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ADAPTIVE FOURIER DECOMPOSITION OF SLICE REGULAR FUNCTIONS

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Abstract. In the slice Hardy space over the unit ball of quaternions, we introduce the slice hyperbolic backward shift operator $\mathcal{S}_a$ with the decomposition process

$$f = e_a\langle f, e_a\rangle + B_a * \mathcal{S}_a f,$$

where e_a denotes the slice normalized Szegö kernel and B_a the slice Blaschke factor. Iterating the above decomposition process, a corresponding maximal selection principle gives rise to the slice adaptive Fourier decomposition. This leads to a adaptive slice Takenaka-Malmquist orthonormal system.

1. Introduction

The purpose of this article is to introduce the quaternionic slice hyperbolic backward shift operators in the slice Hardy space $H^2(\mathbb{B})$ of the unit ball of quaternions. At each of the process we decompose a function $f \in H^2(\mathbb{B})$ into an orthogonal sum of two functions of which one is in the subspace generated by a slice normalized Szegö kernel e_a and the other is in its orthogonal complement expressed by the Blaschke factor and the backward shift operator $\mathcal{S}_a$. By iterating the process we decompose a given function $f \in H^2(\mathbb{B})$ into a slice Takenaka-Malmquist orthonormal system.

Our motivation comes from adaptive Fourier decompositions (AFDs) for the holomorphic Hardy spaces of the unit disc and of a half of the complex plane [18]. The type of decompositions provides approximations by suitable linear combinations of parameterized reproducing kernels in the respective Hardy spaces. The decompositions may result in merely an orthonormal system: It may not be a basis but adaptive to the given signal. It, however, achieves fast decomposition through extracting out the greatest energy portion from the orthogonal remainder at each iterative step. Together with the process there arise the Takenaka-Malmquist (TM) orthonormal systems ([21]). If instead of the maximal selection principle one uses a set of parameters satisfying the hyperbolic non-separability rule $\sum_{k=1}^{\infty}(1-|a_k|)=\infty$, then the decomposition process results in a TM basis. The Fourier basis $\{z^n\}_{n=0}^{\infty}$ is a particular case corresponding to all the parameters a_n being identically zero. Since Takenaka-Malmquist system consists of rational functions, the study falls into the scope of rational approximation ([21]). The Takenaka-Malmquist bases can be thought of hyperbolic versions of the Fourier

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system. The adaptive Fourier decomposition allows repeating selections of the parameters that offers attainability of the best matching pursuit at each step of parameter selection. The adaptive Fourier decomposition methodology facilitates efficient and thus useful sparse representations. It is, in particular, effective when in the underlying Hilbert space there does exist an approximation theory. Adaptive Fourier decomposition with different contexts has undergone substantial developments ([1, 2, 15, 17]) with ample applications, such as digital signal processing [16], image processing [14], and system identification [22].

In this paper we establish the slice adaptive Fourier decomposition of slice regular functions over quaternions. It is a higher dimensional extension of the subject in the one complex variable case. The slice regular function theory of quaternions was initiated by Gentili and Struppa [11], and soon developed by a number of researchers. See, for instance [5, 3, 4, 12, 10]. This theory generalizes the holomorphic theory of one complex variable to quaternions. It is remarkable that the slice regular function theory relies on the slice structure of quaternions, which is different from the monogenic function theory over Clifford algebras. The great difference among the slice analysis, Clifford analysis, several complex variables is due to the canonical topology in slice analysis distinct to the Euclidean topology; see [8]. The slice theory shows vigorous vitality in non-commutative Clifford algebra [5], and non-associative real alternative algebras [12], [20]. It also has significant applications in differential geometry [9], geometric function theory [19], and operator theory [6].

To state our main results we provide some preliminaries of the slice Hardy space which are mostly adopted from [3, 4]. Here we introduce an equivalent definition. The slice Hardy space $H^2(\mathbb{B})$ over the unit ball of the quaternions consists of the slice regular functions $f : \mathbb{B} \to \mathbb{H}$ satisfying

$$\|f\| := \Big(\frac{1}{4\pi^2}\int_{\partial\mathbb{B}} \frac{1}{|Im(q)|^2}|f(q)|^2 d\sigma\Big)^{1/2} < \infty,$$

where $d\sigma$ is the Lebesgue surface measure on $\partial\mathbb{B}$. The polarization identity of this norm provides $H^2(\mathbb{B})$ with an inner product so that it becomes a quaternionic Hilbert space with reproducing kernel. Its normalized reproducing kernel is called the slice normalized Szegö kernel, defined as

$$e_a(q) := \sqrt{1-|a|^2}(1-q\bar{a})^{-*},$$

for a parameter $a \in \mathbb{B}$ and any $q \in \mathbb{B}$, where the $*$-product is defined by

$$(f * g)(q) = \sum_{n=0}^{\infty} q^n \sum_{k=0}^{n} a_k b_{n-k},$$

for any two slice regular functions

$$f = \sum_{n=0}^{\infty} q^n a_n \quad \text{and} \quad g = \sum_{n=0}^{\infty} q^n b_n,$$

where all $a_n, b_n \in \mathbb{H}$. Thus, f^{-*} denote as the inverse of f under this $*$-product.

Based on the $*$-product, we introduce the quaternionic slice hyperbolic backward shift operators $\mathcal{S}_a$, which is uniquely determined by the identity

$$f = e_a\langle f, e_a\rangle + B_a * \mathcal{S}_a f, \tag{1.1}$$

for any $a \in \mathbb{B}$. Here B_a is the slice Möbius transformation, or an order-1 Blaschke product:

$$B_a(q) := (1 - q\bar{a})^{-*} * (a - q)\frac{a}{|a|}.$$

It is noted that the notion of the Hardy spaces and the Blaschke factors in relation to slice regular functions have been introduced and studied by researchers, first appearing in [3], and also others, which are summarized in the book [4].

Iterating the construction in (1.1), we achieve an algebraic relation

$$f = \sum_{j=1}^{n} T_j \langle f, T_j \rangle + B_n * \left(\mathcal{S}_{a_n} \circ \cdots \circ \mathcal{S}_{a_1} f\right) \tag{1.2}$$

for arbitrary $a_1, \ldots, a_n \in \mathbb{H}$, where we denote

$$T_n = B_{n-1} * e_{a_n},$$

and

$$B_n = B_{a_1} * B_{a_2} * \cdots * B_{a_n}.$$

Here $\{T_n\}_{j=1}^n$ is defined to be the slice Takenaka-Malmquist system which constitutes an slice orthonormal system in $H^2(\mathbb{B})$ associated with the non-orthogonal set $\{e_{a_j}\}_{j=1}^n$. We point out that in the classical case a TM system is identical with the Gram-Schmidt (GS) orthogonalization applied to the corresponding Szegö kernels. However, in our slice case, the GS process is no longer valid since the product among slice regular functions is now $*$-product. This $*$-product brings challenge of verifying the orthogonality of a TM system.

