{(2)}_{\widetilde{s},t,\widetilde{N}}(f)\right\|_{L^1(\mathbb{R}^D)}\right]\\
&\le \|f\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} < \infty.
\end{aligned} \tag{2.3}$$

From (2.1), (2.2) and (2.3), we deduce that

$$\left\|f_{s_1,N_1}\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \lesssim \sum_{\ell=0}^{D} \|R_\ell(f)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}.$$

Further, the Levi lemma says

$$\|f\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \le \underline{\lim} \left\|f_{s_1,N_1}\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}.$$

These, together with Theorem 1.5, implies that $f \in \dot{F}^0_{1,q}(\mathbb{R}^D)$ and

$$\|f\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \lesssim \left\| \left\{ \sum_{(\lambda,j,k)\in\Lambda_D} \left[2^{Dj}\left|a^\lambda_{j,k}(f)\right|\chi\left(2^jx-k\right)\right]^q \right\}^{1/q} \right\|_{L^1(\mathbb{R}^D)}$$

$$\sim \lim_{N_1,s_1\to\infty} \left\| \left\{ \sum_{\{(\lambda,j,k)\in\Lambda_D:\, s_1-N_1\le j\le s_1\}} \left[2^{Dj}\left|a^\lambda_{j,k}(f)\right|\chi\left(2^jx-k\right)\right]^q \right\}^{1/q} \right\|_{L^1(\mathbb{R}^D)}$$

$$\lesssim \lim_{N_1,s_1\to\infty} \|f_{s_1,N_1}\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \lesssim \sum_{\ell=0}^{D} \|R_\ell(f)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)},$$

which are the desired conclusions for sufficiency.

Thus, to finish the proof of the sufficiency of Theorem 1.15, we still need to prove (2.1). To this end, fix $s_1 \in \mathbb{Z}$ and $N_1 \in \mathbb{N}$. In order to obtain the $\mathrm{WE}^{1,q}(\mathbb{R}^D)$-norms of $\{R_\ell(f_{s_1,N_1})\}_{\ell=0}^D$, by (2.2) and (2.3), it suffices to consider

$$s := s_1 + 1 \quad \text{and} \quad N := N_1 + 2 \text{ in (1.7) and (1.8)}, \tag{2.4}$$

because the proof for the other indices (s, N) can be deduced in a similar but easier way.

For such s and N, there exist $t^{(0)}_{s,N},\ t^{(1)}_{s,N} \in \{0,\ldots,N+1\}$ such that

$$\begin{aligned}&\left\|T^{(1)}_{s,t^{(0)}_{s,N},N}\left(f_{s_1,N_1}\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} + \left\|T^{(2)}_{s,t^{(0)}_{s,N},N}\left(f_{s_1,N_1}\right)\right\|_{L^1(\mathbb{R}^D)}\\ &\quad = \min_{t\in\{0,\ldots,N+1\}}\left[\left\|T^{(1)}_{s,t,N}\left(f_{s_1,N_1}\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} + \left\|T^{(2)}_{s,t,N}\left(f_{s_1,N_1}\right)\right\|_{L^1(\mathbb{R}^D)}\right]\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}&\sum_{\ell=1}^{D}\left[\left\|T^{(1)}_{s,t^{(1)}_{s,N},N}R_\ell\left(f_{s_1,N_1}\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} + \left\|T^{(2)}_{s,t^{(1)}_{s,N},N}R_\ell\left(f_{s_1,N_1}\right)\right\|_{L^1(\mathbb{R}^D)}\right]\\ &\quad = \min_{t\in\{0,\ldots,N+1\}}\sum_{\ell=1}^{D}\left[\left\|T^{(1)}_{s,t,N}R_\ell\left(f_{s_1,N_1}\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} + \left\|T^{(2)}_{s,t,N}R_\ell\left(f_{s_1,N_1}\right)\right\|_{L^1(\mathbb{R}^D)}\right].\end{aligned} \tag{2.6}$$

In the remainder of this proof, to simplify notation, *we let* $g_1 := f_{s_1,N_1}$ *for any fixed* s_1 *and* N_1, $t_j := t^{(j)}_{s,N}$ *and* $T_{i,j} := T^{(i)}_{s,t^{(j)}_{s,N},N}$ *for any* $i \in \{1,2\}$ *and* $j \in \{0,1\}$. With the help of (2.4), (2.5) and (2.6), to prove Eq. (2.1), it suffices to show that

$$\|g_1\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \lesssim \sum_{\ell=0}^{D} \|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} \tag{2.7}$$

□

2.2 The Proof of Eq. (2.7)

To prove Eq. (2.7), we consider the following three cases: $t_0 = t_1$, $t_0 > t_1$ and $t_0 < t_1$.

Case I $t_0 = t_1$. In this case, we write $g_1 = a_1 + a_2$, where

$$a_1 := \sum_{j=s-t_0+1}^{s} Q_j(g_1) \quad \text{and} \quad a_2 := \sum_{j=s-N}^{s-t_0} Q_j(g_1).$$

By (2.5), we have $a_2 = T_{2,0}(g_1) \in L^1(\mathbb{R}^D)$ and

$$\|a_2\|_{L^1(\mathbb{R}^D)} = \left\|T_{2,0}(g_1)\right\|_{L^1(\mathbb{R}^D)} \leq \|g_1\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}, \tag{2.8}$$

which, together with Lemma 2.4 and $H^1(\mathbb{R}^D) \subset L^1(\mathbb{R}^D)$, further implies that

$$Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1) \in H^1(\mathbb{R}^D)$$

and

$$\begin{aligned} &\left\|Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right\|_{H^1(\mathbb{R}^D)} \\ &\quad \leq \left\|Q_{s-t_0}(g_1)\right\|_{H^1(\mathbb{R}^D)} + \left\|Q_{s-t_0-1}(g_1)\right\|_{H^1(\mathbb{R}^D)} \lesssim \|g_1\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}. \end{aligned} \tag{2.9}$$

Thus, by this, $H^1(\mathbb{R}^D) \subset L^1(\mathbb{R}^D)$ and (2.8), we obtain

$$a_2 - \left[Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right] \in L^1(\mathbb{R}^D)$$

and

$$\begin{aligned} &\left\|a_2 - \left[Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right]\right\|_{L^1(\mathbb{R}^D)} \\ &\quad \leq \|a_2\|_{L^1(\mathbb{R}^D)} + \left\|Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right\|_{H^1(\mathbb{R}^D)} \lesssim \|g_1\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}. \end{aligned} \tag{2.10}$$

Moreover, for each $\ell \in \{1, \dots, D\}$, we have

$$\begin{aligned} T_{2,1}R_\ell(g_1) &= T_{2,1}R_\ell\left(a_2 + Q_{s-t_0+1}(g_1)\right) \\ &= T_{2,1}R_\ell\left(a_2 - \left[Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right]\right) \\ &\quad + T_{2,1}R_\ell\left(Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right) + T_{2,1}R_\ell Q_{s-t_0+1}(g_1) \\ &= R_\ell\left(a_2 - \left[Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right]\right) \\ &\quad + T_{2,1}R_\ell\left(Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right) + T_{2,1}R_\ell Q_{s-t_0+1}(g_1) \\ &= R_\ell\left(a_2 - \left[Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right]\right) + T_{2,1}R_\ell Q_{s-t_0+1}(g_1) \\ &\quad + T_{2,1}R_\ell\left(Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right). \end{aligned} \tag{2.11}$$

Denote

$$\begin{aligned}
\mathrm{I}^{(\ell)} &:= R_\ell \left(a_2 - \left[Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right]\right) \\
\mathrm{II}^{(\ell)} &:= \mathrm{I}^{(\ell)} + T_{2,1} R_\ell Q_{s-t_0+1}(g_1) \\
&= T_{2,1} R_\ell (g_1) - T_{2,1} R_\ell \left(Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right)
\end{aligned}$$

