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ANALYTIC PHASE RETRIEVAL BASED ON INTENSITY MEASUREMENTS*

Dedicated to the memory of Professor Jiarong YU

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Abstract This paper concerns the reconstruction of a function f in the Hardy space of the unit disc $\mathbb{D}$ by using a sample value $f(a)$ and certain n-intensity measurements $|\langle f, E_{a_1\cdots a_n}\rangle|$, where $a_1, \cdots, a_n \in \mathbb{D}$, and $E_{a_1\cdots a_n}$ is the n-th term of the Gram-Schmidt orthogonalization of the Szegö kernels $k_{a_1}, \cdots, k_{a_n}$, or their multiple forms. Three schemes are presented. The first two schemes each directly obtain all the function values $f(z)$. In the first one we use Nevanlinna's inner and outer function factorization which merely requires the 1-intensity measurements equivalent to know the modulus $|f(z)|$. In the second scheme we do not use deep complex analysis, but require some 2- and 3-intensity measurements. The third scheme, as an application of AFD, gives sparse representation of $f(z)$ converging quickly in the energy sense, depending on consecutively selected maximal n-intensity measurements $|\langle f, E_{a_1\cdots a_n}\rangle|$.

Key words phase retrieval; Hardy space of the unit disc; Szegö kernel; Takenaka-Malmquist system; Gram-Schmidt orthogonalization; adaptive Fourier decomposition

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1 Introduction

Practical problems in physics, especially optics, and in engineering, have motivated research into phase retrieval. With phase retrieval, one requires that an analytic function f be reconstructed from some sampled amplitude values of the Fourier transform of f. Given the fact that rational orthogonal systems, or, in other words, Takenaka-Malmquist (TM) systems $\{B_{a_1\cdots a_n}\}_{n=1}^{\infty},\ a_1,\cdots,a_n,\cdots\in\mathbb{D}$, are generalizations of the Fourier system, it is natural to ask about the reconstruction of an analytic function f from n-intensity measurements $|\langle f,B_{a_1\cdots a_n}\rangle|$, where $a_1,\cdots,a_n\in\mathbb{D}$, and $B_{a_1\cdots a_n}$ is the n-th entry of the corresponding TM system, the latter being, apart from certain unimodular constants, coincident with the Gram-Schmidt orthogonalization $E_{a_1\cdots a_n}$ of the multiple Szegö kernels $k_{a_1},\cdots,k_{a_n}$ (see §3). The functions $B_{a_1\cdots a_n}$ are called measurement vectors, with the explicit expression

$$B_{a_1\cdots a_n}(z)=\frac{\sqrt{1-|a_n|^2}}{1-\overline{a}_n z}\prod_{k=1}^{n-1}\frac{z-a_l}{1-\overline{a}_l z}=e_{a_n}(z)\phi_{a_1\cdots a_{n-1}}(z), \tag{1.1}$$

where

$$B_a(z)=e_a(z)=\frac{\sqrt{1-|a|^2}}{1-\overline{a}z},\quad a\in\mathbb{D} \tag{1.2}$$

is the normalized Szegö kernel at a, and

$$k_a(z)=\frac{1}{1-\overline{a}z},\quad a\in\mathbb{D} \tag{1.3}$$

is the Szegö or reproducing kernel of the Hardy space, and

$$\phi_{a_1\cdots a_{n-1}}(z)=\prod_{k=1}^{n-1}\frac{z-a_l}{1-\overline{a}_l z} \tag{1.4}$$

is the canonical Blaschke product generated by $a_1,\cdots,a_{n-1}$. When all a_k's are identical to zero, $\{B_{a_1\cdots a_n}\}_{n=1}^{\infty}$ reduces to a half of the Fourier system $\{z^{n-1}\}_{n=1}^{\infty}$. To make this convenient in practice, although it is not strictly necessary, we establish the restriction that the a_k's are all distinct. In [3], phase retrieval for functions in the Hardy space of the unit disc based on the intensity measurements $|\langle f,B_{a_1\cdots a_n}\rangle|$ were studied. In the present paper, we further develop the analytic phase retrieval theme in relation to TM system, in which the question of whether or not $\{B_{a_1\cdots a_n}\}_{n=1}^{\infty}$ forms a basis of the Hardy space is not important, and thus not an issue.

In general, if f is an analytic function in a connected open set Ω, then the condition $|f(z)|=1$ for $z\in\Omega$ implies that $f(z)\equiv c$ in Ω for a unimodular constant c. This is easily concluded from taking the derivative $\frac{\partial}{\partial z}$:

$$0=\frac{\partial}{\partial z}[f(z)\overline{f}(z)]=\frac{\partial f(z)}{\partial z}\overline{f}(z),\quad \text{or}\quad \frac{\partial f(z)}{\partial \overline{z}}=0.$$

As a consequence, if both f and g are analytic in Ω and $|f(z)|=|g(z)|$, then $f(z)=cg(z)$, where c is a constant satisfying $|c|=1$. This observation shows that if we know the function value $f(a)$ for some $a\in\Omega$ and the amplitudes $|f(z)|$ for all $z\in\Omega$, then the analytic function $f(z)$ is uniquely determined. Without a sampled non-zero function value one can only determine an analytic function up to a unimodular multiplicative constant.

In the present paper we work in the unit disc context. For $\mathbb{C}^+$, the upper half complex plane, the theory and the algorithms are similar. Among the several equivalent definitions of the Hardy space $H^2(\mathbb{D})$, in this paper we adopt the one that is expressed in terms of the Fourier coefficients:

$$H^2(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{C} \mid f(z) = \sum_{k=0}^{\infty} c_k z^k, \sum_{k=0}^{\infty} |c_k|^2 < \infty \right\}.$$

Since

$$\langle f, e_z \rangle = \sqrt{1-|z|^2} f(z), \tag{1.5}$$

the 1-intensity measurements are equivalent to the $|f(z)|$ measurements. Given the above observation on unique determination, the basic question is to compute the function values $f(z)$ based on a sample value $f(a) \neq 0$ and the 1-intensity measurements $|\langle f, e_z \rangle|$. The solution of the phase retrieval problem in such a format can be achieved by using the Nevanlinna factorization Theorem involving inner and outer functions. This, as Scheme I, will be presented in Section 2.