Decomposition (1.2) can be restated as a Beurling-Lax type relation:

$$H^2(\mathbb{B}) = \text{span}\{T_1, \ldots, T_n\} \oplus B_n * H^2(\mathbb{B}). \tag{1.3}$$

By applying the maximum selection principle, we obtain the slice AFD for quaternionic slice Hardy space functions.

The paper is organized as follows. In §2, we recall some basic concepts and results of slice regular functions and the foundation of the slice Hardy space. In §3, the slice Takenaka-Malmquist orthonormal system is established. In §4, We introduce the iterative process, the adaptive selecting process, and prove convergence of the slice adaptive Fourier series. In §5 a convergence rate result is proved. The case of slice Hardy space of the right half plane is outlined in §6.

2. **Preliminary**

2.1. **Slice regular functions.** This paper works on slice regular functions over the non-commutative quaternionic field [10]. Firstly, the quaternionic field $\mathbb{H}$ is linearly generated by an orthogonal basis $\{1, e_1, e_2, e_3 := e_1e_2\}$ of $\mathbb{R}^4$ with the following multiplication rule:

$$e_ie_j + e_je_i = -2\delta_{ij}, \qquad i, j = 1, 2, 3,$$

where δ_{ij} equals 1 if $i = j$ and 0 otherwise.

Now we recall some definitions and results of the slice regular function theory. This theory is based on the slice structure of $\mathbb{H}$, i.e.

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}} \mathbb{C}_I,$$

where $\mathbb{S}$ denotes the set of imaginary units of $\mathbb{H}$, namely,

$$\mathbb{S} := \{q \in \mathbb{H} \mid q^2 = -1\},$$

and $\mathbb{C}_I$ denotes the slice of $\mathbb{H}$ made through I, i.e.,

$$\mathbb{C}_I := \{x + yI, \quad x, y \in \mathbb{R}\}.$$

According to the slice structure, any $q \in \mathbb{H}$ can be written as $q = x + yI$ with $x,\ y \in \mathbb{R}$ and $I \in \mathbb{S}$.

Definition 2.1 (slice function)**.** Let Ω be a set in $\mathbb{H}$, and f a quaternion-valued function defined on Ω that satisfies

$$f(x + yJ) = \frac{1}{2}\big(f(x + yI) + f(x - yI)\big) + \frac{JI}{2}\big(f(x - yI) - f(x + yI)\big),$$

provided $x,\ y \in \mathbb{R}$ and $I,\ J \in \mathbb{S}$ such that $x \pm yI$ and $x + yJ$ belong to Ω. Then the function f is said to be a slice function on Ω.

Definition 2.2 (slice regular function)**.** Let f be a slice function defined on a domain $\Omega \subset \mathbb{H}$. For each $I \in \mathbb{S}$, let $\Omega_I := \Omega \cap \mathbb{C}_I$ and $f_I := f|_{\Omega_I}$ be the restriction of f to Ω_I. The restriction f_I is said to be holomorphic if it has continuous partial derivatives and

$$\bar{\partial}_I f_I(x + yI) = \frac{1}{2}\big(\partial_x + I\partial_y\big) f_I\big(x + yI\big) = 0.$$

If for each $I \in \mathbb{S}$, f_I is holomorphic in Ω_I, then f is called a slice (left) regular function.

Remark 2.3. Similarly, we can define slice right regular functions.

Lemma 2.4 (splitting)**.** *Let $I \in \mathbb{S}$ and Ω_I be open in $\mathbb{C}_I$. The function $f_I : \Omega_I \to \mathbb{H}$ is holomorphic if and only if, for all $J \in \mathbb{S}$ with $J \perp I$ and every $z = x + yI$, there holds*

$$f_I(z) = F(z) + G(z)J,$$

where $F,\ G : \Omega_I \to \mathbb{C}_I$ are complex-valued holomorphic functions of one complex variable.

In the slice regular function theory, under the usual multiplication the product of two slice regular functions is no longer slice regular in general. So there comes the $*$-product.

Definition 2.5 ($*$-product)**.** Let $\mathbb{B}$ be the Euclidean unit ball of $\mathbb{H}$. Let $f,\ g : \mathbb{B} \to \mathbb{H}$ be slice regular functions and let $f(q) = \sum_{n\in\mathbb{N}} q^n a_n,\ g(q) = \sum_{n\in\mathbb{N}} q^n b_n$ where $a_n, b_n \in \mathbb{H}$ be their power series expansions. The $*$-product of f and g is the slice regular function defined by ([10], Definition 1.26)

$$(f * g)(q) = \sum_{n\in\mathbb{N}} q^n \sum_{k=0}^{n} a_k b_{n-k}.$$

The regular conjugate of f is the slice regular function defined by

$$f^c(q) = \sum_{n\in\mathbb{N}} q^n \bar{a}_n.$$

The symmetrization of f is defined to be the function

$$f^s = f * f^c = f^c * f.$$

Furthermore, if $f \neq 0$, the regular reciprocal of f is the function defined on $\mathbb{B} \setminus Z_{f^s}$ as

$$f^{-*} = \frac{1}{f^s} f^c,$$

where Z_{f^s} is the zero set of f^s.

The $*$-product of two slice regular functions is slice regular and it is related to the usual multiplication through the following relations (see [10], Theorem 3.4):

$$(f * g)(q) = \begin{cases} f(q)g(\tilde{q}), & \textit{if } f(q) \neq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where $\tilde{q} = f^{-1}(q) q f(q) \in [q]$ with $[q]$ the symmetry of $q = x + yI$ defined by

$$[q] := \{x + yJ \mid J \in \mathbb{S}\}.$$

Furthermore,

$$f^{-*} * g(q) = f^{-1}(\hat{q}) g(\hat{q}), \quad \forall q \in \mathbb{B} \setminus Z_{f^s}, \tag{2.2}$$

where $\hat{q} = f^c(q)^{-1} q f^c(q) \in [q]$.

A set $\Omega \subset \mathbb{H}$ is said to be axially symmetry if for any point $q \in \Omega$, there holds $[q] \subset \Omega$.

