

Integral Representations in Weighted Bergman Spaces on the Tube Domains

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Herein, the Laplace transform representations for functions of weighted holomorphic Bergman spaces on the tube domains are developed. Then a weighted version of the edge-of-the-wedge theorem is derived as a byproduct of the main results.

Key words: Weighted Bergman space, Tube domain, Laplace transform, Integral representation, Regular cone

1 Introduction

The classical Paley–Wiener theorem asserts that functions of the classical Hardy space $H^2(\mathbb{C}^+)$ can be written as the Laplace transforms of L^2 functions supported in the right half of the real axis, see [1]. This theorem has been extended to more general Hardy spaces, including the H^p spaces cases ($0 < p \leq \infty$), higher dimensional cases and weighted spaces, see [11, 13, 15, 14, 12, 9]. Integral representation theorems have also been investigated for Bergman spaces.

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We first introduce some notations and definitions. Let B be a domain (open and connected set) in \mathbb{R}^n and $T_B = \mathbb{R}^n + iB \subset \mathbb{C}^n$ be the tube over B . For any element $z = (z_1, z_2, \dots, z_n)$, $z_k = x_k + iy_k$, by definition, $z \in T_B$ is and only if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in B$. The inner product of $z, w \in \mathbb{C}^n$ is defined as $z \cdot w = z_1 w_1 + z_2 w_2 + \dots + z_n w_n$. The associated Euclidean norm of z is $|z| = \sqrt{z \cdot \bar{z}}$, where $\bar{z} = (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_n)$.

A nonempty subset $\Gamma \subset \mathbb{R}^n$ is called an open cone if it satisfies (i) $0 \notin \Gamma$, and (ii) $\alpha x + \beta y \in \Gamma$ for any $x, y \in \Gamma$ and $\alpha, \beta > 0$. The dual cone of Γ is defined as $\Gamma^* = \{y \in \mathbb{R}^n : y \cdot x \geq 0, \text{ for any } x \in \Gamma\}$, which is clearly a closed convex cone with vertex at 0. We say that the cone Γ is regular if the interior of its dual cone Γ^* is nonempty.

For $\frac{1}{p} + \frac{1}{q} = 1$, define

$$B^p(T_B) = \left\{ F : F \text{ is holomorphic in } T_B \text{ and satisfies } \int_B \left(\int_{\mathbb{R}^n} |F(x + iy)|^p dx \right)^{q-1} dy < \infty \right\}.$$

Among the previous studies, Genchev showed that the function spaces $B^p(1 \leq p \leq 2)$, in the one- and multi-dimensions in [3] and [4], respectively, admit integral representations in the Laplace transform form. These results can be applied to the Bergman spaces

$$A^p(T_\Gamma) = \left\{ F : F \text{ is holomorphic on } T_\Gamma \text{ and satisfies } \int_{T_\Gamma} |F(x + iy)|^p dx dy < \infty \right\}$$

to obtain the corresponding integral representation results for $A^p(T_\Gamma)$ in the range $1 \leq p \leq 2$ ([5]).

In this paper we initiate a study on a class of function spaces, denoted by $A^{p,s}(B, \psi)$, of which each is associated with a weight function of the form $e^{-2\pi\psi(y)}$, where $\psi(y) \in C(B)$ is continuous on B . The space $A^{p,s}(B, \psi)$ ($0 < p \leq \infty, 0 < s \leq \infty$) is the collection of functions $F(z)$ that are holomorphic in T_B and satisfy

$$\|F\|_{A^{p,s}(B, \psi)} = \left(\int_B \left(\int_{\mathbb{R}^n} |F(x + iy)e^{-2\pi\psi(y)}|^p dx \right)^s dy \right)^{\frac{1}{sp}} < \infty, \quad 0 < p, s < \infty,$$

$$\|F\|_{A^{p,\infty}(B, \psi)} = \sup \left\{ e^{-2\pi\psi(y)} \left(\int_{\mathbb{R}^n} |F(x + iy)|^p dx \right)^{\frac{1}{p}}, y \in B \right\} < \infty, \quad 0 < p < \infty, s = \infty$$

and

$$\|F\|_{A^{\infty,\infty}(B, \psi)} = \sup \{ e^{-2\pi\psi(y)} |F(x + iy)|, x \in \mathbb{R}^n, y \in B \} < \infty, \quad p = \infty, s = \infty.$$

This paper is structured as follows. In §2, we introduce our main work on the integral representation for $A^{p,s}(B, \psi)$, which is separated into three cases, namely, $A^{p,s}(B, \psi)$ for

$1 \leq p \leq 2$, $A^{p,s}(B, \psi)$ for $0 < p < 1$ and $A^{p,s}(\Gamma, \psi)$ for $p > 2$, corresponding to Theorem 1, 2 and 3 respectively. The proofs are given in §3. Finally, some results, referring to Corollary 2, Theorem 4 and Theorem 5, are derived as applications of the integral representation theorems claimed in §2.

2 Main results

In order to introduce our main results, we define the set

$$U_\alpha(B, \psi) = \left\{ t \in \mathbb{R}^n : \int_B e^{-2\pi\alpha(t \cdot y + \psi(y))} dy < \infty \right\} \quad (1)$$

for $\alpha \in (0, \infty)$ and

$$U_\infty(B, \psi) = \{ t : \inf_{y \in \Gamma} (y \cdot t + \psi(y)) > -\infty \} \quad (2)$$

for $\alpha = \infty$.

The representation theorem for $A^{p,s}(B, \psi)$, where $1 \leq p \leq 2$ and $0 < s \leq \infty$, is stated as follows.

THEOREM 1. *Assume that $1 \leq p \leq 2$, $0 < s \leq \infty$, then each $F(z) \in A^{p,s}(B, \psi)$ admits an integral representation in the form*

$$F(z) = \int_{\mathbb{R}^n} f(t) e^{2\pi i t \cdot z} dt, \quad z \in T_B, \quad (3)$$

in which, for $p = 1$, $f(t) \in C(\mathbb{R}^n)$ satisfies

$$|f(t)| \left(\int_B e^{-2s\pi(y \cdot t + \psi(y))} dy \right)^{\frac{1}{s}} \leq \|F\|_{A^{1,s}(B, \psi)} \quad (4)$$

and, for $1 < p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, $f(t)$ is a measurable function that satisfies

$$\left(\int_B \left(\int_{\mathbb{R}^n} |f(t) e^{-2\pi(y \cdot t + \psi(y))}|^q dt \right)^{\frac{sp}{q}} dy \right)^{\frac{1}{sp}} \leq \|F\|_{A^{p,s}(B, \psi)}. \quad (5)$$

Moreover, f is supported in $U_s(B, \psi)$ for $p = 1$ and supported in $U_{sp}(B, \psi)$ for $1 < p \leq 2$, $0 < s(p-1) \leq 1$.

As given in the next theorem, integral representations in the form of Laplace transform are also available for $0 < p < 1$ and $0 < s \leq \infty$.

THEOREM 2. Assume that $F(z) \in A^{p,s}(B, \psi)$, where $0 < p < 1$ and $0 < s \leq \infty$. Then there exists a continuous function $f(t)$ such that $f(t)e^{-2\pi y \cdot t} \in L^1(\mathbb{R}^n)$ and (3) hold for $y \in B$.

Considering the property of $f(t)$ for the case of $0 < p < 1$, we let B be a regular open convex cone Γ and let $\psi \in C(\Gamma)$ satisfy

$$R_\psi = \overline{\lim}_{y \in \Gamma, y \rightarrow \infty} \frac{\psi(y)}{|y|} < \infty. \quad (6)$$

Then we obtain the following corollary.

COROLLARY 1. Assume that Γ is a regular open convex cone and $F(z) \in A^{p,s}(\Gamma, \psi)$ for $0 < p < 1$, $0 < s \leq \infty$, where $\psi \in C(\Gamma)$ satisfies (6). Then there exists $f(t)$ supported in $\Gamma^* + \overline{D(0, R_\psi)}$ such that (3) holds and $|f(t)| \left(\int_\Gamma e^{-2s\pi(y \cdot t + R_\psi|y|)} dy \right)^{\frac{1}{s}}$ is slowly increasing.

Similarly, we establish an analogy for $p > 2$ and $0 < s \leq \infty$.

THEOREM 3. Assume that $p > 2$, $0 < s \leq \infty$, Γ is a regular open convex cone in \mathbb{R}^n and $\psi \in C(\Gamma)$ satisfies (6). If $F(z) \in A^{p,s}(\Gamma, \psi)$ satisfying

$$\lim_{y \in \Gamma, y \rightarrow 0} \int_{\mathbb{R}^n} |F(x + iy)|^2 dx < \infty, \quad (7)$$

then there exists $f(t) \in L^2(\mathbb{R}^n)$ supported in $U_{sp}(\Gamma, \psi)$ such that (3) holds for all $z \in T_\Gamma$.

The definition of $A^{p,s}(B, \psi)$ shows that $A^{p,s}(B, \psi)$ is a weighted Hardy space when $s = \infty$ and a weighted Bergman space when $s = 1$. Taking $\psi(y) = 0$, it becomes, for $s = \infty$ and $s = 1$, respectively, the classical Hardy space H^p and the classical Bergman space A^p . Therefore, our results herein can be regarded as generalizations of certain previously obtained results.