Scheme II of analytic phase retrieval, as the main part of the study, is presented in Section 3. The proposed algorithm does not involve deep analytical knowledge but only employs elementary computation based on the Gram-Schmidt orthogonalizations of certain Szegö kernels. Starting from a sample value $f(a) \neq 0, a \in \mathbb{D}$, the crucial technical problem is to decide $f(z)$ at each $z \in \mathbb{D}$ over two solutions of the triangle equation $\cos\alpha = A$ so that the determined $f(z)$ values are coherent and define an analytic function. In the computation, the intensity measurements $|\langle f, B_{az} \rangle|$ are involved. To determine $f(z)$ we need to employ another complex number $b \in \mathbb{D}$, this being different from a and z. Correspondingly, the numerical values of the intensity measurements $|\langle f, B_{ab} \rangle|$ and $|\langle f, B_{azb} \rangle|$ are involved. This algorithm is named the Forward-Backward Algorithm (FB Algorithm).

The Scheme III, as given in Section 4, is based on the selections of $a_1, a_2, \cdots$, where $a_1 = a$, and is under the maximal selection principle; that is, when $a_1, \cdots, a_{n-1}$ are fixed, select

$$a_n = \arg\max\{|\langle f, B_{a_1 \cdots a_{n-1} b} \rangle| \mid b \in \mathbb{D}\}, \quad n > 1.$$

In Scheme III we determine each function value $f(a_n), n = 2, 3, \cdots$, or, equivalently, $\langle f, B_{a_1 \cdots a_n} \rangle$, by using the Forward-Backward Algorithm. In such a way the adaptive Fourier decomposition can be used, and we have a sparse representation of $f(z)$, or a series fast converging to $f(z)$ in energy:

$$f(z) = \sum_{n=1}^{\infty} \langle f, B_{a_1 \cdots a_n} \rangle B_{a_1 \cdots a_n}(z).$$

2 Scheme I: Phase Retrieval Based on 1-Intensity Measurements and Nevanlinna's Factorization Theorem

The scheme is based on the following theorem:

Theorem 2.1 ([2]) Let $f(z) \in H^2(\mathbb{D}),\ f \not\equiv 0$. Then it holds that

$$f(z) = C B_f(z) O_f(z) S_f(z),\ |C| = 1, \tag{2.1}$$

where C is a unimodular constant, $B_f(z)$ is a Blaschke product made of the zeros of f, $O_f(z)$ is an outer function of f, and $S_f(z)$ is a singular inner function of f. Except for the choice of the constant C, $|C| = 1$, the factorization (2.1) is unique.

The outer, Blaschke and singular inner factors of a function $f \in H^2(\mathbb{D})$ are given, respectively, by

$$O_f(z) = C \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}t}+z}{\mathrm{e}^{\mathrm{i}t}-z} \log\left|f\left(\mathrm{e}^{\mathrm{i}t}\right)\right| \mathrm{d}t\right), \quad |C|=1, \tag{2.2}$$

where $f\left(\mathrm{e}^{\mathrm{i}t}\right)$ is the non-tangential boundary limit of $f(z)$,

$$B_f(z) = z^m \prod_{|\alpha_k|\neq 0} \frac{-\bar{\alpha}_k}{|\alpha_k|}\frac{z-\alpha_k}{1-\bar{\alpha}_k z}, \quad m=1,2,\cdots \tag{2.3}$$

and

$$S_f(z) = C\exp\left(-\int_0^{2\pi}\frac{\mathrm{e}^{\mathrm{i}t}+z}{\mathrm{e}^{\mathrm{i}t}-z}\mathrm{d}\mu(t)\right), \quad |C|=1, \tag{2.4}$$

where $\mathrm{d}\mu$ is a finite Borel measure singular to the Lebesgue measure. We note that if f is analytically extendable to an open neighbourhood of the closed unit disc, then f is continuously extendable to the boundary $\partial\mathbb{D}$, and there will only be a trial singular inner function $S_f = C$, and inside $\mathbb{D}$ the function B_f will only have finitely many zero points.

Let $z \in \mathbb{D}$ be given, and let $|z|<1$. Define a Hardy space function $f_r(z') = f(rz'), |z| < r < 1, |z'| < 1$. We require that on $\partial\mathbb{D}_r$ the function f does not have zeros. The function f_r is analytically extended to $\mathbb{D}_{\frac{1}{r}}$ containing $\overline{\mathbb{D}}$. We have $f(z) = f_r(z/r) = f_r(z'), z' = z/r$. We assert the phase of $f_r(z')$. Since we know $|[f_r(z')]_{\partial\mathbb{D}}|$, we can first compute O_{f_r}. Then, by solving the optimization problem

$$\inf\{|f_r(z')| \mid z' \in \mathbb{D}\},$$

we get all the zeros of f_r, constituting a finite subset of $\mathbb{D}$, denoted by Z_r. We can thus construct the Blaschke product B_{f_r}. Since the singular inner function is trivial, we have, for $|z|<r$,

$$f(z) = f_r(z/r) = CO_{f_r}(z/r)B_{f_r}(z/r),$$

where C is an unimodular constant. A sample value $f(a) \neq 0$ helps us to assert the involved unimodular constant as $C = C_0$:

$$C_0 = \frac{f(a)}{O_{f_r}(a/r)B_{f_r}(a/r)}.$$

Then the function values $f(z)$ for all $|z|<r$ are given by the formula

$$f(z) = C_0 \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{e}^{\mathrm{i}t}+\frac{z}{r}}{\mathrm{e}^{\mathrm{i}t}-\frac{z}{r}}\log\left|f\left(r\mathrm{e}^{\mathrm{i}t}\right)\right|\mathrm{d}t\right)\prod_{\alpha_k\in Z_r}\frac{-\bar{\alpha}_k}{|\alpha_k|}\frac{\frac{z}{r}-\alpha_k}{1-\bar{\alpha}_k\frac{z}{r}}, \tag{2.5}$$

where, for $\alpha_k = 0$, we conventionally let $\frac{-\bar{\alpha}_k}{|\alpha_k|} = 1$.

