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The Poisson kernel and the Fourier transform of the slice monogenic Cauchy kernels

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ABSTRACT

The Fueter-Sce-Qian (FSQ for short) mapping theorem is a two-steps procedure to extend holomorphic functions of one complex variable to slice monogenic functions and to monogenic functions. Using the Cauchy formula of slice monogenic functions the FSQ-theorem admits an integral representation for n odd. In this paper we show that the relation $\Delta_{n+1}^{(n-1)/2} S_L^{-1} = \mathcal{F}_n^L$ between the slice monogenic Cauchy kernel S_L^{-1} and the F-kernel $\mathcal{F}_n^L$, that appear in the integral form of the FSQ-theorem for n odd, holds also in the case we consider the fractional powers of the Laplace operator Δ_{n+1} in dimension $n+1$, i.e., for n even. Moreover, this relation is proven computing explicitly the Fourier transform of the kernels S_L^{-1} and $\mathcal{F}_n^L$ as functions of the Poisson kernel. Similar results hold for the right kernels S_R^{-1} and of $\mathcal{F}_n^R$.

Contents

1. Introduction

The Fueter-Sce-Qian (FSQ for short) mapping theorem is one of the deepest results in complex and hypercomplex analysis because it shows how to extend holomorphic functions of one complex variable to

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high dimensions for vector-valued functions. This theorem is due to R. Fueter [30] for the quaternionic setting, it was generalized by M. Sce [47], to Clifford algebra $\mathbb{R}_n$ for n odd while the case of even dimension was proved by T. Qian in [43] (see also the recent monograph [45]). The method of T. Qian requires the use of the Fourier transform in the space of distributions and is deeply different from the method of R. Fueter and M. Sce. In the literature it is less known that the results of M. Sce, in [47], are written in a setting that contains, as particular case, the Clifford algebra $\mathbb{R}_n$ of odd dimension, but, for example it also works for the octonions. For more details see the recent translation of the work of M. Sce with commentaries [23, chapter 5].

Consider holomorphic functions of one complex variable $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ (denoted the set by $\mathcal{O}(\Omega)$). It is well known that the way to extend the function theory of one complex variable to several complex variables is to consider the systems of Cauchy-Riemann equations, for

$$f : \Pi \subseteq \mathbb{C}^n \to \mathbb{C},$$

where Π is an open set. To explain the FSQ mapping theorem we need some preliminary notation for the Clifford algebra setting.

Let $\mathbb{R}_n$ be the real Clifford algebra over n imaginary units $e_1, \ldots, e_n$ satisfying the relations $e_\ell e_m + e_m e_\ell = 0$, $\ell \neq m$, $e_\ell^2 = -1$. An element in the Clifford algebra will be denoted by $\sum_A e_A x_A$ where $A = \{\ell_1 \ldots \ell_r\} \in \mathcal{P}\{1, 2, \ldots, n\}$, $\ell_1 < \ldots < \ell_r$ is a multi-index and $e_A = e_{\ell_1} e_{\ell_2} \ldots e_{\ell_r}$, $e_\emptyset = 1$. An element $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ will be identified with the element $x = x_0 + \underline{x} = x_0 + \sum_{\ell=1}^n x_\ell e_\ell \in \mathbb{R}_n$ called paravector and the real part x_0 of x will also be denoted by $\mathrm{Re}(x)$. The norm of $x \in \mathbb{R}^{n+1}$ is defined as $|x|^2 = x_0^2 + x_1^2 + \ldots + x_n^2$. The conjugate of x is defined by $\bar{x} = x_0 - \underline{x} = x_0 - \sum_{\ell=1}^n x_\ell e_\ell$. We denote by $\mathbb{S}$ the sphere

$$\mathbb{S} = \{\underline{x} = e_1 x_1 + \ldots + e_n x_n \mid x_1^2 + \ldots + x_n^2 = 1\};$$

for $I \in \mathbb{S}$ we obviously have $I^2 = -1$. Given an element $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ let us set $I_x = \underline{x}/|\underline{x}|$ if $\underline{x} \neq 0$, and given an element $x \in \mathbb{R}^{n+1}$, the set

$$[x] := \{y \in \mathbb{R}^{n+1} \ : \ y = x_0 + I|\underline{x}|, \ I \in \mathbb{S}\}$$

is an $(n-1)$-dimensional sphere in $\mathbb{R}^{n+1}$. The vector space $\mathbb{R} + I\mathbb{R}$ passing through 1 and $I \in \mathbb{S}$ will be denoted by $\mathbb{C}_I$ and an element belonging to $\mathbb{C}_I$ will be indicated by $u + Iv$, for u, $v \in \mathbb{R}$. With an abuse of notation we will write $x \in \mathbb{R}^{n+1}$. Thus, if $U \subseteq \mathbb{R}^{n+1}$ is an open set, a function $f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ can be interpreted as a function of the paravector x.

We can now summarize the FSQ mapping theorem highlighting the two steps of the extension procedure as follows. In the first step we obtain slice monogenic functions [15,22,6,20,9,11,12,3,46], see the book [14], while in the second step we get the classical monogenic functions [2,8,24,31,34]. Let $\tilde{f}(z) = f_0(u, v) + i f_1(u, v)$, where $i = \sqrt{-1}$, be a holomorphic function defined in a symmetric domain with respect to the real axis $D \subseteq \mathbb{C}$ and let

$$\Omega_D = \{x = x_0 + \underline{x} \quad : \quad (x_0, |\underline{x}|) \in D\}$$

be the open set, induced by D, in $\mathbb{R}^{n+1}$. Moreover, we assume that

$$f_0(u, -v) = f_0(u, v) \quad \text{and} \quad f_1(u, -v) = -f_1(u, v)$$

namely f_0 and f_1 are, respectively, even and odd functions in the variable v. Additionally the pair (f_0, f_1) satisfies the Cauchy-Riemann system. Then the FSQ extension procedure is as follows.

Step (I). The linear operator T_{FSQ1} defined as

$$T_{FSQ1}(\tilde{f}(z)) := f_0(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} f_1(x_0, |\underline{x}|) \quad \text{on} \quad \Omega_D$$

extends the holomorphic function $\tilde{f}(z)$ to the slice monogenic function

$$f(x) := f_0(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} f_1(x_0, |\underline{x}|).$$

Step (II). Consider the linear operator $T_{FSQ2} := \Delta_{n+1}^{\frac{n-1}{2}}$ where Δ_{n+1} is the Laplace operator in $n+1$ dimensions, i.e., $\Delta_{n+1} = \partial_{x_0}^2 + \sum_{j=1}^n \partial_{x_j}^2$. Then, T_{FSQ2} maps the slice monogenic function $f(x)$ in the monogenic function

$$\breve{f}(x) := T_{FSQ2}\Big(f_0(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} f_1(x_0, |\underline{x}|) \Big),$$

i.e., $\breve{f}(x)$ is in the kernel of the Dirac operator D, i.e.

$$D\breve{f}(x) := \partial_{x_0} \breve{f}(x) + \sum_{i=1}^n e_i \partial_{x_i} \breve{f}(x) = 0 \quad \text{on} \quad \Omega_D.$$

We point out that in the extension procedure the operator T_{FSQ1} maps holomorphic functions into the set of intrinsic slice monogenic functions, denoted by $\mathcal{N}(\Omega_D)$, that is strictly contained in the set of slice monogenic functions $\mathcal{SM}(U)$. Similarly, when we apply the operator T_{FSQ2} to the set of slice monogenic functions, not necessarily intrinsic, we obtain a subclass of the monogenic functions that are called axially monogenic functions and are denoted by $\mathcal{AM}(\Omega_D)$, so we can visualize the FSQ construction by the diagram:

$$\mathcal{O}(D) \xrightarrow{T_{FSQ1}} \mathcal{N}(\Omega_D) \xrightarrow{T_{FSQ2} = \Delta_{n+1}^{(n-1)/2}} \mathcal{AM}(\Omega_D),$$

where T_{FSQ1} denotes the first linear operator and T_{FSQ2} the second one. As it is clear $\Delta_{n+1}^{\frac{n-1}{2}}$ is a fractional operator for n even.

The FSQ-mapping theorem and its generalizations can be found in [29,28,27,36,41,42,48], more recently there has been an intensive research in the direction of the inverse FSQ-mapping theorem, which has been investigated in the papers [17,13,16,18,26,25]. Using the Radon and dual Radon transform, see [19], it is possible to find a different method, with respect to the FSQ-theorem, to relate slice monogenic functions and the monogenic functions.