Definition 2.6. Let Ω be an axially symmetric domain in $\mathbb{H}$. A slice regular function $f : \Omega \to \mathbb{H}$ such that $f(\Omega_I) \subset \mathbb{C}_I$ for all $I \in \mathbb{S}$ is called a slice preserving function.

Remark 2.7. If f is a slice function on an axially symmetric domain Ω, its symmetrization function $f^s : \Omega \to \mathbb{H}$ is a slice preserving function.

Theorem 2.8 (Cauchy's formula)**.** *Let f be a slice regular function on an open set $\Omega \subset \mathbb{H}$. If U is a bounded axially symmetric open set with $\bar{U} \subset \Omega$ where $\bar{U}$ is the closure of U and if ∂U_I for $I \in \mathbb{S}$ is a finite union of disjoint rectifiable Jordan curves, then for $q \in U$,*

$$f(q) = \frac{1}{2\pi} \int_{\partial U_I} (s - q)^{-*} ds_I f(s).$$

where $ds_I = -I ds$.

2.2. **The foundation of the slice Hardy space over $\mathbb{B}$.** In this subsection, we recall the precondition of the slice Hardy space over the unit ball [4]. Let $\mathbb{B}$ be the Euclidean unit ball of $\mathbb{H}$ and $\mathbb{T} := \partial \mathbb{B}$ its boundary. For any $I \in \mathbb{S}$, denote $\mathbb{T}_I := \mathbb{T} \cap \mathbb{C}_I$. Let $L^2(\mathbb{T}_I)$ be the function space consisting of Lebesgue measurable slice functions f defined on $\mathbb{T}$ for which

$$\frac{1}{2\pi} \int_0^{2\pi} |f_I(e^{It})|^2 dt < \infty.$$

The splitting lemma provides a power series expansion of f_I, i.e.

$$f_I(e^{It}) = \sum_{k=-\infty}^{\infty} e^{Ikt} a_k, \tag{2.3}$$

where $a_k \in \mathbb{H}$ satisfies

$$\sum_{k=-\infty}^{\infty} |a_k|^2 < \infty.$$

For any $f,\ g \in L^2(\mathbb{T}_I)$, the inner product

$$\langle \cdot, \cdot \rangle : L^2(\mathbb{T}_I) \times L^2(\mathbb{T}_I) \to \mathbb{H}$$

is defined by

$$\langle f_I, g_I \rangle := \frac{1}{2\pi} \int_0^{2\pi} \overline{g_I(e^{It})} f_I(e^{It}) dt. \tag{2.4}$$

It is easy to verify that $\langle \cdot, \cdot \rangle$ is an inner product. i.e. for any $f, g, h \in L^2(\mathbb{T}_I)$ and $\lambda, \mu \in \mathbb{H}$, there hold

- $\langle f_I \lambda + g_I \mu, h_I \rangle = \langle f_I, h_I \rangle \lambda + \langle g_I, h_I \rangle \mu$.
- $\langle f_I, g_I \rangle = \overline{\langle g_I, f_I \rangle}$.
- $\langle f_I, f_I \rangle \geqslant 0$, where the equality holds if and only if $f_I = 0$.

Furthermore, there holds the Cauchy-Schwarz inequality, i.e.

$$| \langle f_I, g_I \rangle |^2 \leqslant \langle f_I, f_I \rangle \langle g_I, g_I \rangle . \tag{2.5}$$

The power series expansion in (2.3) shows that $L^2(\mathbb{T}_I)$ is a right $\mathbb{H}$-module. Thus, $L^2(\mathbb{T}_I)$ equipped with the inner product $\langle \cdot, \cdot \rangle$ is a right $\mathbb{H}$-module inner product space.

Denote

$$H^2_+(\mathbb{T}_I) := \{ f_I(e^{It}) = \sum_{k=0}^{\infty} e^{Ikt} a_k \ : \ \ a_k \in \mathbb{H}, \ \ \sum_{k=0}^{\infty} |a_k|^2 < \infty \}.$$

It is a closed subspace of $L^2(\partial \mathbb{B}_I)$. Recall that the Hilbert transformation

$$\tilde{H} : L^2(\mathbb{T}_I) \to L^2(\mathbb{T}_I)$$

is defined by

$$\tilde{H} f_I(e^{It}) = \sum_{k=-\infty}^{\infty} (-I) \mathrm{sgn}(k) e^{Ikt} a_k,$$

where a_0 is the coefficient in formula (2.3). Thus, each $f \in H^2(\mathbb{T}_I)$ can be represented as

$$f_I = \frac{a_0 + f_I + I \tilde{H} f_I}{2}.$$

In fact, $L^2(\mathbb{T}_I)$ has the following direct sum decomposition

$$L^2(\mathbb{T}_I) = H^2_+(\mathbb{T}_I) \oplus H^2_-(\mathbb{T}_I), \tag{2.6}$$

where

$$H^2_-(\mathbb{T}_I) := \{ f_I(e^{It}) = \sum_{k=-\infty}^{-1} e^{Ikt} a_k : \ \ a_k \in \mathbb{H}, \ \ \sum_{k=0}^{\infty} |a_k|^2 < \infty \}.$$

Thus, the function space $H^2(\mathbb{B}_I)$ with $I \in \mathbb{S}$ is the function space consisting of slice regular functions f defined in $\mathbb{B}$ for which

$$\|f_I\|^2 := \sup_{0 \leqslant r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f_I(re^{It})|^2 dt < \infty.$$

For any $f \in H^2(\mathbb{B}_I)$, define its radial limit

$$\hat{f}_I(e^{It}) := \lim_{r \to 1} f_I(re^{It}). \tag{2.7}$$

The limit $\hat{f}_I$ exists almost everywhere.

Theorem 2.9. *Let $f \in H^2(\mathbb{B}_I)$ for some $I \in \mathbb{S}$. The radial limit of f_I exists almost everywhere on $\mathbb{T}_I$. Furthermore, there is an isometric isomorphism*

$$\begin{array}{rcl} H^2(\mathbb{B}_I) & \to & H^2_+(\mathbb{T}_I) \\ f & \mapsto & \hat{f}_I. \end{array}$$

Remark 2.10. For any slice regular function f and for any $I, J \in \mathbb{S}$, $f_I \in H^2(\mathbb{B}_I)$ if and only if $f_J \in H^2(\mathbb{B}_J)$.