For example, taking $s = \infty$ and B a regular open convex cone Γ , $A^{p,\infty}(B, \psi) = H^p(\Gamma, \psi)$ is the weighted Hardy spaces investigated in our previous paper [15]. Then Theorem 1, 2 and 3 in [15] can be derived from our main work, including Theorem 1, 2, 3 and Corollary 1 herein. For $s = \infty$ and $\psi(y) = 0$, letting B be some specific domains, some previous studies for the Hardy spaces, see [1, 13, 14, 12, 9], can be also derived from Theorem 1, 2, 3 and Corollary 1.

On the other hand, letting $s = 1$, by using Theorem 1, 2, 3 and Corollary 1, we can obtain the representation theorems for the standard Bergman spaces. Note that for $s = 1$, $B = \Gamma$ and $\psi(y) = 0$, we have $A^{p,s}(B, \psi) = A^p(T_\Gamma)$. We therefore conclude from Theorem 1 that the counterpart results of Theorem 1, 2 and 3 in [5] hold for the classical Bergman spaces

$A^p(T_\Gamma)(1 \leq p \leq 2)$. If we set $\psi(y) = 0$ and $s = q - 1$, where $\frac{1}{p} + \frac{1}{q} = 1$, then $A^{p,s}(B, \psi) = B^p(T_B)$. The integral representation theorems for those function spaces $B^p(T_B)(1 \leq p \leq 2)$ can be derived from Theorem 1 herein, see [4]. Especially, letting $s = 1$, $p = 2$, $\psi(y) = -\frac{\alpha}{4\pi} \log |y|$ and B a regular open convex cone Γ , Theorem 1 implies a higher dimensional generalization of Theorem 1 of [10] in tube domains, which is established as Corollary 2 in the sequel.

3 Proofs

This section is devoted to proving the results stated in §2.

Proof of Theorem 1. We first prove the case of $p = 1$. If $F(z) \in A^{1,s}(B, \psi)$, then $F_y(x) \in L^1(\mathbb{R}^n)$ as a function of x , and $\check{F}_y(x)$ as well, are both well defined. Next we prove that $\check{F}_y(t)e^{-2\pi y \cdot t}$ is independent of $y \in B$. Without loss of generality, assume that $a = (a', a_n)$, $b = (a', b_n) \in B$, and $a + \tau(b - a) \in B$ for $0 \leq \tau \leq 1$, where $a' = (a_1, \dots, a_{n-1})$. The fact $F_y(x) \in L^1(\mathbb{R}^n)$ implies that

$$\int_0^\infty \int_0^1 \int_{\mathbb{R}^{n-1}} (|F((x', x_n) + i(a + \tau(b - a)))| + |F((x', -x_n) + i(a + \tau(b - a)))|) dx' d\tau dx_n < \infty,$$

which implies

$$\lim_{R \rightarrow \infty} \int_0^1 \int_{\mathbb{R}^{n-1}} (|F((x', R) + i(a + \tau(b - a)))| + |F((x', -R) + i(a + \tau(b - a)))|) dx' d\tau = 0.$$

Hence, we have

$$\begin{aligned} & |\check{F}_b(t)e^{-2\pi b \cdot t} - \check{F}_a(t)e^{-2\pi a \cdot t}| \\ &= \left| \int_{\mathbb{R}^n} (F(x + ib)e^{2\pi i(x+ib) \cdot t} - F(x + ia)e^{2\pi i(x+ia) \cdot t}) dx \right| \\ &= \left| \int_{\mathbb{R}^n} \int_0^1 \frac{\partial}{\partial \tau} (F(x + i(a + \tau(b - a)))e^{2\pi i(x+i(a+\tau(b-a))) \cdot t}) d\tau dx \right| \\ &= \left| \int_{\mathbb{R}^n} \int_0^1 \frac{\partial}{\partial y_n} \left(F(x + i((y', y_n))e^{2\pi i(x+i(y', y_n)) \cdot t} \Big|_{y_n=a_n+\tau(b_n-a_n)}(b_n - a_n) \right) d\tau dx \right| \\ &= |b_n - a_n| \left| \int_{\mathbb{R}^n} \int_0^1 i \frac{\partial}{\partial x_n} (F(x + i(a + \tau(b - a)))e^{2\pi i(x+i(a+\tau(b-a))) \cdot t}) d\tau dx \right| \\ &\leq |b_n - a_n| \lim_{R \rightarrow \infty} \int_0^1 \int_{\mathbb{R}^{n-1}} (|F((x', R), (a + \tau(b - a)))| + |F((x', -R), (a + \tau(b - a)))|) \\ &\quad e^{-2\pi |t|(|a|+|b-a|)} dx' d\tau \\ &= 0. \end{aligned}$$

Remark that B is connected and open, by an iteration argument, we can show that $\check{F}_y(t)e^{-2\pi y \cdot t}$ is independent of $y \in B$ and we write it as $g(t)$. Then $g(t) = \check{F}_y(t)e^{-2\pi y \cdot t}$ holds for $y \in B$. Next, we show that $g(t)e^{2\pi y \cdot t} \in L^1(\mathbb{R}^n)$. Let us decompose \mathbb{R}^n into a finite union of non-overlapping polygonal cones, $\Gamma_1, \Gamma_2, \dots, \Gamma_N$ with their very vertexes at the origin, i.e., $\mathbb{R}^n = \bigcup_{k=1}^N \Gamma_k$. Then $\chi_{\Gamma_k}(t)g(t)e^{2\pi y \cdot t} = \chi_{\Gamma_k}(t)\check{F}_{y_k}(t)e^{-2\pi(y_k - y) \cdot t}$. For any $y_0 \in B$, there exists δ such that $\overline{D(y_0, \delta)} \subset B$. Then for any $y \in D(y_0, \frac{\delta}{4})$ and $y_k \in (y_0 + \Gamma_k)$ satisfying $\frac{3\delta}{4} \leq |y_k - y_0| < \delta$, we get $(y_k - y) \cdot t = (y_k - y_0) \cdot t + (y_0 - y) \cdot t$. Since $y_k - y_0, t \in \Gamma_k$, the angle between the segments $O(y_k - y_0)$ and Ot is less than, say $\frac{\pi}{4}$. Then $(y_k - y) \cdot t \geq \frac{|y_k - y_0|}{\sqrt{2}}|t| - |y_0 - y||t| \geq (\frac{3}{4\sqrt{2}} - \frac{1}{4})\delta|t| \geq \frac{1}{4}\delta|t|$. Thus, it follows from Hölder's inequality that

$$\int_{\Gamma_k} |g(t)e^{2\pi y \cdot t}| dt \leq \int_{\Gamma_k} |\check{F}_{y_k}(t)e^{-\pi \frac{\delta}{4}|t|}| dt \leq \|F_{y_k}(x)\|_{L^1(\mathbb{R}^n)} \int_{\Gamma_k} e^{-\pi \frac{\delta}{4}|t|} dt < \infty,$$

which shows that $g(t)e^{2\pi y \cdot t} \in L^1(\Gamma_k)$. Hence $g(t)e^{2\pi y \cdot t} \in L^1(\mathbb{R}^n)$. Together with the relation $g(t) = \check{F}_y(t)e^{-2\pi y \cdot t}$ for $y \in B$, there holds $F(z) = \int_{\mathbb{R}^n} g(t)e^{-2\pi i z \cdot t}$ for all $y \in B$. By letting $f(t) = g(-t)$, we then obtain the desired formula (3) for $p = 1$ and $z \in T_B$.

Thus, $f(t)e^{-2\pi y \cdot t} \in L^1(\Gamma_k)$ implies that

$$\begin{aligned} \sup_{t \in \mathbb{R}^n} |f(t)|e^{-2\pi y \cdot t} &\leq \int_{\mathbb{R}^n} |F(x + iy)| dx \\ |f(t)|e^{-2\pi y \cdot t}e^{-2\pi \psi(y)} &\leq \int_{\mathbb{R}^n} |F(x + iy)|e^{-2\pi \psi(y)} dx \\ |f(t)|^s \int_B e^{-2s\pi(y \cdot t + \psi(y))} dy &\leq \int_B \left(\int_{\mathbb{R}^n} |F(x + iy)|e^{-2\pi \psi(y)} dx \right)^s dy \\ &= \|F\|_{A^{1,s}(B, \psi)}^s, \end{aligned} \tag{8}$$

which implies (4). Next we prove $\text{supp } f \subset U_s(B, \psi)$. Suppose that $t_0 \notin U_s(B, \psi)$, then (1) implies $\int_B e^{-2s\pi(y \cdot t_0 + \psi(y))} dy = +\infty$ for $y \in B$. It then follows from (8) that $f(t) = 0$, which means the support of f , i.e., $\text{supp } f \subset U_s(B, \psi)$.