The FSQ mapping theorem in integral form, introduced in [10], is associated with the second step of the FSQ extension procedure. In fact, the main idea is to apply the linear operator

$$T_{FSQ2} = \Delta_{n+1}^{\frac{n-1}{2}}$$

to the Cauchy kernel

$$S_L^{-1}(s, x) := (s - \bar{x})(s^2 - 2\text{Re}(x)s + |x|^2)^{-1},$$

of left slice monogenic functions when n is odd, similarly we proceed for right slice monogenic functions. For odd dimension applying the operator $\Delta_{n+1}^{\frac{n-1}{2}}$, in the variables in x, to the function $S_L^{-1}(s, x)$ we have obtained a very simple expression given by

$$\Delta_{n+1}^{\frac{n-1}{2}} S_L^{-1}(s,x) = \gamma_n (s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}},$$

where γ_n are

$$\gamma_n := (-1)^{\frac{n-1}{2}} 2^{n-1} \left[\Gamma\left(\frac{n+1}{2}\right)\right]^2 \tag{1}$$

which can be used to obtain the Fueter-Sce mapping theorem in integral form. Precisely, let f be a slice monogenic function defined in an open set that contains $\overline{U}$, where U is a bounded axially symmetric open set. Suppose that the boundary of $U \cap \mathbb{C}_I$ consists of a finite number of rectifiable Jordan curves for any $I \in \mathbb{S}$. Then, if $x \in U$, the monogenic function $\breve{f}(x)$, given by

$$\breve{f}(x) = \Delta_{n+1}^{\frac{n-1}{2}} f(x) \tag{2}$$

admits the integral representation

$$\breve{f}(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} \mathcal{F}_L(s,x) ds_I f(s), \quad ds_I = ds/I, \tag{3}$$

where

$$\mathcal{F}_L(s,x) := \gamma_n (s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}}$$

is called the left F-kernel, and the integral depends neither on U nor on the imaginary unit $I \in \mathbb{S}$.

The main problems studied in this paper can be formulated as follows.

Problem 1.1. *(A) Determine the type of hyperholomorphicity of the map*

$$(s,x) \mapsto (s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-h},$$

for $h \in \mathbb{R}$, with respect to s and x for $s \notin [x]$.

(B) Compute explicitly the Fourier transform of the slice monogenic Cauchy kernels and of the F_n-kernels as functions of the Poisson kernel.

(C) Show that the relation $\Delta_{n+1}^{\frac{n-1}{2}} S_L^{-1}(s,x) = \gamma_n (s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}}$ is true for all dimensions n replacing suitably the constants γ_n.

The main results of this paper can now be summarized in the following steps.

(I) It is a remarkable fact that the function

$$(s,x) \mapsto (s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-h}$$

is slice monogenic in s for any $h \in \mathbb{N}$ but is monogenic in x if and only if $h = (n+1)/2$, namely if and only if h is related with the Sce's exponent also for n odd. In Theorem 2.15 we have shown that this result remains true also in the case of even dimension, that is when we have the fractional powers.

(II) The Fourier transform, denoted by the symbol F, of $S_L^{-1}(s,x)$ is

$$\mathrm{F}[S_L^{-1}(s,\cdot)](\xi) = c_n \, \frac{\bar{\underline{\xi}}}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}} \, e^{-is\xi_0}, \quad \xi = \xi_0 + \underline{\xi} \neq 0,$$

where

$$c_n := i2^n\pi^{\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right).$$

(III) A second fundamental result is the Fourier transform of the F_n-kernels. We proved that

$$\mathrm{F}[\mathcal{F}_n^L(s,\cdot)](\xi) = k_n\ \frac{\bar{\xi}}{\xi_0^2 + |\underline{\xi}|^2}\ e^{-is\xi_0}, \qquad \xi_0 + \underline{\xi} \neq 0,$$

where

$$k_n := i(-1)^{\frac{n-1}{2}}2^n\pi^{\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right).$$

(IV) We show that the relation

$$\Delta_{n+1}^{\frac{n-1}{2}}S_L^{-1}(s,x) = \gamma_n(s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}}$$

holds true also when n is an even number, using the Fourier transform of the kernels S_L^{-1} and $\mathcal{F}_n^L$.

The plan of the paper. The paper contains three sections including the introduction. In Section 2 we show the monogenicity of the Fueter-Sce kernel in even dimension. Section 3 we compute the Fourier transform of the slice monogenic Cauchy kernels. In Section 4 we compute the Fourier transform of the F_n-kernels. Finally in Section 5 we show that the relation

$$\Delta_{n+1}^{(n-1)/2}S_L^{-1}(s,x) = \mathcal{F}_n^L(s,x), \quad \text{for} \quad s \notin [x]$$

also holds for the even dimension of the Clifford algebra $\mathbb{R}_n$. The proof is based on the Fourier transform.

2. Monogenicity of the Fueter-Sce kernel in even dimension

In this paper we use the definition of slice monogenic functions that is the generalization of slice monogenic functions in the spirit of the FSQ mapping theorem and it is slightly different from the one in [14]. This definition is the most appropriate for operator theory and the reason is widely explained in several papers and in the books [5,21]. Keeping in mind the notations previously given for the Clifford algebra $\mathbb{R}_n$ we recall some definitions. For the missing proofs of the results that we recall see for example [4,10].

Definition 2.1. Let $U \subseteq \mathbb{R}^{n+1}$. We say that U is axially symmetric if $[x] \in U$ for every $x \in U$.

Definition 2.2 *(Slice monogenic functions).* Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set and let $\mathcal{U} = \{(u,v) \in \mathbb{R}^2 : u + \mathbb{S}v \subset U\}$. A function $f : U \to \mathbb{R}_n$ is called a left slice function, if it is of the form

$$f(x) = f_0(u,v) + If_1(u,v) \qquad \text{for } x = u + Iv \in U$$

with the two functions $f_0, f_1 : \mathcal{U} \to \mathbb{R}_n$ that satisfy the compatibility conditions

$$f_0(u,-v) = f_0(u,v), \qquad f_1(u,-v) = -f_1(u,v). \tag{4}$$

If in addition f_0 and f_1 are $\mathcal{C}^1$ and satisfy the Cauchy-Riemann equations

$$\begin{aligned} \partial_u f_0(u,v) - \partial_v f_1(u,v) = 0 \\ \partial_v f_0(u,v) + \partial_u f_1(u,v) = 0 \end{aligned} \tag{5}$$

then f is called left slice monogenic. A function $f : U \to \mathbb{R}_n$ is called a right slice function if it is of the form

$$f(x) = f_0(u, v) + f_1(u, v)I \qquad \text{for } x = u + Iv \in U$$

with the two functions $f_0, f_1 : \mathcal{U} \to \mathbb{R}_n$ that satisfy (4). If f_0 and f_1 are $\mathcal{C}^1$ and satisfy the Cauchy-Riemann equations (5) then f is called right slice monogenic.

If f is a left (or right) slice function such that f_0 and f_1 are real-valued, then f is called intrinsic.

We denote the sets of left, right and intrinsic slice monogenic functions on U by $\mathcal{SM}_L(U)$, $\mathcal{SM}_R(U)$ and $\mathcal{N}(U)$, respectively.

For slice monogenic functions we have two equivalent ways to write the Cauchy kernels.

Proposition 2.3. *If $x, s \in \mathbb{R}^{n+1}$ with $x \notin [s]$, then*

$$-(x^2 - 2x\mathrm{Re}(s) + |s|^2)^{-1}(x - \bar{s}) = (s - \bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-1} \tag{6}$$

and

$$(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-1}(s - \bar{x}) = -(x - \bar{s})(x^2 - 2\mathrm{Re}(s)x + |s|^2)^{-1}. \tag{7}$$

So we can give the following definition to distinguish the two representations of the Cauchy kernels.

Definition 2.4. Let $x, s \in \mathbb{R}^{n+1}$ with $x \notin [s]$.

- We say that $S_L^{-1}(s, x)$ is written in the form I if

$$S_L^{-1}(s, x) := -(x^2 - 2\mathrm{Re}(s)x + |s|^2)^{-1}(x - \bar{s}).$$

- We say that $S_L^{-1}(s, x)$ is written in the form II if

$$S_L^{-1}(s, x) := (s - \bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-1}.$$

- We say that $S_R^{-1}(s, x)$ is written in the form I if

$$S_R^{-1}(s, x) := -(x - \bar{s})(x^2 - 2\mathrm{Re}(s)x + |s|^2)^{-1}.$$

- We say that $S_R^{-1}(s, x)$ is written in the form II if

$$S_R^{-1}(s, x) := (s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-1}(s - \bar{x}).$$

Lemma 2.5. *Let $x, s \in \mathbb{R}^{n+1}$ with $s \notin [x]$. The left slice monogenic Cauchy kernel $S_L^{-1}(s, x)$ is left slice monogenic in x and right slice monogenic in s. The right slice monogenic Cauchy kernel $S_R^{-1}(s, x)$ is left slice monogenic in s and right slice monogenic in x.*

Definition 2.6 *(Slice Cauchy domain).* An axially symmetric open set $U \subset \mathbb{R}^{n+1}$ is called a slice Cauchy domain, if $U \cap \mathbb{C}_I$ is a Cauchy domain in $\mathbb{C}_I$ for any $I \in \mathbb{S}$. More precisely, U is a slice Cauchy domain if, for any $I \in \mathbb{S}$, the boundary $\partial(U \cap \mathbb{C}_I)$ of $U \cap \mathbb{C}_I$ is the union a finite number of non-intersecting piecewise continuously differentiable Jordan curves in $\mathbb{C}_I$.

