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Uniform generalizations of Fueter's theorem

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Abstract
Fueter's theorem (1934) asserts that every holomorphic intrinsic function of one complex variable induces an axial quaternionic monogenic function. Sce (Atti Accad Naz Lincei Rend Cl Sci Fis Mat Nat 23:220–225, 1957) generalizes Fueter's theorem to the Euclidean spaces $\mathbb{R}^{n+1}$ for n being odd positive integers. By using pointwise differential computation he asserted that every holomorphic intrinsic function of one complex variable induces an axial Clifford monogenic function for the cases n being odd. Qian (Rend Mat Acc Lincei 8:111–117, 1997) extended Sce's result to both n being odd and even cases by using the corresponding Fourier multiplier operator when the required integrability is guaranteed, and the Kelvin inversion if not. For n being odd, Qian's generalization coincides with Sce's result based on the pointwise differential operator. In this paper, we unify these results in the distribution sense.

Keywords Fourier multipliers · Holomorphic intrinsic functions · Axial monogenic functions · Fueter's theorem

Mathematics Subject Classification Primary 30G35

1 Introduction

The classical Fueter theorem asserts that every holomorphic intrinsic function of one complex variable induces a quaternionic monogenic function [9]. A holomorphic intrinsic function of one variable $f_0(z)$ is defined on an intrinsic set $O \subset \mathbb{C}$ and satisfies $\overline{f_0(z)} = f_0(\overline{z})$. A set $O \subset \mathbb{C}$ is said to be intrinsic if it is open and symmetric with respect to the real axis. Every holomorphic function $f_0(z)$ has the form

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$$f_0(z) = u(x_0, y_0) + \mathrm{i}v(x_0, y_0),$$

where $u(x_0, y_0)$ and $v(x_0, y_0)$ are real-valued functions. The intrinsic condition $\overline{f_0(z)} = f_0(\overline{z})$ is equivalent with $u(x_0, y_0) = u(x_0, -y_0)$ and $v(x_0, y_0) = -v(x_0, -y_0)$. In particular, $v(x_0, 0) = 0$, i.e., f_0 is real-valued if it is restricted to the real line in its domain. So, a characterization of a holomorphic intrinsic function is that the coefficients of its Laurent series expansions at real points are all real.

For any intrinsic holomorphic function $f_0(z)$, first build up the lifting mapping

$$\mathcal{S}_{\mathbb{H}} : f_0 \to \vec{f}_0(q),$$

where the Hamilton quaternionic-valued function $\vec{f}_0(q)$ is defined by

$$\vec{f}_0(q) := u(q_0, |\underline{q}|) + \frac{\underline{q}}{|\underline{q}|} v(q_0, |\underline{q}|), \quad q \notin \mathbb{R},$$

and q_0 and $\underline{q} := q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ denote, respectively, the real and the imaginary parts of the quaternion number q. Then, take the Laplacian operator with respect to the real variable q_0, q_1, q_2, q_3, where the Laplacian is defined

$$(-\Delta)_{\mathbb{H}} = -\left(\partial^2_{q_0} + \partial^2_{q_1} + \partial^2_{q_2} + \partial^2_{q_3}\right).$$

In such formulation there holds Fueter's theorem (1934) stating that the image of f_0 under the composition operator $(-\Delta)_{\mathbb{H}} \circ \mathcal{S}_{\mathbb{H}}$ is a quaternionic axial regular function. That says, for the quaternionic Cauchy–Riemann operator $D_q = \partial_{q_0} + \mathbf{i}\partial_{q_1} + \mathbf{j}\partial_{q_2} + \mathbf{k}\partial_{q_3}$, there holds

$$D_q\left[(-\Delta)_{\mathbb{H}} \circ \mathcal{S}_{\mathbb{H}} f_0\right] = 0$$

and $[(-\Delta)_{\mathbb{H}} \circ \mathcal{S}_{\mathbb{H}}]f_0$ has the axial form

$$\left[(-\Delta)_{\mathbb{H}} \circ \mathcal{S}_{\mathbb{H}}\right] f_0 = A(q_0, |\underline{q}|) + \frac{\underline{q}}{|\underline{q}|} B(q_0, |\underline{q}|),$$

where $A(q_0, |\underline{q}|)$ and $B(q_0, |\underline{q}|)$ are real-valued functions.

Sce's theorem (1957), in [23], generalizes Fueter's theorem to the Euclidean spaces $\mathbb{R}^{n+1}$ for n being odd positive integers. Sce's result asserts that with $\mathcal{S}_{\mathbb{R}^{n+1}}$ being the lifting mapping to get functions in $\mathbb{R}^{n+1}$, being similarly defined as for the quaternionic case, and $(-\Delta)_p^{(n-1)/2}$ being a pointwise differential operator, where the subscript "p" is for "pointwise" to distinguish it from "non-pointwise" operators in the sequel, then the composition operator $(-\Delta)_p^{(n-1)/2} \circ \mathcal{S}_{\mathbb{R}^{n+1}}$, called the Sce mapping, maps f_0 to an axial Clifford monogenic function.

Later, in [16, 18], Qian raises a generalization of the Sce result to any dimension n by using the operator $\mathcal{D}_{\mathbb{R}^{n+1}}$, called the Qian mapping in the sequel, namely,

$$\mathcal{D}_{\mathbb{R}^{n+1}}(x^k) = \begin{cases} \mathcal{F}^{-1}((2\pi|\cdot|)^{n-1}\mathcal{F}(x^k)), & k = -1, -2, \dots, \\ I(\mathcal{D}_{\mathbb{R}^{n+1}}(x^{-k+n-2})), & k = n-1, n, \dots, \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^{n+1}$, $\mathcal{F}$ is the Fourier transformation in $\mathbb{R}^{n+1}$, and I is the Kelvin inversion. For n being odd, the Qian mapping coincides with the Sce mapping in terms of the pointwise differential operator [16]. Further studies on related topics, including ranges and domains of the related lifting and differential operators, manipulations for harmonic functions in higher dimensions, etc., may be found in [2, 7, 12–14, 22, 24].

The Fueter theorem and its generalizations have found crucial applications in functional calculi of a single Dirac operator and n-tuple non-commutative operators ([5, 10, 11, 15, 17, 19–21] and references therein).

To further study the Fueter type correspondence between one complex variable holomorphic intrinsic functions and quaternionic and Clifford algebra monogenic functions, the injectivity and surjectivity properties of the mappings are certainly among the most interesting topics. For every holomorphic intrinsic function f_0, for n being odd, further write the Sce mapping $(-\Delta)_p^{(n-1)/2} \circ \mathcal{S}_{\mathbb{R}^{n+1}}$ as

$$\left[(-\Delta)_p^{\frac{n-1}{2}} \circ \mathcal{S}_{\mathbb{R}^{n+1}}\right](f_0)(x) = (-\Delta)_p^{\frac{n-1}{2}} \vec{f}_0(x),\ x \in \mathbb{R}^{n+1},$$

where $\vec{f}_0$ is induced from f_0 and is of the slice form (see below). This shows that the Sce mapping is essentially a high-order pointwise differential operators, the injectivity does in such case not hold. It would be of more significance to examine the surjectivity. Establishing surjectivity amounts to asserting the range of the mappings. The ranges of the Fueter, the Sce and the Qian mappings are all proved to be monogenic functions of the axial form. Based on these results [3, 4] first show that the Fueter and the Sce mappings (n being odd) are surjective onto the set of axial monogenic functions. Further, in [7, 8], it is shown that for all dimensions the Qian mapping is surjective onto the set of the axial monogenic functions. This result is summarized as the surjectivity theorem of the Fueter–Sce–Qian mapping, or simply the surjectivity theorem. Under the unified form of the three mappings the surjectivity theorem is restated in Sect. 4.

The definition of the operator $\mathcal{D}_{\mathbb{R}^{n+1}}$, in particular, generates the monogenic monomials in Clifford analysis. The monogenic monomials are constructively important. In the present paper we prove two main results. We first prove that in all dimensions the Fourier multiplier operator involved in the Qian mapping applied to the positive power monomials $x^k, k \in \mathbb{N}$, results in monogenic generalized functions of the axial form. For both the positive and negative power monomials cases when n being odd and all negative power monomials when n being even these are already known. Added with this new case the results are summarized as the axial form theorem.

As a consequence of the axial form theorem, we then prove the monomial theorem,

$$\mathcal{D}_{\mathbb{R}^{n+1}}(x^{l+n-2}) = (-\Delta)^{\frac{n-1}{2}}(x^{l+n-2}),\ n, l \in \mathbb{N}. \tag{1.2}$$

This result was previously proved for n being odd through complicated computations [16]. Now we extend it to n being even. As a consequence of (1.2), it is shown that the operator identical relation

$$\mathcal{D}_{\mathbb{R}^{n+1}} \circ \mathcal{S}_{\mathbb{R}^{n+1}} = (-\Delta)^{\frac{n-1}{2}} \circ \mathcal{S}_{\mathbb{R}^{n+1}}$$

holds in the distribution sense for all dimensions.

It was a technical inconvenience of using generalized functions that motivated to use the Kelvin inversion in the definition of the mapping $\mathcal{D}_{\mathbb{R}^{n+1}}$ at the beginning to generalize Sce's result to all dimensions. Based on the results proved in this paper the Fueter mapping, the Sce mapping and the Qian mapping are all unified into one single form, that is, in the distribution sense, $(-\Delta)^{(n-1)/2} \circ \mathcal{S}_{\mathbb{R}^{n+1}}$.

The structure of the paper is as follows. Section 2 contains the related preliminary knowledge of Clifford analysis and Fourier multiplier operators. The two main results of the paper, viz. the axial form theorem and the monomial theorem, are, respectively, proved in Sects. 3 and 4.