Next we prove the case $1 < p \leq 2$. Let $B_0 \subseteq B$ be a bounded connected open set, so there exists a positive constant R_0 such that $B_0 \subseteq D(0, R_0)$. Assume that $l_\varepsilon(z) = (1 + \varepsilon(z_1^2 + \dots + z_n^2))^N$, where N is an integer satisfying $N > n$. Then for $\varepsilon \leq \frac{1}{2R_0^2}$, $z = x + iy$ with $|y| \leq R_0$,

$$\begin{aligned} |l_\varepsilon(z)| &= |((1 + \varepsilon(z_1^2 + \dots + z_n^2))^2)^{\frac{N}{2}}| \\ &= \left((1 + \varepsilon(|x|^2 - |y|^2))^2 + 4\varepsilon^2(x \cdot y)^2 \right)^{\frac{N}{2}} \\ &\geq (1 + \varepsilon(|x|^2 - |y|^2))^N \geq \left(\frac{1}{2} + \varepsilon|x|^2 \right)^N \end{aligned}$$

for $|y| \leq R_0$, i.e., $|l_\varepsilon^{-1}(z)| \leq \frac{1}{(\frac{1}{2} + \varepsilon|x|^2)^N}$. For $F_y(x) = F(x + iy)$, set $F_{\varepsilon,y}(x) = F_y(x)l_\varepsilon^{-1}(z)$, then based on Hölder's inequality,

$$\int_{\mathbb{R}^n} |F_{\varepsilon,y}(x)| dx \leq \left(\int_{\mathbb{R}^n} |F_y(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |l_\varepsilon^{-1}(x + iy)|^q dx \right)^{\frac{1}{q}} < \infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$, which implies that $F_{\varepsilon,y}(x) \in L^1(\mathbb{R}^n)$. Then as in the proof for $p = 1$, $g_{\varepsilon,y}(t) = \check{F}_{\varepsilon,y}(t)e^{-2\pi y \cdot t}$ can be also proved to be independent of $y \in B_0$ when $1 < p \leq 2$. Put $g_{\varepsilon,y}(t) = g_\varepsilon(t)$, then $g_\varepsilon(t)e^{2\pi y \cdot t} = \check{F}_{\varepsilon,y}(t) \in L^1(\mathbb{R}^n)$.

On the other hand, it is obvious that $F_{\varepsilon,y}(x) \rightarrow F_y(x)$ pointwise as $\varepsilon \rightarrow 0$. Now we prove that $\check{F}_y(t)e^{-2\pi y \cdot t}$ is also independent of $y \in B_0$. Indeed, for $a, b \in B_0$ and any compact subset $K \subset \mathbb{R}^n$, let $R_1 = \max\{|z| : z \in K\}$,

$$\begin{aligned} & \left(\int_K |\check{F}_a(t)e^{-2\pi a \cdot t} - \check{F}_b(t)e^{-2\pi b \cdot t}|^q dt \right)^{\frac{1}{q}} \\ & \leq \left(\int_K |\check{F}_a(t)e^{-2\pi a \cdot t} - g_\varepsilon(t)|^q dt \right)^{\frac{1}{q}} + \left(\int_K |g_\varepsilon(t) - \check{F}_b(t)e^{-2\pi b \cdot t}|^q dt \right)^{\frac{1}{q}} \\ & = \left(\int_K |\check{F}_a(t)e^{-2\pi a \cdot t} - \check{F}_{\varepsilon,a}(t)e^{-2\pi a \cdot t}|^q dt \right)^{\frac{1}{q}} + \left(\int_K |\check{F}_{\varepsilon,b}(t)e^{-2\pi b \cdot t} - \check{F}_b(t)e^{-2\pi b \cdot t}|^q dt \right)^{\frac{1}{q}} \\ & \leq e^{2\pi R_0 R_1} \left(\left(\int_K |\check{F}_a(t) - \check{F}_{\varepsilon,a}(t)|^q dt \right)^{\frac{1}{q}} + \left(\int_K |\check{F}_{\varepsilon,b}(t) - \check{F}_b(t)|^q dt \right)^{\frac{1}{q}} \right) \\ & \leq e^{2\pi R_0 R_1} \left(\left(\int_{\mathbb{R}^n} |F_a(t) - F_{\varepsilon,a}(t)|^p dt \right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^n} |F_{\varepsilon,b}(t) - F_b(t)|^p dt \right)^{\frac{1}{p}} \right) \\ & \rightarrow 0, \end{aligned}$$

as $\varepsilon \rightarrow 0$. Hence we obtain that $\check{F}_a(t)e^{-2\pi a \cdot t} = \check{F}_b(t)e^{-2\pi b \cdot t}$ almost everywhere on \mathbb{R}^n and write it as $g(t)$. Then we have $g(t) = \check{F}_y(t)e^{-2\pi y \cdot t}$.

Next, we show that $g(t)e^{2\pi y \cdot t} \in L^1(\mathbb{R}^n)$. As in the proof for $p = 1$, let $\mathbb{R}^n = \bigcup_{k=1}^N \Gamma_k$ and $\overline{D(y_0, \delta)} \subset B_0$. Then for any $y \in D(y_0, \frac{\delta}{4})$ and $y_k \in (y_0 + \Gamma_k)$ satisfying $\frac{3\delta}{4} \leq |y_k - y_0| < \delta$, we have

$$(y_k - y) \cdot t \geq \frac{|y_k - y_0|}{\sqrt{2}}|t| - |y_0 - y||t| \geq \left(\frac{3}{4\sqrt{2}} - \frac{1}{4}\right)\delta|t| \geq \frac{1}{4}\delta|t|$$

for $y_k - y_0, t \in \Gamma_k$. Thus, from Hölder's inequality

$$\int_{\Gamma_k} |g(t)e^{2\pi y \cdot t}| dt \leq \int_{\Gamma_k} |\check{F}_{y_k}(t)e^{-\pi \frac{\delta_0}{4}|t|}| dt \leq \left(\int_{\Gamma_k} |\check{F}_{y_k}(t)|^p dt \right)^{\frac{1}{p}} \left(\int_{\Gamma_k} |e^{-q\pi \frac{\delta_0}{4}|t|}| dt \right)^{\frac{1}{q}} < \infty,$$

which shows that $g(t)e^{2\pi y \cdot t} \in L^1(\Gamma_k)$ and the function $G(z)$ defined by

$$G(z) = \int_{\mathbb{R}^n} g(t)e^{-2\pi i(x+iy) \cdot t} dt$$

is holomorphic in the tube domain $T_{D(y_0, \delta)}$.

Now we can prove that, for $y \in B_0$,

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} g_\varepsilon(t) e^{-2\pi i(x+iy) \cdot t} dt = \int_{\mathbb{R}^n} g(t) e^{-2\pi i(x+iy) \cdot t} dt.$$

In fact, if $y \in B_0$,

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} (g_\varepsilon(t) - g(t)) e^{-2\pi i(x+iy) \cdot t} dt \right| \\ & \leq \int_{\mathbb{R}^n} |(\check{F}_{\varepsilon, y}(t) e^{-2\pi y \cdot t} - \check{F}_y(t) e^{-2\pi y \cdot t}) e^{2\pi i z \cdot t}| dt \\ & = \sum_{k=1}^n \int_{\Gamma_k} |(\check{F}_{\varepsilon, y_k}(x) - \check{F}_{y_k}(x)) e^{-2\pi i(y_k - y) \cdot t}| dt \\ & \leq \sum_{k=1}^n \left(\int_{\Gamma_k} |\check{F}_{\varepsilon, y_k}(x) - \check{F}_{y_k}(x)|^q dt \right)^{\frac{1}{q}} \left(\int_{\Gamma_k} e^{-p\pi \frac{\delta_0}{4} |t|} dt \right)^{\frac{1}{p}} \\ & \leq C_{\delta_0} \sum_{k=1}^n \left(\int_{\Gamma_k} |F_{\varepsilon, y_k}(x) - F_{y_k}(x)|^p dt \right)^{\frac{1}{p}} \\ & \rightarrow 0 \end{aligned}$$

when $\varepsilon \rightarrow 0$, where $C_{\delta_0}^p = \int_{\mathbb{R}^n} e^{-p\pi \frac{\delta_0}{4} |t|} dt$. It follows that $\lim_{\varepsilon \rightarrow 0} F_\varepsilon(z) = G(z)$. Combining with $\lim_{\varepsilon \rightarrow 0} F_\varepsilon(z) = F(z)$, we can state $G(z) = F(z)$ for $y \in B_0$. Then there exists a measurable function $g(t)$ such that $F(z) = \int_{\mathbb{R}^n} g(t) e^{-2\pi i z \cdot t} dt$ holds for $y \in B_0$. Since B is connected, we can choose a sequence of bounded connected open set $\{B_k\}$ such that $B_0 \subset B_1 \subset \dots$ and $B = \bigcup_{k=0}^\infty B_k$. Together with the fact that $g(t) = \check{F}_y(t) e^{-2\pi y \cdot t}$ is independent of $y \in B_k$, then $\check{F}_{y_l}(t) e^{-2\pi y_l \cdot t} = \check{F}_{y_j}(t) e^{-2\pi y_j \cdot t} = \check{F}_y(t) e^{-2\pi y \cdot t}$ for $l \neq j$, $y_l \in B_l$, $y_j \in B_j$ and $y \in B_0$. Hence $g(t) e^{2\pi y \cdot t} = \check{F}_y(t)$ holds for $y \in B_k$, $k = 0, 1, 2, \dots$. In other words, $f(z) = \int_{\mathbb{R}^n} g(t) e^{-2\pi i z \cdot t} dt$ holds for all $y \in B$. By letting $f(t) = g(-t)$, we obtain the desired representation $F(z) = \int_{\mathbb{R}^n} f(t) e^{2\pi i z \cdot t} dt$ for $y \in B$ when $1 < p \leq 2$.