Theorem 2.7 *(Cauchy formulas). Let $U \subset \mathbb{R}^{n+1}$ be a slice Cauchy domain, let $I \in \mathbb{S}$ and set $ds_I = ds(-I)$. If f is a (left) slice monogenic function on a set that contains $\overline{U}$ then*

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S_L^{-1}(s,x)\, ds_I\, f(s), \qquad \text{for any} \quad x \in U. \tag{8}$$

If f is a right slice monogenic function on a set that contains $\overline{U}$, then

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} f(s)\, ds_I\, S_R^{-1}(s,x), \qquad \text{for any} \quad x \in U. \tag{9}$$

These integrals depend neither on U nor on the imaginary unit $I \in \mathbb{S}$.

Even though $S_L^{-1}(s,x)$ written in the form I is more suitable for several applications, for example for the definition of a functional calculus, see [14], it does not allow easy computations of the powers of the Laplacian

$$\Delta_{n+1} = \partial_{x_0}^2 + \sum_{j=1}^{n} \partial_{x_j}^2,$$

with respect to the variable x applied to it. The form II is the one that allows, by iteration, the computation of $\Delta_{n+1}^{\frac{n-1}{2}} S_L^{-1}(s,x)$. In the sequel we will write Δ instead of Δ_{n+1}.

Theorem 2.8. *Let x, $s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. Let $S_L^{-1}(s,x) = (s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-1}$ be the slice-monogenic Cauchy kernel and let $\Delta = \sum_{i=0}^{n} \frac{\partial^2}{\partial x_i^2}$ be the Laplace operator in the variable x. Then, for $h \geq 1$, we have:*

$$\Delta^h S_L^{-1}(s,x) = (-1)^h \prod_{\ell=1}^{h} (2\ell) \prod_{\ell=1}^{h} (n-(2\ell-1))(s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-(h+1)} \tag{10}$$

and

$$\Delta^h S_R^{-1}(s,x) = (-1)^h \prod_{\ell=1}^{h} (2\ell) \prod_{\ell=1}^{h} (n-(2\ell-1))(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-(h+1)}(s-\bar{x}). \tag{11}$$

We recal the definition of Monogenic functions.

Definition 2.9 *(Monogenic functions).* Let U be an open set in $\mathbb{R}^{n+1}$. A real differentiable function $f: U \to \mathbb{R}_n$ is left monogenic if

$$Df(x) := \partial_{x_0} f(x) + \sum_{i=1}^{n} e_i \partial_{x_i} f(x) = 0.$$

It is right monogenic if

$$f(x)D := \partial_{x_0} f(x) + \sum_{i=1}^{n} \partial_{x_i} f(x) e_i = 0.$$

In the sequel, we will also need the property of slice monogenicity of the functions $\Delta^h S_L^{-1}(s,x)$ and $\Delta^h S_L^{-1}(s,x)$ as shown in the following result:

Proposition 2.10. *Let x, $s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. Then we have.*

(I) The function $\Delta^h S_L^{-1}(s,x)$ is a right slice monogenic function in the variable s for all $h \geq 0$ and for all $x \notin [s]$.

(II) The function $\Delta^h S_R^{-1}(s,x)$ is a left slice monogenic function in the variable s for all $h \geq 0$ and for all $x \notin [s]$.

Proposition 2.11. *Let n be an odd number and let x, $s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. Then the function $\Delta^h S_L^{-1}(s,x)$ is a left monogenic function in the variable x, and the function $\Delta^h S_R^{-1}(s,x)$ is a right monogenic function in the variable x, if and only if $h = \frac{n-1}{2}$.*

Definition 2.12 *(The $\mathcal{F}_n$-kernels).* Let n be an odd number. Let x, $s \in \mathbb{R}^{n+1}$. We define, for $s \notin [x]$, the left $\mathcal{F}_n^L$-kernel as

$$\mathcal{F}_n^L(s,x) := \Delta^{\frac{n-1}{2}} S_L^{-1}(s,x) = \gamma_n (s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}},$$

and the right $\mathcal{F}_n^R$-kernel as

$$\mathcal{F}_n^R(s,x) := \Delta^{\frac{n-1}{2}} S_R^{-1}(s,x) = \gamma_n (s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}}(s-\bar{x}),$$

where γ_n are given by (1).

Theorem 2.13 *(The Fueter-Sce mapping theorem in integral form, see [10]). Let n be an odd number. Let $U \subset \mathbb{R}^{n+1}$ be a slice Cauchy domain, let $I \in \mathbb{S}$ and set $ds_I = ds(-I)$.*

(I) Let f is a left slice monogenic function on a set that contains $\overline{U}$, then the monogenic function $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$ admits the integral representation

$$\breve{f}(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} \mathcal{F}_n^L(s,x) ds_I f(s), \quad x \in U. \tag{12}$$

(II) Let f is a right slice monogenic function on a set that contains $\overline{U}$, then the monogenic function $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$ admits the integral representation

$$\breve{f}(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} f(s) ds_I \mathcal{F}_n^R(s,x), \quad x \in U. \tag{13}$$

Moreover, the integrals in (12) and (13) depend neither on U nor on the imaginary unit $I \in \mathbb{S}$.

We now consider the $\mathcal{F}_n$-kernels in Definition 2.12 also for the case of the fractional powers. We recall that given a paravector $y = u + I_y v \in \mathbb{R}^{n+1} \setminus (-\infty, 0]$, for $\alpha \in \mathbb{R}$, we can define the fractional powers as

$$y^\alpha := e^{\alpha \log y} = e^{\alpha(\ln|y| + I\arg(y))}, \tag{14}$$

where $\arg(y) = \arccos \frac{u}{|y|}$. The definition is analogue for the quaternions and the fractional powers so defined are slice monogenic functions. The first natural question that arise is formulated in the following problem.

Problem 2.14. *Proposition 2.11 claims that for n odd number the functions*

$$(s,x) \mapsto (s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-(h+1)}$$

and

$$(s,x) \mapsto (s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-(h+1)}(s-\bar{x}),$$

for $x \notin [s]$, are monogenic in the variable x if and only if $h = \frac{n-1}{2}$. Is it still true in the case n even?

The answer is positive and is formulated in the following theorem.

Theorem 2.15. *Let λ be a real number and $x, s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. Let us define:*

$$k_L(s,x) := (s-\bar{x})(s^2 - 2x_0 s + |x|^2)^{-\lambda}, \qquad \lambda \in \mathbb{R},$$

and

$$k_R(s,x) := (s^2 - 2x_0 s + |x|^2)^{-\lambda}(s-\bar{x}), \qquad \lambda \in \mathbb{R},$$

for

$$s^2 - 2x_0 s + |x|^2 \in \mathbb{R}^{n+1} \setminus (-\infty, 0].$$

Then, the function $k_L(s,x)$ is left monogenic function in the variable x and $k_R(s,x)$ right monogenic in the variable x if and only if $\lambda = \frac{n+1}{2}$.

Proof. We give the details for $k_L(s,x)$, similarly we proceed for $k_R(s,x)$. For simplicity, in the proof, we write $k(s,x)$ for $k_L(s,x)$. We have to compute $(\partial_{x_0} + \partial_{\underline{x}})k(s,x)$, where $\partial_{\underline{x}} = \sum_{j=1}^n e_j \partial_{x_j}$. First, we put $s = u + Iv$, thus

$$s^2 - 2x_0 s + |x|^2 = (u^2 - v^2 - 2x_0 u + |x|^2) + I(2uv - 2x_0 v).$$

Using the formula of fractional powers, in (14), we get

$$k(s,x) = (s-\bar{x})e^{\alpha(u,v)}, \tag{15}$$

where

$$\alpha(u,v) := -\frac{\lambda}{2}\ln[(u^2 - v^2 - 2x_0 u + |x|^2)^2 + (2uv - 2x_0 v)^2] + \\ -\lambda I \arccos \frac{u^2 - v^2 - 2x_0 u + |x|^2}{\sqrt{(u^2 - v^2 - 2x_0 u + |x|^2)^2 + (2uv - 2x_0 v)^2}}.$$

Let us denote

$$\beta(u,v) := \frac{u^2 - v^2 - 2x_0 u + |x|^2}{\sqrt{(u^2 - v^2 - 2x_0 u + |x|^2)^2 + (2uv - 2x_0 v)^2}}.$$

So we have

$$k(s,x)=(s-\bar{x})e^{-\frac{\lambda}{2}\ln[(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2]-\lambda I\arccos\beta(u,v)}.$$