Remark 2.11. The $L^2(\mathbb{T}_I)$ space can be expressed as direct sum of the two corresponding Hardy spaces, the latter consisting of boundary limits of well behaved holomorphic functions. Due to this relation, studies of functions of finite energy may use complex analysis methods. This shows the role and importance of Hardy space theory.

3. **Slice rational orthogonal system**

The slice Hardy space introduced in this section is equivalent to the definition in [4]. The reproducing kernel and Blaschke products of the slice Hardy context have also been studied in the book. Thus, our main result in this section is the slice rational orthogonal system $\{T_k\}_{k\geqslant 1}$, i.e. Theorem 3.6.

Definition 3.1. The slice Hardy space $H^2(\mathbb{B})$ consists of slice regular functions f defined in $\mathbb{B}$ which satisfies

$$\|f\|^2 := \frac{1}{4\pi^2}\int_{\partial\mathbb{B}} \frac{1}{|Im(q)|^2}|f(q)|^2 d\sigma(q) < \infty, \tag{3.1}$$

where $d\sigma$ is the surface area element on $\partial\mathbb{B}$.

Based on the slice technique and cylindrical coordinate transformation [13], we can polarize (3.1) to one slice as following:

$$\begin{aligned} \langle f, g\rangle :=& \frac{1}{4\pi^2}\int_{T^2} \sin\theta_1 d\theta \int_{\partial\mathbb{B}_{I(\theta)}} \overline{g(x+I(\theta)y)} f(x+I(\theta)y)dxdy \\ =& \frac{1}{2\pi}\int_{T^2} \sin\theta_1 d\theta \frac{1}{2\pi}\int_0^{2\pi} \overline{g(e^{I(\theta)t})} f(e^{I(\theta)t})dt \\ =& \frac{1}{2\pi}\int_{T^2} \sin\theta_1 d\theta \left\langle f_{I(\theta)}, g_{I(\theta)} \right\rangle, \end{aligned}$$

where $\theta = (\theta_1, \theta_2) \in T^2 := [0, \pi]^2$ and

$$I(\theta) := (e_1, e_2, e_3)\varphi(\theta) \in \mathbb{S}$$

with

$$\varphi(\theta) = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1\cos\theta_2 \\ \sin\theta_1\sin\theta_2 \end{pmatrix}.$$

Notice that $H^2(\mathbb{B})$ is a reproducing kernel Hilbert space. For any $a \in \mathbb{B}$, define the slice normalized Szegö kernel as

$$e_a(q) := e(a, q) := \sqrt{1-|a|^2}(1-q\bar{a})^{-*}, \qquad \forall q \in \mathbb{B}, \tag{3.2}$$

where $-*$ is the regular reciprocal in Definition 2.5. Since $(1-q\bar{a})^s$ does not have zero points, e_a is a left slice regular function over $\mathbb{B}$. Besides, the property of $*$-product shows the general conjugation of e_a:

$$\overline{e_a(q)} = \sqrt{1-|a|^2}(1-a\bar{q})^{-*}, \qquad \forall q \in \mathbb{B},$$

which is right conjugate slice regular over $\mathbb{B}$. We claim that e_a is the normalized reproducing kernel of $H^2(\mathbb{B})$. In fact,

$$\begin{aligned} \langle f_I, (e_a)_I\rangle =& \frac{1}{2\pi}\int_0^{2\pi} \overline{e_a(e^{It})} f(e^{It})dt \\ =& \frac{\sqrt{1-|a|^2}}{2\pi}\int_{\partial\mathbb{B}_I} (1-ae^{-It})^{-*}\big(e^{-It}(-I)de^{It}\big)f(e^{It}) \\ =& \frac{\sqrt{1-|a|^2}}{2\pi}\int_{\partial\mathbb{B}_I} (1-a\bar{q})^{-*} * \bar{q}(-Ids)f(q) \\ =& \frac{\sqrt{1-|a|^2}}{2\pi}\int_{\partial\mathbb{B}_I} (q-a)^{-*}(-Ids)f(q) \\ =& \sqrt{1-|a|^2}f(a), \end{aligned} \tag{3.3}$$

where the third equality holds because the function $g(\bar{q}) = \bar{q}$ is a (right) conjugate slice preserving function so that the $*$-product reduces to the usual product. The last equality holds owing to the Cauchy integral formula (i.e. Theorem 2.8). Furthermore, since $\sqrt{1-|a|^2}f(a)$ is independent of the imaginary unit of q, we obtain

$$\langle f, e_a\rangle = \sqrt{1-|a|^2}f(a).$$

Remark 3.2. We claim that the operator $\mathcal{S} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ defined by

$$\mathcal{S}f(q) := \langle f, e(\cdot, q)\rangle$$

is an orthogonal projection operator from $L^2(\mathbb{T})$ to $H^2(\mathbb{B})$, so e_a with $a \in \mathbb{B}$ is actually the Szegö kernel of $H^2(\mathbb{B})$. In fact, the space decomposition (2.6) shows

$$f = f^+ + f^-,$$

where $f^+ \in H^2_+(\mathbb{T})$ and $f^- \in H^2_-(\mathbb{T})$ and then the Cauchy formula shows

$$\mathcal{S}f(q) = f^+(q) \in H^2(\mathbb{B}).$$

Besides, the conjugte operator of S is

$$\mathcal{S}^*f(q) := \left\langle f, \overline{e(q,\cdot)}\right\rangle.$$

Since $e(\cdot, q) = \overline{e(q,\cdot)}$, we have

$$\mathcal{S}^* = \mathcal{S}.$$

Denote the class of slice normalized Szegö kernels as:

$$\mathcal{D} := \{e_a | a \in \mathbb{B}\}.$$

Theorem 3.3. *$\mathcal{D}$ is a dictionary of $H^2(\mathbb{B})$, i.e.*

$$\overline{span}_{\mathbb{H}}\{e_a | a \in \mathbb{B}\} = H^2(\mathbb{B}).$$

Here the left-hand-side represents the closure of the right $\mathbb{H}$-module linear subspace spanned by finite linear combinations of elements in $\mathcal{D}$.

Proof. If $f \in \overline{span}_{\mathbb{H}}^{\perp}\{e_a | a \in \mathbb{B}\}$, then the reproducing property of e_a tells us that

$$f(a) = 0$$

for any $a \in \mathbb{B}$. This means $f = 0$. $\square$

Remark 3.4. Notice that for each $I \in \mathbb{S}$,

$$\mathcal{D}_I := \{e_a | a \in \mathbb{B}_I\}$$

is also a dictionary of $H^2(\mathbb{B})$ as the definition of slice function shows. However, its slice Takenaka-Malmquist system given by (3.4) with coefficients in $\mathbb{B}_I$ is not a basis of $H^2(\mathbb{B})$.