3 Scheme II: the Forward-Backward Algorithm Based on Some k-Intensity Measurements, $k = 1, 2, 3$

As in the last section, we assume that $a \in \mathbb{D}$ and $f(a)(\neq 0)$ are known. We assert for any $z \in \mathbb{D}$ the function value of $f(z)$. We use the orthogonal projection operators $P_{a_1\cdots a_n}$ and $Q_{a_1\cdots a_n} = I - P_{a_1\cdots a_n}$, where

$$P_{a_1\cdots a_n}(f) = \sum_{k=1}^{n}\langle f, E_{a_1\cdots a_k}\rangle E_{a_1\cdots a_k},$$

where $(E_{a_1}, E_{a_1a_2}, \cdots, E_{a_1\cdots a_k})$ is the Gram-Schmidt orthogonalization of $(k_{a_1}, \cdots, k_{a_k}), 1 \le k \le n$. We now prove the following result:

Lemma 3.1 Let $a_1, \cdots, a_m$ be distinct complex numbers in $\mathbb{D}$. Then

(i)

$$\begin{aligned} E_{a_1\cdots a_m}(z) = \frac{Q_{a_1\cdots a_{m-1}}(k_{a_m})(z)}{\|Q_{a_1\cdots a_{m-1}}(k_{a_m})\|} &= \frac{k_{a_m}(z) - \sum\limits_{l=1}^{m-1}\langle k_{a_m}, E_{a_1\cdots a_l}\rangle E_{a_1\cdots a_l}(z)}{\|k_{a_m} - \sum\limits_{l=1}^{m-1}\langle k_{a_m}, E_{a_1\cdots a_l}\rangle E_{a_1\cdots a_l}\|} \\ &= \mathrm{e}^{\mathrm{i}c}\phi_{a_1\cdots a_{m-1}}(z)e_{a_m}(z) \\ &= \mathrm{e}^{\mathrm{i}c}B_{a_1\cdots a_m}(z), \end{aligned} \tag{3.1}$$

where

$$\mathrm{e}^{\mathrm{i}c} = \frac{\overline{\phi}_{a_1\cdots a_{m-1}}(a_m)}{|\overline{\phi}_{a_1\cdots a_{m-1}}(a_m)|},$$

where $e_{a_m}, \phi_{a_1\cdots a_{m-1}}$ and $B_{a_1\cdots a_m}$ are as defined in (1.2), (1.4) and (1.3);

(ii)

$$\langle f, B_{a_1\cdots a_{m-1}a_m}\rangle = \frac{Q_{a_1\cdots a_{m-1}}(f)(a_m)}{\phi_{a_1\cdots a_{m-1}}(a_m)}\sqrt{1-|a_m|^2}; \tag{3.2}$$

(iii)

$$\frac{Q_{a_1\cdots a_{m-1}}(f)(z)}{\phi_{a_1\cdots a_{m-1}}(z)} = \left(\frac{Q_{a_{m-1}}}{\phi_{a_{m-1}}} \circ \cdots \circ \frac{Q_{a_1}}{\phi_{a_1}}\right)(f)(z). \tag{3.3}$$

The lemma may be extended to the cases where $a_1, \cdots, a_m$ may have multiplicities. In the extended cases the concept of a multiple reproducing kernel is involved ([1]).

Proof First, we show (i). In (3.1), the first equality is from the definition of $Q_{a_1\cdots a_{m-1}}$, as the complementary projection of $P_{a_1\cdots a_{m-1}}$. For the second equality are refer to [5]. The last equality is by the definition of $B_{a_1\cdots a_m}$. Now we show (ii). Since $Q_{a_1\cdots a_{m-1}}$ is a projection, we have $Q^2_{a_1\cdots a_{m-1}} = Q_{a_1\cdots a_{m-1}}$, and $Q_{a_1\cdots a_{m-1}}$ is self-adjoint. Using the result (i), we have

$$\begin{aligned} \langle f, B_{a_1\cdots a_{m-1}a_m}\rangle &= \langle f, \mathrm{e}^{-\mathrm{i}c}\frac{Q^2_{a_1\cdots a_{m-1}}(e_{a_m})}{\|Q_{a_1\cdots a_{m-1}}(e_{a_m})\|}\rangle \\ &= \langle Q_{a_1\cdots a_{m-1}}(f), \mathrm{e}^{-\mathrm{i}c}\frac{Q_{a_1\cdots a_{m-1}}(e_{a_m})}{\|Q_{a_1\cdots a_{m-1}}(e_{a_m})\|}\rangle \\ &= \langle Q_{a_1\cdots a_{m-1}}(f), e_{a_m}\phi_{a_1\cdots a_{m-1}}\rangle \\ &= \frac{Q_{a_1\cdots a_{m-1}}(f)(a_m)}{\phi_{a_1\cdots a_{m-1}}(a_m)}\sqrt{1-|a_m|^2}, \end{aligned}$$

where we used the unimodular property of Blaschke products on the unit circle. Next we show (iii). Denote by g_k the k-th reduced remainder, as used in [6]; that is,

$$\frac{g_k(z) - \langle g_k, e_{a_k}\rangle e_{a_k}(z)}{\phi_{a_k}(z)} = g_{k+1}(z),\ k \ge 1, \quad g_1 = f,$$

which is to say that

$$\left(\frac{Q_{a_k}}{\phi_{a_k}}\right)(g_k) = g_{k+1}.$$

Iterating this process, we obtain

$$\left(\frac{Q_{a_{m-1}}}{\phi_{a_{m-1}}}\circ\cdots\circ\frac{Q_{a_1}}{\phi_{a_1}}\right)(f)(z)=g_m(z)=\left(\frac{Q_{a_{m-1}\cdots a_1}}{\phi_{a_{m-1}\cdots a_1}}\right)(f)(z),$$

as desired. The proof is complete. □

Next we develop the algorithm. In the process, the 2-intensity measurement $|\langle f,B_{az}\rangle|$ is used. From the above lemma, it holds, for $z\neq a$, that

$$\begin{aligned}\langle f,B_{az}\rangle&=\frac{Q_af(z)}{\phi_a(z)}\sqrt{1-|z|^2}\\&=\frac{f(z)-\langle f,B_a\rangle B_a(z)}{\phi_a(z)}\sqrt{1-|z|^2}.\end{aligned}\tag{3.4}$$