To compute $\partial_{x_0}k(s,x)$ we calculate the derivative of $\beta(u,v)$ with respect to x_0

$$\frac{\partial\beta(u,v)}{\partial x_0}=\frac{(-2u+2x_0)(2uv-2x_0v)^2+(u^2-v^2-2x_0u+|x|^2)(2uv-2x_0v)2v}{[(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2]^{3/2}}.$$

Thus

$$\begin{aligned}\frac{\partial\arccos\beta(u,v)}{\partial x_0}&=-\frac{\partial_{x_0}\beta(u,v)}{\sqrt{1-\beta^2(u,v)}}\\&=-\frac{(-2u+2x_0)(2uv-2x_0v)+(u^2-v^2-2x_0u+|x|^2)2v}{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2}.\end{aligned}$$

So we have

$$\begin{aligned}\frac{\partial k(s,x)}{\partial x_0}&=-e^{\alpha(u,v)}+\frac{(u+Iv-\bar{x})2\lambda\Big[(u^2-v^2-2x_0u+|x|^2)(u-x_0+Iv)}{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2}\\&\quad-\frac{I(2uv-2x_0v)(Iv+u-x_0)\Big]}{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2}e^{\alpha(u,v)}.\end{aligned}\tag{16}$$

Now, we compute the derivative of $k(s,x)$ with respect to x_j, $1\le j\le n$. As before we start from the derivative of $\beta(u,v)$

$$\frac{\partial\beta(u,v)}{\partial x_j}=\frac{2x_j(2uv-2x_0v)^2}{[(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2]^{3/2}}.$$

Thus

$$\frac{\partial\arccos\beta(u,v)}{\partial x_j}=-\frac{2x_j(2uv-2x_0v)}{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2}.$$

Therefore

$$\frac{\partial k(s,x)}{\partial x_j}=e_je^{\alpha(u,v)}-\frac{(u+Iv-\bar{x})2\lambda\Big[(u^2-v^2-2x_0u+|x|^2)-I(2uv-2x_0v)\Big]}{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2}x_je^{\alpha(u,v)}.$$

Now, we are ready to compute $\partial_{\underline{x}}k(s,x)$

$$\begin{aligned}\partial_{\underline{x}}k(s,x)&=\sum_{j=1}^n e_j\frac{\partial k(s,x)}{\partial x_j}\\&=-ne^{\alpha(u,v)}-\frac{(u+Iv-\bar{x})2\lambda\Big[(u^2-v^2-2x_0u+|x|^2)-I(2uv-2x_0v)\Big]}{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2}\cdot\\&\quad\cdot\left(\sum_{j=1}^n e_jx_j\right)e^{\alpha(u,v)}\end{aligned}$$

This implies that

$$\partial_{\underline{x}} k(s,x) = -n e^{\alpha(u,v)} - \frac{(u+Iv-\bar{x})2\lambda\Big[(u^2-v^2-2x_0u+|x|^2)-I(2uv-2x_0v)\Big]}{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2}\underline{x}e^{\alpha(u,v)}. \tag{17}$$

Hence form (16) and (17) we get

$$(\partial_{x_0}+\partial_{\underline{x}})k(s,x) = -(n+1)e^{\alpha(u,v)} + \frac{2\lambda(u+Iv-\bar{x})(u+Iv-x)\Big[(u^2-v^2-2x_0u+|x|^2)}{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2}$$
$$-\frac{I(2uv-2x_0v)\Big]}{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2}e^{\alpha(u,v)}.$$

Now, we observe that

$$(u+Iv-\bar{x})(u+Iv-x) = (u^2-v^2-2ux_0+|x|^2)+I(2uv-2x_0v).$$

Setting

$$\gamma(u,v) := (u^2-v^2-2ux_0+|x|^2)+I(2uv-2x_0v)$$

we therefore obtain

$$(\partial_{x_0}+\partial_{\underline{x}})k(s,x) = \left[-(n+1)+2\lambda\frac{\gamma(u,v)\cdot\overline{\gamma(u,v)}}{|\gamma(u,v)|^2}\right]e^{\alpha(u,v)} = \left[-(n+1)+2\lambda\right]e^{\alpha(u,v)},$$

so we finally get

$$(\partial_{x_0}+\partial_{\underline{x}})k(s,x) = 0$$

if and only if $\lambda = \frac{n+1}{2}$. □

3. The Fourier transform of the slice monogenic Cauchy kernels

The main result of this section is the explicit computation of the Fourier transform of the slice monogenic Cauchy kernels $S_L^{-1}(s,x)$ and $S_R^{-1}(s,x)$ with respect to x when s is a real number. Then by extension we get the Fourier transform when s is a paravector. Firstly, let us introduce the definition of Fourier transform that we will use.

Definition 3.1. Let $f \in \mathcal{S}(\mathbb{R}^{n+1})$. The Fourier transform of the function f is

$$\hat{f}(\xi) := \mathrm{F}[f(x)](\xi) = \int_{\mathbb{R}^{n+1}} f(x)e^{-i(x,\xi)}dx,$$

where

$$(x,\xi) = \sum_{j=0}^{n} x_j\xi_j.$$

Definition 3.2. We define the inverse Fourier transform of the function f in the following way

$$\mathcal{R}[f(\xi)](x) = \frac{1}{(2\pi)^{n+1}}\int_{\mathbb{R}^{n+1}} f(\xi)e^{i(x,\xi)}d\xi.$$

In this paper we will use the following important result

Theorem 3.3 *(Plancherel's Theorem). If $f, g \in \mathcal{S}(\mathbb{R}^{n+1})$ then*

$$\int_{\mathbb{R}^{n+1}} f(x)\overline{g(x)}dx = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} \mathrm{F}(f)(\xi)\overline{\mathrm{F}(g(\xi))}d\xi. \tag{18}$$

Remark 3.4. Using the Plancherel's theorem it is possible to obtain

$$\int_{\mathbb{R}^{n+1}} \mathcal{R}(f)(x)\overline{\mathcal{R}(g(x))}dx = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} f(\xi)\overline{g(\xi)}d\xi. \tag{19}$$

In the sequel we will need this result.

Theorem 3.5. *[33, Sect. B.5] Let $f(|\underline{x}|)$ be a radial function in $\mathcal{S}(\mathbb{R}^n)$ with $n \geq 2$. Then the Fourier transform of f is also radial and has the form*

$$\hat{f}(|\underline{\xi}|) = (2\pi)^{\frac{n}{2}} |\underline{\xi}|^{-\frac{n-2}{2}} \int_0^\infty J_{\frac{n-2}{2}}(|\underline{\xi}|r) r^{\frac{n}{2}} f(r)\, dr,$$

where $r = |\underline{x}|$ and $J_{\frac{n-2}{2}}$ are the Bessel functions.

Remark 3.6. (See [40]) The formula in Theorem 3.5 is also valid for all functions

$$f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

Now, we start our computations.

Theorem 3.7. *Let us assume $s_0 \in \mathbb{R}$ and $x \in \mathbb{R}^{n+1}$. If we consider the slice monogenic Cauchy kernels $S_L^{-1}(s_0, x)$ and $S_R^{-1}(s_0, x)$ written in form II (see Definition 2.4) then their Fourier transforms with respect to x are equal and given by*

$$\mathrm{F}[S_L^{-1}(s_0, \cdot)](\xi) = \mathrm{F}[S_R^{-1}(s_0, \cdot)](\xi) = c_n \frac{\bar{\xi}}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}} e^{-is_0\xi_0}, \qquad \xi_0 + \underline{\xi} \neq 0$$

where

$$c_n := i2^n \pi^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right).$$

Moreover, if $s = s_0 + \underline{s} \in \mathbb{R}^{n+1}$ is a paravector the term $e^{-is_0\xi_0}$ extends to the intrinsic entire slice monogenic function $e^{-is\xi_0}$ and we have the Fourier transforms of the Cauchy kernels:

$$\mathrm{F}[S_L^{-1}(s, \cdot)](\xi) = c_n \; \frac{\bar{\xi}}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}} \; e^{-is\xi_0}, \qquad \xi_0 + \underline{\xi} \neq 0 \tag{20}$$

and

$$\mathrm{F}[S_R^{-1}(s, \cdot)](\xi) = c_n \; e^{-is\xi_0} \; \frac{\bar{\xi}}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}}, \qquad \xi_0 + \underline{\xi} \neq 0. \tag{21}$$

The extension $\mathrm{F}[S_L^{-1}(s,\cdot)](\xi)$ *is right slice monogenic in* s, *while* $\mathrm{F}[S_R^{-1}(s,\cdot)](\xi)$ *is left slice monogenic in* s.