2 Preliminary results

In this section we first review basic notations and knowledge about Clifford algebra. Suppose that $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is an orthonormal basis of Euclidean space $\mathbb{R}^n$, and satisfies the relations $\mathbf{e}_i^2 = -1$ for $i = 1, 2, \ldots, n$, and $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 0$ for $1 \le i \ne j \le n$. Then, the real Clifford algebra $\mathbb{R}_{0,n}$ is the real algebra constructed over the algebraically unreducable linear basis elements, i.e.,

$$\mathbb{R}_{0,n} := \left\{ a = \sum_S a_S \mathbf{e}_S \; : \; a_S \in \mathbb{R}, \; \mathbf{e}_S = \mathbf{e}_{j_1}\mathbf{e}_{j_2} \ldots \mathbf{e}_{j_k} \right\},$$

where $S := \{j_1, j_2, \ldots, j_k\} \subseteq \{1, 2, \ldots, n\}$ with $1 \le j_1 < j_2 < \cdots < j_k \le n$; or $S = \emptyset$, with $\mathbf{e}_\emptyset := 1$. Thus, $\mathbb{R}_{0,n}$ stands as a 2^n-dimensional real linear vector space. Let $|S|$ be the cardinal number of the elements in a set S. For each $k \in \{1, 2, \ldots, n\}$, set

$$\mathbb{R}_{0,n}^{(k)} := \left\{ a \in \mathbb{R}_{0,n} \; : \; a = \sum_{|S|=k} a_S \mathbf{e}_S \right\}.$$

The elements of $\mathbb{R}_{0,n}^{(k)}$ are called k-vectors of $\mathbb{R}_{0,n}$. For example, when $k = 1$, $\mathbb{R}_{0,n}^{(1)}$ is identified with $\mathbb{R}^n$. When $k = 0$, $\mathbb{R}_{0,n}^{(0)}$ is $\mathbb{R}$. A 0-vector element is usually called a scalar. Then for each element $a \in \mathbb{R}_{0,n}$, it has a projection $[a]_k$ in $\mathbb{R}_{0,n}^{(k)}$. A Clifford number a can be written as

$$a = \sum_{k=0}^{n} [a]_k.$$

Thus, we have

$$\mathbb{R}_{0,n} = \bigoplus_{k=0}^{n} \mathbb{R}_{0,n}^{(k)}.$$

In order to define the norm of $\mathbb{R}_{0,n}$, we need to introduce three involutions defined on $\mathbb{R}_{0,n}$: the main involution, the reversion and the conjugation. For each element $a \in \mathbb{R}_{0,n}$ the main involution $\sim : \; a \to \tilde{a}$ is defined by

$$\tilde{a} = \sum_S a_S \tilde{\mathbf{e}}_S,$$

where $\tilde{\mathbf{e}}_S := (-1)^{|S|}\mathbf{e}_S$. The reversion $* : \; a \to a^*$ is defined by

$$a^* = \sum_S a_S \mathbf{e}_S^*,$$

where $\mathbf{e}_S^* := (-1)^{|S|(|S|-1)/2}\mathbf{e}_S$ with $\mathbf{e}_S = \mathbf{e}_{j_1}\mathbf{e}_{j_2}\cdots\mathbf{e}_{j_k}$. The conjugation of a is a combination by the main involution and the reversion of a, i.e., the conjugation $- : a \to \overline{a}$ is defined by

$$\overline{a} = (\tilde{a})^* = \sum_S a_S(\tilde{\mathbf{e}}_S)^*.$$

For any element $a \in \mathbb{R}_{0,n}$, its norm $|a|$ is defined by

$$|a| = \left([a\overline{a}]_0\right)^{\frac{1}{2}} = \left(\sum_S |a_S|^2\right)^{\frac{1}{2}}.$$

Similarly, the complex Clifford algebra $\mathbb{C}_{0,n}$ can be defined by

$$\mathbb{C}_{0,n} := \mathbb{C} \otimes \mathbb{R}_{0,n} = \mathbb{R}_{0,n} \oplus \mathrm{i}\mathbb{R}_{0,n},$$

where i is the imaginary unit of $\mathbb{C}$. An element $a \in \mathbb{C}_{0,n}$ can be written as

$$a = \sum_S a_S\mathbf{e}_S, \ a_S \in \mathbb{C}.$$

All the concepts introduced in $\mathbb{R}_{0,n}$ can be reformulated in the complex Clifford algebra except the conjugation. For each element $a = \sum_S a_S\mathbf{e}_S \in \mathbb{C}_{0,n}$, the conjugate of a is $\overline{a} = \sum_S \overline{a}_S\overline{\mathbf{e}}_S$, where $\overline{a}_S$ is the complex conjugation of a_S.

An important subset of $\mathbb{R}_{0,n}$ is $\mathbb{R}_{0,n}^{(0)} \oplus \mathbb{R}_{0,n}^{(1)}$, as the real-linear span of $1, \mathbf{e}_1, \dots, \mathbf{e}_n$. Its elements, called paravectors, have the form $x := x_0 + \underline{x}$, where $x_0 \in \mathbb{R}$ and $\underline{x} := \sum_{j=1}^n x_j\mathbf{e}_j \in \mathbb{R}^n$. For simplicity, we call x_0 the real part of x and $\underline{x}$ the imaginary part of x. $\mathbb{R}_{0,n}^{(0)} \oplus \mathbb{R}_{0,n}^{(1)}$ is naturally identified with $\mathbb{R}^{n+1}$, through associating a paravector $x = x_0 + \underline{x}$ with the element $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$. With the identification, we will sometimes refer $\mathbb{R}_{0,n}^{(0)} \oplus \mathbb{R}_{0,n}^{(1)}$ to $\mathbb{R}^{n+1}$ for briefness. For each $x \in \mathbb{R}^{n+1}$, the norm in $\mathbb{R}^{n+1}$ is

$$|x| := \left([x\overline{x}]_0\right)^{1/2} = \left(x_0^2 + x_1^2 + \cdots + x_n^2\right)^{1/2},$$

where $\overline{x} := x_0 - \underline{x}$. If $x \in \mathbb{R}^{n+1}\backslash\{0\}$, then the inverse x^{-1} exists and $x^{-1} := \overline{x} \cdot |x|^{-2}$.

Now we turn to the monogenic function concept which is a crucial object in Clifford analysis. Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, $Cl_{0,n}$ means either $\mathbb{R}_{0,n}$ or $\mathbb{C}_{0,n}$. The notation $C^1(\Omega, Cl_{0,n})$ (resp. $C^1(\underline{\Omega}, Cl_{0,n})$) stands for the continuously differentiable functions which are defined on an open set $\Omega \subset \mathbb{R}^{n+1}$ (resp. $\underline{\Omega} \subset \mathbb{R}^n$) and take values in the Clifford algebra $Cl_{0,n}$. For each $f \in C^1(\Omega, Cl_{0,n})$, it has the form

$$f = \sum_S f_S\mathbf{e}_S,$$

where the functions f_S are scalar-valued. For $k \in \mathbb{N}_0$ we denote by ∂_k the derivative for the k-th variables, i.e., $\partial_k := \partial_{x_k}$ for x_k being the k-th variable of $x \in \mathbb{R}^{n+1}$. The Dirac operator is defined by

$$D_{\underline{x}} := \partial_1\mathbf{e}_1 + \partial_2\mathbf{e}_2 + \cdots + \partial_n\mathbf{e}_n, \ \ \underline{x} \in \underline{\Omega}.$$

For each $x \in \Omega$, the generalized Cauchy–Riemann operator is defined by

$$D := \partial_0 + D_{\underline{x}}.$$

Definition 2.1 (Monogenic function) Let $f(x) \in C^1(\Omega, Cl_{0,n})$ (resp. $f(\underline{x}) \in C^1(\underline{\Omega}, Cl_{0,n})$). Then $f(x)$ (resp. $f(\underline{x})$) is called a (left) monogenic function if

$$Df(x) = 0 \text{ (resp. } D_{\underline{x}} f(\underline{x}) = 0).$$

We note that the Cauchy kernel

$$E(x) := \frac{\overline{x}}{\omega_n |x|^{n+1}}, \ x \in \mathbb{R}^{n+1} \backslash \{0\},$$

plays a key role, where $\omega_n := 2\pi^{(n+1)/2}/\Gamma[(n+1)/2]$ is the surface area of the n-dimensional unit sphere in $\mathbb{R}^{n+1}$. Let $S \subset \Omega$ be a compact set with smooth and oriented boundary ∂S in the underlying Euclidean space, and S^o be the interior of S. If f is left monogenic in Ω, then its Cauchy integral formula is

$$\int_{\partial S} E(y-x) \mathrm{d}\sigma(y) f(y) = \begin{cases} f(x), & x \in S^o, \\ 0, & x \in \Omega \backslash S, \end{cases}$$

where the differential form $\mathrm{d}\sigma(y)$ is given by $\mathrm{d}\sigma(y) := \eta(y)\mathrm{d}S(y)$, $\eta(y)$ is the outer unit normal to ∂S at the point y and $\mathrm{d}S(y)$ is the surface measure of ∂S. For more details, the reader is referred to [1] and the references therein.

Next we will introduce axial monogenic functions. Let $\mathbb{S}^{n-1}$ stand for the $n-1$ -dimensional unit sphere in $\mathbb{R}^n$

$$\mathbb{S}^{n-1} := \{\underline{x} \in \mathbb{R}^n : |\underline{x}|^2 = 1\}.$$

For every $\underline{\omega} \in \mathbb{S}^{n-1}$, we know that $\underline{\omega}^2 = -1$ and let

$$\mathbb{C}_{\underline{\omega}} := \mathbb{R} + \underline{\omega}\mathbb{R} := \{u + \underline{\omega} v : u, v \in \mathbb{R}\}. \tag{2.1}$$

For $x \in \mathbb{R}^{n+1}$, let

$$\underline{\omega}_x =: \begin{cases} \frac{\underline{x}}{|\underline{x}|}, & \text{if } \underline{x} \neq 0, \\ \text{any element of } \mathbb{S}^{n-1}, & \text{if } \underline{x} = 0. \end{cases}$$

Then, by (2.1), we know that $x \in \mathbb{C}_{\underline{\omega}_x}$ and $x = x_0 + \underline{\omega}_x |\underline{x}|$. Let $x \in \mathbb{R}^{n+1}$, we also need the following notation

$$[x] := \{y \in \mathbb{R}^{n+1} \mid y = \mathrm{Re}(x) + \mathrm{I}|\underline{x}| \text{ for some } \mathrm{I} \in \mathbb{S}^{n-1}\},$$

where $\mathrm{Re}(x)$ denotes the real part of x. The set $[x]$ is an $(n-1)$-dimensional sphere in $\mathbb{R}^{n+1}$ with radius $|\underline{x}|$ and centered at $\mathrm{Re}(x)$. If $x \in \mathbb{R}$, then $[x] = \{x\}$ is the set containing the sole element x, and the radius of the sphere reduces to zero.