Through this relation, $f(z)$ and $\langle f,B_{az}\rangle$ are mutually determined. If $\langle f,B_{az}\rangle=0$, then $f(z)$ is trivially determined. The above equation can only be solved through taking the complex modulus to both sides of the equation. The equation in the complex modulus is reduced to the form

$$A=|f(z)-v|,\tag{3.5}$$

which gives rise to two solutions for the value of $f(z)$, where A is a positive constant, v is a known non-zero complex number, and the modulus $|f(z)|$ is known. This is a standard method for determination of the position of a triangle with one of its three end points being at the origin, and the lengths of the three sides all being known, and the direction of one side, which is v, being known. In the non-trivial cases, the two solution triangles are mirror symmetric with respect to the direction of v, where the two corresponding values for $f(z)$ are denoted as $f^{\{+\}}(z)$ and $f^{\{-\}}(z)$. In particular, $f^{\{+\}}(z)+f^{\{-\}}(z)$ is a complex number of the same phase direction as v. What is crucial is to determine, at each z, the right value of $f(z)$ between $f^{\{+\}}(z)$ and $f^{\{-\}}(z)$. In the sequel, we will call v the axis of the two solution triangles (3.5). Let $\langle f,B_{az}\rangle^{\{+\}}$ correspond to $f^{\{+\}}(z)$, let $\langle f,B_{az}\rangle^{\{-\}}$ correspond to $f^{\{-\}}(z)$, which are precisely given by

$$\begin{aligned}\langle f,B_{az}\rangle^{\{\pm\}}&=\frac{f^{\{\pm\}}(z)-\langle f,B_a\rangle B_a(z)}{\phi_a(z)}\sqrt{1-|z|^2}\\&\triangleq\left(\frac{Q_af(z)}{\phi_a(z)}\right)^{\{\pm\}}\sqrt{1-|z|^2}.\end{aligned}\tag{3.6}$$

Since $f(z)$ is uniquely determined at each z, we denote the right value of f at z by $f^*(z)$. This notation tells that $f^*(z)$ equals one and only one of $f^{\{+\}}(z)$ and $f^{\{-\}}(z)$, and the corresponding value of $\langle f,B_{az}\rangle$ is denoted by $\langle f,B_{az}\rangle^*$, between $\langle f,B_{az}\rangle^{\{+\}}$ and $\langle f,B_{az}\rangle^{\{-\}}$.

To determine $f^*(z)$, we employ an auxiliary complex number $b\in\mathbb{D}$ such that $b\neq z$ and $b\neq a$, and the two involved intensity measurements $|\langle f,B_{ab}\rangle|$ and $|\langle f,B_{azb}\rangle|$ are non-zero. Replacing z by b in (3.4), we have

$$\langle f,B_{ab}\rangle=\frac{f(b)-\langle f,B_a\rangle B_a(b)}{\phi_a(b)}\sqrt{1-|b|^2}.\tag{3.7}$$

The related equation in the modulus, again, has two solutions for $f(b)$, denoted as $f^{[\pm]}(b)$, which are associated with, respectively, $\langle f,B_{ab}\rangle^{[\pm]}$. We note that the true value $f^*(b)$ of $f(b)$, being identical either with $f^{[+]}(b)$ or $f^{[-]}(b)$, can also be computed from the relation

$$|\langle f,B_{azb}\rangle|=\left|\frac{f(b)-\langle f,B_a\rangle B_a(b)-\langle f,B_{az}\rangle^{\{\pm\}}B_{az}(b)}{\phi_{az}(b)}\right|\sqrt{1-|b|^2}.\tag{3.8}$$

The equation (3.8) in the complex modulus has four solutions for $f(b)$, separated into two groups: one group corresponds to $\langle f, B_{az}\rangle^{\{+\}}$, denoted by $f^{\{+\}+}(b)$ and $f^{\{+\}-}(b)$; the other group corresponds to $\langle f, B_{az}\rangle^{\{-\}}$, denoted by $f^{\{-\}+}(b)$ and $f^{\{-\}-}(b)$. The ground truth value $f^*(b)$ must be among $f^{[+]}(b)$ and $f^{[-]}(b)$, and also among $f^{\{+\}+}(b), f^{\{+\}-}(b), f^{\{-\}+}(b)$ and $f^{\{-\}-}(b)$. Set

$$S_b \triangleq \{f^{[+]}(b), f^{[-]}(b)\} \quad \text{and} \quad T_b \triangleq \{f^{\{+\}+}(b), f^{\{+\}-}(b), f^{\{-\}+}(b), f^{\{-\}-}(b)\}.$$

The above analysis asserts that $f^*(b) \in S_b \cap T_b$. If $S_b \cap T_b$ contains exactly one element, then this element must be $f^*(b)$. Such a value of $f^*(b)$ is either identical with one of $f^{\{+\}+}(b)$ or $f^{\{+\}-}(b)$, or identical with one of $f^{\{-\}+}(b)$ or $f^{\{-\}-}(b)$. This implies that $f^*(b)$ is from $\langle f, B_{az}\rangle^{\{+\}}$ or $\langle f, B_{az}\rangle^{\{-\}}$, indicating that the true value $f^*(z)$ is either equal to $f^{\{+\}}(z)$ or equal to $f^{\{-\}}(z)$. With the tested examples this always happens to be the case where $f^{\{+\}+}(b), f^{\{+\}-}(b), f^{\{-\}+}(b)$ and $f^{\{-\}-}(b)$ are four distinct numbers, and $S_b \cap T_b$ contains exactly one point. Theoretically, we are unable to exclude the cases where $S_b \cap T_b$ contains two points. In this case we can sort out the one that gives the right value $f^*(z)$. A valid algorithm can be established based on the following lemma:

Lemma 3.2 Let $a, z \in \mathbb{D}, a \neq z, \langle f, B_a\rangle \neq 0$ and $\langle f, B_z\rangle \neq 0$. Then for all b in a sufficiently small neighbourhood of z such that $b \neq a, b \neq z, \langle f, B_{ab}\rangle \neq 0, \langle f, B_{azb}\rangle \neq 0$, and where b makes the opening angle between $f^{[\pm]}(b)$ different from that between $f^{\{\pm\}}(z)$, there hold

(1) $f^{\{+\}+}(b), f^{\{+\}-}(b), f^{\{-\}+}(b)$ and $f^{\{-\}-}(b)$ are four distinct numbers;

(2) $S_b \cap T_b$ contains one or two complex numbers. In both cases the right value of $f(z)$ may be determined.