For $\frac{1}{p} + \frac{1}{q} = 1$, based on the Hausdorff-Young Inequality,

$$\left(\int_{\mathbb{R}^n} |f(t) e^{-2\pi y \cdot t}|^q dt \right)^{\frac{1}{q}} \leq \left(\int_{\mathbb{R}^n} |F(x + iy)|^p dx \right)^{\frac{1}{p}}, \quad (9)$$

then

$$\left(\left(\int_{\mathbb{R}^n} |f(t) e^{-2\pi y \cdot t}|^q dt \right)^{\frac{2}{q}} e^{-2p\pi\psi(y)} dy \right)^s \leq \left(\int_{\mathbb{R}^n} |F(x + iy) e^{-2\pi\psi(y)}|^p dx \right)^s.$$

Performing integral about $y \in B$ on both sides, we get

$$\int_B \left(\left(\int_{\mathbb{R}^n} |f(t) e^{-2\pi y \cdot t}|^q dt \right)^{\frac{2}{q}} e^{-2p\pi\psi(y)} \right)^s dy \leq \int_B \left(\int_{\mathbb{R}^n} |F(x + iy) e^{-2\pi\psi(y)}|^p dx \right)^s dy$$

and

$$\int_B \left(\left(\int_{\mathbb{R}^n} |f(t) e^{-2\pi y \cdot t}|^q dt \right)^{\frac{p}{q}} e^{-2p\pi\psi(y)} \right)^s dy \leq \|F\|_{A^{p,s}(B,\psi)}^{sp}. \quad (10)$$

As a result, formulas (3) and (5) hold for $1 < p \leq 2$. Now we prove that $\text{supp} f \subset U_{sp}(B, \psi)$ when $0 < s(p-1) \leq 1$. For $0 < s(p-1) \leq 1$, we have $\frac{q}{sp} \geq 1$. Then Minkowski's inequality and (10) imply that

$$\int_{\mathbb{R}^n} |f(t)|^q \left(\int_B e^{-2\pi p s(y \cdot t + \psi(y))} dy \right)^{\frac{q}{ps}} dt \leq \|F\|_{A^{p,s}(B,\psi)}^q < \infty. \quad (11)$$

Consequently, It follows from (11) and (1) that $f(t) = 0$ for almost every $t \notin U_{sp}(B, \psi)$. Therefore, $\text{supp} f \subset U_{sp}(B, \psi)$. \square

In order to prove Theorem 2, we first introduce a lemma.

LEMMA 1. Suppose that $F(z) \in A^{p,s}(B, \psi)$, where $0 < p < \infty$ and $0 < s \leq \infty$, then for $y_0 \in B$ and positive constant δ such that $D_n(y_0, \delta) \subset B$, there exist constants $N > 1$ and $C_{n,N,p,s}$ depending on n, N, p, s such that

$$|F(z)| \leq C_{n,N,p,s} \delta^{-\frac{n}{p}(1+\frac{1}{s})} e^{2\pi\psi_\delta(y_0)}, \quad (12)$$

where $\psi_\delta(y_0) = \max\{\psi(\eta) : |\eta - y_0| \leq \delta\}$.

Proof. For $y_0 \in B$, there exists $\delta > 0$ such that $B_\delta = D(y_0, \delta) \subset B$. Then for $F(z) = F(x + iy) \in A^{p,s}(B, \psi)$, based on the subharmonic properties of $|F(z)|^t$, we have

$$|F(z)|^t \leq \frac{1}{\Omega_{2n}\delta^{2n}} \int_{D_{2n}(z,\delta)} |F(\xi + i\eta)|^t d\xi d\eta \leq \frac{1}{\Omega_{2n}\delta^{2n}} \int_{D_n(y_0,\delta)} \left(\int_{D_n(x,\delta)} |F(\xi + i\eta)|^t d\xi \right) d\eta$$

for $y \in B_\delta$, where Ω_k is the volume of k -dimensional unit ball $D_k(0, 1)$ centered at the origin with radius 1, $k = n, 2n$. Let $p_1 = N = \frac{p}{t} > \max\{1, \frac{1}{s}\}$ and $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Hölder's Inequality implies that

$$\begin{aligned} |F(z)|^t &\leq \frac{1}{\Omega_{2n}\delta^{2n}} \int_{D_n(y_0,\delta)} \left(\int_{D_n(x,\delta)} |F(\xi + i\eta)|^p d\xi \right)^{\frac{1}{p_1}} d\eta \left(\int_{D_n(x,\delta)} 1^{q_1} d\xi \right)^{\frac{1}{q_1}} \\ &= \frac{(\delta^n \Omega_n)^{\frac{1}{q_1}}}{\delta^{2n} \Omega_{2n}} \int_{D_n(y_0,\delta)} \left(\int_{D_n(x,\delta)} |F(\xi + i\eta)|^p d\xi \right)^{\frac{1}{p_1}} d\eta. \end{aligned}$$

For $0 < s < \infty$, let $p_2 = sN$. Then $p_2 > 1$. Again, by Hölder's Inequality, for $\frac{1}{p_2} + \frac{1}{q_2} = 1$,

$$\begin{aligned}
|F(z)|^t &\leq \frac{(\delta^n \Omega_n)^{\frac{1}{q_1}}}{\delta^{2n} \Omega_{2n}} \left(\int_{D_n(y_0, \delta)} \left(\int_{D_n(x, \delta)} |F(\xi + i\eta)|^p d\xi \right)^s d\eta \right)^{\frac{1}{p_2}} \left(\int_{D_n(y_0, \delta)} 1^{q_2} d\eta \right)^{\frac{1}{q_2}} \\
&\leq \frac{(\delta^n \Omega_n)^{\frac{1}{q_1} + \frac{1}{q_2}}}{\delta^{2n} \Omega_{2n}} \left(\int_{D_n(y_0, \delta)} \left(\int_{D_n(x, \delta)} |F(\xi + i\eta) e^{-2\pi\psi(\eta)}|^p d\xi \right)^s e^{2sp\pi\psi(\eta)} d\eta \right)^{\frac{1}{p_2}} \\
&\leq \frac{(\delta^n \Omega_n)^{2 - \frac{1}{N}(1 + \frac{1}{s})} e^{\frac{2sp}{p_2} \pi\psi_\delta(y_0)}}{\delta^{2n} \Omega_{2n}} \left(\int_{D_n(y_0, \delta)} \left(\int_{D_n(x, \delta)} |F(\xi + i\eta) e^{-2\pi\psi(\eta)}|^p d\xi \right)^s d\eta \right)^{\frac{1}{p_2}} \\
&\leq \frac{(\delta^n \Omega_n)^{2 - \frac{1}{N}(1 + \frac{1}{s})} e^{\frac{2sp}{p_2} \pi\psi_\delta(y_0)}}{\delta^{2n} \Omega_{2n}} \left(\int_B \left(\int_{\mathbb{R}^n} |F(\xi + i\eta) e^{-2\pi\psi(\eta)}|^p d\xi \right)^s d\eta \right)^{\frac{1}{p_2}},
\end{aligned}$$

where $\psi_\delta(y_0) = \max\{\psi(\eta) : |\eta - y_0| \leq \delta\}$. Hence,

$$\begin{aligned}
|F(z)| &\leq \left(\frac{\delta^{-\frac{n}{N}(1 + \frac{1}{s})} \Omega_n^{2 - \frac{1}{N}(1 + \frac{1}{s})} e^{\frac{2sp}{p_2} \pi\psi_\delta(y_0)}}{\Omega_{2n}} \right)^{\frac{1}{t}} \left(\int_B \left(\int_{\mathbb{R}^n} |F(\xi + i\eta) e^{-2\pi\psi(\eta)}|^p d\xi \right)^s d\eta \right)^{\frac{1}{tp_2}} \\
&\leq \frac{\Omega_n^{\frac{2N}{p} - \frac{1}{p}(1 + \frac{1}{s})}}{\Omega_{2n}^{\frac{N}{p}} \delta^{\frac{n}{p}(1 + \frac{1}{s})}} e^{\frac{2sp}{tp_2} \pi\psi_\delta(y_0)} \left(\int_B \left(\int_{\mathbb{R}^n} |F(\xi + i\eta) e^{-2\pi\psi(\eta)}|^p d\xi \right)^s d\eta \right)^{\frac{1}{sp} \frac{sp}{tp_2}}.
\end{aligned}$$

Since $\frac{sp}{tp_2} = 1$, by letting $C_{n,N,p,s} = \frac{\Omega_n^{\frac{2N}{p} - \frac{1}{p}(1 + \frac{1}{s})}}{\Omega_{2n}^{\frac{N}{p}}} \|F(z)\|_{A^{p,s}(B, \psi)}$, we obtain the desired inequality

$$|F(z)| \leq C_{n,N,p,s} \delta^{-\frac{n}{p}(1 + \frac{1}{s})} e^{2\pi\psi_\delta(y_0)}.$$

While $s = \infty$, for $p_2 = sN = \infty$, we have

$$|F(z)|^t \leq \frac{(\delta^n \Omega_n)^{2 - \frac{1}{N}}}{\delta^{2n} \Omega_{2n}} \sup_{\eta \in D_n(y, \delta)} \left| \int_{D_n(x, \delta)} |F(\xi + i\eta)|^p d\xi \right|^{\frac{t}{p}}.$$

Then

$$\begin{aligned}
|F(z)| &\leq \frac{(\delta^n \Omega_n)^{(2 - \frac{1}{N}) \frac{N}{p}}}{(\delta^{2n} \Omega_{2n})^{\frac{N}{p}}} e^{2\pi\psi_\delta(y_0)} \sup_{\eta \in D_n(y, \delta)} \left| \left(\int_{D_n(x, \delta)} |F(\xi + i\eta)|^p d\xi \right)^{\frac{1}{p}} e^{-2\pi\psi(y)} \right| \\
&= \frac{\Omega_n^{\frac{2N}{p} - \frac{1}{p}}}{\Omega_{2n}^{\frac{N}{p}}} \delta^{-\frac{n}{p}} e^{2\pi\psi_\delta(y_0)} \|F(z)\|_{A^{p,\infty}(B, \psi)}.
\end{aligned}$$

Obviously, the inequality (12) is also applicable in the case $s = \infty$. \square

Now we are ready to prove Theorem 2.