Proof. In the following proof we always work with $s = s_0 \in \mathbb{R}$ since we have

$$S_L^{-1}(s,x) = (s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-1} = (s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-1}(s-\bar{x}) = S_R^{-1}(s,x).$$

The extension from $s = s_0$ to $s = s_0 + \underline{s}$ is immediate. So in our computations, we set

$$S^{-1}(s,x) := S_L^{-1}(s,x) = S_R^{-1}(s,x) = \frac{s - x_0 + \underline{x}}{(s-x_0)^2 + |\underline{x}|^2}, \quad s \in \mathbb{R}.$$

We put $x = x_0 + \underline{x}$ and we recall the identification of the paravectors with $(x_0, ..., x_n)$. Since the function $S^{-1}(s,x)$ is not in $L^1(\mathbb{R}^{n+1})$ we have to perform the computations in the distributional sense. Firstly, we consider the following function

$$f_{\underline{x}}(x_0) := \frac{s - x_0}{(s-x_0)^2 + |\underline{x}|^2}.$$

Let $\varphi \in \mathcal{S}(\mathbb{R}^{n+1})$. Then we have

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} f_{\underline{x}}(x_0)\overline{\mathrm{F}(\varphi)(x)}dx &= \int_{\mathbb{R}^{n+1}} f_{\underline{x}}(x_0)\overline{\mathrm{F}_0\left(\mathrm{F}_n\varphi_{\underline{x}}\right)(x_0)}dx_0 d\underline{x} \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}} \left(\mathrm{F}_0 f_{\underline{x}}\right)(\xi_0)\overline{\left(\mathrm{F}_n\varphi_{\underline{x}}\right)(\xi_0)}d\xi_0 d\underline{x},\end{aligned} \tag{22}$$

where $d\underline{x} = dx_1...dx_n$, F_0 is the Fourier transform with respect to the variable x_0 and F_n is the Fourier transform with respect to the other variables. Now, we compute $\mathrm{F}_0 f_{\underline{x}}(\xi_0)$. First of all we make the following change of variables $s + y = x_0$, thus by basic properties of the Fourier transform we have

$$\begin{aligned}\mathrm{F}_0 f_{\underline{x}}(\xi_0) &= -\mathrm{F}_y[y(y^2 + |\underline{x}|^2)^{-1}](\xi_0)e^{-is\xi_0} = -i\frac{d}{d\xi_0}\mathrm{F}_y\left(\frac{1}{|\underline{x}|^2 + y^2}\right)(\xi_0)e^{-is\xi_0} \\ &= -i\frac{\pi}{|\underline{x}|}\left(\frac{d}{d\xi_0}e^{-|\underline{x}||\xi_0|}\right)e^{-is\xi_0} = i\frac{\pi\xi_0}{|\xi_0|}e^{-|\underline{x}||\xi_0|}e^{-is\xi_0}.\end{aligned} \tag{23}$$

Since $\varphi_{\underline{x}}(x_0) = \varphi(x)$ and by Fubini's theorem we have

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} f_{\underline{x}}(x_0)\overline{\mathrm{F}(\varphi)(x)}dx &= \int_{\mathbb{R}}\int_{\mathbb{R}^n} \mathrm{F}_n\left(\mathrm{F}_0 f_{\underline{x}}\right)(\xi_0)\overline{\varphi(\xi)}d\underline{\xi}d\xi_0 \\ &= \int_{\mathbb{R}^{n+1}} \mathrm{F}_n\left(\mathrm{F}_0 f_{\underline{x}}\right)(\xi_0)\overline{\varphi(\xi)}d\xi.\end{aligned}$$

We finish this first part by computing $\mathrm{F}_n\left(\mathrm{F}_0 f_{\underline{x}}\right)(\xi_0)$. By Theorem 3.5 with $r = |\underline{x}|$ we have

$$\mathrm{F}_n\left(\mathrm{F}_0 f_{\underline{x}}\right)(\xi_0) = (2\pi)^{\frac{n}{2}}|\underline{\xi}|^{-\frac{n-2}{2}} i\frac{\pi\xi_0}{|\xi_0|}e^{-is\xi_0}\int_0^\infty J_{\frac{n-2}{2}}(|\underline{\xi}|r)r^{\frac{n}{2}}e^{-r|\xi_0|}dr.$$

Now, we make another change of variables $t = |\underline{\xi}|r$.

$$\mathrm{F}_n\left(\mathrm{F}_0 f_{\underline{x}}\right)(\xi_0) = (2\pi)^{\frac{n}{2}} |\underline{\xi}|^{-n} i \frac{\pi \xi_0}{|\xi_0|} e^{-is\xi_0} \int_0^\infty J_{\frac{n-2}{2}}(t) t^{\frac{n}{2}} e^{-\frac{t|\xi_0|}{|\underline{\xi}|}} dt.$$

From [32, formula 6.623(2)] we know that

$$\int_0^\infty e^{-at} t^{\nu+1} J_\nu(bt) dt = \frac{2a(2b)^\nu \Gamma\left(\nu + \frac{3}{2}\right)}{\sqrt{\pi}(a^2 + b^2)^{\nu + \frac{3}{2}}}, \qquad \nu > -1, \quad a > 0, \quad b > 0.$$

In our case $a := \frac{|\xi_0|}{|\underline{\xi}|}$, $\nu := \frac{n}{2} - 1$, $b := 1$. Since $n \geq 2$ all conditions on the parameters are satisfied. Thus, we have

$$\begin{aligned}
\mathrm{F}_n\left(\mathrm{F}_0 f_{\underline{x}}\right)(\xi_0) &= \frac{i}{2}(2\pi)^{\frac{n}{2}+1} \frac{\xi_0}{|\xi_0|} |\underline{\xi}|^{-n} e^{-is\xi_0} 2 \frac{|\xi_0|}{|\underline{\xi}|} 2^{\frac{n}{2}-1} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\left(\frac{\xi_0^2}{|\underline{\xi}|^2} + 1\right)^{\frac{n+1}{2}}} \\
&= \frac{i 2^n \pi^{\frac{n+1}{2}} \xi_0 |\underline{\xi}|^{-n-1} e^{-is\xi_0} |\underline{\xi}|^{n+1} \Gamma\left(\frac{n+1}{2}\right)}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}} \\
&= \frac{i 2^n \pi^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \xi_0 e^{-is\xi_0}}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}}.
\end{aligned}$$

Hence

$$\int_{\mathbb{R}^{n+1}} f_{\underline{x}}(x_0) \overline{\mathrm{F}(\varphi)(x)} dx = c_n \int_{\mathbb{R}^{n+1}} \frac{\xi_0}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}} e^{-is\xi_0} \overline{\varphi}(\xi) d\xi, \tag{24}$$

where $c_n := i 2^n \pi^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right)$. Now, we compute the Fourier transform of

$$h_{\underline{x}}(x_0) := \frac{\underline{x}}{(s - x_0)^2 + |\underline{x}|^2} = \sum_{j=1}^n e_j x_j u_{\underline{x}}(x_0),$$

where we have set

$$u_{\underline{x}}(x_0) := \frac{1}{(s - x_0)^2 + |\underline{x}|^2}.$$

Let $\varphi \in \mathcal{S}(\mathbb{R}^{n+1})$

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}} h_{\underline{x}}(x_0) \overline{\mathrm{F}(\varphi)(x)} dx &= \int_{\mathbb{R}^{n+1}} \sum_{j=1}^n e_j x_j u_{\underline{x}}(x_0) \overline{\mathrm{F}_0\left(\mathrm{F}_n(\varphi_{\underline{x}})\right)(x_0)} dx_0 d\underline{x} \qquad (25) \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}} \sum_{j=1}^n e_j x_j \mathrm{F}_0(u_{\underline{x}})(\xi_0) \overline{\mathrm{F}_n \varphi_{\underline{x}}(\xi_0)} d\xi_0 d\underline{x}. \qquad (26)
\end{aligned}$$

Now, we compute $\mathrm{F}_0(u_{\underline{x}})(\xi_0)$ using the following change of variable $s + y = x_0$

$$\mathrm{F}_0(u_{\underline{x}})(\xi_0) = \mathrm{F}_y\left(\frac{1}{y^2 + |\underline{x}|^2}\right)(\xi_0) e^{-is\xi_0} = \frac{\pi}{|\underline{x}|} e^{-|\underline{x}||\xi_0|} e^{-is\xi_0}.$$

By Fubini's theorem and the fact that $\varphi_{\underline{x}}(x_0) = \varphi(x)$ we have

$$\int\limits_{\mathbb{R}^{n+1}} h_{\underline{x}}(x_0)\overline{\mathrm{F}(\varphi)(x)}dx = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^n}\sum_{j=1}^{n} e_j \mathrm{F}_n\left(x_j \mathrm{F}_0(u_{\underline{x}})\right)(\xi_0)\overline{\varphi(\xi)}d\underline{\xi}d\xi_0.$$

From the following basic property of the Fourier transform

$$\mathrm{F}[xf(x)](\xi) = i\frac{d}{d\xi}(\mathrm{F}f(x))(\xi), \quad x,\xi \in \mathbb{R}$$

we get

$$\int\limits_{\mathbb{R}^{n+1}} h_{\underline{x}}(x_0)\overline{\mathrm{F}(\varphi)(x)}dx = i\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^n}\sum_{j=1}^{n} e_j \frac{\partial}{\partial \xi_j}\mathrm{F}_n\left(\mathrm{F}_0(u_{\underline{x}})\right)(\xi_0)\overline{\varphi(\xi)}d\underline{\xi}d\xi_0.$$