Definition 2.2 (Axial symmetric open set) An open set $\Omega \subset \mathbb{R}^{n+1}$ is said to be axially symmetric if the $(n-1)$-sphere $[u + \underline{\omega} v]$ is contained in Ω whenever $u + \underline{\omega} v \in \Omega$ for some $u, v \in \mathbb{R}$.

Definition 2.3 (Axial monogenic function) Let Ω be an axial symmetric open set. A function $f(x) \in C^1(\Omega, Cl_{0,n})$ is said to be axially monogenic if it is monogenic and has the form

$$f(x) = A(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} B(x_0, |\underline{x}|),$$

where $A(x_0, |\underline{x}|), B(x_0, |\underline{x}|)$ are real-valued functions.

The Fueter type theorems form a bridge between the analysis of functions of one complex variable to that of functions of quaternionic and Clifford variables. In [9] (1934) Fueter gives a machinery to construct a quaternion-valued monogenic function from a holomorphic function in the complex plane. After around 3 decades, in 1957, the Fueter theorem was extended to $\mathbb{R}^{n+1}$ by Sce [23] for odd integers n through using the pointwise differential operator $\Delta^{(n-1)/2}$. Qian in 1997 extended the result to both n being odd and even cases by using the corresponding Fourier multiplier operator when the required integrability is guaranteed, and the Kelvin inversion if not. The joint use of the Fourier multiplier and the Kelvin inversion is to replace the pointwise differential operator $(-\Delta)_p^{(n-1)/2}$ for n being odd, valid for a wide class of functions not restricted to the functions satisfying the integrability condition. The Fourier multiplier-Kelvin inversion method is proved to be a true generalization of Sce as the two methods are identical for the case n being odd [16].

Denote by $\mathcal{S}(\mathbb{R}^{n+1})$ the Schwarz space and $\mathcal{S}^*(\mathbb{R}^{n+1})$ the dual space of $\mathcal{S}(\mathbb{R}^{n+1})$. Then for every $f \in \mathcal{S}^*(\mathbb{R}^{n+1})$, its Fourier transform and inverse Fourier transform are defined by

$$\langle \mathcal{F}(f), \phi \rangle := \langle f, \hat{\phi} \rangle, \ \forall \ \phi \in \mathcal{S}(\mathbb{R}^{n+1})$$

and

$$\langle \mathcal{F}^{-1}(f), \phi \rangle := \langle f, \check{\phi} \rangle, \ \forall \ \phi \in \mathcal{S}(\mathbb{R}^{n+1}),$$

respectively, where $\hat{\phi}$ and $\check{\phi}$ are the usual Fourier and inverse Fourier transforms of ϕ valid in the Schwarz class $\mathcal{S}(\mathbb{R}^{n+1})$.

We will use Fourier multiplier operators induced by suitable functions g :

$$M_g(f) := \mathcal{F}^{-1}[g\mathcal{F}(f)].$$

We involve several closely related operators. Note that the Fourier multiplier operator M_g is defined for only the functions that satisfy the integrability specified in its definition. The notation $(-\Delta)_p^{(n-1)/2}$ is reserved for the pointwise differential operator that is only for the case n being odd. The corresponding generalization of the Fueter theorem to only odd dimensions n will be called *the Fueter–Sce generalization*. The case $n = 3$ of the Fueter–Sce generalization is a true generalization of the Fueter theorem in the sense that e_3 is not identical with $e_1 e_2$, but $k = ij$ for the quaternionic imaginary elements.

When we mention Fueter–Sce, we always mean n is an odd integer. We reserve the notation $(-\Delta)^{(n-1)/2}$ for the sense of distribution, and we may occasionally remark this, but not always. We have been using the notation $\mathcal{D}_{\mathbb{R}^{n+1}}$ for the joint use of a particular Fourier multiplier and the Kelvin inversion. We note that $\mathcal{D}_{\mathbb{R}^{n+1}}$ is originally defined for Clifford monomials, and then extended to functions with the Clifford Laurent series, or in other words, Clifford monomials expansions of real coefficients at the origin. The corresponding generalization of the Fueter theorem by Qian is also called *the $\mathcal{D}_{\mathbb{R}^{n+1}}$-generalization of the Fueter theorem*.

3 Axial form theorem

When a holomorphic intrinsic function $f_0(z)$ is expanded at $z = 0$, then the coefficients of its Laurent series expansion $\sum_{l\in\mathbb{Z}} a_l z^l$ are real numbers. In the case there formally holds

$$\left[(-\Delta)^{\frac{n-1}{2}} \circ \mathcal{S}_{\mathbb{R}^{n+1}}\right](f_0)(x) = \sum_{l\in\mathbb{Z}} a_l (-\Delta)^{\frac{n-1}{2}}(x^l),\ x \in \mathbb{R}^{n+1}. \tag{3.1}$$

The actual identical relation of (3.1) will be justified later in accordance with the convergence radii of the series on the two sides.

We aim to present that $\{(-\Delta)^{(n-1)/2}(x^l) : n \in \mathbb{N} \text{ and } l \in \mathbb{Z}\}$ is a set of axial monogenic distributions. The monogenicity property of the element in $\{(-\Delta)^{(n-1)/2}(x^l) : n \in \mathbb{N} \text{ and } l \in \mathbb{Z}\}$ is proved by Kou, Qian and Sommen in [12]. So, we just need to show that every element of the set has an axial form. Part of this result has already been proved, including

(i) When n being odd, for negative powers x^l, the distribution $(-\Delta)^{(n-1)/2}(x^l)$, the Fourier multiplier $\mathcal{D}(x^l)$ and pointwise defined $(-\Delta)_p^{(n-1)/2}(x^l)$ are all identical [16, 23], and thus all have the axial from.

(ii) When n being even, for negative powers x^l, the identical relations between $(-\Delta)^{(n-1)/2}(x^l)$ and $\mathcal{D}(x^l)$ can be directly verified from Fourier analysis. Then, the axiality of $(-\Delta)^{(n-1)/2}(x^l)$ follows from the axiality of $\mathcal{D}(x^l)$ [16].

The next theorem further shows that every element in $\{(-\Delta)^{(n-1)/2}(x^l) : n \in \mathbb{N}$ being even and $l \in \mathbb{N}_0\}$ has an axial form. To prove the theorem, we first state a technical lemma.

Lemma 3.1 (see [25, on pages 73 and 117]) *Let $k \in \mathbb{N}_0$, $n \in \mathbb{N}$ and $0 < \alpha < n + 1$. Suppose that $P_k(x)$ be a homogeneous harmonic polynomial of degree k. Then*

$$\int_{\mathbb{R}^{n+1}} \frac{P_k(x)}{|x|^{k+n+1-\alpha}} \hat{\varphi}(x)\mathrm{d}x = \gamma_{k,\alpha} \int_{\mathbb{R}^{n+1}} \frac{P_k(x)}{|x|^{k+\alpha}} \varphi(x)\mathrm{d}x \tag{3.2}$$

for every φ which is sufficiently rapidly decreasing at infinity and

$$\gamma_{k,\alpha} := \mathrm{i}^k \pi^{(n+1)/2-\alpha} \Gamma(k/2 + \alpha/2)/\Gamma(k/2 + (n + 1)/2 - \alpha/2),$$

where i *is the imaginary unit of the complex plane* $\mathbb{C}$ *and* Γ *is the Gamma function.*

Remark 3.2 Equation (3.2) implies, in the distribution sense,

$$\mathcal{F}\left[\frac{P_k(x)}{|x|^{k+n+1-\alpha}}\right](\xi) = \gamma_{k,\alpha} \frac{P_k(\xi)}{|\xi|^{k+\alpha}}$$

or

$$\frac{P_k(x)}{|x|^{k+n+1-\alpha}} = \gamma_{k,\alpha} \mathcal{F}^{-1}\left[\frac{P_k(\xi)}{|\xi|^{k+\alpha}}\right](x).$$

Theorem 3.3 *Let $l \in \mathbb{N}_0$, $n \in \mathbb{N}$ be even and $x \in \mathbb{R}^{n+1}$. Then, $(-\Delta)^{(n-1)/2}(x^l)$ is an axial monogenic generalized function.*

Proof First we prove the cases for $l \in \{0, 1, 2, \ldots, n-2\}$. We show that in such cases $(-\Delta)^{(n-1)/2}(x^l)$ are identical with the zero distribution, and hence are axial.

$$\begin{aligned}\mathcal{F}\Big((-\Delta)^{\frac{n-1}{2}}\big((\cdot)^l\big)\Big)(\xi) &= (2\pi|\xi|)^{n-1}\mathcal{F}\big((\cdot)^l\big)(\xi)\\ &= -\mathrm{i}\frac{\overline{\xi}}{|\xi|}(2\pi\mathrm{i}\xi)(2\pi|\xi|)^{n-2}\mathcal{F}\big((\cdot)^l\big)(\xi)\\ &= \sum_{i=0}^{n}\left[-\mathrm{i}\frac{\xi_i}{|\xi|}\mathcal{F}\Big(D(-\Delta)^{\frac{n-2}{2}}\big((\cdot)^l\big)\Big)(\xi)\right]\overline{\mathbf{e}_i}\\ &= \sum_{i=0}^{n}\left[\mathcal{F}\Big(\mathcal{R}_j\Big(D(-\Delta)^{\frac{n-2}{2}}\big((\cdot)^l\big)\Big)\Big)(\xi)\right]\overline{\mathbf{e}_i},\end{aligned}$$

where the last equation contains the Riesz transforms $\mathcal{R}_j$ in terms of their Fourier multipliers.

So, we have

$$(-\Delta)^{\frac{n-1}{2}}\big((\cdot)^l\big) = \sum_{i=0}^{n}\mathcal{R}_j\Big(D(-\Delta)^{\frac{n-2}{2}}\big((\cdot)^l\big)\Big)\overline{\mathbf{e}_i},$$

which holds in the distribution sense.