Proof of Theorem 2. For $y_0 \in B$, there exists $\delta > 0$ such that $B_\delta = D(y_0, \delta) \subset B$. Then for $F(z) \in A^{p,s}(B, \psi)$ and any $y \in B_\delta$, it follows from Lemma 1 that

$$|F(z)| \leq C_{n,N,p,s} \delta^{-\frac{n}{p}(1 + \frac{1}{s})} e^{2\pi\psi_\delta(y_0)}.$$

Thus,

$$\int_{\mathbb{R}^n} |F(z)|^2 dx = \int_{\mathbb{R}^n} |F(z)|^{p+2-p} dx \leq C_{n,N,p,s}^{2-p} \delta^{-\frac{n(2-p)}{p}(1+\frac{1}{s})} e^{2(2-p)\pi\psi_\delta(y_0)} \int_{\mathbb{R}^n} |F(z)|^p dx.$$

Therefore,

$$\begin{aligned} & \int_{\mathbb{R}^n} |F(z)|^2 e^{-4\pi\psi_\delta(y_0)} dx \\ & \leq C_{n,N,p,s}^{2-p} \delta^{-\frac{n(2-p)}{p}(1+\frac{1}{s})} e^{2(2-p)\pi\psi_\delta(y_0)} \int_{\mathbb{R}^n} |F(z) e^{-2\pi\psi(y)}|^p dx e^{2p\pi\psi(y)} e^{-4\pi\psi_\delta(y_0)} \\ & \leq C_{n,N,p,s}^{2-p} \delta^{-\frac{n(2-p)}{p}(1+\frac{1}{s})} e^{2(2-p)\pi\psi_\delta(y_0)} \int_{\mathbb{R}^n} |F(z) e^{-2\pi\psi(y)}|^p dx e^{2(p-2)\pi\psi_\delta(y_0)} \\ & = C_{n,N,p,s}^{2-p} \delta^{-\frac{n(2-p)}{p}(1+\frac{1}{s})} \int_{\mathbb{R}^n} |F(z) e^{-2\pi\psi(y)}|^p dx. \end{aligned}$$

Taking integral with respect to y to both sides of the inequality, we have

$$\int_{B_\delta} \left(\int_{\mathbb{R}^n} |F(z)|^2 e^{-4\pi\psi_\delta(y_0)} dx \right)^s dy \leq C_{n,N,p,s}^{(2-p)s} \delta^{-\frac{n(2-p)(1+s)}{p}} \int_{B_\delta} \left(\int_{\mathbb{R}^n} |F(z) e^{-2\pi\psi(y)}|^p dx \right)^s dy,$$

which concludes that $F \in A^{2,s}(B_\delta, \psi_\delta)$. Similarly, we can prove that

$$\int_{B_\delta} \left(\int_{\mathbb{R}^n} |F| e^{-2\pi\psi_\delta(y_0)} dx \right)^s dy \leq C_{n,N,p,s}^{(1-p)s} \delta^{-\frac{n(1-p)(1+s)}{p}} \|F(z)\|_{A^{1,s}(B_\delta, \psi)}^{sp}. \quad (13)$$

Then $F(z) \in A^{1,s}(B_\delta, \psi_\delta)$.

Following the proof of the case $p = 1$ in Theorem 1, there exists a continuous function $f(t)$ such that $F_y(x) = \int_{\mathbb{R}^n} f(t) e^{2\pi i z \cdot t} dt$ holds for $y \in B_\delta$ and $f(t) = \hat{F}_y(t) e^{2\pi y \cdot t}$ is independent of $y \in B$. Together with the fact that $f(t) e^{-2\pi y \cdot t} \in L^1(\mathbb{R}^n)$ for all $y \in B$, we see that (3) holds for all $y \in B$. This completes the proof of Theorem 2. \square

Before the proof of Corollary 1, we introduce the following lemma.

LEMMA 2. *Assume that Γ is a regular open convex cone of \mathbb{R}^n . Let $\psi \in C(\Gamma)$ satisfy (6), then $U_\alpha(\psi, \Gamma) \subset \Gamma^* + \overline{D(0, R_\psi)}$, where $U_\alpha(\psi, \Gamma)$ is defined by (1) for $0 < \alpha < \infty$ and by (2) for $\alpha = \infty$.*

Proof. For $t_0 \notin \Gamma^* + \overline{D(0, R_\psi)}$, there exist $\varepsilon > 0$ and $\xi \in \Gamma^*$ such that $d(t_0, \Gamma^*) = |\xi - t_0| \geq R_\psi + 3\varepsilon$ and $\xi \cdot (t_0 - \xi) = 0$. Then for any $\tilde{t} \in \Gamma^*$,

$$(\tilde{t} - t_0) \cdot \frac{(\xi - t_0)}{|\xi - t_0|} \geq |\xi - t_0|.$$

Hence $\tilde{t} \cdot (\xi - t_0) = (\tilde{t} - t_0 + t_0 - \xi + \xi) \cdot (\xi - t_0) \geq |\xi - t_0|^2 - |\xi - t_0|^2 = 0$, which means $\xi - t_0 \in \overline{\Gamma}$. For any $\delta > 0$, it follows from (6) that there exists ρ_0 such that $\psi(y) \leq (R_\psi + \delta)|y|$ for $|y| \geq \rho_0$. Let

$e_0 = \frac{\xi - t_0}{|\xi - t_0|} \in \bar{\Gamma} \cap \partial D(0, 1)$, then for any $\varepsilon_1 > 0$, we can find an $e_1 \in \Gamma$ with $|e_1| = 1$ such that $|e_1 - e_0| < \varepsilon_1$, which means there exists a positive constant $\delta_1 < \varepsilon_1$ such that $D(e_1, \delta_1) \subset \Gamma$. Thus, for any $e \in D(e_1, \delta_1)$ with $|e_1| = 1$, we have $|e - e_0| \leq |e - e_1| + |e_1 - e_0| < 2\varepsilon_1$. Choose ε_1 satisfying $2\varepsilon_1|t_0| \leq \varepsilon$ and let $\Gamma_1 = \{y = \rho e : \rho > 0 \text{ and } e \in D(e_1, \delta) \cap \partial D(0, 1)\} \subset \Gamma$. Then for any $y \in \Gamma_1$, $-\rho e \cdot t_0 = \rho(-e + e_0 - e_0) \cdot t_0 \geq \rho(-2\varepsilon_1|t_0| + |\xi - t_0|) \geq \rho(R_\psi + 2\varepsilon)$ and

$$\begin{aligned} \int_{\Gamma} e^{-2\pi\alpha(t_0 \cdot y + \psi(y))} dy &\geq \int_{\Gamma \cap \{|y| \geq \rho_0\}} e^{-2\pi\alpha(t_0 \cdot y + (R_\psi + \delta)|y|)} dy \\ &\geq \int_{\rho_0}^{\infty} \rho^{n-1} d\rho \int_{\partial D(0,1) \cap D(e_1, \delta_1)} e^{2\pi\alpha\rho(2\varepsilon - \delta)} d\sigma(\zeta) = +\infty, \end{aligned}$$

which implies $t_0 \notin U_\alpha(\psi, \Gamma)$. Therefore, $U_\alpha(\psi, \Gamma) \subset \Gamma^* + \overline{D(0, R_\psi)}$. \square

Now we prove Corollary 1.

Proof of Corollary 1. For $y_0 \in \Gamma$, there exists δ such that $D(y_0, \delta) \subset \Gamma$. It follows from Theorem 2 that there exists $f(t)$ such that (3) holds for $y \in D(y_0, \delta)$. Since Γ is connected, (3) also holds for all $y \in \Gamma$. Applying the methods in the proof of Theorem 1 for $p = 1$, we obtain that such an $f(t)$ is supported in $U_s(\Gamma, \psi_\delta)$. Combing with Lemma 2, we have $\text{supp} f \subset U_s(\Gamma, \psi_\delta) \subset \Gamma^* + \overline{D(0, R_{\psi_\delta})}$, where

$$R_{\psi_\delta} = \overline{\lim_{y \in \Gamma, y \rightarrow \infty}} \frac{\psi_\delta(y)}{|y|}.$$

Since $R_{\psi_\delta} = R_\psi$ for any $y \in \Gamma$, we see that $U_s(\Gamma, \psi_\delta)$ is also a subset of $\Gamma^* + \overline{D(0, R_\psi)}$. Hence, $\text{supp} f \subset \Gamma^* + \overline{D(0, R_\psi)}$.