By (2.6), (2.11), $H^1(\mathbb{R}^D) \subset L^1(\mathbb{R}^D)$, Remarks 1.6 and 2.2, (2.9) and Lemma 2.4, we conclude that, for any $\ell \in \{1, \ldots, D\}$,

$$\mathrm{II}^{(\ell)} \in L^1(\mathbb{R}^D)$$

and

$$\begin{aligned}
\left\|\mathrm{II}^{(\ell)}\right\|_{L^1(\mathbb{R}^D)} &\le \left\|T_{2,1} R_\ell (g_1)\right\|_{L^1(\mathbb{R}^D)} + \left\|T_{2,1} R_\ell \left(Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right)\right\|_{H^1(\mathbb{R}^D)} \\
&\lesssim \sum_{\ell=1}^{D} \|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} + \left\|R_\ell \left(Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right)\right\|_{H^1(\mathbb{R}^D)} \\
&\lesssim \sum_{\ell=1}^{D} \|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} + \left\|Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right\|_{H^1(\mathbb{R}^D)} \\
&\lesssim \sum_{\ell=1}^{D} \|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} + \|g_1\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}.
\end{aligned} \tag{2.12}$$

From (1.5), it follows that, for each $\ell \in \{1, \ldots, D\}$, there exist $\{\tau_{j,k}^{\lambda,\ell}\}_{(\lambda,j,k)\in\Lambda_D} \subset \mathbb{C}$ such that

$$\begin{aligned}
\mathrm{I}^{(\ell)} &:= R_\ell \left(a_2 - \left[Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right]\right) \\
&= \sum_{\{(\lambda,j,k)\in\Lambda_D:\ s-N-1\le j\le s-t_0-1\}} \tau_{j,k}^{\lambda,\ell} \psi_{j,k}^{\lambda}
\end{aligned}$$

and

$$R_\ell Q_{s-t_0+1}(g_1) = \sum_{\{(\lambda,j,k)\in\Lambda_D:\ s-t_0\le j\le s-t_0+2\}} \tau_{j,k}^{\lambda,\ell} \psi_{j,k}^{\lambda}.$$

For any $h \in L^\infty(\mathbb{R}^D)$ and $j_0 \in \mathbb{Z}$, let

$$P_{j_0}(h) := \sum_{k\in\mathbb{Z}^D} \left\langle h, \psi_{j_0,k}^{\vec{0}} \right\rangle \psi_{j_0,k}^{\vec{0}},$$

where $\langle\cdot,\cdot\rangle$ represents the duality between $L^\infty(\mathbb{R}^D)$ and $L^1(\mathbb{R}^D)$. Note $Q_j = P_{j+1} - P_j$ and therefore by a telescoping sum argument $P_{j_0} = \sum_{j\le j_0-1} Q_j$.

We claim that $P_{j_0}(h) \in L^\infty(\mathbb{R}^D)$. Indeed, by $|\langle h, \psi^{\vec{0}}_{j_0,k}\rangle| \lesssim 2^{-Dj_0/2}$ and $\psi^{\vec{0}} \in \mathscr{S}(\mathbb{R}^D)$, we know that, for all $x \in \mathbb{R}^D$,

$$\left|P_{j_0}(h)(x)\right| \lesssim \sum_{k\in\mathbb{Z}^D} 2^{-Dj_0/2}\left|\psi^{\vec{0}}_{j_0,k}(x)\right| \lesssim \sum_{k\in\mathbb{Z}^D}\left|\psi^{\vec{0}}\left(2^{j_0}x-k\right)\right| \lesssim 1.$$

Let

$$h_0 := P_{s-t_0}(h) = \sum_{\{(\lambda,j,k)\in\Lambda_D:\ j\le s-t_0-1\}} a^\lambda_{j,k}(h)\psi^\lambda_{j,k}.$$

Thus, $h_0 \in L^\infty(\mathbb{R}^D)$ and $\|h_0\|_{L^\infty(\mathbb{R}^D)} \lesssim 1$ by the above claim.

Moreover, with (2.12) and by definition of $\mathrm{I}^{(\ell)}$, we observe that, for any $\ell \in \{1,\ldots,D\}$,

$$\left|\left\langle \mathrm{I}^{(\ell)}, h\right\rangle\right| = \left|\left\langle \mathrm{I}^{(\ell)}, h_0\right\rangle\right| = \left|\left\langle \mathrm{II}^{(\ell)}, h_0\right\rangle\right| \le \left\|\mathrm{II}^{(\ell)}\right\|_{L^1(\mathbb{R}^D)}\|h_0\|_{L^\infty(\mathbb{R}^D)},$$

which, combined with $\|h_0\|_{L^\infty(\mathbb{R}^D)} \lesssim 1$ and (2.12), further implies that

$$\left\|\mathrm{I}^{(\ell)}\right\|_{L^1(\mathbb{R}^D)} \lesssim \left\|\mathrm{II}^{(\ell)}\right\|_{L^1(\mathbb{R}^D)} \lesssim \sum_{\ell=0}^{D}\|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}.$$

From this, Theorem 2.3 and (2.10), it follows that

$$a_2 - \left[Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right] \in H^1(\mathbb{R}^D)$$

and

$$\begin{aligned}
&\left\|a_2 - \left[Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right]\right\|_{H^1(\mathbb{R}^D)} \\
&\quad\sim \left\|a_2 - \left[Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right]\right\|_{L^1(\mathbb{R}^D)} \\
&\qquad + \sum_{\ell=0}^{D}\left\|R_\ell\left(a_2 - \left[Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right]\right)\right\|_{L^1(\mathbb{R}^D)} \\
&\quad\lesssim \|g_1\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} + \sum_{\ell=1}^{D}\|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)},
\end{aligned}$$

which, together with Remark 1.2, (2.9) and Lemma 2.4, further implies that

$$\begin{aligned}
\|a_2\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} &\lesssim \|a_2\|_{H^1(\mathbb{R}^D)} \\
&\lesssim \left\|a_2 - \left[Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right]\right\|_{H^1(\mathbb{R}^D)} \\
&\quad + \left\|Q_{s-t_0}(g_1) + Q_{s-t_0-1}(g_1)\right\|_{H^1(\mathbb{R}^D)}
\end{aligned}$$

$$\lesssim \|g_1\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} + \sum_{\ell=1}^{D} \|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}. \tag{2.13}$$

Furthermore, by (2.5), we find that

$$\|a_1\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} = \left\|T_{1,0}(g_1)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \le \|g_1\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)},$$

which, combined with (2.13), implies that $g_1 = a_1 + a_2 \in \dot{F}^0_{1,q}(\mathbb{R}^D)$ and

$$\|g_1\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \le \|a_1\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} + \|a_2\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \lesssim \sum_{\ell=0}^{D} \|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}.$$

This finishes the proof of **Case I**.

Case II $t_0 > t_1$. In this case, we write $g_1 = b_1 + b_2 + b_3$, where

$$b_1 := \sum_{j=s-t_1+1}^{s} Q_j(g_1), \quad b_2 := \sum_{j=s-t_0+1}^{s-t_1} Q_j(g_1) \quad \text{and} \quad b_3 := \sum_{j=s-N}^{s-t_0} Q_j(g_1).$$

By (2.5) and (2.6), we have

$$b_1 = T_{1,1}(g_1), b_2 = T_{1,0}(g_1) - T_{1,1}(g_1) = T_{2,1}(g_1) - T_{2,0}(g_1) \text{ and } b_3 = T_{2,0}(g_1).$$

Similar to (2.11), for any $\ell \in \{1, \dots, D\}$, we know that

$$\begin{aligned} T_{2,1}R_\ell(g_1) &= T_{2,1}R_\ell\left(b_3 + b_2 + Q_{s-t_1+1}(g_1)\right) \\ &= R_\ell\left(b_3 + b_2 - \left[Q_{s-t_1}(g_1) + Q_{s-t_1-1}(g_1)\right]\right) \\ &\quad + T_{2,1}R_\ell\left(Q_{s-t_1}(g_1) + Q_{s-t_1-1}(g_1)\right) + T_{2,1}R_\ell Q_{s-t_1+1}(g_1) \\ &=: \mathrm{I}_1^{(\ell)} + \mathrm{I}_2^{(\ell)} + \mathrm{I}_3^{(\ell)}. \end{aligned}$$