Proof We first show (1). From the assertions (ii) and (iii) of Lemma 3.1, we have

$$\begin{aligned}\langle f, B_{azb}\rangle &= \frac{Q_{az}(b)}{\phi_{az}(b)}\sqrt{1-|b|^2} \\ &= \frac{Q_z}{\phi_z} \circ \left(\frac{Q_a f}{\phi_a}\right)(b)\sqrt{1-|b|^2} \\ &= \left[\left(\frac{Q_a f}{\phi_a}\right)(b) - \left(\frac{Q_a f}{\phi_a}\right)(z)\frac{1-|z|^2}{1-\overline{z}b}\right]\frac{\sqrt{1-|b|^2}}{\phi_z(b)}.\end{aligned}$$

In view of (3.8), to solve the equation for the value $f(b)$, we take the complex modulus, and thus introduce the uncertainty. We have, tentatively,

$$|\langle f, B_{azb}\rangle| = \left|\left[\left(\frac{Q_a f}{\phi_a}\right)^{\{\pm\}\pm}(b) - \left(\frac{Q_a f}{\phi_a}\right)^{\{\pm\}}(z)\frac{1-|z|^2}{1-\overline{z}b}\right]\frac{\sqrt{1-|b|^2}}{\phi_z(b)}\right|,$$

where, in the notation defined by (3.6),

$$\left(\frac{Q_a f}{\phi_a}\right)^{\{\pm\}}(z) = \frac{\langle f, B_{az}\rangle^{\{\pm\}}}{\sqrt{1-|z|^2}} \quad \text{correspond to} \quad f^{\{\pm\}}(z), \text{ respectively.} \tag{3.9}$$

The uncertainty is precisely given by the following formula: for x representing $+$ or $-$,

$$\langle f, B_{azb}\rangle^{\{x\}\pm} = \left[\left(\frac{Q_a f}{\phi_a}\right)^{\{x\}\pm}(b) - \left(\frac{Q_a f}{\phi_a}\right)^{\{x\}}(z)\frac{1-|z|^2}{1-\overline{z}b}\right]\frac{\sqrt{1-|b|^2}}{\phi_z(b)}, \tag{3.10}$$

where

$$\left(\frac{Q_a f}{\phi_a}\right)^{\{x\}\pm}(b) = \frac{f^{\{x\}\pm}(b) - \langle f, B_a\rangle B_a(b)}{\phi_a(b)}. \tag{3.11}$$

For each of the cases $x = +$ and $-$, there are two distinct solutions $f^{\{x\}\pm}(b)$, given by (3.11). Notice that for the fixed z, the difference between the two phases of the two complex numbers $\left(\frac{Q_af}{\phi_a}\right)^{\{x\}}(z)\frac{1-|z|^2}{1-\overline{z}b}$ is the same as that between the two phases of $\left(\frac{Q_af}{\phi_a}\right)^{\{x\}}(z)$, for $x = +$ and $x = -$. The latter, however, is equal to a non-zero number depending only on z (for the fixed a throughout). On the other hand, when x is fixed to be $x = +$ or $x = -$, the difference between the two phases of $\left(\frac{Q_af}{\phi_a}\right)^{\{x\}\pm}(b)$ and $\left(\frac{Q_af}{\phi_a}\right)^{\{x\}}(z)\frac{1-|z|^2}{1-\overline{z}b}$ is an infinitesimal along with $b \to z$. This suggests that the four solutions for $f(b)$ are distinct. The proof of (1) is complete.

Now we prove assertion (2). Since $f^*(b) \in S_b \cap T_b$, we have that $S_b \cap T_b \neq \emptyset$. If $S_b \cap T_b$ contains only the point $f^*(z)$, then $f^*(b) = f^{\{x\}y}(b)$ for a pair x and y, where each of x and y is fixed and can be $+$ or $-$. In such circumstance, if $x = +$, then $f^{\{+\}}(z) = f^*(z)$; and if $x = -$, then $f^{\{-\}}(z) = f^*(z)$, and thus $f(z) = f^*(z)$ is determined. In the sequel this is regarded as the easy case. Next we assume that $S_b \cap T_b$ contains two different complex numbers and accordingly derive a contradiction. In the case, $S_b \cap T_b = \{f^{[+]}(b), f^{[-]}(b)\}$. Let both $f^{[+]}(b)$ and $f^{[-]}(b)$ be from $f^{\{+\}}(z)$. Then on one hand, the relation (3.7) implies that the solutions $f^{[+]}(b)$ and $f^{[-]}(b)$ are with the axis $\langle f, B_a\rangle B_a(b)$. On the other hand, the solutions $f^{\{+\}+}(b)$ and $f^{\{+\}-}(b)$, which are respectively coincident with $f^{[+]}(b)$ and $f^{[-]}(b)$, through the relation (3.8), possess the axis $\langle f, B_a\rangle B_a(b) - \langle f, B_{az}\rangle^{\{\pm\}} B_{az}(b)$. The two axes, therefore, are of the same direction of which one depends on only b and the other depends on b and z. For the prescribed z, a generally chosen auxiliary b rules out this coincidence from happening. The same reasoning also rules out the case where both $f^{[+]}(b)$ and $f^{[-]}(b)$ are from $f^{\{-\}}(z)$. So, if $S_b \cap T_b = \{f^{[+]}(b), f^{[-]}(b)\}$, then it must be the case that $f^{[+]}(b) = f^{\{u\}x}(b)$ and $f^{[-]}(b) = f^{\{-u\}y}(b)$, where each of x, y and u can be $+$ or $-$, but fixed. In the case one can show that the two triples of complex numbers, $(0, f^{\{u\}}(z) - \langle f, B_a\rangle B_a(z), f^{\{u\}x}(b) - \langle f, B_a\rangle B_a(b),)$ and $(0, f^{\{-u\}}(z) - \langle f, B_a\rangle B_a(z), f^{\{-u\}y}(b) - \langle f, B_a\rangle B_a(b))$, representing two congruent triangles, are of opposite orientations. The two triples of points are respectively the images of a, z, b under the mappings

$$\phi_a(w)\left(\frac{Q_a}{\phi_a}\right)^{\{u\}x}(f)(w) \quad \text{and} \quad \phi_a(w)\left(\frac{Q_a}{\phi_a}\right)^{\{-u\}y}(f)(w).$$

Holomorphic mappings, however, necessarily keep the local orientation. As a consequence, if b is close enough to z and a, z, b are positively oriented, the holomorphic images of a, z, b should also be positively oriented. Thus, only the triple with positive orientation corresponds to the holomorphic mapping and gives rise to the right value $f(z)$.