Now we show $D(-\Delta)^{(n-2)/2}\big((\cdot)^l\big) = 0$ in the distribution sense for $l \in \{0, 1, 2, \ldots, n-2\}$. We will use the result

$$\mathcal{F}(x^\alpha)(\xi) = \mathcal{F}\big(x_0^{\alpha_0}x_1^{\alpha_1}\cdots x_n^{\alpha_n}\big)(\xi) = \mathrm{i}^{-|\alpha|}D^\alpha\delta(\xi),$$

where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$, $D^\alpha = (\partial_0)^{\alpha_0}(\partial_1)^{\alpha_1}\cdots(\partial_n)^{\alpha_n}$ and δ is the usual Dirac δ function.

Let $x \in \mathbb{R}^{n+1}$ and $|\alpha| = l$, we obtain, with the self-explanatory notation S_α,

$$\begin{aligned}\mathcal{F}(x^l) &= \mathcal{F}\left(\sum_{|S_\alpha|=0}^{n}x_0^{\alpha_0}x_1^{\alpha_1}\cdots x_n^{\alpha_n}\mathbf{e}_{S_\alpha}\right)\\ &= \sum_{|S_\alpha|=0}^{n}\mathcal{F}\big(x_0^{\alpha_0}x_1^{\alpha_1}\cdots x_n^{\alpha_n}\big)\mathbf{e}_{S_\alpha}\\ &= \sum_{|S_\alpha|=0}^{n}\mathrm{i}^{-l}D^\alpha\delta\mathbf{e}_{S_\alpha}.\end{aligned}$$

For any $\varphi \in \mathcal{S}(\mathbb{R}^{n+1})$, we have

$$\begin{aligned}\Big\langle D(-\Delta)^{\frac{n-2}{2}}(x^l), \varphi(x)\Big\rangle &= \big\langle(2\pi\mathrm{i}\xi)(2\pi|\xi|)^{n-2}\mathcal{F}\big((\cdot)^l\big)(\xi), \check{\varphi}(\xi)\big\rangle\\ &= \big\langle\mathcal{F}\big((\cdot)^l\big)(\xi), (2\pi\mathrm{i}\xi)(2\pi|\xi|)^{n-2}\check{\varphi}(\xi)\big\rangle\\ &= \left\langle\sum_{|S_\alpha|=0}^{n}\mathrm{i}^{-l}D^\alpha\delta(\xi)\mathbf{e}_{S_\alpha}, (2\pi\mathrm{i}\xi)(2\pi|\xi|)^{n-2}\check{\varphi}(\xi)\right\rangle\\ &= \sum_{|S_\alpha|=0}^{n}\big\langle\mathrm{i}^{-l}D^\alpha\delta(\xi), (2\pi\mathrm{i}\xi)(2\pi|\xi|)^{n-2}\check{\varphi}(\xi)\big\rangle\mathbf{e}_{S_\alpha}\\ &= \sum_{|S_\alpha|=0}^{n}(-1)^l\big\langle\mathrm{i}^{-l}\delta(\xi), D^\alpha\big[(2\pi\mathrm{i}\xi)(2\pi|\xi|)^{n-2}\check{\varphi}(\xi)\big]\big\rangle\mathbf{e}_{S_\alpha}.\end{aligned}$$

For any fixed α with $|\alpha| \in \{0, 1, 2, \dots, n-2\}$, we know that each term of the expansion of $D^\alpha\left[(2\pi \mathrm{i}\xi)(2\pi|\xi|)^{n-2}\check{\varphi}(\xi)\right]$ under the Leibniz law contains positive powers of some elements of $\xi_0, \xi_1 \dots, \xi_n$ or $|\xi|$, and no negative powers of them. Hence, we have

$$\left\langle \mathrm{i}^{-l}\delta(\xi), D^\alpha\left[(2\pi \mathrm{i}\xi)(2\pi|\xi|)^{n-2}\check{\varphi}(\xi)\right]\right\rangle = 0$$

and

$$\begin{aligned}\left\langle D(-\Delta)^{\frac{n-2}{2}}\left(x^l\right), \varphi(x)\right\rangle &= \sum_{|S_\alpha|=0}^{n} (-1)^l \left\langle \mathrm{i}^{-l}\delta(\xi), D^\alpha\left[(2\pi \mathrm{i}\xi)(2\pi|\xi|)^{n-2}\check{\varphi}(\xi)\right]\right\rangle \mathbf{e}_{S_\alpha}\\ &= 0,\end{aligned}$$

which gives

$$D(-\Delta)^{\frac{n-2}{2}}\left(x^l\right) = 0.$$

Thus, we conclude

$$\begin{aligned}(-\Delta)^{\frac{n-1}{2}}\left(x^l\right) &= \sum_{i=0}^{n} \mathcal{R}_j\left(D(-\Delta)^{\frac{n-2}{2}}\left((\cdot)^l\right)\right)\overline{\mathbf{e}_i}\\ &= 0,\end{aligned}$$

which obviously has the axial form.

Now let $l \in \mathbb{N}_0 \setminus \{0, 1, 2, \dots, n-2\}$. Monogenicity of such cases $(-\Delta)^{(n-1)/2}(x^l)$ is proved in [12]. Again we only need to show that $(-\Delta)^{(n-1)/2}(x^l)$ is axial.

$$\begin{aligned}\mathcal{F}\left((-\Delta)^{\frac{n-1}{2}}\left((\cdot)^l\right)\right)(\xi) &= (2\pi|\xi|)^{n-1}\mathcal{F}\left((\cdot)^l\right)(\xi)\\ &= \frac{1}{2\pi|\xi|}(2\pi|\xi|)^n\mathcal{F}\left((\cdot)^l\right)(\xi)\\ &= \frac{1}{2\pi|\xi|}\mathcal{F}\left((-\Delta)^{\frac{n}{2}}\left((\cdot)^l\right)\right)(\xi).\end{aligned}$$

By Lemma 3.1 for $k = 0, \alpha = 1$, we have

$$\begin{aligned}\mathcal{F}\left((-\Delta)^{\frac{n-1}{2}}\left((\cdot)^l\right)\right)(\xi) &= \frac{1}{2\pi\gamma_{0,1}}\mathcal{F}\left(\frac{1}{|\cdot|^n}\right)(\xi)\mathcal{F}\left((-\Delta)^{\frac{n}{2}}\left((\cdot)^l\right)\right)(\xi)\\ &= \frac{1}{2\pi\gamma_{0,1}}\mathcal{F}\left((-\Delta)^{\frac{n}{2}}\left((\cdot)^l\right) * \frac{1}{|\cdot|^n}\right)(\xi).\end{aligned}$$

So, we obtain

$$\begin{aligned}(-\Delta)^{\frac{n-1}{2}}\left(x^l\right) &= \frac{1}{2\pi\gamma_{0,1}}\left((-\Delta)^{\frac{n}{2}}\left((\cdot)^l\right) * \frac{1}{|\cdot|^n}\right)(x)\\ &= \frac{1}{2\pi\gamma_{0,1}}\int_{\mathbb{R}^{n+1}} \frac{1}{|x-y|^n}(-\Delta)^{\frac{n}{2}}(y^l)\mathrm{d}y,\end{aligned} \tag{3.3}$$

where the equation holds in the tempered distribution sense.

For $y \in \mathbb{R}^{n+1}$, with $\underline{\omega} := \underline{y}/|\underline{y}| \in \mathbb{S}^{n-1}$, we have

$$y = y_0 + \underline{y} = y_0 + \underline{\omega} t, \; t = |\underline{y}|.$$

By the computations on integer powers of the Laplacian as given in [16] or [23], or [24], we know that $(-\Delta)^{n/2}(y^l)$ has the axial form

$$\begin{aligned}(-\Delta)^{\frac{n}{2}}\left(y^l\right) &= A(y_0, t) + \underline{\omega} B(y_0, t)\\ &= A(y_0, |\underline{y}|) + \frac{\underline{y}}{|\underline{y}|} B(y_0, |\underline{y}|),\end{aligned}$$

where $A(y_0, t)$ and $B(y_0, t)$ are real-valued functions.

For any $x \in \mathbb{R}^{n+1}$, with $\underline{v} := \underline{x}/|\underline{x}| \in \mathbb{S}^{n-1}$, we have

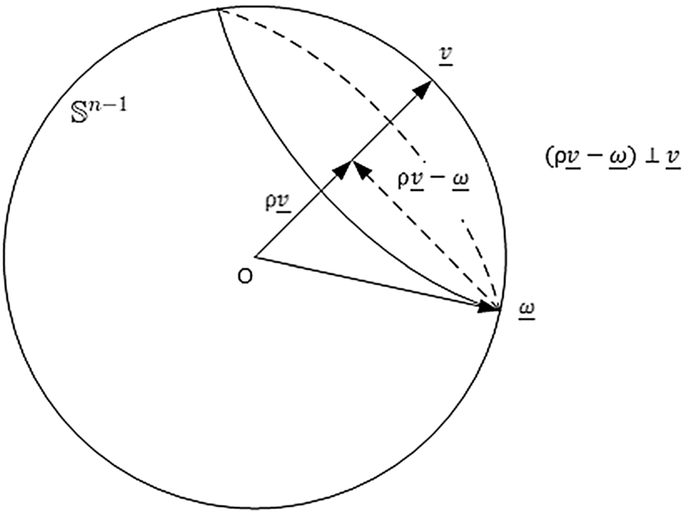
$$x = x_0 + \underline{x} = x_0 + \underline{v} r, \; r = |\underline{x}|.$$

Then,

$$\begin{aligned}(-\Delta)^{\frac{n-1}{2}}\left(x^l\right) =& \frac{1}{2\pi\gamma_{0,1}} \int_{\mathbb{R}^{n+1}} \frac{1}{|x-y|^n}\left(A(y_0, |\underline{y}|) + \frac{\underline{y}}{|\underline{y}|} B(y_0, |\underline{y}|)\right) \mathrm{d}y\\ =& \frac{1}{2\pi\gamma_{0,1}} \int_{\mathbb{R}} \mathrm{d}y_0 \int_{\mathbb{R}^n} \frac{1}{|x-y|^n}\left(A(y_0, |\underline{y}|) + \frac{\underline{y}}{|\underline{y}|} B(y_0, |\underline{y}|)\right) \mathrm{d}\underline{y}\\ =& \frac{1}{2\pi\gamma_{0,1}} \omega_{n-1} \int_{\mathbb{R}} \mathrm{d}y_0 \int_0^{+\infty} t^{n-1} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \frac{A(y_0, t) + \underline{\omega} B(y_0, t)}{\left((x_0 - y_0)^2 + |r\underline{v} - t\underline{\omega}|^2\right)^{\frac{n}{2}}} \mathrm{d}S(\underline{\omega}).\end{aligned}$$

Let $-1 \leqslant \rho \leqslant 1$ and $\underline{\omega}' \in \mathbb{S}^{n-2}$. From the enclosed diagram we can obtain that

(i) $\underline{\omega} = \rho \underline{v} + (1-\rho^2)^{1/2} \underline{\omega}'$;
(ii) $(\rho \underline{v} - \underline{\omega}) \perp \underline{v}$.