With this notation observe that

$$R_\ell(b_2 + b_3) = \mathrm{I}_1^{(\ell)} + R_\ell\left(Q_{s-t_1}(g_1) + Q_{s-t_1-1}(g_1)\right). \tag{2.14}$$

For any $h \in L^\infty(\mathbb{R}^D)$, let

$$h_1 := \sum_{\{(\lambda, j, k)\in\Lambda_D:\ j\le s-t_1-1\}} a^\lambda_{j,k}(h)\psi^\lambda_{j,k} = P_{s-t_1}(h).$$

Similar to the proof of $h_0 \in L^\infty(\mathbb{R}^D)$, we have $h_1 \in L^\infty(\mathbb{R}^D)$ and

$$\|h_1\|_{L^\infty(\mathbb{R}^D)} \lesssim 1. \tag{2.15}$$

By (1.5), we know that, for any $\ell \in \{1, \dots, D\}$, there exists a sequence $\{f_{j,k}^{\lambda,\ell}\}_{(\lambda,j,k)\in\Lambda_D} \subset \mathbb{C}$ such that

$$\mathrm{I}_1^{(\ell)} = \sum_{\{(\lambda,j,k)\in\Lambda_D:\ s-N-1\le j\le s-t_1-1\}} f_{j,k}^{\lambda,\ell}\psi_{j,k}^{\lambda} \quad \text{and} \quad \mathrm{I}_3^{(\ell)} = \sum_{\{(\lambda,j,k)\in\Lambda_D:\ j=s-t_1\}} f_{j,k}^{\lambda,\ell}\psi_{j,k}^{\lambda},$$

which imply that

$$\left|\left\langle \mathrm{I}_1^{(\ell)}, h\right\rangle\right| = \left|\left\langle \mathrm{I}_1^{(\ell)}, h_1\right\rangle\right| = \left|\left\langle \mathrm{I}_1^{(\ell)} + \mathrm{I}_3^{(\ell)}, h_1\right\rangle\right| = \left|\left\langle T_{2,1}R_\ell(g_1) - \mathrm{I}_2^{(\ell)}, h_1\right\rangle\right|.$$

Hence, by this, (2.15), (2.6), $H^1(\mathbb{R}^D) \subset L^1(\mathbb{R}^D)$, Remarks 1.6 and 2.2, and Lemma 2.4, we conclude that

$$\begin{aligned}
\left\|\mathrm{I}_1^{(\ell)}\right\|_{L^1(\mathbb{R}^D)} &\lesssim \left[\left\|T_{2,1}R_\ell(g_1)\right\|_{L^1(\mathbb{R}^D)} + \left\|\mathrm{I}_2^{(\ell)}\right\|_{L^1(\mathbb{R}^D)}\right] \\
&\lesssim \sum_{\ell=1}^{D} \|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} + \left\|\mathrm{I}_2^{(\ell)}\right\|_{H^1(\mathbb{R}^D)} \\
&\lesssim \sum_{\ell=1}^{D} \|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} + \left\|R_\ell\left(Q_{s-t_1}(g_1) + Q_{s-t_1-1}(g_1)\right)\right\|_{H^1(\mathbb{R}^D)} \\
&\lesssim \sum_{\ell=1}^{D} \|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} + \left\|Q_{s-t_1}(g_1) + Q_{s-t_1-1}(g_1)\right\|_{H^1(\mathbb{R}^D)} \\
&\lesssim \sum_{\ell=1}^{D} \|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} + \|g_1\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}.
\end{aligned} \tag{2.16}$$

Thus, $\mathrm{I}_1^{(\ell)} \in L^1(\mathbb{R}^D)$. Moreover, by Remark 2.2 and Lemma 2.4, we have

$$\begin{aligned}
&\left\|R_\ell\left(Q_{s-t_1}(g_1) + Q_{s-t_1-1}(g_1)\right)\right\|_{H^1(\mathbb{R}^D)} \\
&\quad\lesssim \left\|Q_{s-t_1}(g_1) + Q_{s-t_1-1}(g_1)\right\|_{H^1(\mathbb{R}^D)} \lesssim \|g_1\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}.
\end{aligned} \tag{2.17}$$

With this notation, applying $H^1(\mathbb{R}^D) \subset L^1(\mathbb{R}^D)$, from (2.14), (2.16), (2.17) we deduce that

$$\begin{aligned}\|R_\ell(b_2+b_3)\|_{L^1(\mathbb{R}^D)} &\le \left\|\mathrm{I}_1^{(\ell)}\right\|_{L^1(\mathbb{R}^D)} + \left\|R_\ell\left(Q_{s-t_1}(g_1)+Q_{s-t_1-1}(g_1)\right)\right\|_{H^1(\mathbb{R}^D)} \\ &\lesssim \sum_{\ell=0}^{D}\|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}. \end{aligned} \tag{2.18}$$

On the other hand, for any function h satisfying that $\|h\|_{L^\infty(\mathbb{R}^D)} \le 1$, let

$$\widetilde{h}_0 := \sum_{\{(\lambda,j,k)\in\Lambda_D:\ j\ge s-t_0+2\}} a_{j,k}^{\lambda}(h)\psi_{j,k}^{\lambda} = h - P_{s-t_0+2}h.$$

Similar to the proof of $h_0 \in L^\infty(\mathbb{R}^D)$, we have $h-\widetilde{h}_0 \in L^\infty(\mathbb{R}^D)$ and $\|h-\widetilde{h}_0\|_{L^\infty(\mathbb{R}^D)} \lesssim 1$, which further implies that

$$\left\|\widetilde{h}_0\right\|_{L^\infty(\mathbb{R}^D)} \le \|h\|_{L^\infty(\mathbb{R}^D)} + \left\|h-\widetilde{h}_0\right\|_{L^\infty(\mathbb{R}^D)} \lesssim 1. \tag{2.19}$$

By (1.5), we know that

$$R_\ell\left(b_2-\left[Q_{s-t_0+1}(g_1)+Q_{s-t_0+2}(g_1)\right]\right) = \sum_{\{(\lambda,j,k)\in\Lambda_D:\ s-t_0+2\le j\le s-t_1+1\}} f_{j,k}^{\lambda,\ell}\psi_{j,k}^{\lambda},$$

which implies that

$$\begin{aligned}&\left\langle R_\ell\left(b_2-\left[Q_{s-t_0+1}(g_1)+Q_{s-t_0+2}(g_1)\right]\right), h\right\rangle \\ &\quad= \left\langle R_\ell\left(b_2-\left[Q_{s-t_0+1}(g_1)+Q_{s-t_0+2}(g_1)\right]\right), \widetilde{h}_0\right\rangle \\ &\quad= \left\langle R_\ell\left(b_3+b_2-\left[Q_{s-t_0+1}(g_1)+Q_{s-t_0+2}(g_1)\right]\right), \widetilde{h}_0\right\rangle. \end{aligned} \tag{2.20}$$

From an argument similar to that used in (2.17), it follows that

$$\begin{aligned}&\left\|R_\ell\left(Q_{s-t_0+1}(g_1)+Q_{s-t_0+2}(g_1)\right)\right\|_{H^1(\mathbb{R}^D)} \\ &\quad\lesssim \left\|Q_{s-t_0+1}(g_1)+Q_{s-t_0+2}(g_1)\right\|_{H^1(\mathbb{R}^D)} \lesssim \|g_1\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}. \end{aligned} \tag{2.21}$$

Thus, by (2.20), (2.19), (2.21), $H^1(\mathbb{R}^D) \subset L^1(\mathbb{R}^D)$ and (2.18), we conclude that

$$\begin{aligned}&\left\|R_\ell\left(b_2-\left[Q_{s-t_0+1}(g_1)+Q_{s-t_0+2}(g_1)\right]\right)\right\|_{L^1(\mathbb{R}^D)} \\ &\quad\lesssim \left\|R_\ell\left(b_3+b_2-\left[Q_{s-t_0+1}(g_1)+Q_{s-t_0+2}(g_1)\right]\right)\right\|_{L^1(\mathbb{R}^D)} \\ &\quad\lesssim \|R_\ell(b_3+b_2)\|_{L^1(\mathbb{R}^D)} + \left\|R_\ell\left(Q_{s-t_0+1}(g_1)+Q_{s-t_0+2}(g_1)\right)\right\|_{H^1(\mathbb{R}^D)} \\ &\quad\lesssim \sum_{\ell=0}^{D}\|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}. \end{aligned}$$