To conclude the proof we need to show that $(0, f^{\{u\}}(z) - \langle f, B_a\rangle B_a(z), f^{\{u\}x}(b) - \langle f, B_a\rangle B_a(b),)$ and $(0, f^{\{-u\}}(z) - \langle f, B_a\rangle B_a(z), f^{\{-u\}y}(b) - \langle f, B_a\rangle B_a(b))$ are of opposite orientations when b is sufficiently close to z. There hold the relations

$$\left[\left(\frac{Q_af}{\phi_a}\right)^{\{u\}x}(b) - \left(\frac{Q_af}{\phi_a}\right)^{\{u\}}(z)\frac{1-|z|^2}{1-\overline{z}b}\right]\frac{\sqrt{1-|b|^2}}{\phi_z(b)} = \langle f, B_{azb}\rangle^{\{u\}x} \tag{3.12}$$

and

$$\left[\left(\frac{Q_af}{\phi_a}\right)^{\{-u\}y}(b) - \left(\frac{Q_af}{\phi_a}\right)^{\{-u\}}(z)\frac{1-|z|^2}{1-\overline{z}b}\right]\frac{\sqrt{1-|b|^2}}{\phi_z(b)} = \langle f, B_{azb}\rangle^{\{-u\}y}. \tag{3.13}$$

These can be changed into

$$\begin{aligned}&\Big[\frac{f^{\{u\}x}(b)-\langle f,B_a\rangle B_a(b)}{\phi_a(b)}-\frac{f^{\{u\}}(z)-\langle f,B_a\rangle B_a(z)}{\phi_a(z)}\frac{1-|z|^2}{1-\overline{z}b}\Big]\\&=\langle f,B_{azb}\rangle^{\{u\}x}\frac{\phi_z(b)}{\sqrt{1-|b|^2}},\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}&\Big[\frac{f^{\{-u\}y}(b)-\langle f,B_a\rangle B_a(b)}{\phi_a(b)}-\frac{f^{\{-u\}}(z)-\langle f,B_a\rangle B_a(z)}{\phi_a(z)}\frac{1-|z|^2}{1-\overline{z}b}\Big]\\&=\langle f,B_{azb}\rangle^{\{-u\}y}\frac{\phi_z(b)}{\sqrt{1-|b|^2}}.\end{aligned}\tag{3.15}$$

Owing to the relations

$$\Big(\frac{Q_af}{\phi_a}\Big)^{\{u\}x \text{ or } \{-u\}y}(b)=\frac{\langle f,B_{ab}\rangle^{\{\pm u\}}}{\sqrt{1-|b|^2}},\ \Big(\frac{Q_af}{\phi_a}\Big)^{\{\pm u\}}(z)=\frac{\langle f,B_{az}\rangle^{\{\pm u\}}}{\sqrt{1-|z|^2}},$$

$$|\langle f,B_{ab}\rangle^{\{u\}}|=|\langle f,B_{ab}\rangle^{\{-u\}}|,|\langle f,B_{az}\rangle^{\{u\}}|=|\langle f,B_{az}\rangle^{\{-u\}}|,|\langle f,B_{azb}\rangle^{\{u\}x}|=|\langle f,B_{azb}\rangle^{\{-u\}y}|,$$

we have that (3.14) and (3.15) are two solutions of the type of triangle problem described in (3.5). The two solution triangles represent two congruent triangles with different axes each having an end point adherent at the origin. We claim that the two triangles are of opposite orientations. This is based on the following geometric knowledge: if two congruent triangles have the same orientation, then the angles formed by the corresponding sides are all identical. But what we have here,however, is not the case: since $f^{[+]}(b)=f^{\{u\}x}(b), f^{[-]}(b)=f^{\{-u\}y}(b)$, for the fixed a, the phase difference between

$$\frac{f^{[+]}(b)-\langle f,B_a\rangle B_a(b)}{\phi_a(b)}\quad\text{and}\quad\frac{f^{[-]}(b)-\langle f,B_a\rangle B_a(b)}{\phi_a(b)}$$

depends only on b, while the phase difference between the two complex numbers

$$\frac{f^{\{u\}}(z)-\langle f,B_a\rangle B_a(z)}{\phi_a(z)}\frac{1-|z|^2}{1-\overline{z}b}\quad\text{and}\quad\frac{f^{\{-u\}}(z)-\langle f,B_a\rangle B_a(z)}{\phi_a(z)}\frac{1-|z|^2}{1-\overline{z}b}$$

depends only on z. They therefore cannot be equal. This implies that the two congruent triangles are of different orientations. When b is very close to z, the phases of $\phi_a(z)$ and $\phi_a(b)$ are very close, and the phase of $\frac{1-|z|^2}{1-\overline{z}b}$ is very close to zero. Therefore the orientation of

$$(0,\ \frac{f^{\{u\}}(z)-\langle f,B_a\rangle B_a(z)}{\phi_a(z)}\frac{1-|z|^2}{1-\overline{z}b},\ \frac{f^{\{u\}x}(b)-\langle f,B_a\rangle B_a(b)}{\phi_a(b)})$$

is the same as that of

$$(0,\ f^{\{u\}}(z)-\langle f,B_a\rangle B_a(z),\ f^{\{u\}x}(b)-\langle f,B_a\rangle B_a(b)).$$

The latter triple is the image of the triple (a,z,b) under the mapping

$$\phi_a(w)\Big(\frac{Q_a}{\phi_a}\Big)^{\{u\}x}(f)(w).\tag{3.16}$$

Similarly, the orientation of

$$(0,\ \frac{f^{\{-u\}}(z)-\langle f,B_a\rangle B_a(z)}{\phi_a(z)}\frac{1-|z|^2}{1-\overline{z}b},\ \frac{f^{\{-u\}y}(b)-\langle f,B_a\rangle B_a(b)}{\phi_a(b)})$$

is the same as that of

$$(0,\ f^{\{-u\}}(z)-\langle f,B_a\rangle B_a(z),\ f^{\{-u\}y}(b)-\langle f,B_a\rangle B_a(b)).$$