From (i) and (ii) we have

$$\frac{r}{t}\underline{v}-\underline{\omega}=\left(\frac{r}{t}-\rho\right)\underline{v}+(\rho\underline{v}-\underline{\omega}).$$

and

$$\begin{aligned}|r\underline{v}-t\underline{\omega}|^2&=t^2\left|\frac{r}{t}\underline{v}-\underline{\omega}\right|^2\\&=t^2\left[\left(\frac{r}{t}-\rho\right)^2+(1-\rho^2)\right]\\&=t^2+r^2-2rt\rho.\end{aligned}$$

Then

$$\begin{aligned}(-\Delta)^{\frac{n-1}{2}}\left(x^l\right)=&\frac{1}{2\pi\gamma_{0,1}}\omega_{n-1}\int_{\mathbb{R}}\mathrm{d}y_0\int_0^{+\infty}t^{n-1}\mathrm{d}t\int_{-1}^{1}(1-\rho^2)^{\frac{n-3}{2}}\mathrm{d}\rho\\&\int_{\mathbb{S}^{n-2}}\frac{A(y_0,t)+[\rho\underline{v}+(1-\rho^2)^{1/2}\underline{\omega}']B(y_0,t)}{\left((x_0-y_0)^2+t^2+r^2-2rt\rho\right)^{\frac{n}{2}}}\mathrm{d}S(\underline{\omega}')\\=&\frac{1}{2\pi\gamma_{0,1}}\omega_{n-1}\omega_{n-2}\int_{\mathbb{R}}\mathrm{d}y_0\int_0^{+\infty}t^{n-1}\mathrm{d}t\\&\int_{-1}^{1}\frac{\left[A(y_0,t)+\underline{v}\rho B(y_0,t)\right](1-\rho^2)^{\frac{n-3}{2}}}{\left((x_0-y_0)^2+t^2+r^2-2rt\rho\right)^{\frac{n}{2}}}\mathrm{d}\rho.\end{aligned}$$

Let

$$\begin{aligned}U(x_0,r):=&\frac{\omega_{n-1}\omega_{n-2}}{2\pi\gamma_{0,1}}\int_{\mathbb{R}}\mathrm{d}y_0\int_0^{+\infty}t^{n-1}\mathrm{d}t\int_{-1}^{1}\frac{A(y_0,t)(1-\rho^2)^{\frac{n-3}{2}}}{\left((x_0-y_0)^2+t^2+r^2-2rt\rho\right)^{\frac{n}{2}}}\mathrm{d}\rho,\\V(x_0,r):=&\frac{\omega_{n-1}\omega_{n-2}}{2\pi\gamma_{0,1}}\int_{\mathbb{R}}\mathrm{d}y_0\int_0^{+\infty}t^{n-1}\mathrm{d}t\int_{-1}^{1}\frac{B(y_0,t)\rho(1-\rho^2)^{\frac{n-3}{2}}}{\left((x_0-y_0)^2+t^2+r^2-2rt\rho\right)^{\frac{n}{2}}}\mathrm{d}\rho.\end{aligned}$$

We obtain

$$(-\Delta)^{\frac{n-1}{2}}\left(x^l\right)=U(x_0,r)+\underline{v}V(x_0,r),$$

which shows that $(-\Delta)^{(n-1)/2}(x^l)$ is of the axial form. The proof is complete. □

The above treated distributional composition operator $(-\Delta)^{(n-1)/2}\circ\mathcal{S}_{\mathbb{R}^{n+1}}$ maps complex monomials to monogenic distributions of the axial form. The result can be easily extended to *real-value-shifted monomials* $z_a^l=(z-a)^l, z_a=z-a,\ a\in\mathbb{R}, l\in\mathbb{Z}$. The operator can thus be further extended to holomorphic intrinsic functions of one complex variable via Laurent series expansions. We have the following.

Theorem 3.4 (Axial form theorem) *Let $n\in\mathbb{N}$ and $f_0(z)$ be a holomorphic intrinsic function defined on a connected intrinsic set in $\mathbb{C}$ having non-empty intersection with the real* *line. Then, the distributional mapping* $(-\Delta)^{(n-1)/2} \circ \mathcal{S}_{\mathbb{R}^{n+1}}$, *preliminarily defined for real-value-shifted monomials, can be extended to* f_0 *via its Laurent series expansion:*

$$\left[(-\Delta)^{\frac{n-1}{2}} \circ \mathcal{S}_{\mathbb{R}^{n+1}}\right](f_0)(x) = \sum_{l\in\mathbb{Z}} a_l (-\Delta)^{\frac{n-1}{2}} (x_a^l),$$

where $x_a = x - a$. *Moreover,* $[(-\Delta)^{(n-1)/2} \circ \mathcal{S}_{\mathbb{R}^{n+1}}](f_0)(x)$ *is an axial monogenic distribution.*

Proof Holomorphic intrinsic functions correspond to Laurent series of one-complex variable centered at points of the real axis with real coefficients. Such Laurent series through the mapping $(-\Delta)^{(n-1)/2} \circ \mathcal{S}_{\mathbb{R}^{n+1}}$ is formally mapped into a series as a real-linear combination of the terms $(-\Delta)^{(n-1)/2}(x_a^l)$. Those terms, inherited from the pointwise differentiation computation for the odd dimensions n; and through the Fourier integral computation against the test functions for the even dimensions n, as given in the proof of the last theorem, have the singularity $|x_a|^{l-(n-1)/2}$ at the infinity or alternatively at the point a, and those singularities do not change the convergence radius from the one for the original one-complex variable Laurent series [12]. We thus obtain that the Clifford monogenic distributional series are of the same convergence radius at the infinity and the local. The axial form assertion is a consequence of the Theorem 3.3. □

Remark 3.5 Theorem 3.4 is a reformulation and refined results of those in [12, 16, 23].

4 The monomial theorem

In this section, for $n \in \mathbb{N}$, we will give a useful formula of the fractional Laplacian $(-\Delta)^{(n-1)/2}\left(x^l\right)$ with $l \in \mathbb{Z}$. Before proving it, we need to prove several auxiliary lemmas. The first one is referred to [7, 16].

Lemma 4.1 *Let* $n, k \in \mathbb{N}$.

(i) *Let* $f \in \mathcal{S}(\mathbb{R}^{n+1})$, *then*

$$\partial_{x_0}\left[(-\Delta)^{\frac{n-1}{2}} f(x)\right] = (-\Delta)^{\frac{n-1}{2}}\left[\partial_{x_0} f(x)\right];$$

$$D\left[(-\Delta)^{\frac{n-1}{2}} f(x)\right] = (-\Delta)^{\frac{n-1}{2}}\left[D f(x)\right].$$

(ii) *For all* $x \in \mathbb{R}^{n+1}\backslash\{0\}$,

$$(-\Delta)^{\frac{n-1}{2}}\left(x^{-k}\right) = \frac{(-1)^{k-1}\omega_n\lambda_n}{(k-1)!}\cdot\left((\partial_0)^{k-1}E\right)(x),$$

where the constant $\lambda_n = 2^{n-1}\gamma_{1,n}/\gamma_{1,1} = 2^{n-1}\Gamma^2((n+1)/2)$.

The next lemma is crucial to prove the equivalence of two axial monogenic functions. Before introducing it, we need some notations. Let $k \in \mathbb{N}_0$. A monogenic homogeneous polynomial P_k of degree k in $\mathbb{R}^n$ is called a solid inner spherical monogenic of degree k. We denote by M_k the totality of all solid inner spherical monogenics of degree k.

Lemma 4.2 (see [6, Theorem 2.1]) *Let $n \in \mathbb{N}$, $P_k(\underline{x}) \in M_k$ be fixed, and $W_0(x_0)$ a real analytic function in $\widetilde{\Omega} \subset \mathbb{R}$. Then there exists a unique sequence of analytic functions, $\{W_s(x_0)\}_{s>0}$, such that the series*

$$f(x_0, \underline{x}) = \sum_{s=0}^{\infty} \underline{x}^s W_s(x_0) P_k(\underline{x})$$

is convergent in an open set U in $\mathbb{R}^{n+1}$ containing the set $\widetilde{\Omega}$, where the sum f is monogenic in U. The function $W_0(x_0)$ is determined by the relation

$$P_k(\underline{\omega}) W_0(x_0) = \lim_{|\underline{x}| \to 0} \frac{1}{|\underline{x}|^k} f(x_0, \underline{x}), \ \underline{\omega} = \frac{\underline{x}}{|\underline{x}|} \in \mathbb{S}^{n-1}.$$

The series $f(x_0, \underline{x})$ is the generalized axial CK-extension of the function $W_0(x_0)$.

Remark 4.3 In the present study what we concern with Lemma 4.2 here is the case $k = 0$, when $P_k(\underline{x})$ is a constant. In the case the axial monogenic extension $f(x_0, \underline{x})$ is uniquely determined by its restriction to $|\underline{x}| = 0$, that is, $f(x_0, 0)$, identical with its restriction to the real axis. Moreover, due to the continuity, the function values $f(x_0, 0)$ can be obtained from the above presented limit for $k = 0$, i.e., $\lim_{|\underline{x}| \to 0} f(x_0, \underline{x})$.

We also need the following two functions which was defined in [3].

Definition 4.4 Let $n \in \mathbb{N}$ and $E(x)$ be the Cauchy kernel. For all $x \in \mathbb{R}^{n+1} \setminus \mathbb{S}^{n-1}$,

$$\mathcal{K}_n^+(x) := \int_{\mathbb{S}^{n-1}} E(x - \underline{\omega}) \, \mathrm{d}S(\underline{\omega})$$

and

$$\mathcal{K}_n^-(x) := \int_{\mathbb{S}^{n-1}} E(x - \underline{\omega}) \, \underline{\omega} \, \mathrm{d}S(\underline{\omega}),$$

where $\mathrm{d}S(\underline{\omega})$ is the surface measure on $\mathbb{S}^{n-1}$.