Therefore, by this, $H^1(\mathbb{R}^D) \subset L^1(\mathbb{R}^D)$ and (2.21), we obtain

$$\|R_\ell(b_2)\|_{L^1(\mathbb{R}^D)} \le \left\|R_\ell\left(b_2-\left[Q_{s-t_0+1}(g_1)+Q_{s-t_0+2}(g_1)\right]\right)\right\|_{L^1(\mathbb{R}^D)}$$

$$
\begin{aligned}
&+\left\|R_\ell\left(Q_{s-t_0+1}(g_1)+Q_{s-t_0+2}(g_1)\right)\right\|_{H^1(\mathbb{R}^D)}\\
&\lesssim \sum_{\ell=0}^{D}\|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)},
\end{aligned}
$$

which, together with (2.18), implies that

$$
\begin{aligned}
\|R_\ell(b_3)\|_{L^1(\mathbb{R}^D)} &\le \|R_\ell(b_3+b_2)\|_{L^1(\mathbb{R}^D)}+\|R_\ell(b_2)\|_{L^1(\mathbb{R}^D)}\\
&\lesssim \sum_{\ell=0}^{D}\|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}. \qquad (2.22)
\end{aligned}
$$

Furthermore, by (2.5), we know that

$$
\|b_3\|_{L^1(\mathbb{R}^D)}=\left\|T_{2,0}(g_1)\right\|_{L^1(\mathbb{R}^D)}\lesssim \|g_1\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)},
$$

which, combined with (2.22) and Theorem 2.3, implies that $b_3\in H^1(\mathbb{R}^D)$ and

$$
\|b_3\|_{H^1(\mathbb{R}^D)}\sim \|b_3\|_{L^1(\mathbb{R}^D)}+\sum_{\ell=1}^{D}\|R_\ell(b_3)\|_{L^1(\mathbb{R}^D)}\lesssim \sum_{\ell=0}^{D}\|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}. \qquad (2.23)
$$

By (2.23) and Remark 1.2, we know that $b_3\in \dot{F}^0_{1,q}(\mathbb{R}^D)$ and

$$
\|b_3\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}\lesssim \|b_3\|_{H^1(\mathbb{R}^D)}\lesssim \sum_{\ell=0}^{D}\|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}. \qquad (2.24)
$$

Moreover, by (2.5), we obtain

$$
\|b_1+b_2\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}=\left\|T_{1,0}(g_1)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}\lesssim \|g_1\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)},
$$

which, together with (2.24) and Remark 1.2, further implies that

$$
\|g_1\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}\le \|b_1+b_2\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}+\|b_3\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}\lesssim \sum_{\ell=0}^{D}\|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}.
$$

This finishes the proof of **Case II**.

Case III $t_0<t_1$. In this case, we write $g_1=e_1+e_2+e_3$, where

$$
e_1:=\sum_{j=s-t_0+1}^{s}Q_j(g_1),\quad e_2:=\sum_{j=s-t_1+1}^{s-t_0}Q_j(g_1)\quad \text{and}\quad e_3:=\sum_{j=s-N}^{s-t_1}Q_j(g_1).
$$

By (2.5) and (2.6), we have

$$e_1 = T_{1,0}(g_1),\, e_2 = T_{1,1}(g_1) - T_{1,0}(g_1) = T_{2,0}(g_1) - T_{2,1}(g_1) \text{ and } e_3 = T_{2,1}(g_1).$$

Moreover note that

$$e_2 + e_3 = T_{2,0}(g_1) \quad \text{and} \quad e_2 + e_1 = T_{1,1}(g_1).$$

Similar to (2.11), for any $\ell \in \{1, \dots, D\}$, we have

$$\begin{aligned} T_{2,1} R_\ell\,(g_1) &= T_{2,1} R_\ell\left(e_3 + Q_{s-t_1+1}\,(g_1)\right) \\ &= R_\ell\left(e_3 - \left[Q_{s-t_1}\,(g_1) + Q_{s-t_1-1}\,(g_1)\right]\right) \\ &\quad + T_{2,1} R_\ell\left(Q_{s-t_1}\,(g_1) + Q_{s-t_1-1}\,(g_1)\right) + T_{2,1} R_\ell Q_{s-t_1+1}\,(g_1) \\ &=: \mathrm{II}_1^{(\ell)} + \mathrm{II}_2^{(\ell)} + \mathrm{II}_3^{(\ell)}. \end{aligned} \tag{2.25}$$

Note that with this notation

$$\mathrm{II}_1^{(\ell)} + \mathrm{II}_3^{(\ell)} = T_{2,1} R_\ell\,(g_1) - \mathrm{II}_2^{(\ell)}.$$

For any $h \in L^\infty(\mathbb{R}^D)$, let

$$h_2 := \sum_{\{(\lambda,\, j,\, k)\in\Lambda_D:\ j\le s-t_1-1\}} a_{j,k}^\lambda(h)\psi_{j,k}^\lambda = P_{s-t_1}(h).$$

By an argument similar to that used in the proof of $h_0 \in L^\infty(\mathbb{R}^D)$, we conclude that $h_2 \in L^\infty(\mathbb{R}^D)$ and $\|h_2\|_{L^\infty(\mathbb{R}^D)} \lesssim 1$. By (1.5), we know that, for any $\ell \in \{1, \dots, D\}$, there exists a sequence $\{f_{j,k}^{\lambda,\ell}\}_{(\lambda,\, j,\, k)\in\Lambda_D} \subset \mathbb{C}$ such that

$$\mathrm{II}_1^{(\ell)} = \sum_{\{(\lambda,\, j,\, k)\in\Lambda_D:\ s-N-1\le j\le s-t_1-1\}} f_{j,k}^{\lambda,\ell}\psi_{j,k}^\lambda$$

and

$$T_{2,1} R_\ell Q_{s-t_1+1}\,(g_1) = \sum_{\{(\lambda,\, j,\, k)\in\Lambda_D:\ j=s-t_1\}} f_{j,k}^{\lambda,\ell}\psi_{j,k}^\lambda,$$

which, together with (2.25), imply that

$$\left|\left\langle \mathrm{II}_1^{(\ell)}, h\right\rangle\right| = \left|\left\langle \mathrm{II}_1^{(\ell)}, h_2\right\rangle\right| = \left|\left\langle \mathrm{II}_1^{(\ell)} + \mathrm{II}_3^{(\ell)}, h_2\right\rangle\right| = \left|\left\langle T_{2,1} R_\ell\,(g_1) - \mathrm{II}_2^{(\ell)}, h_2\right\rangle\right|.$$

Hence, by this, $\|h_2\|_{L^\infty(\mathbb{R}^D)} \lesssim 1$, (2.6), $H^1(\mathbb{R}^D) \subset L^1(\mathbb{R}^D)$, Remarks 1.2 and 2.2, and Lemma 2.4, we conclude that

$$\begin{aligned}\left\|\mathrm{II}_1^{(\ell)}\right\|_{L^1(\mathbb{R}^D)} &\lesssim \left\|T_{2,1}R_\ell(g_1)\right\|_{L^1(\mathbb{R}^D)} + \left\|\mathrm{II}_2^{(\ell)}\right\|_{L^1(\mathbb{R}^D)}\\ &\lesssim \sum_{\ell=1}^{D}\|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} + \left\|\mathrm{II}_2^{(\ell)}\right\|_{H^1(\mathbb{R}^D)}\\ &\lesssim \sum_{\ell=1}^{D}\|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} + \left\|R_\ell\left(Q_{s-t_1}(g_1)+Q_{s-t_1-1}(g_1)\right)\right\|_{H^1(\mathbb{R}^D)}\\ &\lesssim \sum_{\ell=1}^{D}\|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} + \left\|Q_{s-t_1}(g_1)+Q_{s-t_1-1}(g_1)\right\|_{H^1(\mathbb{R}^D)}\\ &\lesssim \sum_{\ell=1}^{D}\|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} + \|g_1\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}. \end{aligned} \tag{2.26}$$

Thus, $\mathrm{II}_1^{(\ell)} \in L^1(\mathbb{R}^D)$.