Now we show that $|f(t)| \left(\int_{\Gamma} e^{-2s\pi(y \cdot t + R_\psi|y|)} dy \right)^{\frac{1}{s}}$ is slowly increasing. For $y_0, y \in \Gamma$, $y_0 + y \in \Gamma$, $F_{y_0}(z) = F(x + i(y + y_0)) \in A^{p,s}(\Gamma, \psi)$. As in Theorem 1, we have $f(t) = g(-t) = \check{F}_{y_0+y}(-t)e^{2\pi(y_0+y) \cdot t}$. Due to the relation $R_\psi = \overline{\lim_{y \in B, y \rightarrow \infty}} \frac{\psi(y)}{|y|}$, we have $\psi_\delta(y) \leq R_\psi(1 + |y_0| + |y|)$, where R_ψ is a positive constant independent of $y_0, y \in \Gamma$. Then

$$\begin{aligned} |f(t)| &= |\check{F}_{y_0+y}(-t)e^{2\pi(y_0+y) \cdot t}| = \left| \int_{\mathbb{R}^n} F_{y_0+y}(x) e^{-2\pi i x \cdot t} e^{-2\pi\psi_\delta(y)} dx \right| e^{2\pi(\psi_\delta(y) + (y_0+y) \cdot t)} \\ &\leq \int_{\mathbb{R}^n} |F_{y_0}(z)| e^{-2\pi\psi_\delta(y)} dx e^{2\pi(R_\psi(1+|y_0|+|y|) + (y_0+y) \cdot t)}. \end{aligned}$$

Combining with (13), it follows that

$$\begin{aligned} \left(\int_{\Gamma} |f(t)|^s e^{-2s\pi(y \cdot t + R_\psi|y|)} dy \right)^{\frac{1}{s}} &\leq \left(\int_{\Gamma} \left(\int_{\mathbb{R}^n} |F_{y_0}(z)| e^{-2\pi\psi_\delta(y)} dx \right)^s dy \right)^{\frac{1}{s}} e^{2\pi(R_\psi(1+|y_0|) + y_0 \cdot t)} \\ &\leq C_{n,N,p,s}^{1-p} \delta^{-\frac{n(1-p)(1+s)}{sp}} \|F_{y_0}\|_{A^{1,s}(B,\psi)}^p e^{2\pi(R_\psi(1+|y_0|) + y_0 \cdot t)} \\ &= C \exp\{J(y_0, t)\}, \end{aligned}$$

where $C = C_{n,N,p,s}^{1-p} \|F_{y_0}\|_{A^{1,s}(\Gamma,\psi)}^p$ and $J(y_0) = -\frac{n(1-p)(1+s)}{sp} \log \delta + 2\pi(R_\psi(1+|y_0|) + y_0 \cdot t)$. Let $J(t) = \inf\{J(y_0, t) : y_0 \in \Gamma\}$, then

$$|f(t)| \left(\int_{\Gamma} e^{-2s\pi(y \cdot t + R_\psi|y|)} dy \right)^{\frac{1}{s}} \leq C \exp\{J(t)\}.$$

Take $y_0 = \rho v$ with $\rho > 0$ and a fixed $v \in \Gamma$ with $|v| = 1$, then $\delta = d(\rho v, \partial\Gamma)/2 = \rho\varepsilon$, where $\varepsilon = d(v, \partial\Gamma)/2$. Therefore,

$$J(t) = \inf_{\rho>0} \left\{ -\frac{n(1-p)(1+s)}{sp} \log(\varepsilon\rho) + 2\pi R_\psi(1+\rho) + 2\pi\rho|t| \right\},$$

in which the infimum can be attained when $\rho = \frac{n(1-p)(1+s)}{2sp\pi(R_\psi+|t|)}$. It follows that

$$J(t) \leq 2\pi R_\psi + n \left(\frac{1}{p} - 1 \right) \left(\frac{1}{s} + 1 \right) \left(1 - \log \varepsilon - \log n \left(\frac{1}{p} - 1 \right) \left(\frac{1}{s} + 1 \right) + \log 2\pi(R_\psi + |t|) \right).$$

Thus, there exists a positive constant $M_{n,p,s,v}$ such that

$$|f(t)| \left(\int_{\Gamma} e^{-2s\pi(y \cdot t + R_\psi|y|)} dy \right)^{\frac{1}{s}} \leq C e^{J(t)} \leq M_{n,p,s,v} (1+|t|)^{n(\frac{1}{p}-1)(\frac{1}{s}+1)}.$$

The proof is complete. \square

Proof of Theorem 3. We first prove the case when $2 < p < \infty$. Since Γ is a regular open convex cone, $\text{int}\Gamma \neq \emptyset$, where $\text{int}\Gamma$ is denoted as the interior of Γ . Then for $y \in \Gamma$, we can find a basis $\{e_j\} \subset \text{int}\Gamma^*$ such that $y = \sum_{j=1}^n e_j y_j$ and $e_j \cdot y \geq 0$. For $\varepsilon > 0$, let $l_\varepsilon(z) = \left(\prod_{j=1}^n (1 - i\varepsilon e_j \cdot z) \right)^{2N}$ with $N > \frac{n}{2} \left(1 - \frac{1}{p} \right)$ and choose two positive constant A, B such that $B|x|^2 \leq \varepsilon^2 \sum_{j=1}^n (e_j \cdot x)^2 \leq A|x|^2$ for all $x \in \mathbb{R}^n$. Thus,

$$\begin{aligned} |l_\varepsilon(z)| &= \left(\prod_{j=1}^n |1 - i\varepsilon e_j \cdot z|^2 \right)^N = \left(\prod_{j=1}^n ((1 + \varepsilon e_j \cdot y)^2 + \varepsilon^2 (e_1 \cdot x)^2) \right)^N \\ &\geq \left(\prod_{j=1}^n (1 + \varepsilon^2 (e_j \cdot x)^2) \right)^N \geq \left(1 + \varepsilon^2 \sum_{j=1}^n (e_j \cdot x)^2 \right)^N \geq (1 + \varepsilon^2 B|x|^2)^N, \end{aligned}$$

i.e., $|l_\varepsilon^{-1}(z)| \leq (1 + \varepsilon^2 B|x|^2)^{-N}$. For $F(x + iy) \in A^{p,s}(\Gamma, \psi)$, $F_y(x) = F(x + iy) \in L^p(\mathbb{R}^n)$ as a function of x . Let $F_\varepsilon(z) = F_{\varepsilon,y}(x) = F_y(x) l_\varepsilon^{-1}(z)$, then $F_{\varepsilon,y}(x) \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Indeed, Hölder's inequality implies that

$$\int_{\mathbb{R}^n} |F_{\varepsilon,y}(x)| dx \leq \left(\int_{\mathbb{R}^n} |F_y(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |l_\varepsilon^{-1}(x + iy)|^q dx \right)^{\frac{1}{q}} \leq C_{1,\varepsilon} \|F_y\|_{L^p(\mathbb{R}^n)} \quad (14)$$

and

$$\int_{\mathbb{R}^n} |F_{\varepsilon,y}(x)|^2 dx \leq \left(\int_{\mathbb{R}^n} |F_y(x)|^{\frac{p}{2}} dx \right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^n} |l_{\varepsilon}^{-1}(x+iy)|^{\frac{p}{p-2}} dx \right)^{\frac{p-2}{p}} \leq C_{2,\varepsilon} \|F_y\|_{L^p(\mathbb{R}^n)},$$

$$\text{where } C_{1,\varepsilon} = \left(\int_{\mathbb{R}^n} \frac{dx}{(1+\varepsilon^2 B|x|^2)^{qN}} \right)^{\frac{1}{q}} < \infty, \quad C_{2,\varepsilon} = \left(\int_{\mathbb{R}^n} \frac{dx}{(1+\varepsilon^2 B|x|^2)^{\frac{p}{p-2}N}} \right)^{\frac{p-2}{p}} < \infty.$$

As the proof of $p = 1$ in Theorem 1, we can show $g_{\varepsilon}(t)e^{2\pi y \cdot t} = \check{F}_{\varepsilon,y}(t) \in L^1(\mathbb{R}^n)$. Thus,

$$g_{\varepsilon}(t)e^{2\pi y \cdot t} = \int_{\mathbb{R}^n} F_{\varepsilon,y}(x)e^{2\pi i x \cdot t} dx, \quad (15)$$

then $|g_{\varepsilon}(t)|e^{2\pi y \cdot t} \leq \int_{\mathbb{R}^n} |F_{\varepsilon,y}(x)| dx$. Together with (14), there hold

$$\begin{aligned} |g_{\varepsilon}(t)|e^{2\pi(y \cdot t - \psi(y))} &\leq C_{1,\varepsilon} \left(\int_{\mathbb{R}^n} |F(x+iy)e^{-2\pi\psi(y)}|^p dx \right)^{\frac{1}{p}}, \\ \int_{\Gamma} |g_{\varepsilon}(t)|^{sp} e^{2sp\pi(y \cdot t - \psi(y))} dy &\leq C_{1,\varepsilon} \int_{\Gamma} \left(\int_{\mathbb{R}^n} |F(x+iy)e^{-2\pi\psi(y)}|^p dx \right)^s dy, \\ |g_{\varepsilon}(t)|^{sp} \int_{\Gamma} e^{2sp\pi(y \cdot t - \psi(y))} dy &\leq C_{1,\varepsilon} \|F\|_{A^{p,s}(\Gamma,\psi)}^{sp}. \end{aligned}$$

Now we prove that $\text{supp} g_{\varepsilon}(t) \subset -U_{ps}(\Gamma, \psi)$. Note that $g_{\varepsilon}(t)$ is continuous in \mathbb{R}^n . Then for $t_0 \notin -U_{ps}(\Gamma, \psi)$, formula (1) shows that $\int_{\Gamma} e^{2ps\pi(y \cdot t_0 - \psi(y))} dy = \infty$ for $y \in \Gamma$. It follows from the above inequality that $g_{\varepsilon}(t_0) = 0$ for $t_0 \notin -U_{ps}(\Gamma, \psi)$. As a result, $\text{supp} g_{\varepsilon}(t) \subset -U_{ps}(\Gamma, \psi)$.