Lemma 4.5 *Let $n \in \mathbb{N}$ and $x \in \mathbb{R}^{n+1} \setminus \mathbb{S}^{n-1}$. Then the kernels $\mathcal{K}_n^+(x)$ and $\mathcal{K}_n^-(x)$ are determined by their restrictions $\lim_{|\underline{x}| \to 0} \mathcal{K}_n^+(x)$ and $\lim_{|\underline{x}| \to 0} \mathcal{K}_n^-(x)$, respectively. Specifically, if $x_0 \in \mathbb{R}$,*

$$\lim_{|\underline{x}| \to 0} \mathcal{K}_n^+(x) = C_n \frac{x_0}{(x_0^2 + 1)^{(n+1)/2}},$$

$$\lim_{|\underline{x}| \to 0} \mathcal{K}_n^-(x) = -C_n \frac{1}{(x_0^2 + 1)^{(n+1)/2}},$$

where

$$C_n := \frac{\Gamma[(n+1)/2]}{\sqrt{\pi} \Gamma(n/2)}.$$

Proof $\mathcal{K}_n^+(x)$ and $\mathcal{K}_n^-(x)$ are axial monogenic functions because the Cauchy kernel $E(x)$ is axially monogenic. By Lemma 4.2, we can conclude that $\mathcal{K}_n^+(x)$ and $\mathcal{K}_n^-(x)$ are determined by their restrictions $\lim_{|\underline{x}|\to 0}\mathcal{K}_n^+(x)$ and $\lim_{|\underline{x}|\to 0}\mathcal{K}_n^-(x)$, respectively.

Let $x = x_0 + \underline{v}r$, where $r = |\underline{x}|$, $\underline{v} \in \mathbb{S}^{n-1}$. We have

$$\begin{aligned}\mathcal{K}_n^+(x) &= \int_{\mathbb{S}^{n-1}} E(x-\underline{\omega})\mathrm{d}S(\underline{\omega})\\ &=: I_1(x_0,r,\underline{v}) + I_2(x_0,r,\underline{v}),\end{aligned}$$

where

$$I_1(x_0,r,\underline{v}) := \frac{1}{\omega_n}\int_{\mathbb{S}^{n-1}} \frac{x_0}{\left(x_0^2+|r\underline{v}-\underline{\omega}|^2\right)^{(n+1)/2}}\mathrm{d}S(\underline{\omega})$$

and

$$I_2(x_0,r,\underline{v}) := -\frac{1}{\omega_n}\int_{\mathbb{S}^{n-1}} \frac{r\underline{v}-\underline{\omega}}{\left(x_0^2+|r\underline{v}-\underline{\omega}|^2\right)^{(n+1)/2}}\mathrm{d}S(\underline{\omega}).$$

Because $r\underline{v}-\underline{\omega} = (r-\rho)\underline{v} + (\rho\underline{v}-\underline{\omega})$, we have

$$|r\underline{v}-\underline{\omega}|^2 = (r-\rho)^2 + (1-\rho^2) = 1 + r^2 - 2r\rho.$$

So

$$\begin{aligned}I_1(x_0,r,\underline{v}) &= \frac{1}{\omega_n}\int_{-1}^{1}\int_{\mathbb{S}^{n-2}} \frac{x_0}{\left(x_0^2+1+r^2-2r\rho\right)^{(n+1)/2}}(1-\rho^2)^{\frac{n-3}{2}}\mathrm{d}S(\underline{\omega}')\mathrm{d}\rho\\ &= \frac{x_0\omega_{n-2}}{\omega_n}\int_{-1}^{1} \frac{1}{\left(x_0^2+1+r^2-2r\rho\right)^{(n+1)/2}}(1-\rho^2)^{\frac{n-3}{2}}\mathrm{d}\rho,\end{aligned}$$

and

$$\begin{aligned}\lim_{r\to 0} I_1(x_0,r,\underline{v}) &= \frac{x_0\omega_{n-2}}{\omega_n}\lim_{r\to 0}\int_{-1}^{1} \frac{1}{\left(x_0^2+1+r^2-2r\rho\right)^{(n+1)/2}}(1-\rho^2)^{\frac{n-3}{2}}\mathrm{d}\rho\\ &= \frac{\omega_{n-2}}{\omega_n}\frac{x_0}{\left(x_0^2+1\right)^{(n+1)/2}}\int_{-1}^{1}(1-\rho^2)^{\frac{n-3}{2}}\mathrm{d}\rho.\end{aligned}$$

Note that, for every positive real number α, there holds

$$\int_{-1}^{1}(1-\rho^2)^{\alpha}\mathrm{d}\rho = \sqrt{\pi}\frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{3}{2}\right)}.$$

Hence, we have

$$\lim_{r\to 0} I_1(x_0,r,\underline{v}) = C_n\frac{x_0}{\left(x_0^2+1\right)^{(n+1)/2}}.$$

Now we compute $I_2(x_0,r,\underline{v})$. Again using the relation

$$r\underline{v} - \underline{\omega} = (r-\rho)\underline{v} + (\rho\underline{v} - \underline{\omega}),$$

the integral is separated into two parts. By the anti-symmetry of

$$-(1-\rho^2)^{1/2}\underline{\omega}' = \rho\underline{v} - \underline{\omega}$$

over the $n-2$-dimensional sphere centered at zero with radius $(1-\rho^2)^{1/2}$, we have

$$\int_{-1}^{1}\int_{\mathbb{S}^{n-2}} \frac{-(1-\rho^2)^{1/2}\underline{\omega}'}{\left(x_0^2+1+r^2-2r\rho\right)^{(n+1)/2}}(1-\rho^2)^{\frac{n-3}{2}}\mathrm{d}S(\underline{\omega}')\mathrm{d}\rho = 0.$$

Thus, we have

$$\begin{aligned} I_2(x_0, r, \underline{v}) &= -\frac{1}{\omega_n}\underline{v}\int_{-1}^{1}\int_{\mathbb{S}^{n-1}} \frac{\rho - r}{\left(x_0^2+1+r^2-2r\rho\right)^{(n+1)/2}}(1-\rho^2)^{\frac{n-3}{2}}\mathrm{d}S(\underline{\omega}')\mathrm{d}\rho \\ &= -\frac{\underline{v}\omega_{n-2}}{\omega_n}\int_{-1}^{1} \frac{\rho - r}{\left(x_0^2+1+r^2-2r\rho\right)^{(n+1)/2}}(1-\rho^2)^{\frac{n-3}{2}}\mathrm{d}\rho, \end{aligned}$$

and

$$\begin{aligned} \lim_{r\to 0} I_2(x_0, r, \underline{v}) &= -\frac{\underline{v}\omega_{n-2}}{\omega_n}\lim_{r\to 0}\int_{-1}^{1} \frac{\rho - r}{\left(x_0^2+1+r^2-2r\rho\right)^{(n+1)/2}}(1-\rho^2)^{\frac{n-3}{2}}\mathrm{d}\rho \\ &= -\frac{\underline{v}\omega_{n-2}}{\omega_n}\frac{x_0}{\left(x_0^2+1\right)^{(n+1)/2}}\int_{-1}^{1}\rho(1-\rho^2)^{\frac{n-3}{2}}\mathrm{d}\rho \\ &= 0. \end{aligned}$$

Therefore,

$$\lim_{|\underline{x}|\to 0}\mathcal{K}_n^+(x) = C_n\frac{x_0}{(x_0^2+1)^{(n+1)/2}}.$$

A similar computation gives

$$\lim_{|\underline{x}|\to 0}\mathcal{K}_n^-(x) = -C_n\frac{1}{(x_0^2+1)^{(n+1)/2}}.$$

□

Inspired by the works in [3, 4, 7, 8], the holomorphic intrinsic functions $\mathcal{P}_n^+(z)$ and $\mathcal{P}_n^-(z)$ are constructed with the restrictions of $\mathcal{K}_n^+(x)$ and $\mathcal{K}_n^-(x)$, respectively. First, let $z \in \mathbb{C}\backslash\{z\in\mathbb{C} : \mathrm{Re}\, z = 0 \text{ and } |\mathrm{Im}\, z| \geqslant 1\}$, replacing x_0 by z in the restrictions of $\mathcal{K}_n^+(x)$ and $\mathcal{K}_n^-(x)$, we have

$$C_n\frac{z}{(1+z^2)^{(n+1)/2}} \text{ and } -C_n\frac{1}{(1+z^2)^{(n+1)/2}}.$$

Secondly, denote, for $z \in \mathbb{C}\backslash\{z\in\mathbb{C} : \mathrm{Re}\, z = 0 \text{ and } |\mathrm{Im}\, z| \geqslant 1\}$,

$$\mathcal{P}_n^+(z) := \frac{C_n}{\lambda_n'} \cdot D_z^{-(n-1)} \left\{ \frac{z}{(1+z^2)^{(n+1)/2}} \right\},$$
$$\mathcal{P}_n^-(z) := -\frac{C_n}{\lambda_n'} \cdot D_z^{-(n-1)} \left\{ \frac{1}{(1+z^2)^{(n+1)/2}} \right\},$$

where $\lambda_n' := (-1)^{n-1}\lambda_n/(n-1)!$, and $D_z^{-(n-1)}$ stands for the $(n-1)$-fold principal antiderivative operation with respect to variable z.

The onefold principal antiderivative operation on the function

$$\frac{z^r}{(1+z^2)^{(n+1)/2}}, \quad r = 1 \text{ or } 0,$$

for instance, can be uniquely defined as the path integral

$$\int_{\gamma(0,z)} \frac{\xi^r}{(1+\xi^2)^{(n+1)/2}} d\xi,$$

where $\gamma(0,z)$ is any smooth path connecting 0 and z that does not intersect with the set $\{z \in \mathbb{C} : \text{Re}\, z = 0 \text{ and } |\text{Im}\, z| \geqslant 1\}$. The used higher-order principal antiderivative operations are defined similarly.

The computation shows that such defined functions $\mathcal{P}_n^+(z)$ and $\mathcal{P}_n^-(z)$ are holomorphic intrinsic functions on $\mathbb{C}\backslash\{z \in \mathbb{C} : \text{Re}\, z = 0 \text{ and } |\text{Im}\, z| \geqslant 1\}$.