By (2.6), we have $e_2 + e_3 \in L^1(\mathbb{R}^D)$ and

$$\|e_2 + e_3\|_{L^1(\mathbb{R}^D)} = \left\|T_{2,0}(g_1)\right\|_{L^1(\mathbb{R}^D)} \lesssim \|g_1\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}. \tag{2.27}$$

For any $h \in L^\infty(\mathbb{R}^D)$, let

$$\widetilde{h}_1 := \sum_{\{(\lambda,j,k)\in\Lambda_D:\ j\le s-t_1-2\}} a_{j,k}^{\lambda}(h)\psi_{j,k}^{\lambda} = P_{s-t_1-1}h.$$

Similar to the proof of $h_0 \in L^\infty(\mathbb{R}^D)$, we have $\|\widetilde{h}_1\|_{L^\infty(\mathbb{R}^D)} \lesssim 1$. We notice that

$$\begin{aligned}&\left\langle e_3 - \left[Q_{s-t_1}(g_1) + Q_{s-t_1-1}(g_1)\right], h\right\rangle\\ &\quad = \left\langle e_3 - \left[Q_{s-t_1}(g_1) + Q_{s-t_1-1}(g_1)\right], \widetilde{h}_1\right\rangle = \left\langle e_3, \widetilde{h}_1\right\rangle = \left\langle e_3 + e_2, \widetilde{h}_1\right\rangle.\end{aligned}$$

Therefore, by this, (2.27) and $\|\widetilde{h}_1\|_{L^\infty(\mathbb{R}^D)} \lesssim 1$, we obtain

$$\left\|e_3 - \left[Q_{s-t_1}(g_1) + Q_{s-t_1-1}(g_1)\right]\right\|_{L^1(\mathbb{R}^D)} \lesssim \|e_3 + e_2\|_{L^1(\mathbb{R}^D)} \lesssim \|g_1\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}.$$

Hence $e_3 - [Q_{s-t_1}(g_1) + Q_{s-t_1-1}(g_1)] \in L^1(\mathbb{R}^D)$. From this, (2.26) and Theorem 2.3, we deduce that $e_3 - [Q_{s-t_1}(g_1) + Q_{s-t_1-1}(g_1)] \in H^1(\mathbb{R}^D)$ and

$$\begin{aligned}&\left\|e_3 - \left[Q_{s-t_1}(g_1) + Q_{s-t_1-1}(g_1)\right]\right\|_{H^1(\mathbb{R}^D)}\\ &\quad \sim \left\|e_3 - \left[Q_{s-t_1}(g_1) + Q_{s-t_1-1}(g_1)\right]\right\|_{L^1(\mathbb{R}^D)}\end{aligned}$$

$$+\sum_{\ell=1}^{D}\left\|R_\ell\left(e_3-\left[Q_{s-t_1}\left(g_1\right)+Q_{s-t_1-1}\left(g_1\right)\right]\right)\right\|_{L^1(\mathbb{R}^D)}$$
$$\lesssim\sum_{\ell=0}^{D}\left\|R_\ell\left(g_1\right)\right\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}.$$

Then, by this, $H^1(\mathbb{R}^D)\subset L^1(\mathbb{R}^D)$ and Lemma 2.4, we know that $e_3\in L^1(\mathbb{R}^D)$ and

$$\begin{aligned}\|e_3\|_{L^1(\mathbb{R}^D)}&\le\left\|e_3-\left[Q_{s-t_1}\left(g_1\right)+Q_{s-t_1-1}\left(g_1\right)\right]\right\|_{L^1(\mathbb{R}^D)}\\&\quad+\left\|Q_{s-t_1}\left(g_1\right)+Q_{s-t_1-1}\left(g_1\right)\right\|_{H^1(\mathbb{R}^D)}\\&\lesssim\sum_{\ell=0}^{D}\left\|R_\ell\left(g_1\right)\right\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}.\end{aligned}\tag{2.28}$$

For each $\ell\in\{1,\ldots,D\}$, we observe that

$$T_{1,1}R_\ell\left(g_1\right)=T_{1,1}R_\ell\left(e_1+e_2+Q_{s-t_1}\left(g_1\right)\right).$$

By this and (2.6), we know that, for any $\ell\in\{1,\ldots,D\}$,

$$T_{1,1}R_\ell\left(e_1+e_2+Q_{s-t_1}\left(g_1\right)\right)\in\dot{F}^0_{1,q}(\mathbb{R}^D)$$

and

$$\left\|T_{1,1}R_\ell\left(e_1+e_2+Q_{s-t_1}\left(g_1\right)\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}\lesssim\sum_{\ell=1}^{D}\left\|R_\ell\left(g_1\right)\right\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}.$$

This, together with

$$\|e_1\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}=\left\|T_{1,0}\left(g_1\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}\le\|g_1\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}\quad(\text{see (2.5)}),\tag{2.29}$$

Theorems 1.5 and 2.1, and (2.29), further implies that, for each $\ell\in\{1,\ldots,D\}$,

$$\begin{aligned}&\left\|T_{1,1}R_\ell\left(e_2+Q_{s-t_0}\left(g_1\right)\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}\\&\quad\le\left\|T_{1,1}R_\ell\left(e_1+e_2+Q_{s-t_0}\left(g_1\right)\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}+\left\|T_{1,1}R_\ell\left(e_1\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}\\&\quad\lesssim\sum_{\ell=1}^{D}\left\|R_\ell\left(g_1\right)\right\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}+\left\|R_\ell\left(e_1\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}\end{aligned}$$

$$\lesssim \sum_{\ell=1}^{D} \|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} + \|e_1\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}$$
$$\lesssim \sum_{\ell=1}^{D} \|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} + \|g_1\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}. \tag{2.30}$$

Furthermore, for any $\ell \in \{1, \ldots, D\}$, we notice that

$$T_{1,1}R_\ell\left(e_2 + Q_{s-t_1}(g_1)\right) = R_\ell\left(e_2 - \left[Q_{s-t_1+1}(g_1) + Q_{s-t_0}(g_1)\right]\right) + T_{1,1}R_\ell\left(Q_{s-t_1+1}(g_1) + Q_{s-t_1}(g_1) + Q_{s-t_0}(g_1)\right). \tag{2.31}$$

By Theorems 1.5 and 2.1, Remark 1.2 and Lemma 2.4, we conclude that

$$\begin{aligned}
&\left\|T_{1,1}R_\ell\left(Q_{s-t_1+1}(g_1) + Q_{s-t_1}(g_1) + Q_{s-t_0}(g_1)\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}\\
&\lesssim \left\|R_\ell\left(Q_{s-t_1+1}(g_1) + Q_{s-t_1}(g_1) + Q_{s-t_0}(g_1)\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}\\
&\lesssim \left\|Q_{s-t_1+1}(g_1) + Q_{s-t_1}(g_1) + Q_{s-t_0}(g_1)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}\\
&\lesssim \left\|Q_{s-t_1+1}(g_1) + Q_{s-t_1}(g_1) + Q_{s-t_0}(g_1)\right\|_{H^1(\mathbb{R}^D)} \lesssim \|g_1\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)},
\end{aligned}$$

which, together with (2.30) and (2.31), implies that

$$\begin{aligned}
&\left\|R_\ell\left(e_2 - \left[Q_{s-t_1+1}(g_1) + Q_{s-t_0}(g_1)\right]\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}\\
&\leq \left\|T_{1,1}R_\ell\left(e_2 + Q_{s-t_1}(g_1)\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}\\
&\quad + \left\|T_{1,1}R_\ell\left(Q_{s-t_1+1}(g_1) + Q_{s-t_1}(g_1) + Q_{s-t_0}(g_1)\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}\\
&\lesssim \|g_1\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} + \sum_{\ell=1}^{D} \|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}.
\end{aligned} \tag{2.32}$$