For every $a \in \mathbb{B}$, the Blaschke factor (or the Möbius transfrom) B_a is a slice regular function in $\mathbb{B}$ defined as

$$B_a(q) := (1 - q\bar{a})^{-*} * (a - q)\frac{a}{|a|}.$$

Proposition 3.5 ([4])**.** *Let $a \in \mathbb{B}$. The Blaschke factor B_a has the following properties:*

- *it takes the unit ball $\mathbb{B}$ to itself;*
- *it takes the boundary of the unit ball to itself;*
- *it has a unique zero point a.*

A Blaschke product is defined to be the *- product of a finite number of Blaschke factors (also see ([4]):

$$B_k(q) := \prod_{j=1}^{*\ k} (1 - q\bar{a}_j)^{-*} * (a_j - q)\frac{a_j}{|a_j|},$$

where $a_k \in \mathbb{B}$ for any $k \in \{1, 2, \cdots\}$.

In the unit ball $\mathbb{B}$, the slice rational orthogonal system, i.e., the slice TM system, consists of weighted Blaschke products, i.e. for any $k \geqslant 1$,

$$T_k := B_{k-1} * e_{a_k}. \tag{3.4}$$

Theorem 3.6. *$\{T_k\}_{k \geqslant 1}$ is an orthonormal system.*

Proof. By definition, for any $k \geqslant 1$ and $I \in \mathbb{S}$,

$$\langle (T_k)_I, (T_k)_I \rangle = \frac{1}{2\pi}\int_0^{2\pi} \overline{B_{k-1} * e_k(e^{It})} B_{k-1} * e_k(e^{It}) dt. \tag{3.5}$$

Equation (2.1) shows

$$B_{k-1} * e_k(e^{It}) = B_{k-1}(e^{It}) e_k(e^{Jt}),$$

where $J \in \mathbb{S}$ such that $e^{Jt} = B_k^{-1}(e^{It}) e^{It} B_k(e^{It})$. So equation (3.5) becomes

$$\langle (T_k)_I, (T_k)_I \rangle = \frac{1}{2\pi}\int_0^{2\pi} \overline{e_k(e^{Jt})}\ \overline{B_{k-1}(e^{It})} B_{k-1}(e^{It}) e_k(e^{Jt}) dt.$$

Proposition 3.5 implies that

$$\overline{B_{k-1}(e^{It})} B_{k-1}(e^{It}) = |B_{k-1}(e^{It})|^2 = 1.$$

By change of variables, the Cauchy integral formula shows

$$\begin{aligned}\langle (T_k)_I, (T_k)_I\rangle =&\frac{1}{2\pi}\int_0^{2\pi}\overline{e_k(e^{Jt})}e_k(e^{Jt})dt\\ =&\frac{\sqrt{1-|a_k|^2}}{2\pi}\int_{\partial\mathbb{B}_J}(q-a_k)^{-*}(-Jds)e_k(q)\\ =&\sqrt{1-|a_k|^2}e_k(a_k)\\ =&1.\end{aligned}$$

Since the result is independent of imaginary I, we obtain

$$\langle T_k, T_k\rangle = 1.$$

Now we consider the case of different indices where $1 \leqslant l < k$

$$\langle (T_k)_I, (T_l)_I\rangle = \frac{1}{2\pi}\int_0^{2\pi}\overline{B_{l-1} * e_l(e^{It})}B_{k-1} * e_k(e^{It})dt. \tag{3.6}$$

There we reformulate $B_{k-1} * e_k$ as

$$B_{k-1} * e_k := B_{l-1} * g,$$

where

$$g(q) = \Big(\prod_{j=l}^{k-1}{}^{*} (1-q\bar{a}_j)^{-*} * (a_j - q)\frac{a_j}{|a_j|}\Big) * e_k(q).$$

As before,

$$B_{l-1} * g(e^{It}) = B_{l-1}(e^{It})g(e^{Kt}), \tag{3.7}$$

where $K \in \mathbb{S}$ such that $e^{Kt} = B_l^{-1}(e^{It})e^{It}B_l(e^{It})$. Notice that

$$B_{l-1} * e_l(e^{It}) = B_{l-1}(e^{It})e_l(e^{Kt}).$$

Then equation (3.6) becomes

$$\langle (T_k)_I, (T_l)_I\rangle = \frac{1}{2\pi}\int_0^{2\pi}\overline{e_l(e^{Kt})}\;\overline{B_{l-1}(e^{It})}B_{l-1}(e^{It})g(e^{Kt})dt.$$

Similarly, we have

$$|B_{l-1}(e^{It})|^2 = 1,$$

Again by the change of variables, we apply the Cauchy integral theorem to get

$$\begin{aligned}\langle (T_k)_I, (T_l)_I\rangle =&\frac{1}{2\pi}\int_0^{2\pi}\overline{e_l(e^{Kt})}g(e^{Kt})dt\\ =&\frac{\sqrt{1-|a|^2}}{2\pi}\int_{\partial\mathbb{B}_K}(q-a_l)^{-*}(-Kds)g(q)\\ =&\sqrt{1-|a|^2}g(a_l)\\ =&0,\end{aligned}$$

which deduces that

$$\langle T_k, T_l\rangle = 0.$$

This completes the proof. □

4. **Slice Adaptive Fourier Decomposition**

We have known that the slice Hardy space $H^2(\mathbb{B})$ is a reproducing kernel Hilbert space and T_k for any $k \geqslant 1$ is a slice rational orthogonal system in the unit ball $\mathbb{B}$.