Let $\mathbb{S}_{\geqslant}^{n-1} := \{x \in \mathbb{R}^{n+1} : x_0 = 0 \text{ and } |\underline{x}|^2 \geqslant 1\}$. We have the following lemma.

Lemma 4.6 *Let $n \in \mathbb{N}$, $\mathcal{P}_n^+(z)$ and $\mathcal{P}_n^-(z)$ be the holomorphic intrinsic functions defined on* $\mathbb{C}\backslash\{z \in \mathbb{C} : \text{Re}\, z = 0 \text{ and } |\text{Im}\, z| \geqslant 1\}$. *Then, for each $x \in \mathbb{R}^{n+1}\backslash\mathbb{S}_{\geqslant}^{n-1}$,*

$$\left[(-\Delta)^{\frac{n-1}{2}} \circ \mathcal{S}_{\mathbb{R}^{n+1}}\right]\left(\mathcal{P}_n^+\right)(x) = \mathcal{K}_n^+(x),$$
$$\left[(-\Delta)^{\frac{n-1}{2}} \circ \mathcal{S}_{\mathbb{R}^{n+1}}\right]\left(\mathcal{P}_n^-\right)(x) = \mathcal{K}_n^-(x).$$

Proof We first prove $\left[(-\Delta)^{(n-1)/2} \circ \mathcal{S}_{\mathbb{R}^{n+1}}\right]\left(\mathcal{P}_n^+\right)(x) = \mathcal{K}_n^+(x)$ for $x \in \mathbb{R}^{n+1}\backslash\mathbb{S}_{\geqslant}^{n-1}$. The case $\left[(-\Delta)^{(n-1)/2} \circ \mathcal{S}_{\mathbb{R}^{n+1}}\right]\left(\mathcal{P}_n^-\right)(x) = \mathcal{K}_n^-(x)$ for $x \in \mathbb{R}^{n+1}\backslash\mathbb{S}_{\geqslant}^{n-1}$ is similar. Due to Corollary 10.7 in [1], we only need to prove

$$\left[(-\Delta)^{\frac{n-1}{2}} \circ \mathcal{S}_{\mathbb{R}^{n+1}}\right]\left(\mathcal{P}_n^+\right)(x) = \mathcal{K}_n^+(x), \quad |x| > 1, \tag{4.1}$$

because $\left[(-\Delta)^{(n-1)/2} \circ \mathcal{S}_{\mathbb{R}^{n+1}}\right]\left(\mathcal{P}_n^+\right)(x)$ and $\mathcal{K}_n^+(x)$ are axially monogenic functions on $x \in \mathbb{R}^{n+1}\backslash\mathbb{S}_{\geqslant}^{n-1}$. By Lemma 4.2, Eq. (4.1) holds if we can show the following equation

$$\lim_{|\underline{x}|\to 0} \left[(-\Delta)^{\frac{n-1}{2}} \circ \mathcal{S}_{\mathbb{R}^{n+1}}\right]\left(\mathcal{P}_n^+\right)(x) = \lim_{|\underline{x}|\to 0} \mathcal{K}_n^+(x), \quad |x| > 1. \tag{4.2}$$

So the key step is to prove Eq. (4.2).

When $|z| > 1$, we have

$$\begin{aligned}\mathcal{P}_n^+(z) =& \frac{C_n}{\lambda_n'} \cdot D_z^{-(n-1)}\left\{\frac{z}{(1+z^2)^{(n+1)/2}}\right\}\\ =& \frac{C_n}{\lambda_n'} \cdot D_z^{-(n-1)}\left\{zz^{-(n+1)}\sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k}z^{-2k}\right\}\\ =& \frac{C_n}{\lambda_n'} \cdot D_z^{-(n-1)}\left\{\sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k}z^{-(2k+n)}\right\}\\ =& \frac{C_n}{\lambda_n'} \cdot \sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k}\frac{(-1)^{n-1}(2k)!}{(2k+n-1)!}z^{-(2k+1)}.\end{aligned}$$

Because the power series expressing $\mathcal{P}_n^+(z)$ converges absolutely, and uniformly on the set $\{z: \ |z| \geq \rho, \ \rho > 1\}$, we can apply the operator $(-\Delta)^{\frac{n-1}{2}} \circ \mathcal{S}_{\mathbb{R}^{n+1}}$ to the both sides of the power series expressing $\mathcal{P}_n^+(z)$ and have

$$\begin{aligned}&\left[(-\Delta)^{\frac{n-1}{2}} \circ \mathcal{S}_{\mathbb{R}^{n+1}}\right]\left(\mathcal{P}_n^+\right)(x)\\ &= \frac{C_n}{\lambda_n'} \cdot \sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k}\frac{(-1)^{n-1}(2k)!}{(2k+n-1)!}\left[(-\Delta)^{\frac{n-1}{2}} \circ \mathcal{S}_{\mathbb{R}^{n+1}}\right]\left((\cdot)^{-(2k+1)}\right)(x).\end{aligned}$$

By Lemma 4.1,

$$\left[(-\Delta)^{\frac{n-1}{2}} \circ \mathcal{S}_{\mathbb{R}^{n+1}}\right]\left((\cdot)^{-(2k+1)}\right)(x) = \frac{\omega_n \lambda_n}{(2k)!} \cdot \left((\partial_0)^{2k}E\right)(x). \tag{4.3}$$

Taking the limit $|\underline{x}| \to 0$ to the both sides of (4.3) we have

$$\begin{aligned}\lim_{|\underline{x}|\to 0}\left[(-\Delta)^{\frac{n-1}{2}} \circ \mathcal{S}_{\mathbb{R}^{n+1}}\right]\left((\cdot)^{-(2k+1)}\right)(x) &= \frac{\omega_n \lambda_n}{(2k)!} \cdot \lim_{|\underline{x}|\to 0}\left((\partial_0)^{2k}E\right)(x)\\ &= \frac{\omega_n \lambda_n}{(2k)!} \cdot \left((\partial_0)^{2k}E\right)(x_0)\\ &= \frac{\lambda_n (n+2k-1)!}{(2k)!(n-1)!}x_0^{-(2k+n)},\end{aligned}$$

where the second equality is valid due to the continuous differentiability of the function E and the third equality is due to the definition of the partial derivative.

Let $\lambda_n' = (-1)^{n-1}\lambda_n/(n-1)!$, we have

$$\begin{aligned}
&\lim_{|\underline{x}|\to 0}\left[(-\Delta)^{\frac{n-1}{2}}\circ\mathcal{S}_{\mathbb{R}^{n+1}}\right]\left(\mathcal{P}_n^+\right)(x)\\
&=\frac{C_n}{\lambda_n'}\cdot\sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k}\frac{(-1)^{n-1}(2k)!}{(2k+n-1)!}\lim_{|\underline{x}|\to 0}\left[(-\Delta)^{\frac{n-1}{2}}\circ\mathcal{S}_{\mathbb{R}^{n+1}}\right]\left((\cdot)^{-(2k+1)}\right)(x)\\
&=\frac{C_n}{\lambda_n'}\cdot\sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k}\frac{(-1)^{n-1}\lambda_n(2k)!(n+2k-1)!}{(2k+n-1)!(2k)!(n-1)!}x_0^{-(2k+n)}\\
&=C_n\cdot x_0^{-n}\sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k}x_0^{-2k}\\
&=C_n\cdot\frac{x_0}{(1+x_0^2)^{(n+1)/2}}\\
&=\lim_{|\underline{x}|\to 0}\mathcal{K}_n^+(x).
\end{aligned}$$

This concludes the proof. □

Now we give an important formula of the fractional Laplacian $(-\Delta)^{(n-1)/2}\left(x^l\right)$ with $l\geq 0$. For a function $f(x)$ defined on $\mathbb{R}^{n+1}$, the Kelvin inversion I is defined by

$$I(f)(x):=(-1)^{n-1}\omega_n E(x)f(1/x).$$

Let $n,k\in\mathbb{N}$, denote the monogenic monomials $P^{(-k)}$ and $P^{(k-1)}$ respectively by

$$\begin{aligned}
P^{(-k)} &:=\frac{(-1)^{k-1}\omega_n\lambda_n}{(k-1)!}\cdot\left((\partial_0)^{k-1}E\right),\\
P^{(k-1)} &:=I(P^{(-k)}).
\end{aligned}$$

In 1997, Qian obtained the following theorem through complicated computation for odd n [16]. The next result is based on a different method and for all n.

Theorem 4.7 (Monomial theorem) *Let $n\in\mathbb{N}$ and $l\in\mathbb{N}_0$. Then, for $x\in\mathbb{R}^{n+1}$,*

$$(-\Delta)^{\frac{n-1}{2}}(x^l)=\begin{cases}P^{(l+1-n)}(x) & \text{if } l\in\mathbb{N}_0\setminus\{0,1,2,\dots,n-2\},\\ 0 & \text{if } l\in\{0,1,2,\dots,n-2\}.\end{cases}$$

Proof From the proof of Theorem 3.3, we know that

$$(-\Delta)^{\frac{n-1}{2}}(x^l)=0,$$

where $l\in\{0,1,2,\dots,n-2\}$ and $n\in\mathbb{N}$ being even. Below, we only need to prove the theorem for $l\in\mathbb{N}_0\setminus\{0,1,2,\dots,n-2\}$. We first deal with the case when $l-(n-1)$ is an odd number, that is $l-(n-1)=2k+1\geq 0$ with $k\in\mathbb{N}_0$. We are to show

$$(-\Delta)^{\frac{n-1}{2}}\left(x^{2k+n}\right)=P^{(2k+1)}(x),\ x\in\mathbb{R}^{n+1}.$$

Similarly with the proof of Lemma 4.6, we have the power series of $\mathcal{P}_n^+(z)$ for $|z|<1$:

$$\mathcal{P}_n^+(z)=\frac{C_n}{\lambda_n'}\cdot\sum_{k=0}^{\infty}\binom{-\frac{n+1}{2}}{k}\frac{(2k+1)!}{(2k+n)!}z^{2k+n}. \tag{4.4}$$