Now we need a useful identity from [18, p. 224, (2.9)] that, for all $f \in L^2(\mathbb{R}^D)$,

$$\sum_{\ell=1}^{D} R_\ell^2(f) = -f. \tag{2.33}$$

From $e_2 - [Q_{s-t_1+1}(g_1) + Q_{s-t_0}(g_1)] \in L^2(\mathbb{R}^D)$ and (2.33), we deduce that

$$\begin{aligned}
&e_2 - \left[Q_{s-t_1+1}(g_1) + Q_{s-t_0}(g_1)\right]\\
&\quad = \sum_{\ell=1}^{D} R_\ell^2\left(e_2 - \left[Q_{s-t_1+1}(g_1) + Q_{s-t_0}(g_1)\right]\right) \in \dot{F}^0_{1,q}(\mathbb{R}^D),
\end{aligned}$$

which, combined with Theorem 2.1 and (2.32), implies that

$$\begin{aligned}
&\left\|e_2 - \left[Q_{s-t_1+1}(g_1) + Q_{s-t_0}(g_1)\right]\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \\
&\quad\le \sum_{\ell=1}^{D} \left\|R^2_\ell \left(e_2 - \left[Q_{s-t_1+1}(g_1) + Q_{s-t_0}(g_1)\right]\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \\
&\quad\lesssim \sum_{\ell=1}^{D} \left\|R_\ell \left(e_2 - \left[Q_{s-t_1+1}(g_1) + Q_{s-t_0}(g_1)\right]\right)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \\
&\quad\lesssim \sum_{\ell=0}^{D} \|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} .
\end{aligned} \tag{2.34}$$

Again, by Remark 1.2 and Lemma 2.4, we obtain

$$\begin{aligned}
\left\|Q_{s-t_1+1}(g_1) + Q_{s-t_0}(g_1)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} &\lesssim \left\|Q_{s-t_1+1}(g_1) + Q_{s-t_0}(g_1)\right\|_{H^1(\mathbb{R}^D)} \\
&\lesssim \|g_1\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} ,
\end{aligned}$$

which, together with (2.34), implies that

$$\begin{aligned}
\|e_2\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} &\le \left\|e_2 - \left[Q_{s-t_1+1}(g_1) + Q_{s-t_0}(g_1)\right]\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \\
&\quad + \left\|Q_{s-t_1+1}(g_1) + Q_{s-t_0}(g_1)\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \lesssim \sum_{\ell=0}^{D} \|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} .
\end{aligned} \tag{2.35}$$

Combining with (2.29), (2.35) and (2.28), we obtain

$$\|g_1\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \le \sum_{j=1}^{3} \left\|e_j\right\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \lesssim \sum_{\ell=0}^{D} \|R_\ell(g_1)\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)} ,$$

which completes the proof of **Case III** and hence Theorem 1.15.

3 Proof of Theorem 1.17

Proof of Theorem 1.17 (i) Suppose that $q \in [2, \infty)$, ϕ and Φ are defined as in the construction of the 1-dimensional Meyer wavelets. Moreover, we assume that the 1-dimensional Meyer wavelet ψ satisfies $\psi(0) \neq 0$.

For any $x := (x_1, \ldots, x_D) \in \mathbb{R}^D$, let $\psi^{\vec{0}}(x) := \phi(x_1) \cdots \phi(x_D)$. From [11, (5.1)], we deduce that, for all $x \in \mathbb{R}^D$,

$$\sum_{k\in\mathbb{Z}^D}\left|\psi^{\vec{0}}(x-k)\right| = \prod_{\ell=1}^{D}\sum_{k_\ell\in\mathbb{Z}}|\phi\,(x_\ell-k_\ell)| \lesssim 1. \tag{3.1}$$

For any $j\in\mathbb{Z}, k\in\mathbb{Z}^D$ and $x\in\mathbb{R}^D$, we write $\psi^{\vec{0}}_{j,k}(x) := 2^{Dj/2}\psi^{\vec{0}}\left(2^j x-k\right)$. Let $f\in L^1(\mathbb{R}^D)$. For any $j\in\mathbb{Z}$, define $P_j(f) := \sum_{k\in\mathbb{Z}^D}\langle f,\psi^{\vec{0}}_{j,k}\rangle\psi^{\vec{0}}_{j,k}$. Then, for any $j\in\mathbb{Z}$, by estimate in (3.1),

$$\begin{aligned}\|P_j(f)\|_{L^1(\mathbb{R}^D)} &\le \int_{\mathbb{R}^D}\int_{\mathbb{R}^D}|f(y)|\sum_{k\in\mathbb{Z}^D}\left|\psi^{\vec{0}}(2^j y-k)\right|\left|2^{Dj}\psi^{\vec{0}}(2^j x-k)\right|\,dx\,dy\\ &\lesssim \int_{\mathbb{R}^D}|f(y)|\sum_{k\in\mathbb{Z}^D}\left|\psi^{\vec{0}}(y-k)\right|\,dy \lesssim \|f\|_{L^1(\mathbb{R}^D)}. \end{aligned}\tag{3.2}$$

The proper inclusion relations in (i) of Theorem 1.17 are contained in Remark 1.12 except for the first inclusion. In order to prove (i) of Theorem 1.17, it suffices to show that $\dot{F}^0_{1,q}(\mathbb{R}^D) \subsetneqq L^1(\mathbb{R}^D)\cup\dot{F}^0_{1,q}(\mathbb{R}^D)$. We first observe that $\psi^{\vec{0}}\in L^1(\mathbb{R}^D)$. Indeed,

$$\left\|\psi^{\vec{0}}\right\|_{L^1(\mathbb{R}^D)} = \prod_{\ell=1}^{D}\|\phi\|_{L^1(\mathbb{R}^D)} = \|\phi\|^D_{L^1(\mathbb{R}^D)} < \infty.$$

To show $\psi^{\vec{0}}\notin\dot{F}^0_{1,q}(\mathbb{R}^D)$, let $a_{j,k}(\psi^{\vec{0}}) := (\psi^{\vec{0}},\psi_{j,k})$ for any $j\in\mathbb{Z}$ and $k\in\mathbb{Z}^D$, where $(\cdot,\cdot)$ represents the $L^2(\mathbb{R}^D)$ inner product. Let $\xi := (\xi_1,\dots,\xi_D)$, $\eta := (\eta_1,\dots,\eta_D)\in\mathbb{R}^D$. Then, by the multiplication formula (see [18, p. 8, Thm. 1.15]), (1.2), (1.1), (1.3) and the assumption $\psi(0)\neq 0$, we obtain

$$\begin{aligned}\left|a_{j,0}\left(\psi^{\vec{0}}\right)\right| &= \prod_{\ell=1}^{D}\left|\int_{\mathbb{R}}\widehat{\phi}\,(-\xi_\ell)\,2^{-j/2}\widehat{\psi}\left(2^{-j}\xi_\ell\right)\,d\xi_\ell\right|\\ &= \prod_{\ell=1}^{D}\left|\int_{-4\pi/3}^{4\pi/3}\Phi\,(\xi_\ell)\,2^{-j/2}\widehat{\psi}\left(2^{-j}\xi_\ell\right)\,d\xi_\ell\right|\\ &= \prod_{\ell=1}^{D}\left|\int_{-2^{2-j}\pi/3}^{2^{2-j}\pi/3}\Phi\left(2^j\eta_\ell\right)2^{j/2}\widehat{\psi}\,(\eta_\ell)\,d\eta_\ell\right|\\ &= \prod_{\ell=1}^{D}\left|\int_{-8\pi/3}^{8\pi/3}\Phi\left(2^j\eta_\ell\right)2^{j/2}\widehat{\psi}\,(\eta_\ell)\,d\eta_\ell\right|\\ &\sim 2^{Dj/2}\prod_{\ell=1}^{D}\left|\int_{-8\pi/3}^{8\pi/3}\widehat{\psi}\,(\eta_\ell)\,d\eta_\ell\right| \sim 2^{Dj/2}|\psi(0)|^D \gtrsim 2^{Dj/2}, \end{aligned}\tag{3.3}$$