Since $g_{\varepsilon}(t)e^{2\pi y \cdot t} \in L^1(\mathbb{R}^n)$, we can rewrite (15) as

$$F_{\varepsilon,y}(x) = \int_{\mathbb{R}^n} g_{\varepsilon}(t)e^{-2\pi i z \cdot t} \chi_{-U_{ps}(\Gamma,\psi)}(t) dt. \quad (16)$$

Plancherel's Theorem implies that $\int_{\mathbb{R}^n} |g_{\varepsilon}(t)e^{2\pi y \cdot t}|^2 dt = \int_{\mathbb{R}^n} |F_{\varepsilon,y}(x)|^2 dx$. Then based on Fatou's lemma,

$$\int_{\mathbb{R}^n} |g_{\varepsilon}(t)|^2 \leq \liminf_{y \in \Gamma, y \rightarrow 0} \int_{\mathbb{R}^n} |F(x+iy)|^2 dx < \infty.$$

Thus, there exist $g(t) \in L^2(\mathbb{R}^n)$ and a sequence $\{\varepsilon_k\}$ tending to zero as $k \rightarrow \infty$ such that $\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} g_{\varepsilon_k}(t)h(t)dt = \int_{\mathbb{R}^n} g(t)h(t)dt$ for $h(t) \in L^2$. In fact, for $t \in -U_{ps}(\Gamma, \psi)$, lemma 2 implies that $t \in -\Gamma_k^* + \overline{D(0, R_{\psi})}$. Then t can always be written as $t_1 + t_2$ with $t_1 \in -\Gamma_k^*$ and $|t_2| < R_{\psi}$. Hence, for $y \in \Gamma$,

$$y \cdot t = y \cdot (t_1 + t_2) \leq -|t_1|k + |t_2||y| \leq -(|t| - |t_2|)k + R_{\psi}|t| \leq (R_{\psi} - k)|t| + R_{\psi}k,$$

implying that $\int_{\mathbb{R}^n} |e^{2\pi y \cdot t} \chi_{-U_{ps}(B_k, \psi)}(t)|^2 dt < \infty$. Therefore, on the right hand side of (16),

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} g_{\varepsilon_k}(t)e^{-2\pi i z \cdot t} \chi_{-U_{ps}(\Gamma,\psi)}(t) dt = \int_{\mathbb{R}^n} g(t)e^{-2\pi i z \cdot t} \chi_{-U_{ps}(\Gamma,\psi)}(t) dt$$

for $e^{2\pi y \cdot t} \chi_{-U_{ps}(\Gamma, \psi)}(t) \in L^2(\mathbb{R}^n)$. Whilst it is obvious that $F_\varepsilon(z) \rightarrow F(z)$ when $\varepsilon \rightarrow 0$. Sending k to ∞ on both sides of (16) and letting $f(t) = g(-t)$, we obtain that $f \in L^2(\mathbb{R}^n)$ and the support $\text{supp} f$ is contained in $U_{ps}(\Gamma, \psi)$, as well as the desired representation (3) holds for all $z \in T_\Gamma$.

We now prove the case when $p = \infty$. For $z = (z_1, \dots, z_n) \in T_\Gamma$ and $\varepsilon > 0$, we can also construct a function $F_{\varepsilon, y}(x) = F_\varepsilon(z) = F_y(x) l_\varepsilon^{-1}(z)$, where $l_\varepsilon(z) = \left(\prod_{j=1}^n (1 - i\varepsilon e_j \cdot z) \right)^{2N}$ with $N > \frac{n}{2}$. Then

$$\int_{\mathbb{R}^n} |F_{\varepsilon, y}(x)| dx \leq \sup_{x \in \mathbb{R}^n} |F_y(x)| \int_{\mathbb{R}^n} |l_\varepsilon^{-1}(x + iy)| dx \leq \tilde{C}_{1, \varepsilon} \|F_y\|_{L^\infty(\mathbb{R}^n)} < \infty \quad (17)$$

and

$$\int_{\mathbb{R}^n} |F_{\varepsilon, y}(x)|^2 dx \leq \sup_{x \in \mathbb{R}^n} |F_y(x)| \int_{\mathbb{R}^n} |l_\varepsilon^{-1}(x + iy)|^2 dx \leq \tilde{C}_{2, \varepsilon} \|F_y\|_{L^\infty(\mathbb{R}^n)} < \infty,$$

where $\tilde{C}_{1, \varepsilon} = \int_{\mathbb{R}^n} \frac{dx}{(1 + \varepsilon^2 B|x|^2)^N}$ and $\tilde{C}_{2, \varepsilon} = \left(\int_{\mathbb{R}^n} \frac{dx}{(1 + \varepsilon^2 B|x|^2)^{2N}} \right)^{\frac{1}{2}} < \infty$. Hence $F_{\varepsilon, y} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. In this case, we also have $g_\varepsilon(t) e^{2\pi y \cdot t} = \tilde{F}_{\varepsilon, y}(t) \in L^1(\mathbb{R}^n)$. Then $g_\varepsilon(t) e^{2\pi y \cdot t} = \int_{\mathbb{R}^n} F_{\varepsilon, y}(x) e^{2\pi i x \cdot t} dx$. Therefore, together with (17),

$$\begin{aligned} |g_\varepsilon(t)| e^{2\pi(y \cdot t - \psi(y))} &\leq \tilde{C}_{1, \varepsilon} \sup_{x \in \mathbb{R}^n} |F_y(x)| e^{-2\pi \psi(y)}, \\ \sup_{y \in \Gamma} |g_\varepsilon(t)| e^{2\pi(y \cdot t - \psi(y))} &\leq \tilde{C}_{1, \varepsilon} \sup_{x \in \mathbb{R}^n, y \in \Gamma} |F(x + iy)| e^{-2\pi \psi(y)} \\ &= \tilde{C}_{1, \varepsilon} \|F\|_{A^{\infty, \infty}(\Gamma, \psi)} < \infty. \end{aligned}$$

Then we can similarly show that $\text{supp} g_\varepsilon(t) \subset -U_\infty(\Gamma, \psi) \subset -\Gamma^* + \overline{D(0, R_\psi)}$. Applying the same method for $2 < p < \infty$, we obtain the desired formula (3) holds for all $z \in T_\Gamma$ and the support $\text{supp} f$ is contained in $U_\infty(\Gamma, \psi) \subset \Gamma^* + \overline{D(0, R_\psi)}$. \square

4 Applications

In [10], denoting by $A_\alpha^2(\mathbb{C}^+)$ a weighted Bergman space of functions holomorphic in \mathbb{C}^+ satisfying $\|F\|_{A_\alpha^2(\mathbb{C}^+)}^2 = \int_{\mathbb{C}^+} |F(x + iy)|^2 y^\alpha dx dy < \infty$, and by $L_\beta^2(\mathbb{R}^+)$ the space of complex-valued measurable functions f on \mathbb{R}^+ satisfying $\|f\|_{L_\beta^2(\mathbb{R}^+)}^2 = \frac{\Gamma(\beta)}{(4\pi)^\beta} \int_0^\infty |f(t)|^2 t^{-\beta} dt < \infty$, Duren stated an analogy of the Paley–Wiener theorem for Bergman space.

Theorem A ([10]) For each $\alpha > -1$, the space $A_\alpha^2(\mathbb{C}^+)$ is isometrically isomorphic under the Fourier transform to the space $L_{\alpha+1}^2(\mathbb{R}^+)$. More precisely, $F \in A_\alpha^2(\mathbb{C}^+)$ if and only if it is the Fourier transform $F(z) = \int_0^\infty f(t) e^{2\pi i z \cdot t} dt$ of some function $f \in L_{\alpha+1}^2(\mathbb{R}^+)$, in which case $\|F\|_{A_\alpha^2(\mathbb{C}^+)} = \|f\|_{L_{\alpha+1}^2(\mathbb{R}^+)}$.

Based on Theorem 1, letting $s = 1$, $p = 2$, $\psi(y) = -\frac{\alpha}{4\pi} \log |y|$ and B be a regular open convex cone Γ , we establish Corollary 2, which can be regarded as a higher dimensional and tube domain generalization of Theorem A.