In this section, we intend to adaptively decompose functions in the slice Hardy space into the subspace spanned by $\{T_k\}_{k\geq 1}$ as following: For any $f \in H^2(\mathbb{B})$ and $a_1 \in \mathbb{B}$, there is an equality

$$f(q) = e_{a_1}(q) \langle f, e_{a_1} \rangle + B_{a_1} * \mathcal{S}_{a_1} f,$$

where

$$\mathcal{S}_{a_1} f := B_{a_1}^{-*} * \big(f(q) - e_{a_1}(q) \langle f, e_{a_1} \rangle \big).$$

Denote

$$r_2(q) := f(q) - e_{a_1}(q) \langle f, e_{a_1} \rangle$$

as the standard remainder and

$$f_2(q) := S_{a_1} f$$

as the reduced remainder. By setting $f_1 = f$, we have

$$f_1(q) = e_{a_1}(q) \langle f_1, e_{a_1} \rangle + B_{a_1} * f_2(q).$$

Notice that a_1 is a removable singularity of r_2. This is because a_1 is a common zero point of function r_2 and Blaschke factor B_{a_1}. In fact, Proposition 3.5 shows B_{a_1} has the unique zero a_1. Hence, f_2 is a slice regular function in $\mathbb{B}$. Meanwhile, by the right $\mathbb{H}$-linear properties of the inner product $\langle \cdot, \cdot \rangle$, we have

$$\begin{aligned} \langle e_{a_1} \langle f_1, e_{a_1} \rangle, r_2 \rangle &= \langle e_{a_1} \langle f_1, e_{a_1} \rangle, f_1 - e_{a_1} \langle f_1, e_{a_1} \rangle \rangle \\ &= \langle e_{a_1} \langle f_1, e_{a_1} \rangle, f_1 \rangle - \langle e_{a_1} \langle f_1, e_{a_1} \rangle, e_{a_1} \langle f_1, e_{a_1} \rangle \rangle \\ &= \langle e_{a_1}, f_1 \rangle \langle f_1, e_{a_1} \rangle - \overline{\langle f_1, e_{a_1} \rangle} \langle e_{a_1}, e_{a_1} \rangle \langle f_1, e_{a_1} \rangle \\ &= 0. \end{aligned} \tag{4.1}$$

This deduces that

$$\| f_2(q) \|^2 = \| r_2(q) \|^2 = \| f_1(q) \|^2 - | \langle f_1, e_{a_1} \rangle |^2 < +\infty,$$

where the first equality holds as shown in the proof of Theorem 3.6. Thus, we have $f_2 \in H^2(\mathbb{B})$.

Now we can apply the iterative process:

$$\begin{aligned} f_1(q) =& e_{a_1}(q) \langle f_1, e_{a_1} \rangle + B_{a_1} * f_2(q) \\ =& e_{a_1}(q) \langle f_1, e_{a_1} \rangle + B_{a_1} * (e_{a_2}(q) \langle f_2, e_{a_2} \rangle + B_{a_2} * f_3(q)) \\ =& T_1(q) \langle f_1, e_{a_1} \rangle + T_2(q) \langle f_2, e_{a_2} \rangle + B_{a_1} * B_{a_2} * f_3(q) \\ =& T_1(q) \langle f_1, e_{a_1} \rangle + T_2(q) \langle f_2, e_{a_2} \rangle + B_{a_1} * B_{a_2} * (e_{a_3}(q) \langle f_3, e_{a_3} \rangle + B_{a_3} * f_4(q)) \\ =& T_1(q) \langle f_1, e_{a_1} \rangle + T_2(q) \langle f_2, e_{a_2} \rangle + T_3(q) \langle f_3, e_{a_3} \rangle + B_{a_1} * B_{a_2} * B_{a_3} * f_4(q) \\ =& \cdots \end{aligned} \tag{4.2}$$

Theorem 3.6 shows that in the decomposition (1.2) the first n terms are orthogonal to each other, so we just need to show the orthogonality between each of the n summed terms and the remainder term. This can be down by following the same method as in the proof of Theorem 3.6.

The orthogonality implies the following energy equality:

$$\|f_1(q)\|^2 = |\langle f_1, e_{a_1}\rangle|^2 + |\langle f_2, e_{a_2}\rangle|^2 + \cdots + |\langle f_k, e_{a_k}\rangle|^2 + \|f_{k+1}(q)\|^2 \tag{4.3}$$

Now we have had a decomposition of $f \in H^2(\mathbb{B})$. We want to know that whether it is convergent and how fast it converges as $k \to \infty$. Clearly, the answer relies on the choice of a_n. Our purpose is to find at every n-th step a suitable parameter a_n such that the corresponding normalized Szegö kernel e_{a_n} extracts out the largest possible energy portion from the reduced remainder f_n. The premise is that the maximal choice must exist.

Theorem 4.1 (maximum selection principle)**.** *For any $f \in H^2(\mathbb{B})$, there exists an element $a \in \mathbb{B}$ such that*

$$|\langle f, e_a\rangle| = \max_{b\in\mathbb{B}} |\langle f, e_b\rangle|.$$

Proof. We only need to prove that

$$\lim_{|a|\to 1} |\langle f, e_a\rangle| = \sqrt{1-|a|^2}|f(a)| = 0.$$

In fact, Theorem 2.9 implies that there exists a polynomial g defined in the closure of $\mathbb{B}$ such that

$$\|f - g\| < \frac{\varepsilon}{2}.$$

The inner product is then divided into two parts:

$$\langle f, e_a\rangle = \langle f - g, e_a\rangle + \langle g, e_a\rangle.$$

The Cauchy-Schwarz inequality (2.5) implies

$$|\langle f - g, e_a\rangle| \leqslant \|f - g\| < \frac{\varepsilon}{2}.$$

Now let C be any but fixed bound of g on the closed unit disc. When $|a|$ is sufficiently close to 1, there follows

$$|\langle g, e_a\rangle| = \sqrt{1-|a|^2}|g(a)| \leqslant C\sqrt{1-|a|^2} < \frac{\varepsilon}{2}.$$

Combining the above two estimates the proof is complete.

□

Lemma 4.2. *With the notation $f_n,\ r_n,\ T_n, e_{a_n}$ defined in the text for $n \geqslant 1$ there hold*

$$\langle f_n, e_{a_n}\rangle = \langle r_n, T_n\rangle = \langle f, T_n\rangle.$$

Proof. For the first equality, recall that

$$r_n = B_{n-1} * f_n, \quad T_n = B_{n-1}\surd * e_n.$$

Following the proof of Theorem 3.6, we obtain

$$\begin{aligned}\langle r_n, T_n\rangle &= \langle B_{n-1} * f_n, B_{n-1} * e_n\rangle \\ &= \langle f_n, e_{a_n}\rangle.\end{aligned}$$

For the second equation, Theorem 3.6 shows

$$\langle T_k, T_n\rangle = 0, \quad 1 \leqslant k < n.$$

Hence, the iteration formula (4.2) deduce

$$\langle f, T_n\rangle = \langle r_n, T_n\rangle.$$

□

Theorem 4.3. *Let $f \in H^2(\mathbb{B})$. If for every $n \geqslant 1$, the parameters a_n is chosen according to the maximal selection principle in Theorem 4.1, then*

$$f = \sum_{n=1}^{\infty} T_n \langle f_n, e_{a_n} \rangle = \sum_{n=1}^{\infty} T_n \langle f, T_n \rangle .$$