We complete the proof of this theorem by computing $\mathrm{F}_n\left(\mathrm{F}_0(u_{\underline{x}})\right)(\xi_0)$. By Theorem 3.5, with $r = |\underline{x}|$, we get

$$\mathrm{F}_n\left(\mathrm{F}_0(u_{\underline{x}})\right)(\xi_0) = (2\pi)^{\frac{n}{2}}\pi e^{-is\xi_0}|\underline{\xi}|^{-\frac{n-2}{2}}\int\limits_0^{\infty} J_{\frac{n-2}{2}}(|\underline{\xi}|r)r^{\frac{n}{2}-1}e^{-r|\xi_0|}dr.$$

Now we put $t = r|\underline{\xi}|$

$$\mathrm{F}_n(\mathrm{F}_0(u_{\underline{x}}))(\xi_0) = \frac{1}{2}(2\pi)^{\frac{n}{2}+1}e^{-is\xi_0}|\underline{\xi}|^{1-n}\int\limits_0^{\infty} J_{\frac{n-2}{2}}(t)t^{\frac{n}{2}-1}e^{-t\frac{|\xi_0|}{|\underline{\xi}|}}dt.$$

From [32, formula 6.623 (1)] we know that

$$\int\limits_0^{\infty} e^{-at}t^{\nu}J_{\nu}(bt)dt = \frac{(2b)^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}(a^2+b^2)^{\nu+\frac{1}{2}}}, \qquad \nu > -\frac{1}{2}, \quad a>0, \quad b>0.$$

Thus by putting $b := 1$, $\nu := \frac{n}{2}-1$ and $a := \frac{|\xi_0|}{|\underline{\xi}|}$ we obtain

$$\begin{aligned}
\mathrm{F}_n\left(\mathrm{F}_0(u_{\underline{x}})\right)(\xi_0) &= \frac{1}{2}(2\pi)^{\frac{n}{2}+1}e^{-is\xi_0}\left(\frac{|\underline{\xi}|^{1-n}2^{\frac{n}{2}-1}\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\left(\frac{\xi_0^2}{|\underline{\xi}|^2}+1\right)^{\frac{n-1}{2}}}\right)\\
&= 2^{n-1}\pi^{\frac{n+1}{2}}e^{-is\xi_0}\Gamma\left(\frac{n-1}{2}\right)\left(\frac{|\underline{\xi}|^{1-n}|\underline{\xi}|^{n-1}}{(\xi_0^2+|\underline{\xi}|^2)^{\frac{n-1}{2}}}\right)\\
&= 2^{n-1}\pi^{\frac{n+1}{2}}e^{-is\xi_0}\Gamma\left(\frac{n-1}{2}\right)\left(\frac{1}{(\xi_0^2+|\underline{\xi}|^2)^{\frac{n-1}{2}}}\right).
\end{aligned}$$

We compute the derivative

$$\frac{\partial}{\partial \xi_j}\left(\frac{1}{(\xi_0^2+|\underline{\xi}|^2)^{\frac{n-1}{2}}}\right) = -\frac{\left(\frac{n-1}{2}\right)(\xi^2+|\underline{\xi}|^2)^{\frac{n-3}{2}}2\xi_j}{(\xi_0^2+|\underline{\xi}|^2)^{n-1}} = -\frac{2\xi_j\left(\frac{n-1}{2}\right)}{(\xi_0^2+|\underline{\xi}|^2)^{\frac{n+1}{2}}}. \tag{27}$$

Now, by using the following property of the Gamma function $\Gamma(x+1) = x\Gamma(x)$, for $x > 0$, we obtain

$$\sum_{j=1}^{n} e_j \frac{\partial}{\partial \xi_j} \mathrm{F}_n \left(\mathrm{F}_0(u_{\underline{x}})\right)(\xi_0) = -\frac{2^{n-1}\pi^{\frac{n+1}{2}} e^{-is\xi_0} \left(\frac{n-1}{2}\right) \Gamma\left(\frac{n-1}{2}\right) 2 \sum_{j=1}^{n} e_j \xi_j}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}}$$
$$= -\frac{2^n \pi^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right)}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}} \underline{\xi} e^{-is\xi_0}.$$

Hence we get

$$\int_{\mathbb{R}^{n+1}} h_{\underline{x}}(x_0) \overline{\mathrm{F}(\varphi)(x)} dx = -c_n \int_{\mathbb{R}^{n+1}} \frac{\underline{\xi}}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}} e^{-is\xi_0} \overline{\varphi(\xi)} d\xi, \tag{28}$$

where $c_n := i2^n \pi^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right)$. Finally from (24) and (28) we get

$$\int_{\mathbb{R}^{n+1}} S^{-1}(s,x) \overline{\mathrm{F}(\varphi)(x)} dx = \int_{\mathbb{R}^{n+1}} f_{\underline{x}}(x_0) \overline{\mathrm{F}(\varphi)(x)} dx - \int_{\mathbb{R}^{n+1}} h_{\underline{x}}(x_0) \overline{\mathrm{F}(\varphi)(x)} dx$$
$$= c_n \int_{\mathbb{R}^{n+1}} \frac{\bar{\xi}}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}} e^{-is\xi_0} \overline{\varphi(\xi)} d\xi.$$

This proves (20) and (21), respectively.

We are now in the position to observe that the term $e^{-is\xi_0}$, for $s = s_0$ extends to the entire intrinsic slice monogenic function $e^{-i(s_0+\underline{s})\xi_0}$. So the function

$$s_0 \mapsto c_n \frac{\bar{\xi}}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}} e^{-is_0\xi_0}$$

has the right slice monogenic extension in $s \in \mathbb{R}^{n+1}$

$$\mathrm{F}[S_L^{-1}(s,\cdot)](\xi) = c_n \ \frac{\bar{\xi}}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}} \ e^{-is\xi_0}$$

while the left slice monogenic extension in $s \in \mathbb{R}^{n+1}$ is

$$\mathrm{F}[S_R^{-1}(s,\cdot)](\xi) = c_n \ e^{-is\xi_0} \ \frac{\bar{\xi}}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}}$$

and this concludes the proof. □

4. The Fourier transform of the F_n-kernels

Thanks to Theorem 2.15 the F_n-kernels are meaningful also with n odd where we interpret the fractional powers of paravectors are slice monogenic functions.

As we have shown when n be an odd number and x, $s \in \mathbb{R}^{n+1}$, for $s \notin [x]$, the relations

$$\Delta^{\frac{n-1}{2}} S_L^{-1}(s,x) = \gamma_n (s - \bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}},$$

and

$$\Delta^{\frac{n-1}{2}} S_R^{-1}(s,x) = \gamma_n (s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}} (s - \bar{x}),$$

where γ_n are given by (1) are obtained by a long, but direct computation. We now let n be any natural number and when n is even we interpret the terms $(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}}$ as fractional power for x, $s \in \mathbb{R}^{n+1}$. We define, for $s \notin [x]$, the left $\mathcal{F}_n^L$-kernel as

$$\mathcal{F}_n^L(s,x) := \gamma_n(s - \bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}},$$

and the right $\mathcal{F}_n^R$-kernel as

$$\mathcal{F}_n^R(s,x) := \gamma_n(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}}(s - \bar{x}),$$

where γ_n, are given by (1), are now interpreted in terms of the Euler's Gamma function

$$\gamma_n := (-1)^{\frac{n-1}{2}} 2^{n-1} \left[\Gamma\left(\frac{n+1}{2}\right)\right]^2.$$

We will compute the Fourier transforms of $\mathcal{F}_L(s,x)$ and $\mathcal{F}_R(s,x)$ with respect to x.