COROLLARY 2. *For each $\alpha > -1$, $F \in A_\alpha^2(T_\Gamma)$ if and only if there exists $f(t) \in L_{\alpha+1}^2(\Gamma^*)$ such that*

$$F(z) = \int_{\Gamma^*} f(t) e^{2\pi i z \cdot t} dt$$

holds for $z \in T_\Gamma$ and $\|F\|_{A_\alpha^2(T_\Gamma)} = \|f\|_{L_{\alpha+1}^2(\Gamma^)}$.*

Proof. By restricting the base B to be a regular open convex cone Γ and letting $\psi(y) = \psi_\alpha(y) = -\frac{\alpha}{4\pi} \log |y|$, $F \in A_\alpha^2(T_\Gamma)$ is also an element of $A^{2,1}(\Gamma, \psi_\alpha)$. Applying Theorem 1 to such an F , we can show that there exists $f(t)$ satisfying (5) such that $F(z) = \int_{\mathbb{R}^n} f(t) e^{2\pi i z \cdot t} dt$ and $\text{supp } f \subset U_1(\Gamma, \psi_\alpha)$. Based on (6), we have

$$R_{\psi_\alpha} = \overline{\lim_{y \in \Gamma, y \rightarrow \infty}} \frac{\psi_\alpha(y)}{|y|} = 0.$$

Thus, together with Lemma 2, the supporter of $f(t)$ is contained in Γ^* and $F(z) = \int_{\Gamma^*} f(t) e^{2\pi i z \cdot t} dt$. Moreover, $\int_\Gamma \int_{\Gamma^*} |f(t)|^2 e^{-4\pi(y \cdot t + \psi_\alpha(y))} dt dy \leq \|F\|_{A^{2,1}(\Gamma, \psi_\alpha)}$. Thus,

$$\int_\Gamma \int_{\Gamma^*} |f(t)|^2 e^{-4\pi(y \cdot t + \psi_\alpha(y))} dt dy = \int_{\Gamma^*} \int_\Gamma |f(t)|^2 e^{-4\pi y \cdot t} y^\alpha dy dt = \int_{\Gamma^*} |f(t)|^2 \frac{\Gamma(\alpha)}{(4\pi t)^{\alpha+1}} dt,$$

which shows $f \in L_{\alpha+1}^2(\Gamma^*)$. And Plancherel's Theorem assures that $\|F\|_{A_\alpha^2(T_\Gamma)} = \|f\|_{L_{\alpha+1}^2(\Gamma^*)}$.

Conversely, note that $F(z) = \int_{\Gamma^*} f(t) e^{2\pi i t \cdot z} dt$. For $f(t) \in L_{\alpha+1}^2(\Gamma^*)$, Plancherel's theorem implies that

$$\begin{aligned} \int_{\mathbb{R}^n} |F(x + iy)|^2 dx &= \int_{\Gamma^*} e^{-4\pi y \cdot t} |f(t)|^2 dt, \\ \int_\Gamma \int_{\mathbb{R}^n} |F(x + iy)|^2 e^{-4\pi \psi_\alpha(y)} dx dy &= \int_\Gamma \int_{\Gamma^*} |f(t)|^2 e^{-4\pi(y \cdot t + \psi_\alpha(y))} dt dy < \infty, \end{aligned}$$

in which $\psi_\alpha(y) = -\frac{\alpha}{4\pi} \log |y|$. Hence, $F(z) \in A^{2,1}(\Gamma, \psi_\alpha) = A_\alpha^2(T_\Gamma)$. The proof is complete. \square

By restricting the base B to be a regular open convex cone Γ , we establish the following weighted version of the edge-of-the-wedge theorem (see [2]) as an application of Theorem 1.

THEOREM 4. *Assume that Γ is a regular open convex cone in \mathbb{R}^n , $\psi_1 \in C(\Gamma)$ and $\psi_2 \in C(-\Gamma)$ satisfy*

$$R_{\psi_1} = \overline{\lim_{y \in \Gamma, y \rightarrow \infty}} \frac{\psi_1(y)}{|y|} < \infty \tag{18}$$

and

$$R_{\psi_2} = \overline{\lim}_{y \in \Gamma, y \rightarrow \infty} \frac{\psi_2(-y)}{|y|} < \infty \quad (19)$$

respectively. If $1 < p \leq 2$, $0 < s(p-1) \leq 1$, $F_1 \in A^{p,s}(\Gamma, \psi_1)$ and $F_2 \in A^{p,s}(-\Gamma, \psi_2)$, satisfying

$$\lim_{y \rightarrow 0} \int_{\mathbb{R}^n} |F_1(x+iy) - F_2(x-iy)|^p dx = 0, \quad (20)$$

then F_1 and F_2 can be analytically extended to each other and further form an entire function F . Furthermore, there exists a function $f \in L^1(\mathbb{R}^n)$ supported in a bounded convex set K such that $F(z) = \int_K f(t) e^{2\pi i t \cdot z} dt$.

Proof. Theorem 1 implies that there exists a function $f_j (j = 1, 2)$ such that

$$F_j = \int_{\mathbb{R}^n} f_j(t) e^{2\pi i t \cdot z} dt$$

holds, in which the supporter of f_j is contained in $U_{sp}((-1)^{j+1}\Gamma, \psi_j)$ for $1 < p \leq 2$. Based on lemma 2, $\text{supp} f_j \subset (-1)^{j+1}\Gamma^* + \overline{D(0, R_{\psi_j})}$. By the Hausdorff-Young inequality,

$$\left(\int_{\mathbb{R}^n} |f_1(t) e^{2\pi i y \cdot t} - f_2(t) e^{-2\pi i y \cdot t}|^q dt \right)^{\frac{1}{q}} \leq \left(\int_{\mathbb{R}^n} |F_1(x+iy) - F_2(x-iy)|^p dt \right)^{\frac{1}{p}}.$$

Then it follows from Fatou's lemma and (20) that $\|f_1 - f_2\|_{L^q(\mathbb{R}^n)} = 0$. Thus, $f_1 = f_2$ almost everywhere on \mathbb{R}^n . Let $f_1(t) = f_2(t) = f(t)$, and $R = \max\{R_{\psi_1}, R_{\psi_2}\}$, then $\text{supp} f \subset K \subset (\Gamma^* + \overline{D(0, R)}) \cap (-\Gamma^* + \overline{D(0, R)})$. Thus, K is a bounded convex set. Consequently, $F(z) = \int_K e^{2\pi i z \cdot t} f(t) dt$ is an entire function, where $F(z) = F_1(z)$ for $z \in T_\Gamma$ and $F(z) = F_2(z)$ for $z \in T_{-\Gamma}$. \square

Similarly, we can prove the weighted version of the edge-of-the-wedge theorem for $p > 2$.

THEOREM 5. Suppose that Γ is a regular open convex cone in \mathbb{R}^n , $\psi_1 \in C(\Gamma)$ and $\psi_2 \in C(-\Gamma)$ satisfy (18) and (19) respectively. If $F_1 \in A^{p,s}(\Gamma, \psi_1)$ and $F_2 \in A^{p,s}(-\Gamma, \psi_2)$, where $p > 2$, satisfying

$$\lim_{y \in \Gamma, y \rightarrow 0} \int_{\mathbb{R}^n} |F_1(x+iy) - F_2(x-iy)|^2 dx = 0, \quad (21)$$

then F_1 and F_2 can be analytically extended to each other and further form an entire function F . Furthermore, there exists a measurable function $f(t)$ supported in a bounded convex set K such that $F(z) = \int_K f(t) e^{2\pi i t \cdot z} dt$.

Proof. For $F_j \in A^{p,s}((-1)^{j+1}\Gamma, \psi_j)$ and $\frac{1}{p} + \frac{1}{q} = 1$, exists a measurable function f_j such that $F_j = \int_{\mathbb{R}^n} f_j(t) e^{2\pi i t \cdot z} dt$ and $\text{supp} f_j \subset U_{sp}((-1)^{j+1}\Gamma, \psi_j)$, where $j = 1, 2$. It then follows from

Lemma 2 that $\text{supp} f_j \subset (-1)^{j+1}\Gamma^* + \overline{D(0, R_{\psi_j})}$. Plancherel's Theorem implies that

$$\left(\int_{\mathbb{R}^n} |f_1(t)e^{2\pi y \cdot t} - f_2(t)e^{-2\pi y \cdot t}|^2 dt \right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^n} |F_1(x + iy) - F_2(x - iy)|^2 dx \right)^{\frac{1}{2}}.$$

Then based on (21) and Fatou's Lemma, $\|f_1 - f_2\|_{L^2(\mathbb{R}^n)} = 0$, which means $f_1 = f_2$ almost everywhere on \mathbb{R}^n . Let $f_1(t) = f_2(t) = f(t)$ and $R = \max\{R_{\psi_1}, R_{\psi_2}\}$, then $\text{supp} f(t) \subset K = (\Gamma^* + \overline{D(0, R)}) \cap (-\Gamma^* + \overline{D(0, R)})$. Thus, K is a bounded convex set. As a result, $F(z) = \int_K e^{2\pi iz \cdot t} f(t) dt$ is an entire function, where $F(z) = F_1(z)$ for $z \in T_\Gamma$ and $F(z) = F_2(z)$ for $z \in T_{-\Gamma}$. \square

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