Because the power series (4.4) converges absolutely and uniformly on the set $\{z : |z| \leq \rho,\ 0 < \rho < 1\}$, we apply the operator $(-\Delta)^{\frac{n-1}{2}} \circ \mathcal{S}_{\mathbb{R}^{n+1}}$ to the both sides of (4.4) and have

$$\left[(-\Delta)^{\frac{n-1}{2}} \circ \mathcal{S}_{\mathbb{R}^{n+1}}\right]\left(\mathcal{P}_n^+\right)(x)$$
$$= \frac{C_n}{\lambda_n'} \cdot \sum_{k=0}^{\infty} \binom{-\frac{n+1}{2}}{k} \frac{(2k+1)!}{(2k+n)!} \left[(-\Delta)^{\frac{n-1}{2}} \circ \mathcal{S}_{\mathbb{R}^{n+1}}\right]\left((\cdot)^{2k+n}\right)(x).$$

By Lemma 4.6, for each $x \in \mathbb{R}^{n+1} \backslash \mathbb{S}_{\geqslant}^{n-1}$, we have

$$\left[(-\Delta)^{\frac{n-1}{2}} \circ \mathcal{S}_{\mathbb{R}^{n+1}}\right]\left(\mathcal{P}_n^+\right)(x) = \mathcal{K}_n^+(x).$$

Then

$$\mathcal{K}_n^+(x) = \frac{C_n}{\lambda_n'} \cdot \sum_{k=0}^{\infty} \binom{-\frac{n+1}{2}}{k} \frac{(2k+1)!}{(2k+n)!} (-\Delta)^{\frac{n-1}{2}} \left(x^{2k+n}\right). \tag{4.5}$$

Taking the limit $|\underline{x}| \to 0$ on the both sides of (4.5) we have

$$\frac{C_n}{\lambda_n'} \cdot \sum_{k=0}^{\infty} \binom{-\frac{n+1}{2}}{k} \frac{(2k+1)!}{(2k+n)!} \lim_{|\underline{x}| \to 0} \left((-\Delta)^{\frac{n-1}{2}} \left(x^{2k+n}\right)\right)$$
$$= \lim_{|\underline{x}| \to 0} \mathcal{K}_n^+(x)$$
$$= C_n \frac{x_0}{(x_0^2 + 1)^{(n+1)/2}}$$
$$= C_n \sum_{k=0}^{\infty} \binom{-\frac{n+1}{2}}{k} x_0^{2k+1}.$$

Because $(-\Delta)^{\frac{n-1}{2}} \left(x^{2k+n}\right)$ is homogeneous of degree $2k+1$, being referred to Eq. (3.3), we have

$$\lim_{|\underline{x}| \to 0} \left((-\Delta)^{\frac{n-1}{2}} \left(x^{2k+n}\right)\right) = \frac{\lambda_n'(n+2k)!}{(2k+1)!} x_0^{2k+1}.$$

On the other hand, for every $k, n \in \mathbb{N}$ and $x \neq 0$,

$$P^{(2k+1)}(x) = I\left(P^{(-(2k+2))}\right)(x)$$
$$= -\frac{(-1)^{n-1}\omega_n^2 \lambda_n}{(2k+1)!} E(x) \cdot \left((\partial_0)^{2k+1} E\right)(x^{-1}).$$

Taking the limit $|\underline{x}| \to 0$ on the both sides of the formula:

$$\begin{aligned}\lim_{|\underline{x}|\to 0} P^{(2k+1)}(x) &= -\frac{(-1)^{n-1}\omega_n^2\lambda_n}{(2k+1)!}\lim_{|\underline{x}|\to 0}\left\{E(x)\cdot\left((\partial_0)^{2k+1}E\right)(x^{-1})\right\}\\ &= -\frac{(-1)^{n-1}\omega_n^2\lambda_n}{(2k+1)!}E(x_0)\cdot\left((\partial_0)^{2k+1}E\right)\left(\frac{x_0}{|x_0|^2}\right)\\ &= -\frac{(-1)^{n-1}\omega_n\lambda_n}{(2k+1)!}E(x_0)\cdot\left(-\frac{(n+2k)!}{(n-1)!}|x_0|^{n+2k+1}\right)\\ &= \frac{(-1)^{n-1}\lambda_n(n+2k)!}{(2k+1)!(n-1)!}x_0^{2k+1},\end{aligned}$$

where the second equality is due to the continuous differentiability of the function E.

To sum up, we have

$$\begin{aligned}\lim_{|\underline{x}|\to 0}\left(P^{(2k+1)}(x)\right) &= \lim_{|\underline{x}|\to 0}\left((-\Delta)^{\frac{n-1}{2}}\left(x^{2k+n}\right)\right)\\ &= \lim_{|\underline{x}|\to 0}\left[(-\Delta)^{\frac{n-1}{2}}\circ\mathcal{S}_{\mathbb{R}^{n+1}}\right]\left((\cdot)^{2k+n}\right)(x).\end{aligned}$$

By Lemma 4.2, we have

$$\left[(-\Delta)^{\frac{n-1}{2}}\circ\mathcal{S}_{\mathbb{R}^{n+1}}\right]\left((\cdot)^{2k+n}\right)(x) = (-\Delta)^{\frac{n-1}{2}}\left(x^{2k+n}\right) = P^{(2k+1)}(x),\ |x| < 1.$$

Since $\left[(-\Delta)^{(n-1)/2}\circ\mathcal{S}_{\mathbb{R}^{n+1}}\right]\left((\cdot)^{2k+n}\right)(x)$ and $P^{(2k+1)}(x)$ are monogenic functions in the whole $\mathbb{R}^{n+1}$, we obtain

$$\left[(-\Delta)^{\frac{n-1}{2}}\circ\mathcal{S}_{\mathbb{R}^{n+1}}\right]\left((\cdot)^{2k+n}\right)(x) = (-\Delta)^{\frac{n-1}{2}}\left(x^{2k+n}\right) = P^{(2k+1)}(x),\ x\in\mathbb{R}^{n+1}.$$

Now we consider the case when $l-(n-1) = 2k \geq 0$ with $k\in\mathbb{N}_0$. We are to prove

$$(-\Delta)^{\frac{n-1}{2}}\left(x^{2k+n-1}\right) = P^{(2k)}(x),\ x\in\mathbb{R}^{n+1}.$$

Invoking the relation $\left[(-\Delta)^{(n-1)/2}\circ\mathcal{S}_{\mathbb{R}^{n+1}}\right]\left(\mathcal{P}_n^-\right)(x) = \mathcal{K}_n^-(x)$, the proof is similar to the case $l-(n-1) = 2k+1$. We omit the details here, and the proof is complete. □

Remark 4.8 Let $n\in\mathbb{N}$, $l\in\mathbb{Z}$ and $x\in\mathbb{R}^{n+1}$. By Theorem 3.3, Lemma 4.1 and Theorem 4.7, we have

$$(-\Delta)^{\frac{n-1}{2}}(x^l) = \begin{cases} P^{(l)}(x) & \text{if } l\in\mathbb{Z}\backslash\mathbb{N}_0,\\ P^{(l+1-n)}(x) & \text{if } l\in\mathbb{N}_0\setminus\{0,1,2,\ldots,n-2\},\\ 0 & \text{if } l\in\{0,1,2,\ldots,n-2\}.\end{cases}$$

As a consequence, it is shown that the relation $\mathcal{D}_{\mathbb{R}^{n+1}} = (-\Delta)^{(n-1)/2}$ holds in the distribution sense in all dimensions. Thus, $\mathcal{D}_{\mathbb{R}^{n+1}}$-generalization of the Fueter theorem, in fact, enjoys the same form as that of Fueter–Sce generalization.

To the end of the whole paper we want to state the surjectivity theorems. The ranges of the Fueter, the Sce and the Qian mappings so far have all been proved to be monogenic functions of the axial form. Conversely, in [3, 4], it is shown that the Fueter and the Sce mappings (n being odd) are surjective onto the set of axial monogenic functions. In [7] it is shown that for all dimensions the Qian mapping is surjective onto the set of the axially monogenic functions. This truly extends the surjectivity to all dimensions. The results of this paper unify the form of the Sce mapping, restricted to odd dimensions, and that of the Qian mapping into the same distributional form in all dimensions. Based on this uniform distributional form the surjectivity theorem is restated as follows.

Theorem 4.9 (Surjectivity theorem) *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^{n+1}$ be an axially symmetric open set, and $f(y) = f(y_0 + \underline{\omega} r) = A(y_0, r) + \underline{\omega} B(y_0, r)$ an axially monogenic function defined on Ω. For each $\underline{\omega} \in \mathbb{S}^{n-1}$ let $\Gamma_{\underline{\omega}}$ be the boundary of an open and bounded set $\mathcal{V}_{\underline{\omega}} \subset \mathbb{R} + \underline{\omega}\mathbb{R}^+$, and $V := \cup_{\underline{\omega} \in \mathbb{S}^{n-1}} \mathcal{V}_{\underline{\omega}} \subset \Omega$. Suppose that $\Gamma_{\underline{\omega}}$ is a regular curve whose parametric equations in the upper complex plane $\mathbb{C}^+_{\underline{\omega}} = \{y_0 + \underline{\omega} r,\ y_0 \in \mathbb{R}, r \in \mathbb{R}^+\}$ are parameterized by the arc length $s, s \in [0, L], L > 0$, as $y_0 = y_0(s), r = r(s)$. Then, there exists a holomorphic intrinsic function $f_0(z)$ defined on $\mathbb{C} \backslash \{\mathrm{i}, -\mathrm{i}\}$ such that for all $x \in V$,*

$$\left[(-\Delta)^{\frac{n-1}{2}} \circ \mathcal{S}_{\mathbb{R}^{n+1}}\right](f_0)(x) = f(x),$$

where

$$\begin{aligned} f_0(z) := &\int_{\Gamma_{\underline{\omega}}} \mathcal{P}_n^-\Big(\frac{z - y_0}{r}\Big) r^{n-2} [\mathrm{d}y_0 A(y_0, r) - \mathrm{d}r B(y_0, r)] \\ &- \int_{\Gamma_{\underline{\omega}}} \mathcal{P}_n^+\Big(\frac{z - y_0}{r}\Big) r^{n-2} [\mathrm{d}y_0 B(y_0, r) + \mathrm{d}r A(y_0, r)], \end{aligned}$$

and $\mathcal{P}_n^{\pm}(z)$ were defined in Lemma 4.6.

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