Proof. In the last two sections, we have introduced the slice Hardy space $H^2(\mathbb{B})$ of the slice regular functions over the quaternion field. When equipped with the inner product $\langle \cdot, \cdot \rangle$, it is a quaternion Hilbert space. By virtue of the Cauchy formula, we obtained the slice normalized Szegö kernel $\{e_a\}_{a\in\mathbb{B}}$. Together with e_a, the slice Blaschke products and $*$-product, we established the theory of TM systems $\{T_k\}_{k\geq 1}$ in the slice regular Hardy space. With these preparations and the maximum selection principle we can translate the proof of Theorem 2.2 in [18] word by word to get the counterpart convergence result in the slice regular Hardy space context. □

5. The convergence rate

In this section, we prove a convergence rate result for slice adaptive Fourier decomposition. We consider the convergence rate issue in a subclass of $H^2(\mathbb{B})$, defined by:

$$H^2(\mathcal{D}, M) := \{f \in H^2(\mathbb{B}) : f = \sum_{k=1}^{\infty} c_k e_{b_k},\ e_{b_k} \in \mathcal{D},\ \sum_{k=1}^{\infty} |c_k| \leqslant M\},$$

where $\mathcal{D}$ is the dictionary consisting of the slice normalized Szegö kernels and M is a positive constant.

Lemma 5.1. *If $f \in H^2(\mathcal{D}, M)$, then $\|f\| \leqslant M$.*

Proof. Since $f \in H^2(\mathcal{D}, M)$, there exist a quaternion series $\{c_k\}_{k\geqslant 1}$ and a function series $\{e_{b_k}\}_{k\geqslant 1} \in \mathcal{D}$ such that

$$f = \sum_{k=1}^{\infty} c_k e_{b_k}, \quad \textit{with} \quad \sum_{k=1}^{\infty} |c_k| \leqslant M.$$

Thus we obtain

$$\begin{aligned}
\|f\|^2 =& | \langle f, f \rangle | \\
=& \left| \left\langle f, \sum_{k=1}^{\infty} c_k e_{b_k} \right\rangle \right| \\
\leqslant& \sum_{k=1}^{\infty} |c_k| | \langle f, e_{b_k} \rangle | \\
\leqslant& \|f\| \sum_{k=1}^{\infty} |c_k| \\
\leqslant& M\|f\|,
\end{aligned}$$

where the second inequality holds due to the Cauchy-Schwarz inequality (2.5). □

Lemma 5.2 ([7]). *Let A be a positive constant and $\{d_m\}_{m=1}^{\infty}$ be a series of non-negative numbers satisfying*

$$d_1 \leqslant A, \quad d_{m+1} \leqslant d_m(1-\frac{d_m}{A}),$$

then for every positive integer m, we have

$$d_m \leqslant \frac{A}{m}.$$

Now we can show that the convergence rate is about $O(m^{-1/2})$ in the space $H^2(\mathcal{D}, M)$.

Theorem 5.3. *If $f \in H^2(\mathcal{D}, M)$, then*

$$\|r_m\| \leqslant \frac{M}{\sqrt{m}}.$$

Proof. The proof follows the same route as for the classical complex Hardy space. However, we write down the proof since there are details which make it not a direct translation.

Since $f \in H^2(\mathcal{D}, M)$, there exist $\{c_k\}_{k\geqslant 1} \in \mathbb{H}$ and $\{b_k\}_{k=1}^{\infty} \in \mathbb{B}$ such that

$$f = \sum_{k=1}^{\infty} e_{b_k} c_k, \quad \textit{with} \quad \sum_{k=1}^{\infty} |c_k| \leqslant M. \tag{5.1}$$

It follows from (4.2), (4.3), and Lemma 4.2 that

$$\|r_{m+1}\|^2 = \|r_m\|^2 - |\langle f_m, e_{a_m}\rangle|^2 = \|r_m\|^2 - |\langle r_m, T_m\rangle|^2. \tag{5.2}$$

Now we consider the second term in the right side of equality (5.2). By applying the maximum selection principle of the m-th step, Lemma 4.2 and the reproducing property of e_a, we have

$$\begin{aligned} |\langle r_m, T_m\rangle| &= \sup_{a\in\mathbb{B}} |\langle r_m, T_{\{a_1,\cdots,a_{m-1},a\}}\rangle| \\ &= \sup_{a\in\mathbb{B}} |\langle f_m, e_a\rangle| \\ &= \sup_{a\in\mathbb{B}} \sqrt{1-|a|^2}|B_{m-1}^{-*} * r_m(a)| \\ &= \sup_{a\in\mathbb{B}} \sqrt{1-|\hat{a}|^2}|B_{m-1}^{-1}(\hat{a})||r_m(\hat{a})| \\ &\geqslant \sup_{[b_k]} \sqrt{1-|\hat{b}_k|^2}|B_{m-1}^{-1}(\hat{b_k})||r_m(\hat{b_k})| \\ &\geqslant \sup_{[b_k]} \sqrt{1-|b_k|^2}|r_m(b_k)| \\ &\geqslant \sup_{b_k} \sqrt{1-|b_k|^2}|r_m(b_k)|, \end{aligned} \tag{5.3}$$

where the fourth equality holds because of formula (2.2) with

$$\hat{a} = B_{m-1}^c(a)^{-1} a B_{m-1}^c(a) \in [a].$$

In the first inequality, we consider the set $\{[b_k]_{k\geqslant 1}\}$ which is the spherical extension of the set $\{b_k\}_{k\geqslant 1}$.

We claim that

$$\sup_{b_k} \sqrt{1-|b_k|^2}|r_m(b_k)| \geqslant \frac{1}{M}\|r_m\|^2.$$

In fact, by applying the orthogonal iterative process of f, we obtain

$$\langle r_m, r_m\rangle = \left|\langle r_m, f\rangle - \langle r_m, \sum_{k=1}^{m-1} T_k \langle f_k, e_{a_k}\rangle\rangle\right| = |\langle r_m, f\rangle|.$$

The equation (5.1) and the reproducing property of e_{b_k} give

$$\begin{aligned} |\langle r_m, f\rangle| =& \left|\langle r_m, \sum_{k=1}^{\infty} e_{b_k} c_k\rangle\right| \\ \leqslant& M \sup_{b_k} |\langle r_m, e_{b_k}\rangle| \\ =& M \sup_{b_k} \sqrt{1-|b_k|^2}|r_m(b_k)|, \end{aligned} \tag{5.4}$$

The claim is hence verified.