provided that $j < -M$ for some positive integer M large enough. Therefore, we have

$$\begin{aligned}
&\int_{\mathbb{R}^D}\int_{\mathbb{R}^D}\left\{\sum_{j\in\mathbb{Z},\,k\in\mathbb{Z}^D}\left[2^{Dj/2}\left|a_{j,k}\left(\psi^{\vec{0}}\right)\right|\chi\left(2^j x-k\right)\right]^q\right\}^{1/q}dx\\
&\geq \int_{\mathbb{R}^D}\left\{\sum_{j=-\infty}^{-M-1}2^{Djq/2}\left|a_{j,0}\left(\psi^{\vec{0}}\right)\right|^q\chi\left(2^j x-k\right)\right\}^{1/q}dx\\
&\gtrsim \int_{\mathbb{R}^D}\left\{\sum_{j=-\infty}^{-M-1}2^{Djq}\chi\left(2^j x-k\right)\right\}^{1/q}dx\\
&\gtrsim \sum_{m=M}^{\infty}\int_{B(0,\,2^{m+1})\setminus B(0,\,2^m)}\left\{\sum_{j=-\infty}^{-m-1}2^{Djq}\right\}^{1/q}dx=\infty,
\end{aligned}$$

which, combined with Theorem 1.5, implies that $\psi^{\vec{0}}\notin \dot{F}^0_{1,q}(\mathbb{R}^D)$. This finishes the proof of (i) of Theorem 1.17.

(ii) Then we use Daubechies wavelets to prove (ii) of Theorem 1.17. The goal is to show that there is function $g\in \dot{F}^0_{\infty,q'}(\mathbb{R}^D)$ such that $g\notin WE^{\infty,q'}(\mathbb{R}^D)$. We will realize this g as the Riesz transform of a suitable function f, more precisely, $g=R_1 f$. We know that there exist some integer M and a Daubechies scale function $\Phi^0(x)\in C_0^{D+2}([-2^M,2^M]^D)$ satisfying

$$C_D=\int\frac{-y_1}{|y|^{n+1}}\Phi^0(y-2^{M+1}e)dy<0, \tag{3.4}$$

where $e=(1,1,\ldots,1)$. Let $\Phi(x)=\Phi^0(x-2^{M+1}e)$ and let f be defined as

$$f(x)=\sum_{j\in 2\mathbb{N}}\Phi(2^j x). \tag{3.5}$$

For $j,j'\in 2\mathbb{N}$, $j\neq j'$, the supports of $\Phi(2^j x)$ and $\Phi(2^{j'}x)$ are disjoint. Hence the above $f(x)$ in (3.5) belongs to $L^\infty(\mathbb{R}^D)$. The same reasoning gives, for any $j'\in\mathbb{N}$,

$$\sum_{j\in\mathbb{N},2j>j'}\Phi(2^{2j}x)\in L^\infty(\mathbb{R}^D).$$

Now we compute the wavelet coefficients of $f(x)$ in (3.5). For $(\lambda',j',k')\in\Lambda_D$, let $f^{\lambda'}_{j',k'}=\langle f,\ \Phi^{\lambda'}_{j',k'}\rangle$. We divide two cases: $j'<0$ and $j'\geq 0$.

For $j'<0$, since the support of f is contained in $[-3\cdot 2^M,3\cdot 2^M]^D$, we know that if $|k'|>2^{2M+5}$, then $f^{\lambda'}_{j',k'}=0$. If $|k'|\leq 2^{2M+5}$, we have

$$|f^{\lambda'}_{j',k'}|\leq C2^{Dj'}\int|f(x)|dx\leq C2^{Dj'}.$$

For $j' \geq 0$, by orthogonality of the wavelets, we have

$$f^{\lambda'}_{j',k'} = \left\langle f,\ \Phi^{\lambda'}_{j',k'} \right\rangle = \left\langle \sum_{j\in\mathbb{N},2j>j'} \Phi(2^{2j}\cdot),\ \Phi^{\lambda'}_{j',k'} \right\rangle.$$

By the same reasoning, for the case $j' \geq 0$, we know that if $|k'| > 2^{2M+5}$, then $f^{\lambda'}_{j',k'} = 0$. Since $\sum_{j\in\mathbb{N},2j>j'} \Phi(2^{2j}x) \in L^\infty$, if $|k'| \leq 2^{2M+5}$, we have

$$|f^{\lambda'}_{j',k'}| \leq C \int |\Phi^{\lambda'}_{j',k'}(x)|dx \leq C.$$

By the above estimation of wavelet coefficients of $f(x)$ and by the wavelet characterization of $\dot{F}^0_{\infty,q'}(\mathbb{R}^D)$ in (ii) of Theorem 1.5, we conclude that $f \in \dot{F}^0_{\infty,q'}(\mathbb{R}^D)$. Hence,

$$f \in L^\infty(\mathbb{R}^D) \cap \dot{F}^0_{\infty,q'}(\mathbb{R}^D). \tag{3.6}$$

Since $\Phi^0 \in C_0^{D+2}([-2^M, 2^M]^D)$, we know that

$$\Phi(x) = \Phi^0(x - 2^{M+1}e) \in C_0^{D+2}([2^M, 3\cdot 2^M]^D).$$

Further, if $|x| \leq 2^{M-1}$ and $y \in [2^M, 3\cdot 2^M]^D$, then $|x-y| > 2^{M-1}$. Hence $R_1\Phi(x)$ is smooth in the ball $\{x : |x| \leq 2^{M-1}\}$.

Applying (3.4), there exists a positive $\delta > 0$ such that for $|x| < \delta$, there holds $R_1\Phi(x) < C_D/2 < 0$. That implies, if $2^{2j}|x| < \delta$, then $R_1\Phi(2^{2j}x) < C_D/2 < 0$. Hence

$$R_1 f(x) \notin L^\infty(\mathbb{R}^D). \tag{3.7}$$

The Eqs. (3.6), (3.7) and the continuity of Riesz operators on $\dot{F}^0_{\infty,q'}(\mathbb{R}^D)$ implies $g = R_1(f) \in \dot{F}^0_{\infty,q'}(\mathbb{R}^D)$ but $g \notin WE^{\infty,q'}(\mathbb{R}^D)$, which is precisely what we set out to prove. □

4 The Proofs of Theorems 1.14, 1.18 and 1.19

We first prove Theorem 1.14; note that this proof is independent of Theorem 1.15.

Proof of Theorem 1.14 If $l \in (L^1(\mathbb{R}^D) \cup \dot{F}^0_{1,q}(\mathbb{R}^D))'$, then

$$\sup_{f\in\mathscr{S}_\infty(\mathbb{R}^D),\|f\|_{L^1\cup\dot{F}^0_{1,q}}\leq 1} |\langle l, f\rangle| < \infty.$$

In other words,

$$\sup_{f\in\mathscr{S}_\infty(\mathbb{R}^D),\|f\|_{L^1}\leq 1} |\langle l, f\rangle| < \infty \text{ and} \tag{4.1}$$

$$\sup_{f\in\mathscr{S}_\infty(\mathbb{R}^D),\|f\|_{\dot F^0_{1,q}}\le 1} |\langle l, f\rangle| < \infty. \tag{4.2}$$

The condition (4.1) means $l \in L^\infty(\mathbb{R}^D)$, the condition (4.2) means $l \in \dot F^0_{\infty,q'}(\mathbb{R}^D)$. Hence we have the following inclusion relation:

$$\left(L^1(\mathbb{R}^D) \cup \dot F^0_{1,q}(\mathbb{R}^D)\right)' \subset L^\infty(\mathbb{R}^D) \cap \dot F^0_{\infty,q'}(\mathbb{R}^D). \tag{4.3}$$