Remark 4.1. Observe that when $s = s_0 \in \mathbb{R}$, for $s \notin [x]$, then we have

$$\begin{aligned}\mathcal{F}_n^L(s_0,x) &= \gamma_n(s_0 - \bar{x})(s_0^2 - 2\mathrm{Re}(x)s_0 + |x|^2)^{-\frac{n+1}{2}} \\ &= \gamma_n(s_0^2 - 2\mathrm{Re}(x)s_0 + |x|^2)^{-\frac{n+1}{2}}(s_0 - \bar{x}) \\ &= \mathcal{F}_n^R(s_0,x).\end{aligned}$$

So for simplicity in the following when $s = s_0 \in \mathbb{R}$ we use the notation

$$\mathcal{F}_n(s_0,x) := \mathcal{F}_n^L(s_0,x) = \mathcal{F}_n^R(s_0,x).$$

Theorem 4.2. *Let us assume $x \in \mathbb{R}^{n+1}$ and s a real number. The Fourier transform of $\mathcal{F}_n(s,x)$ with respect to x is*

$$\widehat{\mathcal{F}_n}(s,\xi) = k_n \frac{\bar{\xi}}{\xi_0^2 + |\underline{\xi}|^2} e^{-is\xi_0},$$

where

$$k_n := i(-1)^{\frac{n-1}{2}} 2^n \pi^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right).$$

Moreover, if $s = s_0 + \underline{s} \in \mathbb{R}^{n+1}$ is a paravector the term $e^{-is_0\xi_0}$ extends to the intrinsic entire slice monogenic function $e^{-is\xi_0}$ and we have the Fourier transforms of the kernels $\mathcal{F}_n^L$ and $\mathcal{F}_n^R$:

$$\mathrm{F}[\mathcal{F}_n^L(s,\cdot)](\xi) = k_n \ \frac{\bar{\xi}}{\xi_0^2 + |\underline{\xi}|^2} \ e^{-is\xi_0}, \qquad \xi_0 + \underline{\xi} \neq 0$$

and

$$\mathrm{F}[\mathcal{F}_n^R(s,\cdot)](\xi) = k_n \ e^{-is\xi_0} \ \frac{\bar{\xi}}{\xi_0^2 + |\underline{\xi}|^2}, \qquad \xi_0 + \underline{\xi} \neq 0.$$

The extension $\mathrm{F}[\mathcal{F}_n^L(s,\cdot)](\xi)$ is right slice monogenic in s, while $\mathrm{F}[\mathcal{F}_n^R(s,\cdot)](\xi)$ is right slice monogenic in s.

Proof. We observe that the Fourier transform of the kernel $\mathcal{F}_n(s,\cdot)$ is meaningful and from similar computations at the beginning of Theorem 3.7 we obtain

$$\begin{aligned}\widehat{\mathcal{F}_n}(s,\xi) &= \gamma_n s \int_{\mathbb{R}} e^{-ix_0\xi_0}(2\pi)^{\frac{n}{2}}|\underline{\xi}|^{-\frac{n-2}{2}} \int_0^\infty (s^2-2x_0s+x_0^2+r^2)^{-\frac{n+1}{2}} J_{\frac{n-2}{2}}(|\underline{\xi}|r)r^{\frac{n}{2}}\,dr\,dx_0 \\ &- \gamma_n \int_{\mathbb{R}} x_0 e^{-ix_0\xi_0}(2\pi)^{\frac{n}{2}}|\underline{\xi}|^{-\frac{n-2}{2}} \int_0^\infty (s^2-2x_0s+x_0^2+r^2)^{-\frac{n+1}{2}} J_{\frac{n-2}{2}}(|\underline{\xi}|r)r^{\frac{n}{2}}\,dr\,dx_0 \\ &+ \gamma_n i\sum_{j=1}^n e_j \int_{\mathbb{R}} e^{-ix_0\xi_0}\frac{\partial}{\partial \xi_j}\left((2\pi)^{\frac{n}{2}}|\underline{\xi}|^{-\frac{n-2}{2}} \int_0^\infty (s^2-2x_0s+x_0^2+r^2)^{-\frac{n+1}{2}} J_{\frac{n-2}{2}}(|\underline{\xi}|r)r^{\frac{n}{2}}\,dr\,dx_0\right) \\ &:= \gamma_n\left(\widehat{\mathcal{F}_{n,1}}(s,\xi)+\widehat{\mathcal{F}_{n,2}}(s,\xi)+\widehat{\mathcal{F}_{n,3}}(s,\xi)\right).\end{aligned}$$

Now, we focus on the first two members

$$\begin{aligned}&\widehat{\mathcal{F}_{n,1}}(s,\xi)+\widehat{\mathcal{F}_{n,2}}(s,\xi) \\ &\quad = s\int_{\mathbb{R}} e^{-ix_0\xi_0}(2\pi)^{\frac{n}{2}}|\underline{\xi}|^{-\frac{n-2}{2}} \int_0^\infty (s^2-2x_0s+x_0^2+r^2)^{-\frac{n+1}{2}} J_{\frac{n-2}{2}}(|\underline{\xi}|r)r^{\frac{n}{2}}\,dr\,dx_0 \\ &\quad - \int_{\mathbb{R}} x_0 e^{-ix_0\xi_0}(2\pi)^{\frac{n}{2}}|\underline{\xi}|^{-\frac{n-2}{2}} \int_0^\infty (s^2-2x_0s+x_0^2+r^2)^{-\frac{n+1}{2}} J_{\frac{n-2}{2}}(|\underline{\xi}|r)r^{\frac{n}{2}}\,dr\,dx_0 \\ &\quad = (2\pi)^{\frac{n}{2}}|\underline{\xi}|^{-\frac{n-2}{2}}\int_{\mathbb{R}}(s-x_0)e^{-ix_0\xi_0}\int_0^\infty [(s-x_0)^2+r^2]^{-\frac{n+1}{2}} J_{\frac{n-2}{2}}(|\underline{\xi}|r)r^{\frac{n}{2}}\,dr\,dx_0.\end{aligned}$$

Firstly, we solve the integral in the variable r. From [32, formula 6.565 (3)] we know that

$$\int_0^\infty x^{\nu+1}(x^2+a^2)^{-\nu-\frac{3}{2}}J_\nu(bx)dx = \frac{b^\nu\sqrt{\pi}}{2^{\nu+1}|a|e^{|a|b}\Gamma\left(\nu+\frac{3}{2}\right)} \qquad b>0, \quad \nu>-1. \tag{29}$$

In our case $b:=|\underline{\xi}|$, $\nu=\frac{n}{2}-1$ and $a=s-x_0$. Then

$$\begin{aligned}\widehat{\mathcal{F}_{n,1}}(s,\xi)+\widehat{\mathcal{F}_{n,2}}(s,\xi) &= \frac{(2\pi)^{\frac{n}{2}}2^{-\frac{n}{2}}\sqrt{\pi}|\underline{\xi}|^{-\frac{n}{2}+1}|\underline{\xi}|^{\frac{n}{2}-1}}{\Gamma\left(\frac{n+1}{2}\right)}\int_{\mathbb{R}} e^{-ix_0\xi_0}\frac{(s-x_0)}{|s-x_0|}e^{-|\underline{\xi}||s-x_0|}dx_0 \\ &= \frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}\int_{\mathbb{R}} e^{-ix_0\xi_0}\frac{(s-x_0)}{|s-x_0|}e^{-|\underline{\xi}||s-x_0|}dx_0.\end{aligned}$$

Now, we put $s+y=x_0$. Thus we have

$$\widehat{\mathcal{F}_{n,1}}(s,\xi)+\widehat{\mathcal{F}_{n,2}}(s,\xi) = -\frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}e^{-is\xi_0}\int_{\mathbb{R}} e^{-iy\xi_0}\frac{y}{|y|}e^{-|\underline{\xi}||y|}dy.$$

From [39, formula 3.2 pag 11] we know that

$$\int_0^\infty \cos(xu)\frac{e^{-ax}}{x}dx = -\frac{\log(a^2+u^2)}{2}.$$

Thus by the Euler's formula we get

$$\mathrm{F}\left(\frac{1}{|y|}e^{-|\underline{\xi}||y|}\right)(\xi_0) = 2\int_0^\infty \cos(y\xi_0)\frac{1}{y}e^{-|\underline{\xi}|y}dy = -\log(\xi_0^2+|\underline{\xi}|^2). \tag{30}$$

Using basic properties of the Fourier transform we obtain

$$\begin{aligned}\int_{\mathbb{R}} e^{-iy\xi_0}\frac{y}{|y|}e^{-|\underline{\xi}||y|}dy &= \mathrm{F}\left(\frac{y}{|y|}e^{-|\underline{\xi}||y|}\right)(\xi_0) = i\frac{d}{d\xi_0}\mathrm{F}\left(\frac{1}{|y|}e^{-|\underline{\xi}||y|}\right)(\xi_0)\\ &= -i\frac{d}{d\xi_0}\left(\log(\xi_0^2+|\underline{\xi}|^2)\right) = -\frac{2i\xi_0}{\xi_0^2+|\underline{\xi}|^2}.\end{aligned}$$

Therefore

$$\widehat{\mathcal{F}_{n,1}}(s,\xi)+\widehat{\mathcal{F}_{n,2}}(s,\xi) = \frac{2i\pi^{\frac{n+1}{2}}e^{-is\xi_0}\xi_0}{\Gamma\left(\frac{n+1}{2}\right)(\xi_0^2+|\underline{\xi}|^2)}.$$

Finally, we multiply by $\gamma_n := (-1)^{\frac{n-1}{2}}2^{n-1}\left[\Gamma\left(\frac{n+1}{2}\right)\right]^2$

$$\gamma_n(\widehat{\mathcal{F}_{n,1}}(s,\xi)+\widehat{\mathcal{F}_{n,2}}(s,\xi)) = \frac{i(-1)^{\frac{n-1}{2}}2^n\pi^{\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right)}{\xi_0^2+|\underline{\xi}|^2}\xi_0 e^{-is\xi_0}. \tag{31}$$