The latter triple is the image of the triple (a,z,b) under the mapping

$$\phi_a(w)\left(\frac{Q_a}{\phi_a}\right)^{\{-u\}y}(f)(w). \tag{3.17}$$

From the above analysis, when b is close to z, only one of the functions (3.16) and (3.17) keeps the local orientation, and can thus be a holomorphic function. We therefore can determine $f^*(z)$ accordingly. The proof of (2) is complete. □

Remark 3.3 In practice for any $b\neq a, b\neq z, \langle f,B_{ab}\rangle\neq 0, \langle f,B_{azb}\rangle\neq 0$ we observe that $f^{\{+\}+}(b), f^{\{+\}-}(b), f^{\{-\}+}(b)$ and $f^{\{-\}-}(b)$ are always four distinct numbers, and that $S\cap T$ contain only one point, corresponding to the easy case. In summary, the forward-backward algorithm involves that for any $z\in\mathbb{D}$ one selects $b\in\mathbb{D}, b\neq z$, and b satisfies all the non-orthogonality conditions set in Lemma (3.2), and that it is close to z, if necessary. This allows one to use the points in the set $S\cap T$ to go backwards to determine the true value $f(z)$ between $f^{\{+\}}(z)$ and $f^{\{-\}}(z)$.

4 Scheme III: Phase Retrieval based on Sparse Representation: The AFD Type Methods

We are able to give an approximation representation formula of the phase retrieval problem. We will employ the so called adaptive Fourier decomposition (AFD, or Core AFD) or alternatively the n-best rational approximation method (Cyclic AFD). Based on a sequence of intensity measurements the AFD type methods practically give rise to approximation formulas to the solution function. We first illustrate the Core AFD method.

We still assume that we know a non-zero sample value $f(a)$ at some point $a\in\mathbb{D}$. The AFD method involves a sequence of maximal selections of the parameters $a_k,\ k=1,2,\cdots$ to define the related Takenaka-Malmquist system. Let

$$a_1=\arg\max\{|\langle f,B_z\rangle| \mid z\in\mathbb{D}\}.$$

Based on the value $f(a)$ Scheme II (i.e., the FB algorithm) can be used to assert the function value $f(a_1)$. Then the simple relation (1.5) can be used to assert the value $\langle f,B_{a_1}\rangle$. Next, we select

$$a_2=\arg\max\{|\langle f,B_{a_1z}\rangle| \mid z\in\mathbb{D}\}.$$

Theoretically there will be no problem if we encounter $a_2=a_1$. To simplify the discussion we can select $a_2\neq a_1$, for instance, to satisfy

$$|\langle f,B_{a_1a_2}\rangle|\geq\frac{1}{2}\max\{|\langle f,B_{a_1z}\rangle| \mid z\in\mathbb{D}\}.$$

By using Scheme II the function value $f(a_2)$ is computable. In formula (3.7) replacing a by a_1 and z by a_2, we obtain the value $\langle f,B_{a_1a_2}\rangle$. After inductively obtaining $a_1,\cdots,a_n$, and the related $\langle f,B_{a_1\cdots a_k}\rangle, k=1,\cdots,n$, we next find

$$a_{n+1}=\arg\max\{|\langle f,B_{a_1\cdots a_nz}\rangle| \mid z\in\mathbb{D}\}.$$

If it happens that a_{n+1} coincides with some $a_k, k=1,\cdots,n$, then we can select a_{n+1} not equal to any $a_1,\cdots,a_n$, such that

$$|\langle f, B_{a_1\cdots a_n a_{n+1}}\rangle| \geq \frac{1}{2}\max\{|\langle f, B_{a_1\cdots a_n z}\rangle| \mid z\in\mathbb{D}\}.$$

We have

$$\begin{aligned}\langle f, B_{a_1\cdots a_n a_{n+1}}\rangle &= \frac{Q_{a_1\cdots a_n} f(a_{n+1})}{\phi_{a_1\cdots a_n}(a_{n+1})}\sqrt{1-|a_{n+1}|^2}\\ &= \frac{f(a_{n+1})-\sum\limits_{k=1}^{n}\langle f, B_{a_1\cdots a_k}\rangle B_{a_1\cdots a_k}(a_{n+1})}{\phi_{a_1\cdots a_n}(a_{n+1})}\sqrt{1-|a_{n+1}|^2}. \qquad (4.1)\end{aligned}$$

The FB algorithm established in Section 3 for Scheme II can be used to compute the function value $f(a_{n+1})$ and thus the value of $\langle f, B_{a_1\cdots a_n a_{n+1}}\rangle$ too. In such way we obtain all the values $\langle f, B_{a_1\cdots a_m}\rangle$. Then the Weak AFD theory guarantees that

$$f^*(z)=\sum_{n=1}^{\infty}\langle f, B_{a_1\cdots a_n}\rangle B_{a_1\cdots a_n}(z)$$

converges in fast speed. Usually, for numerical purposes one needs to compute the series up to a term n. In both the infinite or finite approximation cases the error is estimated by the L^2-norm of f^*-f over the boundary.

Cyclic AFD offers more accurate results. Let n be fixed. When we have found, consecutively, $a_1,\cdots,a_n$, and accordingly formed an n-term AFD series to approximate $f(z)$, that n-term series is usually not the optimal one over all the n-series of the same kind. To improve it, we can, for instance, let $\{a_2,\cdots,a_n\}$ be an $(n-1)$-set, say $\{b_1,\cdots,b_{n-1}\}$, and find a b_n, more optimal than a_1. This process of improvement is based on the fact that the orthogonal projection of f into the span of n vectors is irrelevant with the order of the vectors listed. In [4] and [7] the detailed algorithms are studied. The result of n-Cyclic AFD is of the form

$$\sum_{k=1}^{n} c_l\tilde{k}_{a_k},$$

where

$$\tilde{k}_{a_k}=\left[\left(\frac{\partial}{\partial\overline{a}}\right)^{l_k-1}k_a\right]_{a=a_k},$$

where l_k is the repeating number of a_k in the k-tuple $(a_1,\cdots,a_k)$. The result of n-Cyclic AFD coincides with the result of best approximation by rational functions of degree not exceeding n.

Examples of the AFD methods will be given in Section 5.