Further, by Theorem 1.17(i), we have that

$$L^1(\mathbb{R}^D) \cup \dot F^0_{1,q}(\mathbb{R}^D) \subset \mathrm{WE}^{1,q}(\mathbb{R}^D) \subset L^1(\mathbb{R}^D) + \dot F^0_{1,q}(\mathbb{R}^D).$$

Hence we have

$$\left(L^1(\mathbb{R}^D) + \dot F^0_{1,q}(\mathbb{R}^D)\right)' \subset \left(\mathrm{WE}^{1,q}(\mathbb{R}^D)\right)' \subset \left(L^1(\mathbb{R}^D) \cup \dot F^0_{1,q}(\mathbb{R}^D)\right)'. \tag{4.4}$$

Moreover, by the fact $\left(L^1(\mathbb{R}^D)\right)' = L^\infty(\mathbb{R}^D)$ and $\left(\dot F^0_{1,q}(\mathbb{R}^D)\right)' = \dot F^0_{\infty,q'}$ and the definition of $L^1(\mathbb{R}^D) + \dot F^0_{1,q}(\mathbb{R}^D)$ in (1.9), we know that

$$\left(L^1(\mathbb{R}^D) + \dot F^0_{1,q}(\mathbb{R}^D)\right)' = L^\infty(\mathbb{R}^D) \cap \dot F^0_{\infty,q'}(\mathbb{R}^D). \tag{4.5}$$

The Eqs. (4.3), (4.4) and (4.5) implies Theorem 1.14. □

Using Theorems 1.14 and 1.15, we prove Theorem 1.18.

Proof of Theorem 1.18 By the continuity of Riesz operators on the $\dot F^0_{\infty,q}(\mathbb{R}^D)$, we know that if $f_l \in \dot F^0_{\infty,q}(\mathbb{R}^D) \cap L^\infty(\mathbb{R}^D)$, then

$$\sum_{0\le l\le D} R_l f_l(x) \in \dot F^0_{\infty,q}(\mathbb{R}^D).$$

We now prove the converse result. Let

$$B = \left\{(g_0, g_1, \ldots, g_D) : g_l \in \mathrm{WE}^{1,q'}(\mathbb{R}^D), l = 0, \ldots, D\right\},$$
$$\tilde B = \left\{(g_0, g_1, \ldots, g_D) : g_l \in L^1(\mathbb{R}^D) + \dot F^0_{1,q'}(\mathbb{R}^D), l = 0, \ldots, D\right\},$$

where $B \subset \tilde B$. The norm of B and $\tilde B$ is defined respectively as follows

$$\|(g_0, g_1, \ldots, g_D)\|_B = \sum_{l=0}^{D} \|g_l\|_{\mathrm{WE}^{1,q'}},$$

$$\|(g_0, g_1, \ldots, g_D)\|_{\tilde{B}} = \sum_{l=0}^{D} \|g_l\|_{L^1+\dot{F}^0_{1,q'}}.$$

We define

$$S = \{(g_0, g_1, \ldots, g_D) \in B : g_l = R_l g_0, l = 0, 1, \ldots, D\},$$
$$\tilde{S} = \left\{(g_0, g_1, \ldots, g_D) \in \tilde{B} : g_l = R_l g_0, l = 0, 1, \ldots, D\right\},$$

where $S \subset \tilde{S}$.

By Theorem 1.15, $g_0 \to (g_0, R_1 g_0, \ldots, R_D g_0)$ defines a norm preserving map from $\dot{F}^0_{1,q'}(\mathbb{R}^D)$ to S. Hence the set of continuous linear functionals f on $\dot{F}^0_{1,q'}(\mathbb{R}^D)$ is equivalent to the set of bounded linear maps on the set S. According to the definition of B and $\tilde{B}$ and the second equality in the equations of Theorem 1.14, the set of continuous linear functionals f on $\dot{F}^0_{1,q'}(\mathbb{R}^D)$ is equivalent also to the set of continuous linear maps on the Banach space $\tilde{S}$.

According to Theorem 1.14, the continuous linear functionals on B belong to

$$\mathrm{WE}^{\infty,q}(\mathbb{R}^D) + \cdots + \mathrm{WE}^{\infty,q}(\mathbb{R}^D).$$

For all $f \in \dot{F}^0_{\infty,q}(\mathbb{R}^D)$, f defines a continuous linear functional l on $\dot{F}^0_{1,q'}(\mathbb{R}^D)$ and also on $\tilde{S}$. Hence there exist $\tilde{f}_l \in \mathrm{WE}^{\infty,q}(\mathbb{R}^D), l = 0, 1, \ldots, D$, such that for any $g_0 \in \dot{F}^0_{1,q'}(\mathbb{R}^D)$,

$$\begin{aligned}
&\int_{\mathbb{R}^D} f(x) g_0(x) dx \\
&= \int_{\mathbb{R}^D} \tilde{f}_0(x) g_0(x) dx + \sum_{l=1}^{D} \int_{\mathbb{R}^D} \tilde{f}_l(x) R_l g_0(x) dx \\
&= \int_{\mathbb{R}^D} \tilde{f}_0(x) g_0(x) dx - \sum_{l=1}^{D} \int_{\mathbb{R}^D} R_l(\tilde{f}_l)(x) g_0(x) dx.
\end{aligned}$$

Hence $f(x) = \tilde{f}_0(x) - \sum_{l=1}^{D} R_l(\tilde{f}_l)(x)$. □

Finally, as a consequence of Theorem 1.18, we deduce Theorem 1.19.

Proof of Theorem 1.19 By the continuity of Riesz operators on $\dot{F}^0_{1,q}(\mathbb{R}^D)$, there exists a positive constant C such that, for all $f \in \dot{F}^0_{1,q}(\mathbb{R}^D)$,

$$\sum_{\ell=0}^{D} \|R_\ell(f)\|_{L^1(\mathbb{R}^D)+\dot{F}^0_{1,q}(\mathbb{R}^D)} \le C\|f\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)}.$$

To prove

$$\frac{1}{C}\|f\|_{\dot{F}^0_{1,q}(\mathbb{R}^D)} \le \sum_{\ell=0}^{D} \|R_\ell(f)\|_{L^1(\mathbb{R}^D)+\dot{F}^0_{1,q}(\mathbb{R}^D)},$$

it is sufficient to prove

$$|\langle f, g\rangle| \le C\left\{\sum_{l=0}^{D}\|R_l f\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}\right\}\|g\|_{\dot{F}^0_{\infty,q'}(\mathbb{R}^D)}$$
$$\text{for all } g \in \mathscr{S}_\infty(\mathbb{R}^D)\cap \dot{F}^0_{\infty,q'}(\mathbb{R}^D).$$

But, by Theorem 1.18, for all $g \in \mathscr{S}_\infty(\mathbb{R}^D)\cap \dot{F}^0_{\infty,q'}(\mathbb{R}^D)$ there exists g_l such that

$$\|g_l\|_{\mathrm{WE}^{\infty,q'}(\mathbb{R}^D)} \le C\|g\|_{\dot{F}^0_{\infty,q'}(\mathbb{R}^D)} \quad \text{and} \quad g = \sum_{l=0}^{D} R_l g_l.$$

Hence, we have

$$\begin{aligned}|\langle f, g\rangle| = |\langle f, \sum_{l=0}^{D} R_l g_l\rangle| &\le \sum_{l=0}^{D}|\langle f, R_l g_l\rangle| = \sum_{l=0}^{D}|\langle R_l f, g_l\rangle|\\ &\le C\sum_{l=0}^{D}\|R_l f\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}\|g_l\|_{\mathrm{WE}^{\infty,q'}(\mathbb{R}^D)}\\ &\le C\sum_{l=0}^{D}\|R_l f\|_{\mathrm{WE}^{1,q}(\mathbb{R}^D)}\|g\|_{\dot{F}^0_{\infty,q'}(\mathbb{R}^D)}.\end{aligned}$$

□

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