Now we compute $\widehat{\mathcal{F}_{n,3}}(s,\xi)$.

$$\widehat{\mathcal{F}_{n,3}}(s,\xi) = i(2\pi)^{\frac{n}{2}}\sum_{j=1}^n e_j\frac{\partial}{\partial\xi_j}\left(|\underline{\xi}|^{-\frac{n-2}{2}}\int_{\mathbb{R}} e^{-ix_0\xi_0}\int_0^\infty[(s-x_0)^2+r^2]^{-\frac{n+1}{2}}J_{\frac{n-2}{2}}(|\underline{\xi}|r)r^{\frac{n}{2}}drdx_0\right).$$

As before we compute the integral in the variable r using (29) with $b := |\underline{\xi}|$, $\nu = \frac{n}{2}-1$ and $a = s-x_0$. Thus we have

$$\begin{aligned}\widehat{\mathcal{F}_{n,3}}(s,\xi) &= \frac{i(2\pi)^{\frac{n}{2}}2^{-\frac{n}{2}}\sqrt{\pi}}{\Gamma\left(\frac{n+1}{2}\right)}\sum_{j=1}^n e_j\frac{\partial}{\partial\xi_j}\left(|\underline{\xi}|^{-\frac{n}{2}+1}|\underline{\xi}|^{\frac{n}{2}-1}\int_{\mathbb{R}} e^{-ix_0\xi_0}|s-x_0|^{-1}e^{-|\underline{\xi}||s-x_0|}dx_0\right)\\ &= \frac{i\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}\sum_{j=1}^n e_j\frac{\partial}{\partial\xi_j}\left(\int_{\mathbb{R}} e^{-ix_0\xi_0}|s-x_0|^{-1}e^{-|\underline{\xi}||s-x_0|}dx_0\right).\end{aligned}$$

We put $s+y=x_0$.

$$\widehat{\mathcal{F}_{n,3}}(s,\xi) = \frac{i\pi^{\frac{n+1}{2}}e^{-is\xi_0}}{\Gamma\left(\frac{n+1}{2}\right)}\sum_{j=1}^n e_j\frac{\partial}{\partial\xi_j}\left(\int_{\mathbb{R}} e^{-iy\xi_0}|y|^{-1}e^{-|\underline{\xi}||y|}dy\right).$$

From (30) we know that

$$\int_{\mathbb{R}} e^{-iy\xi_0}|y|^{-1}e^{-|\underline{\xi}||y|}dy = -\log(\xi_0^2 + |\underline{\xi}|^2).$$

Therefore

$$\begin{aligned}
\widehat{\mathcal{F}_{n,3}}(s,\xi) &= -\frac{i\pi^{\frac{n+1}{2}}e^{-is\xi_0}}{\Gamma\left(\frac{n+1}{2}\right)}\sum_{j=1}^{n} e_j\frac{\partial}{\partial\xi_j}\left(\log(|\underline{\xi}|^2+\xi_0^2)\right)\\
&= -\frac{2i\pi^{\frac{n+1}{2}}e^{-is\xi_0}}{\Gamma\left(\frac{n+1}{2}\right)(\xi_0^2+|\underline{\xi}|^2)}\sum_{j=1}^{n} e_j\xi_j\\
&= -\frac{2i\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)(\xi_0^2+|\underline{\xi}|^2)}\,\underline{\xi}e^{-is\xi_0}.
\end{aligned}$$

Finally, multiplying by $\gamma_n := (-1)^{\frac{n-1}{2}}2^{n-1}\left[\Gamma\left(\frac{n+1}{2}\right)\right]^2$ we have

$$\gamma_n\widehat{\mathcal{F}_{n,3}}(s,\xi) = -\frac{i(-1)^{\frac{n-1}{2}}2^n\pi^{\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right)}{\xi_0^2+|\underline{\xi}|^2}\,\underline{\xi}e^{-is\xi_0}. \tag{32}$$

Putting together (31) and (32) we get

$$\widehat{\mathcal{F}_n}(s,\xi) = k_n\frac{\bar{\xi}}{\xi^2+|\underline{\xi}|^2}e^{-is\xi_0},$$

where $k_n := i(-1)^{\frac{n-1}{2}}2^n\pi^{\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right)$. Finally, the slice monogenic extensions are obtained reasoning as in the case of the Cauchy kernel. □

5. The relation of the kernels S^{-1} and F_n via the Fourier transform

Before to prove a fundamental result we recall that when n is even the operator $\Delta^{\frac{n-1}{2}}$ is defined by the Fourier multipliers

$$\Delta^{\frac{n-1}{2}}f(x) = \mathcal{R}[(i|\xi|)^{n-1}\mathrm{F}(f(x))(\xi)](x), \tag{33}$$

where F and $\mathcal{R}$, are respectively, the Fourier and the inverse Fourier transformations, given, respectively in Definition 3.1 and Definition 3.2.

Theorem 5.1. *For $x \in \mathbb{R}^{n+1}$ and $s \in \mathbb{R}$ we have that*

$$\Delta^{\frac{n-1}{2}}S^{-1}(s,x) = \gamma_n(s-\bar{x})(s^2-2x_0s+|x|^2)^{-\frac{n+1}{2}} \tag{34}$$

Proof. If n is odd, this can be proved through pointwise differential computation, see [10, Thm. 3.3]. While for the case n even the result will be showed for any $\varphi \in \mathcal{S}(\mathbb{R}^{n+1})$. Firstly we prove the equality for $s \in \mathbb{R}$. The formula (19) and Theorem 3.3 imply that we can pass the factional Laplacian to the test function, so we have

$$\int_{\mathbb{R}^{n+1}} \Delta^{\frac{n-1}{2}}S^{-1}(s,x)\overline{\varphi(x)}dx = \int_{\mathbb{R}^{n+1}} S^{-1}(s,x)\overline{\Delta^{\frac{n-1}{2}}\varphi(x)}dx.$$

Using another time Theorem 3.3 we get

$$\int_{\mathbb{R}^{n+1}} \Delta^{\frac{n-1}{2}} S^{-1}(s,x)\overline{\varphi(x)}dx = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} \mathrm{F}\left(S^{-1}(s,.)\right)(\xi)\overline{\mathrm{F}\left(\Delta^{\frac{n-1}{2}}\varphi(x)\right)(\xi)}d\xi$$

From Theorem 3.7 and Theorem 4.2 we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}} \Delta^{\frac{n-1}{2}} S^{-1}(s,x)\overline{\varphi(x)}dx \\
&= \frac{1}{(2\pi)^{n+1}} i(-1)^{\frac{n-1}{2}} 2^n \pi^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \int_{\mathbb{R}^{n+1}} \frac{\bar{\xi} e^{-is\xi_0}}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}} |\xi|^{n-1} \overline{\hat{\varphi}(\xi)} d\xi \\
&= \frac{1}{(2\pi)^{n+1}} i(-1)^{\frac{n-1}{2}} 2^n \pi^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \int_{\mathbb{R}^{n+1}} \frac{\bar{\xi} e^{-is\xi_0}}{\xi_0^2 + |\underline{\xi}|^2} \overline{\hat{\varphi}(\xi)} d\xi \\
&= \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} \widehat{\mathcal{F}_n}(s,\xi)\overline{\hat{\varphi}(\xi)}d\xi.
\end{aligned}$$

Finally by applying another time the Theorem 3.3 we get

$$\int_{\mathbb{R}^{n+1}} \Delta^{\frac{n-1}{2}} S^{-1}(s,x)\overline{\varphi(x)}dx = \int_{\mathbb{R}^{n+1}} \mathcal{F}_n(s,x)\overline{\varphi(x)}dx. \quad \square$$

Corollary 5.2. *The relation (34) extends to $s \in \mathbb{R}^{n+1}$, considering the left and the right slice monogenic extensions.*

Proof. The extension of the equation (34) to $s \in \mathbb{R}^{n+1}$ follows from the Identity principle, because the function $e^{-is\xi_0}$ is trivially intrinsic slice monogenic. $\square$

We conclude with some remarks.

Remark 5.3. Both classes of hyperholomorphic functions have a Cauchy formula that can be used to define functions of quaternionic operators or of n-tuples of operators that do not necessarily commute.

Remark 5.4. The Cauchy formula of slice hyperholomorphic functions generates the S-functional calculus for quaternionic linear operators or for n-tuples of not necessarily commuting operators, this calculus is based on the notion of S-spectrum. The spectral theorem for quaternionic operators is also based on the S-spectrum. The S-spectrum is used in quaternionic and in Clifford operators theory, see [1,5,21,14].

Remark 5.5. The Cauchy formula of monogenic functions generates the monogenic functional calculus that is based on the monogenic spectrum. For monogenic operator theory and related topics see [35,37,38,45,44] and the references therein. The F-functional calculus is a bridge between the spectral theory on the S-spectrum and the monogenic spectral theory and it is studied in [4,7,10].

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