By substituting (5.4) and (5.3) into equality (5.2), we have

$$\|r_{m+1}\|^2 \leqslant \|r_m\|^2(1 - \frac{\|r_m\|^2}{M^2}).$$

Applying Lemma 5.2, we obtain

$$\|r_m\|^2 \leqslant \frac{M^2}{m}.$$

□

6. The slice Hardy space over $\mathbb{H}^+$

The slice Hardy space over $\mathbb{H}^+$ has also been studied in [4]. Again, we use an equivalent definition. Let H^+ be the right half plane of $\mathbb{H}$, i.e.

$$H^+ := \{q \in \mathbb{H} \mid Re(q) > 0\}.$$

In this section, we just list the corresponding results of the slice Hardy space over H^+, for which the proofs are similar to those for the slice Hardy space over $\mathbb{B}$.

Definition 6.1. The slice Hardy space $H^2(\mathbb{H}^+)$ consists of slice regular functions f, which satisfies

$$\|f\|^2 := \frac{1}{2\pi}\int_{T^2} \sin\theta_1 d\theta \int_{-\infty}^{+\infty} |f(I(\theta)y)|^2 dy < \infty,$$

where $\theta = (\theta_1, \theta_2) \in T^2 = [0, \pi]^2$. Furthermore, $H^2(\mathbb{H}^+)$ is a Hilbert space.

Denote by $\langle\cdot,\cdot\rangle$ the inner product with the induced norm $\|\cdot\|$.

For any $a \in \mathbb{H}^+$, the slice normalized Szegö kernel is a left slice regular function over $\mathbb{H}^+$, defined as

$$e_a(q) := \sqrt{\frac{Re(a)}{\pi}}(q+\bar{a})^{-*},$$

where $-*$ is the regular reciprocal in Definition 2.5 and $Re(a)$ is the real part of $a \in \mathbb{H}$. Then e_a is a reproducing kernel of $H^2(\mathbb{H}^+)$, i.e.

$$\langle f, e_a\rangle = \sqrt{4\pi Re(a)} f(a). \tag{6.1}$$

Besides, the slice normalized Szegö kernel e_a is a dictionary of $H^2(\partial\mathbb{H}^+)$, i.e.

$$\overline{span}_{\mathbb{H}}\{e_a | a \in \mathbb{H}\} = H^2(\partial\mathbb{H}^+),$$

The Blaschke product is

$$B_k(q) := \prod_{j=1}^{k}{}^{*} (q + \bar{a}_j)^{-*} * (q - a_j),$$

where $a_k \in \mathbb{H}_+$ for any $k \in \{1, 2, \cdots\}$.

In the right half plane $\mathbb{H}_+$, the slice rational orthogonal system (i.e. the slice Takenaka-Malmquist system) consists of weighted Blaschke product, i.e. for any $k \geqslant 1$,

$$T_k := B_{k-1} * e_k.$$

Theorem 6.2. *$\{T_k\}_{k\geqslant 1}$ is a normal orthogonal system, i.e.*

$$\begin{cases} \langle T_k, T_k \rangle = 1, & k \geqslant 1, \\ \langle T_k, T_l \rangle = 0, & 1 \leqslant l < k. \end{cases}$$

Now we consider the slice adaptive Fourier decomposition. For any $f \in H^2(\mathbb{H}^+)$ and $a_1, a_2, a_3 \cdots \in \mathbb{H}^+$, there is an iterative process:

(6.2)

$$\begin{aligned} f_1(q) =& e_{a_1}(q) \langle f_1, e_{a_1} \rangle + B_{a_1} * f_2(q) \\ =& e_{a_1}(q) \langle f_1, e_{a_1} \rangle + B_{a_1} * (e_{a_2}(q) \langle f_2, e_{a_2} \rangle + B_{a_2} * f_3(q)) \\ =& T_1(q) \langle f_1, e_{a_1} \rangle + T_2(q) \langle f_2, e_{a_2} \rangle + B_{a_1} * B_{a_2} * f_3(q) \\ =& T_1(q) \langle f_1, e_{a_1} \rangle + T_2(q) \langle f_2, e_{a_2} \rangle + B_{a_1} * B_{a_2} * (e_{a_3}(q) \langle f_3, e_{a_3} \rangle + B_{a_3} * f_4(q)) \\ =& T_1(q) \langle f_1, e_{a_1} \rangle + T_2(q) \langle f_2, e_{a_2} \rangle + T_3(q) \langle f_3, e_{a_3} \rangle + B_{a_1} * B_{a_2} * B_{a_3} * f_4(q) \\ =& \cdots \end{aligned}$$

Orthogonality of the Takenaka-Malmquist system $\{T_k\}_{k\geqslant 1}$ implies the following energy relation:

$$|f_1(q)|^2 = |\langle f_1, e_{a_1} \rangle|^2 + |\langle f_2, e_{a_2} \rangle|^2 + \cdots + |\langle f_k, e_{a_k} \rangle|^2 + |f_{k+1}(q)|^2.$$

Theorem 6.3. *For any $f \in H^2(\mathbb{H}^+)$, there exists an element $a \in \mathbb{H}^+$ such that*

$$|\langle f, e_a \rangle| = \max_{b \in \mathbb{H}^+} \{|\langle f, e_b \rangle|\}.$$

Theorem 6.4. *Let $f \in H^2(\mathbb{H}^+)$. If for every $n \geqslant 1$ the parameters a_n in relation to the reduced remainder function f_n is chosen according to Theorem 6.3, then*

$$f = \sum_{n=1}^{\infty} T_n \langle f, e_{a_n} \rangle = \sum_{n=1}^{\infty} T_n \langle f, T_n \rangle.$$

Denote

$$H^2(\mathcal{D}, M) := \{f \in H^2(\mathbb{H}^+) : f = \sum_{k=1}^{\infty} e_{b_k} c_k,\ e_{b_k} \in \mathcal{D},\ \sum_{k=1}^{\infty} |c_k| \leqslant M\},$$

where $\mathcal{D}$ is the dictionary consisting of the slice normalized Szegö kernels and M is a positive constant. Then we have the following

Theorem 6.5. *If $f \in H^2(\mathcal{D}, M)$, then*

$$\|r_m\| \leqslant \frac{M}{\sqrt{m}}.$$

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