5 Experiments

We will test our methods by using two examples. One of them is taken from [3], and the other is a Blaschke product with finite zeros. In both of the examples we employ the initial value $f(a)$ for $a=0.32-0.16i$ generated by a random process. With the Scheme I and II methods we measure the error by the discrete relative error r_n, where

$$r_n = \sqrt{\frac{\sum\limits_{k=1}^{n} |f^*(z_k) - f(z_k)|^2}{\sum\limits_{k=1}^{n} |f(z_k)|^2}},$$

where $f^*(z_k)$ and $f(z_k)$ are, respectively, the recovered and ground truth values of the functions. With the Scheme III method, we estimate the error by the L^2 relative error $\frac{\|f^*-f\|}{\|f\|}$.

Example 5.1 Let

$$f(z) = \frac{0.1867z^6 - 0.00869z^5}{(1-0.7842z)(1-0.2669z)}.$$

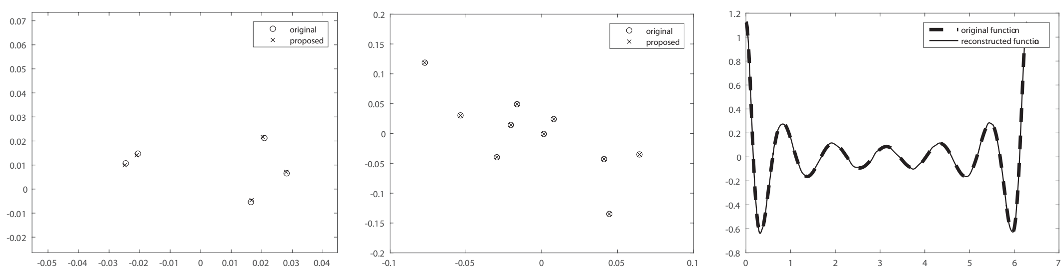
The initial value is $f(0.32-0.16\mathrm{i}) = 0.0013 - 0.0009\mathrm{i}$.

By Scheme I the error r_5 is $r_5 = 0.0279$.

By Scheme II the error r_{10} is 8.1813×10^{-9}.

By Scheme III (AFD) with an iteration number 23, the relative error is 0.0009.

The re-constructions using the three schemes are illustrated in Figure 1.



By scheme I By scheme II By AFD

Figure 1 Three schemes for reconstruction

Table 1 The part of function values reconstructed by Scheme I

z	0.41-0.49i	0.60-0.22i	0.64-0.02i	0.64-0.07i	0.60-0.24i
$f(z)$	0.0164-0.0054i	-0.0206+0.0147i	0.0281+0.0065i	0.0208+0.0212i	-0.0246+0.0107i
Reconstructed $f(z)$	0.0166-0.0046i	-0.0209+0.0141i	0.0280+0.0070i	0.0204+0.0216i	-0.0249+0.0100i

Table 2 The part of function values reconstructed by Scheme II

Re(z)	-0.44	0.42	-0.82	0.44	0.5	0.6	-0.6	0.72	-0.82	-0.92
Im(z)	-0.82	-0.04	-0.44	-0.82	-0.44	-0.42	0.5	-0.44	0.36	0.14
Re($f(z)$)	0.0644	0.0013	-0.0537	0.0446	0.0079	-0.0163	-0.0204	-0.0772	-0.0297	0.0410
Im($f(z)$)	-0.0348	-0.0009	0.032	-0.1348	0.0240	0.0491	0.0144	0.1188	-0.0396	-0.0427

Example 5.2 Let

$$f(z) = \prod_{k=1}^{4} \frac{z - a_k}{1 - \overline{a}_k z},$$

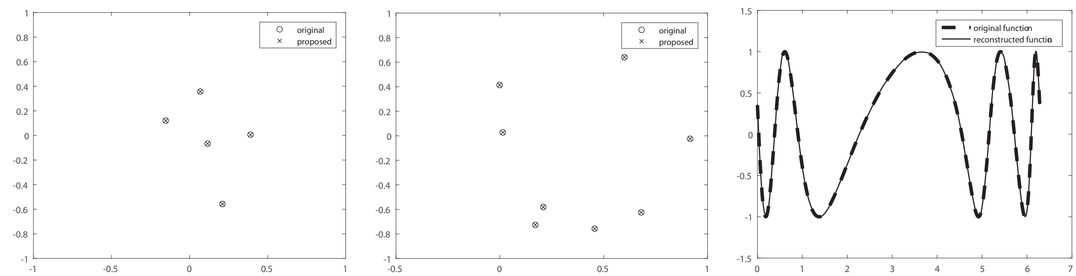
where $a_1 = 0.51 + 0.22\mathrm{i}$, $a_2 = 0.83 + 0.1\mathrm{i}$, $a_3 = 0.39 + 0.13\mathrm{i}$, $a_4 = 0.25 + 0.64\mathrm{i}$. The initial value is $f(0.32 - 0.16\mathrm{i}) = 0.0534 + 0.0377\mathrm{i}$.

By Scheme I the error r_5 is 0.3576×10^{-8}.

By Scheme II the error r_8 is 1.0678×10^{-7}.

By Scheme III (AFD) with an iteration number 18 the relative error is 0.0004.

The re-constructions using the three schemes are illustrated in Figure 2.



By scheme I By scheme II By AFD

Figure 2 Three schemes for reconstruction

Table 3 The part of function values reconstructed by Scheme I

z	-0.02i	-0.35-0.28i	-0.52+0.36i	0.75-0.26i	0.68-0.45i
$f(z)$	0.1151-0.0653i	0.3911+0.0057i	0.2091-0.5566i	-0.1532+0.1197i	0.0693+0.3576i
Reconstructed $f(z)$	0.1151-0.0653i	0.3911+0.0057i	0.2091-0.5566i	-0.1532+0.1197i	0.0693+0.3576i

Table 4 The part of function values reconstructed by Scheme II

Re(z)	-0.44	0.42	-0.82	0.44	-0.6	-0.82	-0.92	0.72
Im(z)	-0.82	-0.04	-0.44	-0.82	0.5	0.36	0.14	-0.44
Re($f(z)$)	0.5990	0.0148	0.9156	0.2091	0.1715	0.4574	0.6818	-0.0007
Im($f(z)$)	0.6407	0.0270	-0.0237	-0.5805	-0.7258	-0.7576	-0.6257	0.4140

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